WAVES AND STRUCTURES

By

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(Suggestions to improve/modify contents are welcome)

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CHAPTER 1

INTRODUCTION

1.1 Introduction

The knowledge of magnitude and behavior of ocean waves at site is an essential prerequisite for almost all activities in the ocean including planning, design, construction and operation related to harbor, coastal and structures.

The waves of major concern to a harbor engineer are generated by the action of wind. The wind creates a disturbance in the sea which is restored to its calm equilibrium position by the action of gravity and hence resulting waves are called wind generated gravity waves. The height of a wave is the vertical distance between its crest and trough, the period of a wave is the time required to complete one cycle of its oscillation, while the horizontal spread of one wave (distance between any two points with the same phase) is called wave length. See Fig. 1.1. The height of waves (H), in practice varies from a few cm to over 30 m while the variation in corresponding wave period (T), is from 3 seconds to about 25 seconds. The length of a wave (L) may range from a few meters to over a kilometer.

1.2 Generation

Continuous changes in the temperature over the rotating earth produce corresponding changes in the atmospheric pressure all over it. The wind produced by these pressure gradients blows with different energies at different places and at different times. The relatively higher energies associated with the wind are transferred to calm water by the pressure acting normal to the sea surface as well as by the shear exerted tangential to it.

Many investigators (like Helmholtz, Jeffery, Phillips, Miles, Hasselmann) in the past have attempted to explain the process of wind energy transfer through factors like pressure gradient across the wake (Fig 1.2), resonance of turbulent eddies in the atmosphere (Fig 1.3), shear forces based logarithmic wind profile (Fig 1.4) and resonant interactions between different wave components. However the exact nature of the process of wave generation still eludes the scientists owing to its complexity. The formation and growth of waves is influenced by wind pressure, its speed, fetch (the distance over which the wind, blowing over the sea surface, remains the same) and wind duration (the time over which the storm prevails) together with depth of water at the site.

In the initial stages of wave generation, high frequency and short length waves are formed (Stage 1 of Fig 1.5). These, being unstable, break and supply energies thereby to the lower frequency waves which in turn get developed (Stage 2 of Fig 1.5). The process continues till a 'fully developed sea' is formed (Stage 3 of Fig 1.5) where all wave component reach a saturation stage (Brebbia and Walker, 1979).

As the wave height and period increases from Stage 1, (Fig 1.5) waves start moving faster and faster, and when the increasing wave speed matches the speed of the generating wind the transfer of wind energy ceases and so also the growth of the waves. This process requires availability of certain time duration as well as that of fetch distance. If either time or fetch is less, a 'partially developed sea' is formed. The fully developed sea on the other hand (Stage 3, Fig. 1.5) is associated with unlimited fetch and duration.

1.3 Decay

After generation, waves travel along different directions and their energies get spent due to factors, like, the air or water turbulence, bottom friction, besides spreading over wider areas due to angular dispersion. This gives rise to the decay in the height of waves as they travel out of their area of generation.

Opposition to wave movement lowers its length and increases its height making it steep and unstable. Higher steepness (wave height to wave length ratio) is associated with higher water particle velocities. When such velocities exceed the speed of the wave motion, the water particles come out and the wave breaks. This happens when the steepness becomes as high as 1/7 or when the angle at the crest lowers to 120^0 (Fig 1.6). This is the case in the deep water. In shallow water the waves break when they arrive in a region where the depth is anywhere in between about 0.8 to 1.4 times their height. The exact value of the water depth at breaking depends on the sea bed slope and wave steepness. Section 4.7 gives more details on wave breaking.

1.4 Classification

Depending on the repetition of wave form, the waves can be r*egular*, if the same wave form repeats in time as well as space or i*rregular* or r*andom*, if it does not repeat. (Fig 1.7). Actual waves found in nature are basically random; but for the sake of analytical simplicity they are many times assumed to be regular.

With the wave period (or frequency – number of oscillations per second) as basis the waves can be l*ong period*, g*ravity* or s*hort period* waves where the normal gravity waves correspond to periods ranging from 1 to 30 seconds. They are generated by wind and restored by gravity. Fig 1.8 shows energy content in waves of different periods.

The waves can be generated by *wind, tectonic activities, sun* and *moon's attraction* or *ship movements* while they are restored to their equilibrium position by s*urface tension, gravity* or *Coriolis force*.

As per the shape of their profile the waves can be *Sinusoidal, Trochoidal, Cnoidal, Solitary* and *Random* (Fig 1.7).

If the whole profile moves in the forward direction the wave is a *Progressive Wave*; otherwise, simple up and down oscillations of the water particles at fixed position constitute a *Standing* or a *Clapotis wave* (Fig 1.7).

If the water particles show back and forth movement with open or close orbits wave is an *Oscillatory wave* otherwise it is a *Translatory wave* (Fig 1.7) in which there is no backward motion of particles.

If wave steepness is small (say less than 0.02) the wave is called *Small Amplitude* wave otherwise it is a *Finite Amplitude Wave*.

1.5 Measurement

Wave measurements can be made with different types of recorders kept either at the sea surface or over and below it. The airborne devices include the satellite based sensing of the surface using a radar altimeter. The floating recorders could be either of electrical resistance gauges, ship borne pressure sensors or wave rider buoys. The submerged category involves the pressure gauges and the echo sounders. Out of all above types the wave rider buoy (Fig. 1.9) is most commonly employed in routine wave data collection. It is in the form of a spherical buoy that is kept floating on the sea surface. It undergoes accelerations in accordance with the wave motion. The vertical accelerations are continuously recorded by an accelerometer located inside the buoy. These are further integrated twice electronically to obtain records of the sea surface elevations which in turn are sent to a shore based receiving station. Commonly, a 20-minute record collected once in every 3 hours, as a true statistical sample during the period, is practiced to optimize the data collection.

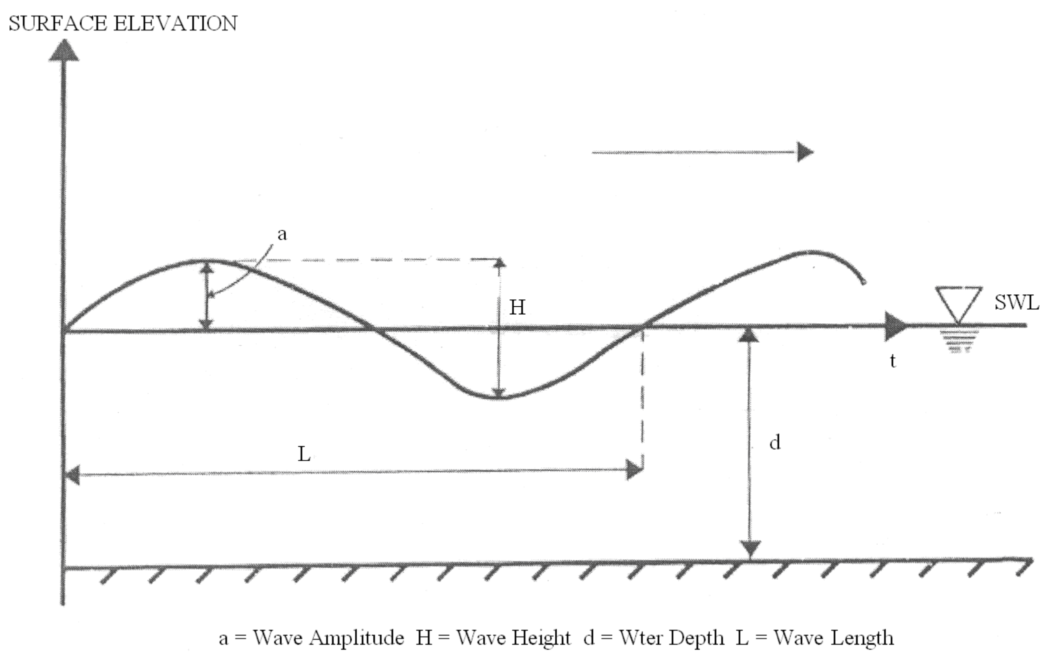
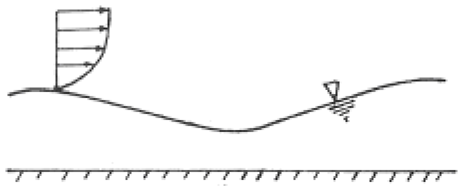


Fig 1.1 Definition Sketch Of a Propagating Wave

Fig 1.2 Flow Separation at Crest

Fig 1.3 Turbulent Eddies



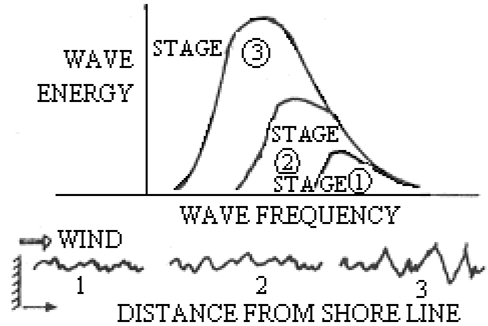


Fig 1.4 Logarithmic Wind Profile

Fig 1.5 Wave Growths

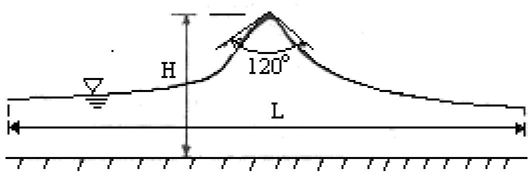


Fig 1.6 Wave Breaking

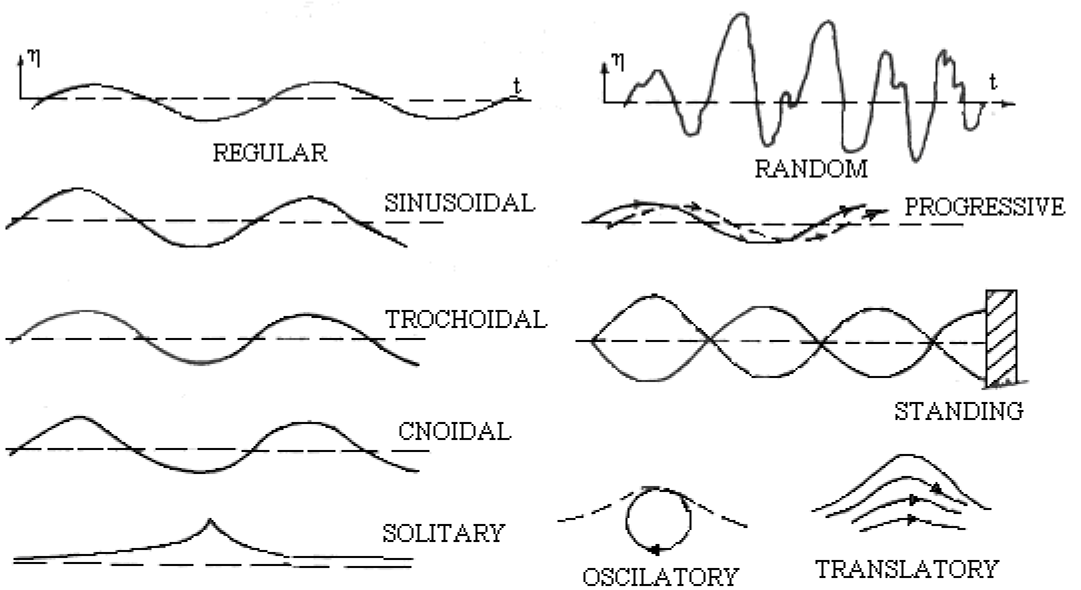


Fig 1.7 Wave Types

Legend:	1. Capillary waves	A. Wind
	2. Ultra-gravity waves	B. Wind + Ordinary grav. waves
	3. Ordinary grav. waves	C. Storm & earthquakes
	4. Infra-grav. waves	D. Sun & Moon
	5. Long period waves	E. Storm + Sun & Moon

6. Ordinary tide waves
7. Trans-tidal waves

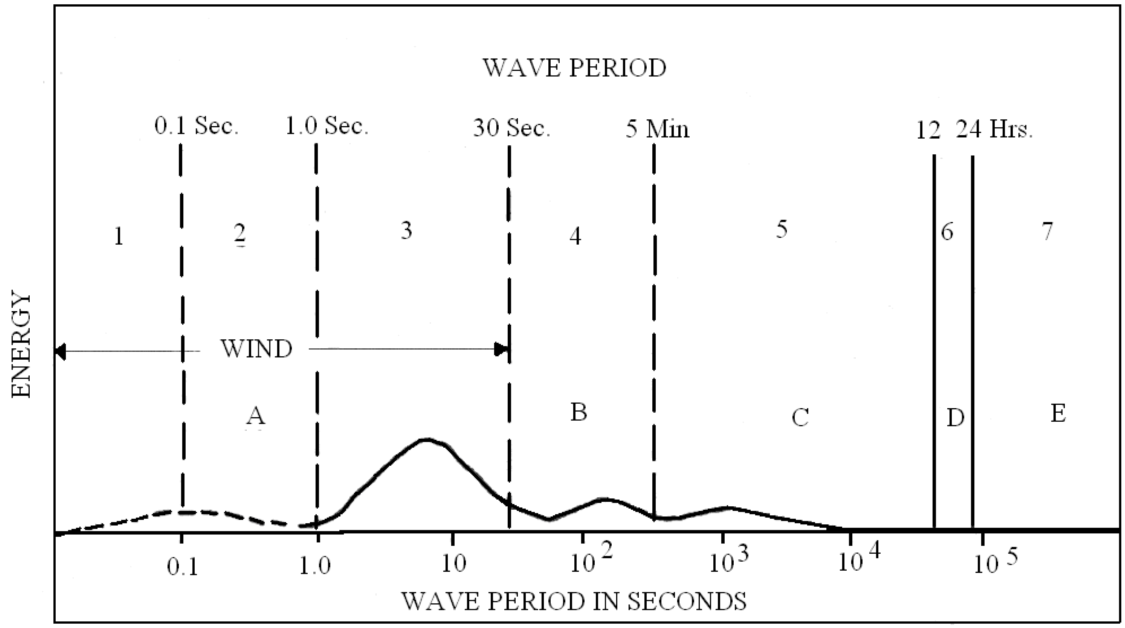


Fig. 1.8 Wave Energy versus Period **(Gaythwaite, 1981)**

Fig 1.9 Wave Rider Buoy **(www.niot.res.in)**

1.6 Wave Forecasting

1.6.1 The Significant Wave

Forecasting of waves for operational or design purpose needs to be made by measuring and analyzing the actual wave observations at a given location. But considering the difficulties and costs involved in getting large scale wave data, many times, the readily available wind information is gathered and then converted into corresponding wave information although this procedure is less accurate than the actual wave analysis.

The wind information required to forecast the waves can be obtained by making direct observations at the specific ocean site or at a nearby land site. The latter observations require projection to the actual location by applying some overland observation corrections. Wind speed and its direction can be observed at regular intervals and hourly wind vectors can be recorded. Alternatively use of synoptic surface weather maps can also be made to extract the wind information. These maps may give Geostrophic or free air speed, which is defined as the one undisturbed by effects of the boundary layer prevalent at the interface of air and sea. Instead of this speed, which may exist at a very large height from the sea surface, the wind prediction formulae incorporate the wind speed value at a standard height of 10 m above the mean sea level (U_{10}) which can be obtained by multiplying the geostrophic speed by a varying correction factor. This value of U_{10} so deduced needs further corrections as below before it can be used as input in the wave prediction formulae.

(i) **Correction for overland observations:** This is necessary when wind is observed overland and not over water in which case the roughness of the sea surface is different. If wind speed overland (U_L) is greater than 1.85 m/sec, i.e., 41.5 mph, the correction factor $R_L = U_{10}/U_L$ may be taken as 0.9. If $U_L \sim 15$ m/sec, $R_L = 1.0$. If $U_L < 15$ m/sec, $R_L = 1.25$.

(ii) **Correction for the difference in air and sea temperature**: This difference affects the boundary layer. The correction factor can be substantial - varying from 1.21 for a temperature difference of -20 degrees to about 0.78 for the temperature difference of +20 degrees.

(iii) **Correction for shortness of observations duration:** Since the wind is observed for a very short duration of say 2 minutes at a time, its stable value over duration of an hour or so is required to be calculated. Empirical curves are available to obtain the corrections (i), (ii) and (iii) above. (SPM, 1984).

(iv) **Correction to account for the non-linear relation between the measured wind speed and its stress on the seawater:** This correction is given by,

$$U_{corrected} = (0.71)\ U_{10}^{\ 1.23} \qquad 1.6.1$$

If the wind speed in a given region does not change by about ± 2.5 m/sec with corresponding direction changes of about ± 15 degrees then such a region can be regarded as fetch region. Its horizontal dimension expressed in distance scale, called Fetch, is required as another input in the wave prediction formulae. For coastal sites the upwind distance along the wind direction would give the required fetch value. Alignment (curvature or spreading) of the isobars in weather maps also yields the wind fetch.

Constant wind duration forms an additional input in the formulae of wave prediction. This is obtained by counting the time after allowing deviations of 5 percent in speed and 15 degrees in the directions.

The problem of wave forecasting aims at arriving at the values of the significant wave height (H_s) and the significant wave period (T_s) from given wind speed, duration and fetch distances over which the speed remains constant.

If we have a collection of pairs of individual wave heights and wave periods (or zero cross periods, meaning thereby that the crests should necessarily cross the mean zeroth line. (Fig 1.10), then an average height of the highest one third of all the waves (like H_1, H_2, H_3...of Fig 1.10) would give the significant height. (H_s) while an average of all wave periods (like T_1, T_2, T_3...of Fig 1.10) would yield significant wave periods (T_s). These definitions are empirical in origin.

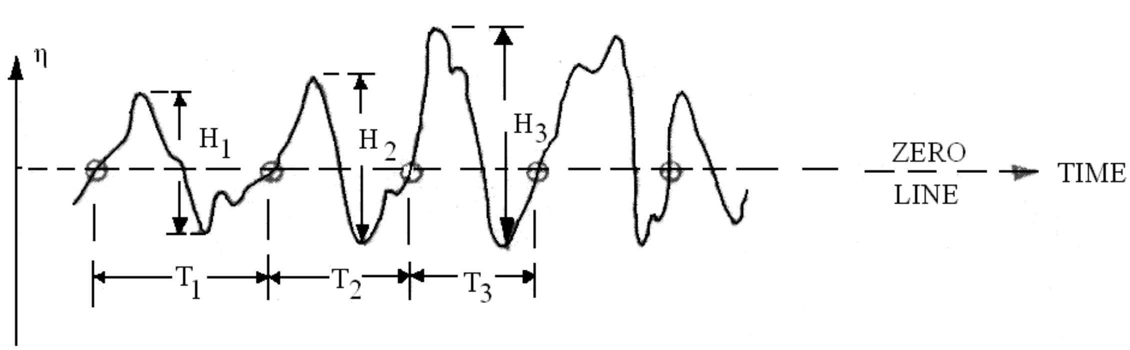


Fig 1.10 Individual Waves

1.6.2 Empirical versus numerical techniques

The wave forecasting techniques can be classified into two broad types viz., (i) empirical, simplified or parametric and (ii) numerical or elaborate. The former methods explicitly give wave height and period from the knowledge of wind-speed, fetch and duration while the later ones require numerical solution of the equation of wave growth. The numerical methods are far more accurate than the parametric and give information over a number of locations simultaneously. They however require a number of oceanographic and meteorological parameters. They are more justified when the wind speed varies considerably along with its direction in a given time duration and area.

When the wind field can be assumed to be fairly stationary and when accurate and elaborate wind data are not available, simplified parametric wind-wave relationships, involving an empirical treatment, could be a workable alternative to the elaborate techniques. Common methods under this category are Darbyshire, Pierson-Neumann-James, Sverdrup-Munk-Bretschneider and Hasselmann, methods of prediction of wave characteristics. The latter two techniques are more common and are described below:

1.6.3 Empirical Methods

SMB Method

The Sverdrup-Munk and Bretschneider (SMB) equations are based on dimensional analysis considerations. These equations are suggested for deep water (where depth may exceed about 90 m) are given below (SPM 1984). The wind of speed (u) blowing over fetch (F) will produce the Hs and Ts values according to following equation:

$$\frac{gH_s}{u^2} = 0.283\tanh\left[0.0125\left(\frac{gF}{u^2}\right)^{0.42}\right] \qquad 1.6.2$$

$$\frac{gT_s}{u} = 2.4\pi \tanh\left[0.077\left(\frac{gF}{u^2}\right)^{0.25}\right] \qquad 1.6.3$$

The above H_s, T_s values would occur only if the wind blows for a duration t_{min} given in terms of fetch 'F' as follows:

$$\frac{gt_{\min}}{u} = 68.8\left(\frac{gF}{u^2}\right)^{0.67} \qquad 1.6.4$$

If actual duration t<t_{min}, then find F from equation (1.6.4) for the given t and then substitute the new F value in equation (1.6.2) and (1.6.3). This is duration limited sea (with fetch controlled by duration). If $t \geq t_{min}$, the wave heights and periods are controlled by the given fetch. A graphical representation of the above equation is known as SMB curves (SPM, 1984). In shallower water of depth (d) the three equations equivalent to equation (1.6.2), (1.6.3) and (1.6.4) are, respectively

$$\frac{gH_s}{u^2} = 0.283\tanh\left[0.53\left(\frac{gd}{u^2}\right)^{0.75}\right]\tanh\left\{\frac{0.0125\left(\frac{gF}{u^2}\right)^{0.42}}{\tanh\left[0.53\left(\frac{gd}{u^2}\right)^{0.75}\right]}\right\} \qquad 1.6.5$$

$$\frac{gT_s}{u} = 7.54\tanh\left[0.833\left(\frac{gd}{u^2}\right)^{0.375}\right]\tanh\left[\frac{0.077\left(\frac{gF}{u^2}\right)^{0.25}}{\tanh\left[0.833\left(\frac{gd}{u^2}\right)^{0.375}\right]}\right] \qquad 1.6.6$$

$$\frac{gt_{\min}}{u} = 6.5882\exp\left\{\left[0.016\left(\ln\left(\frac{gF}{u^2}\right)\right)^2 - 0.3692\ln\left(\frac{gF}{u^2}\right) + 2.2024\right]^{0.5} + 0.8798\ln\left(\frac{gF}{u^2}\right)\right\}$$

1.6.7

The curves for the shallow water, these equations (each drawn for a separate water depth), are available in the graphical forms. Fig 1.12 shows an example corresponding to water of depth 10.5m.

Hasselmann Method

A group of investigators led by Hasselmann developed a simplified parametric model of wave growth to obtain the H_s and T_s values for given quantities of u and Fathers equations along with the one that gives the minimum duration necessary to produce these values of H_s and T_s are given below:

For Deep Water:

$$\frac{gH_s}{u^2} = 0.0016\sqrt{\frac{gF}{u^2}}$$

1.6.8

$$\frac{gT_s}{u} = 0.2857\left(\frac{gF}{u^2}\right)^{1/3}$$

1.6.9

$$\frac{gt_{\min}}{u} = 68.8\left(\frac{gF}{u^2}\right)^{2/3} \qquad 1.6.10$$

For Shallow water:

$$\frac{gH_s}{u^2} = 0.283\tanh\left[0.53\left(\frac{gd}{u^2}\right)^{0.75}\right]\tanh\left\{\frac{0.00565\sqrt{\frac{gF}{u^2}}}{\tanh\left[0.53\left(\frac{gd}{u^2}\right)^{0.75}\right]}\right\} \qquad 1.6.11$$

$$\frac{gT_s}{u} = 7.54\tanh\left[0.833\left(\frac{gd}{u^2}\right)^{0.375}\right]\tanh\left[\frac{0.0379\left(\frac{gF}{u^2}\right)^{0.33}}{\tanh\left[0.833\left(\frac{gd}{u^2}\right)^{0.375}\right]}\right] \qquad 1.6.12$$

$$\frac{gt_{\min}}{u} = 537\left(\frac{gF_s}{u}\right)^{7/3} \qquad 1.6.13$$

The graphical forms of these equations are also available.

Darbyshire and Draper's Technique:

Another widely used alternative wave prediction technique is that developed to Darbyshire and Draper (1963). This can be conveniently given in terms of the curves. (Brebbia and Walker, 1978).

1.6.4 Forecasting in Hurricanes:

Above referred simple parametric forecasting models fail when wind conditions like, its speed, direction and profile rapidly change with time as in case of the cyclones. It becomes difficult to forecast waves using simple equations in such situations. However, tropical cyclones, called hurricanes in U.S.A., exhibit relatively stable wind profile and hence they can be tackled by parametric modeling. For slowly moving hurricanes, waves in deep water can be predicted by knowing (i) forward speed of the hurricane, U_F, (ii) radial distance from the hurricane center to the point of maximum wind on isobar map & (iii) air pressure at the hurricane center. At the point of maximum wind, the H_s and T_s values are given by, (SPM 1984):

$$H_s = 5.03 \exp\left(R\Delta P / 4700\right)\left\{1 + \left[0.29\alpha U_F \,/ (U_R)^{1/2}\right]\right\} \qquad 1.6.15$$

$$T_s = 8.60 \exp(R\Delta P / 9400)\left\{1 + \left[0.145\alpha U_F \,/ (U_R)^{1/2}\right]\right\} \qquad 1.6.16$$

where R = radius of maximum wind (km)

Δ P = normal pressure (= 760 mm of mercury) at hurricane center (mm)

U_F = wind speed along hurricane forward direction

U_R = wind speed at radius R corresponding to maximum wind (at 10 m above MSL – sustained)

= 0.865 U_{max} (if hurricane is stationary)

= 0.865 U_{max} + 0.5 U_F (if hurricane is moving)

where

U_{max} = maximum gradient wind speed at 10 m above MSL

= $0.447[14.5(\Delta P)^{1/2} – R (0.31 f)]$ where

f = Coriolis parameter = $2\ \omega \sin\phi$ where

ω = angular speed of earth's rotation = $(2\ \pi)/ 24$

α = 1 (if hurricane is slowly moving) or

= f (U_F, fetch) otherwise.

Above equations give the values of H_s and T_s at the point of maximum wind. To find the significant wave height value at any other point, say $H_s^{'}$, same H_s is required to be reduced by a reduction factor shown in Fig 1.17 while corresponding T_s' value is to be obtained by using,

$$T_s^{'} = (H_s^{'} / g)^{1/2} \qquad 1.6.17$$

1.6.5 Numerical Wave modeling

The numerical wave models deal with a spectrum of waves rather than unique wave height and period values of the above simplified schemes. They involve a detailed modeling of wave generation, propagation and dissipation mechanisms. They basically solve a differential wave energy balance equation given below in terms of a directional spectrum G_η :

$$\frac{\partial}{\partial t} G_\eta(f,\theta,\bar{x},t) + \overline{C}_g(f,\theta).\nabla G_\eta(f,\theta,\bar{x},t) = S \qquad 1.6.18$$

where

$G_\eta(f,\theta,\bar{x},t)$= Directional wave spectrum at wave frequency f and direction θ at given position $\bar{x}$ and time 't'. (Note: directional wave energy spectrum gives energy of a wave component of certain frequency along a certain direction)

$\overline{C}_g(f,\theta)$ = Group velocity vector for wave frequency (f) and direction (θ).

∇ = Operator; $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

S = Source function = S_{in} + S_{ds} +S_{nl}

S_{in} = Wind energy input

S_{ds} = Wave energy dissipation in bottom friction and wave breaking

S_{nl} = Wave energy input being transferred from one wave frequency component to the other in a non-linear way.

Many numerical models employ a net source function (S) rather than its separation into three parts as above. The source functions are based on some theoretical understanding and may require modifications based on measurements. The above governing differential equation, (1.6.18), is generally solved using finite difference schemes so as to obtain wave directional spectrum over a number of locations and over a series of time instants. This requires specification of initial temporal conditions and spatial boundary conditions. The directional spectrum may typically be resolved into finite number of frequencies and directions. Equation (1.6.18) is applicable for deep water and can be modified to account for shallow water effects life refraction and diffraction.

Resolution of the directional spectrum into discrete frequencies and directions is laborious and can be substituted by parametering it into assumed forms of wave spectrum and energy spreading function.

Actual waves at site may result from a combination of wind waves and swells arriving from a distant storm. In that case separate governing equations are required to be written.

There is a variety of numerical wave models used worldwide to obtain spatial wave forecasts with lead time of 1 to typically 72 hours. They can be classified as First Generation, Second Generation and Third Generation models- each indicating significant improvement in the wave modeling technique. The First Generation models, evolved in 1960s and 1970s,are the simplest. They assume growth of each wave spectral component independently. They are useful mainly in constant wind field. In the Second Generation model, the concept of a non-linear interaction between different wave components was introduced with simplified terms. These simplified terms are substituted by their exact solution in the Third Generation models. These can be usefully employed when the wind field is rapidly changing.

CHAPTER 2

WAVE THEORIES

2.1 Basic Hydrodynamic Equations

Continuity Equation

Consider an element of fluid with its mass centre at (x, y, z). Let (*u*, *v*, *w*) be the fluid velocity components at (x, y, z) respectively along the x, y, z axes. According to the 'conservation of mass' principle, within time Δt, net mass of the fluid flowing into the element from the *x, y, z* directions must be the same as the increase in the mass of the element. Starting with this consideration and using taylor's series to denote small increments involved the following equation of continuity can be obtained: (streeter and wylie, 1986):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad 2.1$$

(Note: for any function *f(x)* of variable '*x*', Taylor's series indicates:

$$f(x+\Delta x)= f(x)+\frac{\partial f(x)}{\partial x}.\Delta x+\frac{\partial^2 f(x)}{\partial x^2}.\Delta x^2+\text{...........})$$

Rotational Flow:

With the passage of time an element of fluid may undergo a rotation due to the moment produced by shear forces at its mass centre as shown above (Fig. 2.1) for a simple 2-D case. Such rotations about the *x, y, z* axes respectively are given by,

$$\omega_x = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \qquad 2.2$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \qquad 2.3$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \qquad 2.4$$

Note: The rotation is often measured in terms of vorticity, which is a vector whose components along the coordinate axes are ($-2\omega_x$, $-2\omega_y$ and $-2\omega_z$) respectively.

Irrotational Flow:

Many complex fluid problems become tractable if we assume that the net rotation of the fluid element about any of the x, y and z axes is zero, i.e.

$$\omega_x = \omega_y = \omega_z = 0 \qquad 2.5$$

Such a flow is called Irrotational Flow. Here the vorticity is zero.

Since rotation can be produced by the moment of shear forces acting tangential on the fluid element, inviscid or frictionless fluid (implying absence of shear forces) would make the flow irrotational. Wave motion is normally assumed to involve negligible viscosity and internal friction.

Velocity potential

In accordance with preceding Equation (2.5), for an irrotational flow,

$\omega_x = \omega_y = \omega_z = 0$

Typically then,

$$\therefore \omega_y = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = 0 \text{ (from equation (2.3))}$$

$$\therefore \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \qquad 2.6$$

If we define a scalar function 'ϕ' of position x, y, z and time 't' having continuous derivative, i.e

$$\frac{\partial^2 \phi}{\partial x \partial z} = \frac{\partial^2 \phi}{\partial z \partial x} \qquad 2.7$$

or,

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)=\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right), \qquad 2.8$$

then comparing with equation (2.6) we get

$$w=\frac{\partial \phi}{\partial z}; u=\frac{\partial \phi}{\partial x} \qquad 2.9$$

Similarly it can be shown, from other rotation equations, that

$$v=\frac{\partial \phi}{\partial y} \qquad 2.10$$

The assumption of flow irrotationality thus leads to the establishment of the velocity potential 'ϕ'. Putting u, v and w by equations (2.9) and (2.10), into the continuity equation (2.1), we get

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0 \qquad 2.11$$

i.e. $\nabla^2 \phi = 0$

This is called the Laplace form of continuity equation.

Stream Function

For any 2-D flow a function $\psi(x, y)$ can be defined such that its partial derivative along any direction would give the flow velocity along the clockwise normal orientation, i.e.

$$\frac{\partial \psi}{\partial x}=-v; \frac{\partial \psi}{\partial y}=u \qquad 2.12$$

This function ψ is called the stream function.

Equations of Motion

The forces acting on a fluid element can be classified as (i) body forces, which act per unit fluid mass, e. g., gravity, magnetic forces, and, (ii) surface forces, which act per unit surface area, e. g. pressure. As shown below the surface force can act in any direction, but can be resolved into tangential and normal forces:

The normally acting force is typically a pressure distribution while the tangential force may represent shear (or frictional, viscous) force.

Euler assumed the shear forces to be zero and taking the lead from the earlier specified (~ 1700) Newton's equation of motion (force = mass x acceleration), that was aimed at rigid bodies, applied now to fluids. The resulting equations are called **Euler's equations of motion** and are given as below along the x, y and z axes: (While doing so, the changes in forces across the fluid element were denoted by Taylor's series mentioned earlier):

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

(2.13)

Where, 'D' indicates a total derivative consisting of a time-dependent local component as well as a space-dependent convective part, typically,

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

ρ = mass density of fluid; p = pressure force per unit area, X, Y, Z = abstract body forces per unit mass along the x, y, z directions, respectively.

Navier and Stokes around the year 1840 stated that when the shear forces are not negligible (viscous fluid) the above Euler's equations should be replaced (for Newtonian and incompressible fluid) by the following equations, now known as the Navier-Stokes equations of motion:

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}) + \nu\nabla^2 v$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w$$

Where, ν = kinematic viscocity = absolute viscocity, μ / fluid mass density ρ, and,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}$$

Consider the Euler's equation of motion along the 'x' direction:

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

This gives:

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

(Note: '*D*' stands for total derivative having a convective (x, y, z dependent) component as well as a local (time dependent) component).

Now, $u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\frac{u^2}{2})$

From the irrotationality equation (2.4): $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

$$\therefore v\frac{\partial u}{\partial y} = v\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(\frac{v^2}{2})$$

From the irrotationality equation (2.3): $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$

$$\therefore w\frac{\partial u}{\partial z} = w\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(\frac{w^2}{2})$$

Also, $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right)$

The Euler's equation along the x direction thus is:

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{v^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{w^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)=\frac{1}{X}-\frac{1}{\rho}\frac{\partial p}{\partial X}$$

$$\therefore X=\frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}+\frac{\partial \phi}{\partial t}+\frac{p}{\rho}\right)$$

$$\therefore Y=\frac{\partial}{\partial y}\left(\frac{u^2+v^2+w^2}{2}+\frac{\partial \phi}{\partial t}+\frac{p}{\rho}\right)$$

$$\therefore Z=\frac{\partial}{\partial z}\left(\frac{u^2+v^2+w^2}{2}+\frac{\partial \phi}{\partial t}+\frac{p}{\rho}\right)$$

If gravity is considered as the only body force acting on the unit mass of the fluid then X=0, Y=0 and Z= -*g*.1= $\frac{\partial}{\partial z}(-gz)$

Hence from the preceding equation:

$$\frac{\left(u^2+v^2+w^2\right)}{2}+\frac{p}{\rho}+gz+\frac{\partial \phi}{\partial t}=0 \qquad 2.14$$

Where, *g* is the acceleration due to gravity and *z* is the elevation of the point. This is the Bernoulli's equation.

2.2 Wave Theories

Wave theories yield the information on the wave motion such as the water particles kinematics and wave speed, using the input of wave height, its period and depth of water at the site. There are more than a dozen different theories available in this regard. However, only a few of them are common in use and these are described below: All wave theories involve some common assumptions, viz,

1. The waves have regular profiles.
2. The flow is two-dimensional (in vertical x, z plane).
3. The wave propagation is unidirectional (or long crested).
4. The fluid is ideal i.e. inviscid, incompressible and irrotational.
5. The sea bed is impermeable and horizontal.

The wave theories can be categorized into two types:

(i) Linear or Airy's (or sinusoidal or small amplitude) wave theory

(ii) Non-linear (or finite amplitude) wave theories.

The former is distinguished from the latter in that it assumes that the waves are flatter with a small steepness ratio (typically < 2%).

Linear Wave Theory

Summary of derivation for velocity potential: ϕ

Figure 2.2 shows the definition sketch for the linear wave theory. It is assumed that the velocity potential (ϕ) depends on position (x, y) and time t and this is given by,

$\phi(x, y, z) =$ X(x) Z(z) T(t)

= XZT say

where X, Z, T are initially unknown functions of x, z and t respectively, assumed to be independent of each other.

These unknown functions can be determined by making ϕ to satisfy

(i) Laplace Equation (2.11).

(ii) Linearised form of Bernoulli's dynamic Equation (2.13) at the free surface (z = η), which is,

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + g\eta = 0 \qquad 2.15$$

[In deriving above Equation (2.15) we assume that the partial differential terms (like $\frac{\partial \phi}{\partial x}$, etc.) are small so that the product of any two such terms negligible]

(iii) Dynamic Free Surface Boundary Condition (DFSBC)

This states that the pressure at the free surface p is zero (atmospheric).

(iv) Kinematic Free Surface Boundary Condition (KFSBC)

It ensures that the free surface is continuous.

(v) Bottom Boundary Condition (BBC)

This means that the velocity normal to the sea bottom is zero.

The expression for ϕ determined in this way (Ippen 1965) is

$$\phi = \frac{gH \cosh(k(d+z))}{2\omega \cosh(kd)} \sin(kx - \omega t) \qquad 2.16$$

Where, H = wave height; ω = Circular wave frequency = $\frac{2\pi}{T}$; T = Wave period; k = Wave number = $\frac{2\pi}{L}$; L = Wave length; z = Vertical co-ordinate of the point at which ϕ is being considered (from the SWL); d = Water depth; x = Horizontal co-ordinate of the point (w.r.t any arbitrary origin at SWL); t = Time instant.

Derivation of ϕ

Refer to Fig. 2.2. Writing eq. (2.13) in 2-dimensions (vertical plane) at any point affected by the wave motion:

$$\frac{p}{\rho} + \frac{1}{2}q^2 + gz + \frac{\partial \phi}{\partial t} = 0 \qquad (2.16)$$

Where, p = pressure at the concerned point in the wave motion, ρ = mass density of the sea water, q = velocity = $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} k$; i, j = unit vectors along the x and z directions; z = vertical coordinate of the concerned point; t = time instant.

In linear theory we assume that the partial derivatives are small and hence their product is negligible compared to the other terms. Hence,

$$q^2 = (\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 = 0$$

Hence the above equation (2.16) becomes

:

$$\frac{p}{\rho} + gz + \frac{\partial \phi}{\partial t} = 0$$

Consider a point on the free surface. For this point, $z = \eta$; Let us apply the dynamic free surface boundary condition (DFSBC): at the surface, p = 0.

Hence, $g\eta + \frac{\partial \phi}{\partial t} = 0$

$\eta = -\frac{1}{g}(\frac{\partial \phi}{\partial t})_{z=\eta}$

Assuming that the wave height, H, is very small compared to wave length, L, the above equation can be approximated as:

$$\eta = -\frac{1}{g}(\frac{\partial \phi}{\partial t})_{z=0} \qquad (2.17)$$

Let us apply the kinematic free surface boundary condition (KFSBC) indicating that the particle on the free surface must always remain on the free surface (and not come out):

$w.dt = d\eta$; $w = \frac{d\eta}{dt}$; i. e., $\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial t}$

The last term involves a product of two differentials and hence can be neglected as per our assumption. Hence approximately (assuming that the evaluation at z=0 is same as the one at z=η):

$$\frac{\partial \eta}{\partial t} \approx (\frac{\partial \phi}{\partial z})_{z=0=\eta} \qquad (2.18)$$

(2.1.2) and (2.1.3) give:

$-\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial \phi}{\partial z}$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \qquad (2.19)$$

This has solution:

ϕ (x, y, z, t) = X(x).Z(z).T(t)

= XZT say

Putting this in Laplace's continuity equation

$\Delta^2 \phi = 0$

i. e., $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

$$\frac{\partial^2 X}{\partial x^2}(ZT) + \frac{\partial^2 Z}{\partial z^2}(XT) = 0$$

Dividing by (X Z T): $\frac{\partial^2 X}{\partial x^2}(\frac{1}{X}) + \frac{\partial^2 Z}{\partial z^2}(\frac{1}{Z}) = 0$

Because the first term is a pure function of x, the second one is a pure function of z, each has to be a constant (unchanging with x and z) if their sum is to be constant (or zero).

Hence let $\frac{\partial^2 X}{\partial x^2}(\frac{1}{X}) = -k^2$ and

$$\frac{\partial^2 Z}{\partial z^2}(\frac{1}{Z}) = k^2$$

We have two equations:

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0 \quad \text{and}$$

$$\frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0$$

Having standard solutions: X = $c_1 \cos kx + c_2 \sin kx$, and

Z = $c_3 e^{kz} + c_4 e^{-kz}$

$\phi = (c_1 \cos kx + c_2 \sin kx).(c_3 e^{kz} + c_4 e^{-kz})$ T

As ϕ has to be harmonic (repetitious) in time period T, we should write

T(t) = $\cos (2\pi/T) = \cos \omega t$ or $= \sin (2\pi/t) = \sin \omega t$

The above ϕ can have two alternative general solutions (valid for all x, z, t):

$$\phi = (c_1 \cos kx + c_2 \sin kx)(\mathrm{c}_3 \mathrm{e}^{kz} + \mathrm{c}_4 \mathrm{e}^{-kz}) \cos \omega t$$

and, $$\phi = (c_1 \cos kx + c_2 \sin kx)(\mathrm{c}_3 \mathrm{e}^{kz} + \mathrm{c}_4 \mathrm{e}^{-kz}) \sin \omega t$$

According to the property of the Laplace equation: if we can express $\phi_{\text{general}} = \phi_a + \phi_b$, then each of these ϕ_a and ϕ_b, are particular solutions, also satisfying the Laplace equation. From the above two equations thus we can form four particular solutions, namely,

$$\phi_1 = c_1 \cos kx \cos \omega t (\mathrm{c_3 e^{kz}} + \mathrm{c_4\ e^{-kz}})$$

$$\phi_2 = c_1 \cos kx \cos \omega t (\mathrm{c_3 e^{kz}} + \mathrm{c_4\ e^{-kz}})$$

$$\phi_3 = c_2 \sin kx \cos \omega t (\mathrm{c_3 e^{kz}} + \mathrm{c_4 e^{-kz}})$$

$$\phi_4 = c_2 \sin kx \sin \omega t (\mathrm{c_3 e^{kz}} + \mathrm{c_4 e^{-kz}})$$

Let us work with the particular solution ϕ_1. Let us apply the bottom boundary conditiona that states that

$$\frac{\partial \phi_1}{\partial z} = 0 \qquad \text{at the sea bed (z = -d)}$$

Hence, ϕ_1 above gives, $\quad c_1 \cos kx \cos \omega t (c_3 k e^{kz} - c_4 k e^{-kz}) = 0$

$$c_3 e^{kz} = c_4 e^{-kz} \quad \text{at z=-d}$$

$$c_3 = c_4 e^{2kd} \qquad \text{for all x and t}$$

$$\phi_1 = c_1 \cos kx \cos \omega t \{c_4 e^{2kd} e^{kz} + c_4 e^{-kz}\}$$

$$= c_1 c_4 \{e^{kd} e^{kd} e^{kz} + e^{kd} e^{-kd} e^{-kz}\} \cos kx \cos \omega t$$

$$= 2c_1 c_4 e^{kd} \{\frac{e^{kd} e^{kz} + e^{-kd} e^{-kz}}{2}\} \cos kx \cos \omega t$$

$$= 2c_1 c_4 e^{kd} \{\frac{e^{k(d+z)} + e^{-kd+z}}{2}\} \cos kx \cos \omega t$$

$$= 2c_1 c_4 e^{kd} \cosh k(d+z) \cos kx \cos \omega t$$

From eq. (2.1.2), $\quad \eta = -\frac{1}{g}(\frac{\partial \phi}{\partial t})_{z=0}$

Hence, $\quad \eta = -\frac{1}{g}\{2c_1 c_4 e^{kd} \cosh k(d+z) \cos kx(\omega) \sin \omega t\}(-)$

$$= \frac{2c_1 c_4 \omega}{g} e^{kd} \cosh k(d+z) \cos ks \sin \omega t$$

Maximum η (=amplitude A) will occur **at z=0** when *cos kx sin ωt = 1*

In that case $\; A = 2c_1 c_4 \frac{\omega}{g} e^{kd} \cosh kd \qquad$ at z=0

Hence $\qquad \frac{Ag}{\omega \cosh kd} = 2c_1 c_4 e^{kd}$

This gives $\phi_1 = \frac{Ag}{\omega}\frac{\cosh k(d+z)}{\cosh kd}\cos kx \cos \omega t$

$$\phi_2 = -\frac{Ag}{\omega}\frac{\cosh k(d+z)}{\cosh kd}\cos kx \sin \omega t$$

$$\phi_3 = \frac{Ag}{\omega}\frac{\cosh k(d+z)}{\cosh kd}\sin kx \cos \omega t$$

$$\phi_4 = \frac{-Ag}{\omega}\frac{\cosh k(d+z)}{\cosh kd}\sin kx \sin \omega t$$

Each of them is a solution of the Laplace equation, separately as well as in linear combination. For getting a general solution however we linearly combine two particular solutions. Thus

$\phi_1 - \phi_4$ is one solution. This equals $\phi = \frac{Ag}{\omega}\frac{\cosh k(d+z)}{\cosh kd}\cos(kx - \omega t)$

Similarly $\phi_3 + \phi_2$ is another solution. This equals $\phi = \frac{Ag}{\omega}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx - \omega t)$

In linear theory the amplitude A = H/2, where H = wave height. Hence

$$\phi = \frac{gH}{2\omega}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx - \omega t)$$

eq. (2.16) referred to earlier

Noting that when we write $\omega = 2\pi/T$ we ensure that ϕ is harmonic (repeating) in time, it is clear that if we write $k = 2\pi/L$ then we will make ϕ harmonic in space as well. The value of k so defined is called wave number.

Expression for wave profile:

From (2.17), $\eta = -\frac{1}{g}(\frac{\partial \phi}{\partial t})_{z=0}$

$$= -\frac{1}{g}\{\frac{gH}{2\omega}\frac{\cosh k(d+z)}{\cosh kd}\cos(kx-\omega t)\}(-\omega) \qquad \text{at } z=0$$

Hence

$$\eta = \frac{H}{2}\cos(kx-\omega t) \qquad 2.20$$

Expression for wave Celerity:

If we move with the same speed as that of the wave, the wave form (η) will appear stationary, i.e., from equation (2.20)

kx- ω t = constant

$\therefore$ wave speed or celerity

$$C = \frac{dx}{dt} - = \frac{\omega}{k} = \frac{L}{T} \qquad 2.21$$

We therefore interpret eq. (2.16) as the velocity potential of a progressive wave travelling in +x direction. If the wave travels in –x direction then C becomes –C and working backwards,

kx+ ω t = constant

and thus velocity potential of a negative left running wave is:

$$\phi_- = \frac{gH}{2\omega}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx+\omega t)$$

From eq. (2.19)

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z = 0$$

Let us derive the two partial differential terms as follows from (2.16).

$$\frac{\partial\phi}{\partial t} = (-\omega)\frac{2H}{2\omega}\frac{\cosh k(d+z)}{\cosh kd}\cos(kx-\omega t)$$

$$\frac{\partial^2\phi}{\partial t^2} = (-\omega)\frac{gH}{2}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx-\omega t)$$

$$\frac{\partial\phi}{\partial z} = \frac{gH}{2\omega} k \frac{\sinh k(d+z)}{\cosh kd} \sin(kx - \omega t)$$

Substituting in (2.19) for z = 0,

$$\omega^2 = \text{gk tanh (kd)} \qquad 2.22$$

which is called the linear dispersion relationship and is useful to obtain 'k' (or L) from the wave frequency ω (or T). Substituting C = ω/k in this equation, we get

$$C^2 = \frac{g}{k} \tanh(kd)$$

$$C^2 = \frac{g}{2\pi} \frac{L}{T} T \tanh(kd)$$

$$C = \frac{gT}{2\pi} \tanh(kd) \qquad 2.23$$

Further, we can write ϕ alternatively as follows:

$$\omega^2 = gk \tanh (kd)$$

$$\omega^2 \cosh kd = gk \sinh kd$$

Thus from $\phi = \frac{gH}{2\omega} \frac{\cosh k(d+z)}{\cosh kd} \sin(kx - \omega t)$

We get, $\phi = \frac{gH}{2gk} \frac{\omega}{\sinh kd} \frac{\cosh k(d+z)}{} \sin(kx - \omega t)$

$$\phi = \frac{\pi H}{kT} \frac{\cosh k(d+z)}{\sinh kd} \sin(kx - \omega t)$$

Simplification in shallow and deep water:

In typically shallow water (d < L/20 or kd < π/10) and deep water (d > L/2 or kd > π) cosh, sinh and tanh terms involved in the above equations take limiting values and thus we may use the approximation as follows to simplify the equations:

for small 'kd': sinh (kd) $\approx$ kd; cosh (kd) $\approx$ 1; tanh (kd) $\approx$ kd 2.21

for large 'kd': sinh (kd) $\approx$ cosh (kd) $\approx \frac{e^{kd}}{2}$: tanh (kd) ≈ 1 2.22

Hence the above equations (2.16) to (2.20) can be simplified. In deep water from equations (2.20) and (2.22), the deep water wave celerity, C_0 is

$$C_0 = \frac{gT}{2\pi} \qquad 2.23$$

If L_0 is the wavelength in deep water, $C_0 = L_0/T$ gives,

$$L_0 = \frac{gT^2}{2\pi} \qquad 2.24$$

Similarly in shallow water, Equation (2.20) after a little modification becomes

$$C_s = \sqrt{gd} \qquad 2.25$$

where C_s = wave speed in shallow water

Substituting $C = \frac{L}{T}$ in equation (2.20)

$$L = \frac{gT^2}{2\pi}\tanh(kd)$$

$= L_0$ tanh (kd)

$$\therefore \frac{d}{L_0} = \frac{d}{L}\tanh(kd) \qquad 2.26$$

Solution of this equation in graphical/tabular form is used to obtain wave length (L) in any given depth ‘d’ from the wave period ‘T’ [since $L_0 = \frac{gT^2}{2\pi}$].

Expression for particle kinematics

From equation (2.16) and using equation (2.19) an alternative expression for ϕ is:

$$\phi = \frac{\pi H \cosh(k(d+z))}{kT\sinh(kd)}\sin(kx - \omega t) \qquad 2.27$$

The horizontal and vertical components of water particle velocity (u and w) as well as those of accelerations,($\dot{u}$ and $\dot{w}$) are then given by:

$$u = \frac{\partial \phi}{\partial x} = \frac{\pi H \cosh(k(d+z))}{T\sinh(kd)}\cos(kx - \omega t) \qquad 2.28$$

$$w = \frac{\partial \phi}{\partial z} = \frac{\pi H \sinh(k(d+z))}{T \sinh(kd)} \sin(kx - \omega t) \qquad 2.29$$

$$\dot{u} = \frac{\partial u}{\partial t} = \frac{2\pi^2 H \cosh(k(d+z))}{T^2 \sinh(kd)} \sin(kx - \omega t) \qquad 2.30$$

$$\dot{w} = \frac{\partial w}{\partial t} = \frac{2\pi^2 H \sinh(k(d+z))}{T^2 \sinh(kd)} \cos(kx - \omega t) \qquad 2.31$$

Note that the four preceding equations represent the particle velocities that are different than the wave velocity or wave speed which indicates the speed with which the entire wave motion advances and is a single value for given H, T and d; while the particle kinematics vary from point to point and from time to time (at a point).

Expression for water particle displacement:

Referring to Figure 2.3 if the displacement vector of any particle of water, $\bar{r}$, is:

$$\bar{r} = \xi \hat{i} + \zeta \hat{k} \qquad 2.32$$

$$\xi = \int_0^t u dt = \frac{-H \cosh(k(d+z))}{2\sinh(kd)} \sin(kx - \omega t) \qquad 2.33$$

$$\xi = \int_0^t w dt = \frac{-H \sinh(k(d+z))}{2\sinh(kd)} \cos(kx - \omega t)$$

From equations (2.28) and (2.29) respectively,

If $a_1 = \dfrac{H \cosh(k(d+z))}{2\sinh(kd)}$ and $b_1 = \dfrac{H \sinh(k(d+z))}{2\sinh(kd)}$

$$\xi = -a_1 \sin(kx - \omega t)$$

$$\zeta = b_1 \cos(kx - \omega t)$$

squaring and adding,

$$\frac{\xi^2}{a_1^2} + \frac{\zeta^2}{b_1^2} = 1 \qquad 2.34$$

This shows that the locus of wave particles is an ellipse in any general water depth as shown in Figure 2.3.

In deep water, using equation (2.22)

$$a_1 = b_1 \frac{H}{2} e^{kz}$$

This indicates that the water particles trace out a circle. Further, at free surface, z=0. Hence,

$$\therefore a_1 = b_1 = \frac{H}{2}$$

at z = - $\frac{L}{2}$, $a_1 = b_1 = \frac{H}{2} e^{\left(\frac{2\pi}{L}\right)\left(-\frac{L}{2}\right)} = 0.02H$ which is a negligible quantity.

In shallow water, using equation (2.21)

$a_1 = \frac{H}{2kd}$ (independent of z)

$$b_1 = \frac{H(d+z)}{2d}$$

$= \frac{H}{2}$ at z = 0

$= 0$ at z = - d

The paths followed by water particles in different depths are therefore as shown in Figure 2.3.

Expression for the pressure below the sea surface:

From the linearised from of the dynamic equation (equation (2.15)).

$$p = -\rho g z - \rho \frac{\partial \phi}{\partial t}$$

Using equation (2.17) and (2.27),

$$p = -\rho g z + \rho g \eta \frac{\cosh(k(d+z))}{\cosh(kd)} \qquad 2.35$$

The ratio $\frac{\cosh(k(d+z))}{\cosh(kd)}$ is called the pressure response factor.

Expression for wave energy:

Since the wave particles are disturbed from their equilibrium positions and since they move with some velocity they possess potential as well as kinetic energy. The total energy (E), per unit plan (also called specific energy or energy density) is:

$$E = \int_0^{\eta}\int_0^{L} \rho gz dx dz + \int_{-d}^{\eta}\int_0^{L} \frac{\rho}{2}\left(u^2 + w^2\right) dx dz \qquad 2.36$$

Substituting the values of u, w, η and simplifying, we get,

$$E = \frac{\gamma H^2}{8} \qquad 2.37$$

or

$$E = \frac{\gamma a^2}{2} \qquad 2.38$$

Where,

a = Wave amplitude = H/2

γ = Specific Weight of Sea Water

Expression for group velocity:

The velocity with which a group of waves moves, C_g is different than that of an isolated individual wave.

$$C_g = \lim_{\Delta k \to 0} \frac{\Delta \omega}{\Delta k} = \frac{d\omega}{dk} \qquad 2.39$$

Using the linear dispersion equation (2.19), above equation reduces to

$C_g = nC$ 2.40

where,

$$n = f(kd) = \frac{1}{2}\left[1 + \frac{2kd}{\sinh(2kd)}\right] \qquad 2.41$$

With the help of shallow and deep water approximations, Equations (2.21) and (2.22) respectively, it is easy to see that in deep water and in shallow water

$$C_{g_0} = \frac{C_0}{2} \qquad 2.42$$

and in shallow water

$C_g = C_s = \sqrt{gd}$ 2.43

Expression for wave power or energy flux:

This is defined as the average rate of transmission of wave energy per unit lateral width along the direction of the wave propagation.

$$\therefore P = \frac{1}{T}\int_{-d}^{\eta}\int_{0}^{T}\left[p + \rho g z + \frac{\rho}{2}\left(u^2 + w^2\right)\right]u\,dt\,dz \qquad 2.44$$

Substituting the value of η, p, u and w from the previous expressions,

$$P = n\rho g \frac{H^2}{8}\frac{L}{T} = nCE = C_g E \qquad 2.45$$

Finite Amplitude Wave Theories

General Method of Solution:

In preceding small amplitude wave theory the wave steepness (H/L) was assumed to be small (so that its higher powers became negligible) and the expressions for ϕ and η turned out to be

$$\phi = C_1 \sin(0) \qquad 2.46$$

and

$$\eta = a\cos(\theta) \qquad 2.47$$

where,

C_1 = f (H, T, d) (see equation (2.27))

$\theta = (kx - \omega t)$, phase angle

a = H/2

When wave steepness value is high, or finite, above assumption becomes no longer valid and a different solution for ϕ results. As per different alternative methods to formulate 'ϕ' we have different theories like Stokes, Cnoidal, Solitary, Dean's, etc.under the Finite Amplitude category. A general common procedure to obtain the wave properties, in these theories, is as discussed below:

First, the ϕ (or ψ) is formulated as some unknown function of given H, T, d values containing unknown coefficients. This ϕ (or ψ) is then made to satisfy the continuity equation, dynamic equation, irrotationality equation and various boundary conditions discussed in the previous section. By solving all such equations simultaneously the unknown coefficients are established and ϕ (or ψ) in turn is obtained. Once ϕ (or ψ) is known its derivatives like $u, \dot{u}, w, \dot{w}, \phi$ etc. automatically follow.

Stokes Wave Theory:

The ϕ and η are modelled in Stokes theory using perturbation parameters b and a respectively as below:

$$\phi = \sum_{n=1}^{M} b^n \phi_n (H, T, d) \sin(n\theta) \qquad 2.48$$

$$\eta = \sum_{n=1}^{M} a^n f_n (H, T, d) \cos(n\theta) \qquad 2.49$$

where b_n and a_n are initially unknown functions of H, T, d and so also ϕ_n and f_n.

The above represented series can be explained to any order (considering powers of (H/L) only upto that order) to obtain the Stokes theory of the corresponding order. e.g. in the Stokes second order theory,

$$\phi = b_1 \sin(\theta) + b_2 \sin(2\theta)$$

$$\eta = a_1 \cos(\theta) + a_2 \cos(2\theta)$$

where, (b_1,b_2,a_1,a_2 are functions of H, T and d)

The fifth order theory is popular owing to its better prediction of the actual water particle kinematics.

The procedure followed in the fifth order theory to arrive at the values of particle kinematics is as below:

1. From the given values of H, T, d obtain the unknowns λ and kd by the two expressions given below that result from the application of KFSBC and FSBCs.

$$\frac{1}{kd}\left[\lambda + B_{33}\lambda^2 + (B_{35} + B_{55})\lambda^5\right] = \frac{H}{2d} \qquad 2.50$$

$$\text{kd tanh (kd)} \left[1 + C_1\lambda^2 + C_2\lambda^4\right] = 4\pi^2 \frac{d}{gT^2} \qquad 2.51$$

where B_{ij},C_i (for various values of i and j) are initially unknown functions of kd as listed in Appendix 2.1, which also explains additional symbol A_{ij} used below:

2. Obtain ϕ using,

$$\phi = \frac{\overline{C}}{k}\sum_{n=1}^{5}\phi_n' \cosh(nk(d+z))\sin(n\theta) \quad 2.52$$

where,

$$\phi_1' = \lambda A_{11} + \lambda^3 A_{13} + \lambda^5 A_{15}$$

$$\phi_2' = \lambda^2 A_{22} + \lambda^4 A_{24}$$

$$\phi_3' = \lambda^3 A_{33} + \lambda^5 A_{35}$$

$$\phi_4' = \lambda^4 A_{44}$$

$$\phi_5' = \lambda^5 A_{55}$$

$$\bar{c} = \frac{g}{k}\tanh(kd)\left[1 + C_1\lambda^2 + C_2\lambda^4\right]$$

3. $$u = \bar{c}\sum_{n=1}^{5} n\phi_n' \cosh(nk(d+z))\cos(n\theta) \quad 2.53$$

$$w = \bar{c}\sum_{n=1}^{5} n\phi_n' \sinh(nk(d+z))\sin(n\theta) \quad 2.54$$

$$\dot{u} = \bar{c}\,\omega\sum_{n=1}^{5} n^2\phi_n' \cosh(nk(d+z))\sin(n\theta) \quad 2.55$$

$$\dot{w} = -\bar{c}\,\omega\sum_{n=1}^{5} n^2\phi_n' \sinh(nk(d+z))\cos(n\theta) \quad 2.56$$

4. $$\eta = \frac{1}{k}\sum_{n=1}^{5}\eta_n' \cos(n\theta) \quad 2.57$$

where,

$$\eta_1' = \lambda$$

$$\eta_2' = \lambda^2 B_{22} + \lambda^4 B_{24}$$

$$\eta_3' = \lambda^3 B_{33} + \lambda^5 B_{35}$$

$$\eta_4' = \lambda^4 B_{44}$$

$$\eta_5' = \lambda^5 B_{55}$$

A computer program carrying out all above steps can be easily developed [Chaudhari (1985), Kankarej (1992)].

Cnoidal Theory:

The general definitions of Cnoidal theory are given in the Figure 2.4.This theory involves formulating ϕ in terms of the elliptic cosine or Cnoidal function:

$$\frac{\phi}{L'\sqrt{gd}} = \cos\left(\sqrt{\sigma}SD\right)f(x) \qquad 2.58$$

$$= \left[1 - \sigma S^2 \frac{D^2}{2!} + \sigma^2 S^4 \frac{D^4}{4!} + \ldots\ldots\ldots\ldots\ldots\right] f(x) \qquad 2.59$$

where,

$$\sigma = \left[\frac{WaterDepth, d}{achosentypicallength, 'L' along the horizontal 'x' direction}\right]^z << 1 \text{ (very small)}$$

$$S = \frac{d+z}{d}$$ [z = vertical co-ordinate from the sea bed; positive upwards.(Figure 2.4)]

$$D = \frac{d}{dx} \; ; \; X = \frac{x - Ct}{L'}$$

Solution ϕ of involves elliptic functions, typically the complete Elliptic integral of first kind, K (k), where k is the argument depending upon H, T, d and ranging from 0 to 1.The application of them to obtain water particle kinematics involves elaborate computer programming. However, for certain application, like determination of wave profile and wave length, graphical solutions are available (Wiegel (1965).

Solitary Wave Theory:

When the value of the argument k of K (k) tends to its upper limit 1,K (k) approaches sech (k) and a great simplification in the resulting values emerges. The resulting theory is called the solitary wave theory.(See Figure 2.5 for the definition sketch).Solitary theory of second order is found to be simple and satisfactory for steep waves in shallow water.(See Sarpkaya and Issacson,(1981)).

Dean Stream Function Theory:

Herein, in contrast to previous theories a solution for stream function ψ is sought for as expressed below:

A reference frame advancing with the same speed is taken so that the flow becomes steady and the steady state Bernoulli's equation becomes applicable. (See Figure 2.6)

$$\psi = -Cz + \sum_{n=1}^{M} X_n \sinh\left(n\left[\frac{2\pi}{L}(d+z) \right] \right) \cos\left(n\left(\frac{2\pi x}{L} \right) \right) \qquad 2.60$$

with M representing the desired order of expression.

X_n are the coefficients that are obtained by following a numerical procedure. The resulting computer program is complicated. But tabular aids are available (SPM (1984)).

Trochoidal Wave Theory:

In this wave theory, the wave profile is idealized to that of a trochoid which is a curve generated by the locus of any point on a circle as the circle is imagined to be translating along a horizontal line.

If (x_0, z_0) are the co-ordinates of the mean position of a water particle then its trajectories at any time are given by,

$$x = x_0 + (H/2)\exp(kz_0)\sin(kx_0 - \omega t)$$

$$z = z_0 + (H/2)\exp(kz_0)\cos(kx_0 - \omega t)$$

These quantities can be further differentiated to yield the velocity components as $u = \partial x/\partial t$ and $w = \partial z/\partial t$.

Method of Complex variables:

Another technique of getting the wave parameters involves transforming the physical region of the fluid bounded by the ocean bottom and the free surface wave profile into an annulus region bounded by an outer circle of unit radius representing the free surface and an inner circle corresponding to the ocean bottom. The flow in this annulus complex plane is potential clockwise vortex whose properties are known and can therefore be mapped on the complex plane of ϕ and ψ plane. The mapping of the physical plane and the ϕ-ψ plane can as well be done using a perturbation parameter technique.

Non-linear versus Linear Theory

The profile of a linear wave is symmetrical with respect to the undisturbed or the still water level (SWL) whereas in case of a typical non-linear wave height of the crest is greater than the depth of the through as shown in Figure 2.7.

As the order of a non-linear theory increases, the crests become more and more steep and the troughs become more and more flat.

For design purpose it is assumed that due to wave nonlinearity, the SWL is below the level of symmetry (drawn midway, horizontally, between crest and trough) by an amount h_0 given by,

$$h_0 = \left[\pi H^2/(4L)\right]\left[\coth(2\pi d/L)\right] + \text{higher order terms} \qquad 2.61$$

The higher order terms in the above equation are many times neglected as an approximation.

1. In general the non-linear theories produce larger values of the wavelength, speed as well as the particle kinematics.
2. The paths followed by the water particles is closed, in linear waves while it is open, producing a 'drift', or 'mass transport' (as shown in Figure 2.8) in case of a non-linear wave. This gives rise to a 'mass transport velocity' and the wave celerity needs to be redefined with respect to it.

Choice of Wave Theory

Uniformly acceptable criteria for choosing a particular wave theory are not available owing to the fact that no simple theory predicts all wave properties (like, $u, \dot{u}, w, \dot{w}, p, \eta$) satisfactorily. Further, steep waves near breaking are not amenable to any wave theory.

In general the simple Air's linear theory is preferable if the wave has a small steepness, the sea is multi-directional, the wave spectrum is broad banded or the structural dimensions are such that the inertial forces are dominant than the drag forces. Experiments in the laboratory and those in the sea have shown the adequacy of the linear theory in general and that of the Stokes fifth order theory in deep water to predict the particle kinematics. The steeper waves, however, fit better into the Dean's higher order analysis. Considering the convergence of the series terms, the Stokes theory is useful when water is deeper than 10% of the wave length while the Solitary theory is good if it is shallower than 20% of the wave length. In between, the Cnoidal theory would give converging results. Experimentally based guidelines are given in Sarpkaya and Issacson (1981). Figure 2.9 shows the corresponding chart to select an appropriate wave theory among the Linear, Stokes, Cnoidal and Dean's theories. Starting from the given values of wave height (H), period (T) and water depth (d), determine the non-dimensional quantities, viz., (H/gT^2) and $d/(gT^2)$. From the former quantity proceed horizontally till you get the point of intersection with the vertical line corresponding to the known $d/(gT^2)$ value. The location of this intersection point indicates the appropriate theory to be chosen.

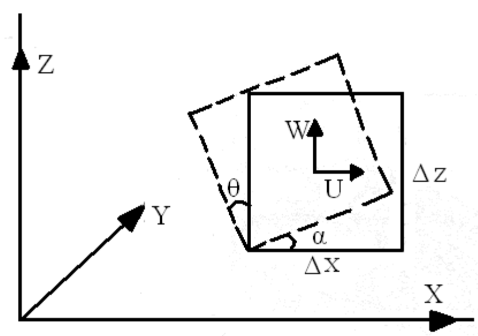


Fig 2.1 Fluid Rotation

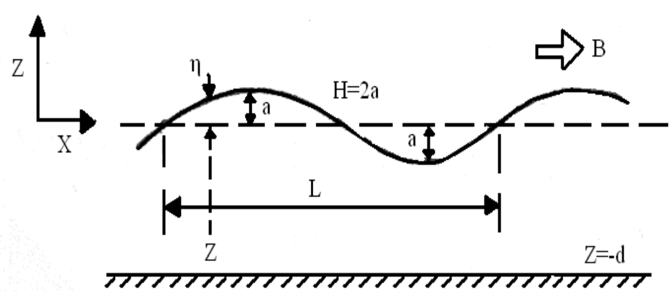


Fig 2.2 Linear Theory

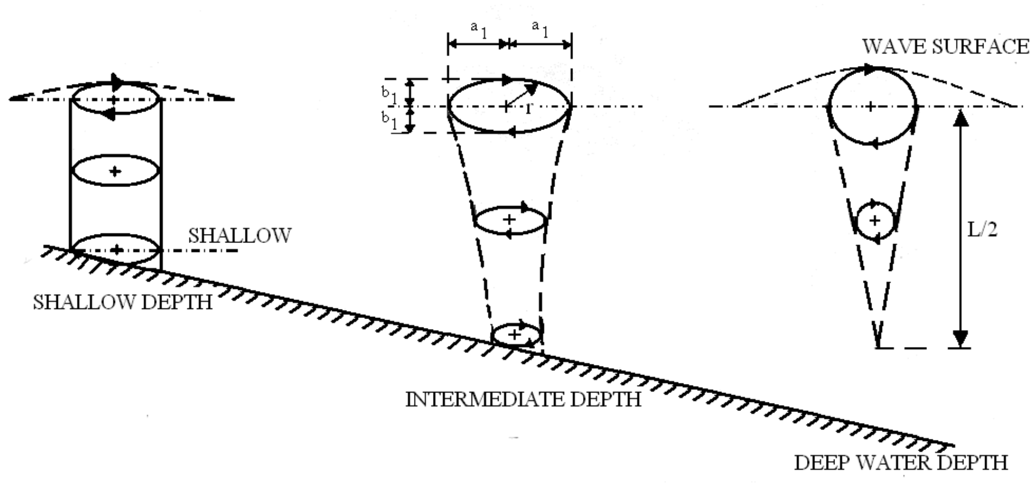
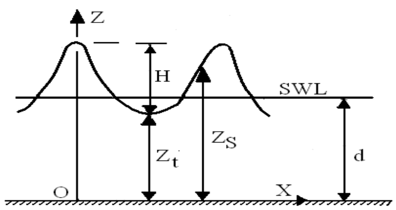
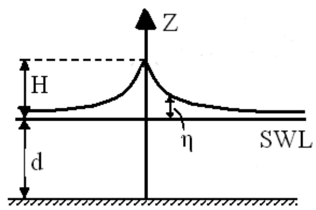


Fig 2.3 Water Particle Displacements





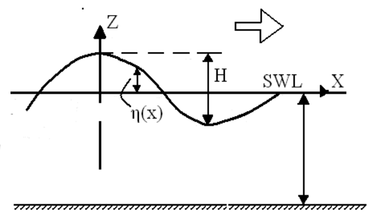
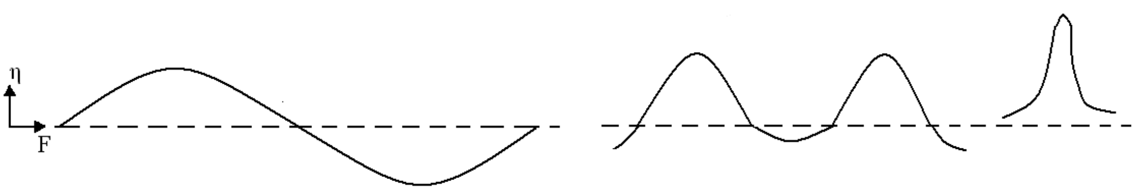


Fig 2.4 Cnoidal Theory Definitions

Fig 2.5 Solitary Theory

Fig 2.6 Dean's Theory



Linear

Non Linear

Solitary

Fig 2.7 Comparison of Wave Profiles

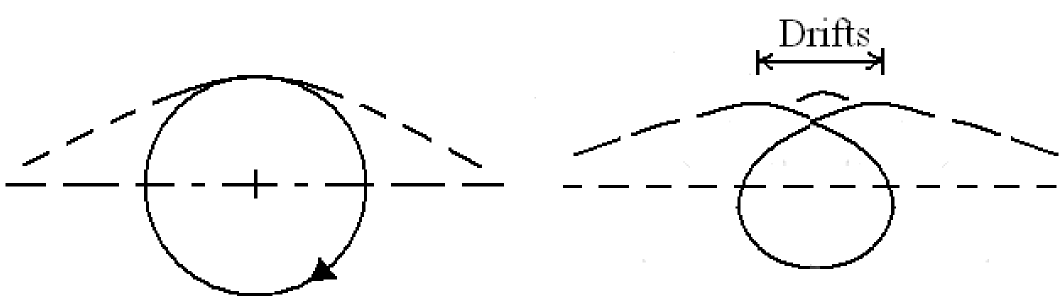


Fig 2.8 Closed and Open Orbits

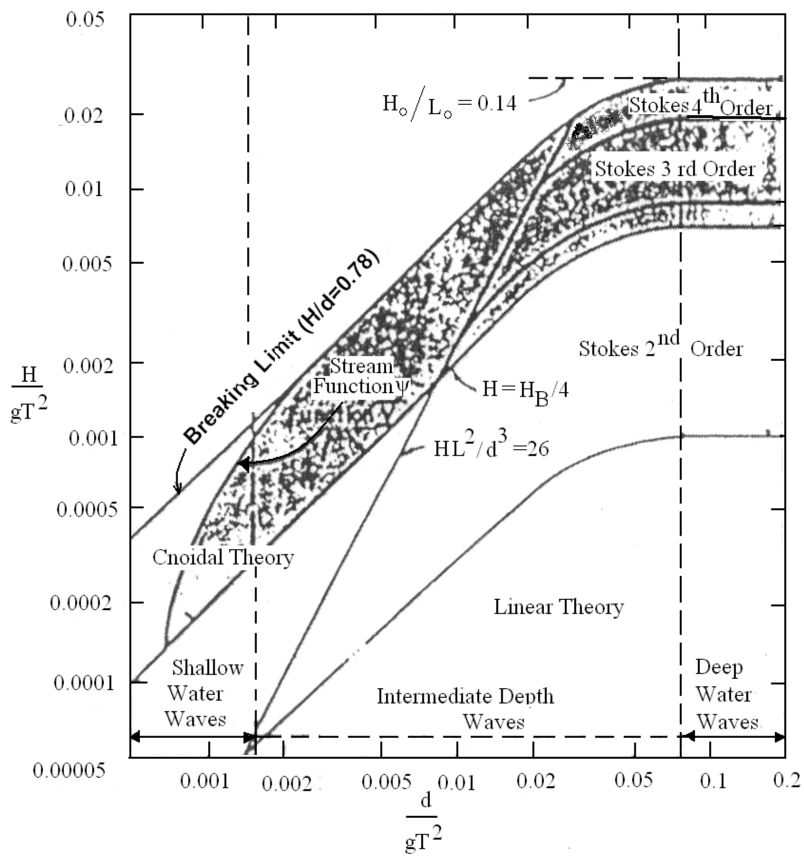


Fig 2.9 Choice of a Wave Theory **(Sarpakaya and Issacson, 1981)**

Appendix 2.1

COEFFICIENTS OF STOKE FIFTH ORDER THEORY

s = sinh (kd)

c = cosh (kd)

$$A_{11} = \frac{1}{s}$$

$$A_{13} = \frac{-c^2\left(5c^2+1\right)}{8s^5}$$

$$A_{15} = \frac{-\left(1184c^{10} - 1440c^8 - 1992c^6 + 2641c^4 - 249c^2 + 18\right)}{1536s^{11}}$$

$$A_{22} = \frac{3}{8s^4}$$

$$A_{24} = \frac{\left(192c^8 - 424c^6 - 312c^4 + 480c^2 - 17\right)}{748s^{10}}$$

$$A_{33} = \frac{\left(13 - 4c^2\right)}{64s^7}$$

$$A_{35} = \frac{\left(512c^{12} + 4224c^{10} - 6800c^8 - 12808c^6 + 16704c^4 - 3154c^2 107\right)}{4096s^{13}(6c^2-1)}$$

$$A_{44} = \frac{\left(80c^6 - 816c^4 + 1338c^2 - 197\right)}{1536s^{10}(6c^2-1)}$$

$$A_{55} = \frac{-\left(2880c^{10} - 72480c^8 + 324000c^6 - 432000c^4 + 163470c^2 - 16245\right)}{61440s^{11}(6c^2-1)(8c^4-11c^2+3)}$$

$$B_{22} = \frac{\left(2c^2+1\right)}{4s^3}c$$

$$B_{24} = \frac{c\left(272c^8 - 504c^6 - 192c^4 + 322c^2 + 21\right)}{384s^9}$$

$$B_{33} = \frac{3\left(8c^6+1\right)}{64s^6}$$

$$B_{35} = \frac{\left(88128c^{14} - 208224c^{12} + 70848c^{10} + 54000c^8 - 21816c^6 + 6264c^4 - 52c^2 - 81\right)}{12288s^{12}(6c^2-1)}$$

$$B_{44} = \frac{c\left(768c^{10} - 448c^{8} - 48c^{6} + 48c^{4} + 106c^{2} - 21\right)}{384s^{10}(6c^{2} - 1)}$$

$$B_{55} = \frac{\left(192000c^{16} - 262720c^{14} + 83680c^{12} + 20160c^{10} - 7280c^{8}\right)}{12288s^{10}(6c^{2} - 1)(8c^{4}11c^{2} + 3)}$$

$$+ \frac{(7160c^{6}) - 1800c^{4} - 1050c^{2} + 225}{12288s^{10}(6c^{2} - 1)(8c^{4}11c^{2} + 3)}$$

$$C_{1} = \frac{\left(8c^{4} - 8c^{2} + 9\right)}{8s^{4}}$$

$$C_{2} = \frac{\left(3840c^{12} - 4096c^{10} + 2592c^{8} - 1008c^{6} + 5944c^{4} - 1830c^{2} + 147\right)}{512s^{10}(6c^{2} - 1)}$$

$$C_{3} = -\frac{1}{4sc}$$

$$C_{4} = \frac{\left(12c^{8} + 36c^{6} - 162c^{4} + 141c^{2} - 27\right)}{192cs^{9}}$$

CHAPTER 3

RANDOM WAVES

For the purpose of simplification in the analysis we assume that the waves are regular in nature. However, in actual they are random or irregular in their occurrence as well as behavior. Successive waves observed at any given location have varying heights, periods and lengths. Analysis of such irregular waves measured at a particular site is necessary both from the point of knowing more about them over a single sea state as well as for the purpose of deriving the largest wave height expected in the lifetime of say a coastal or a harbor facility. The former analysis is called short term analysis while the latter is termed as long term analysis.

Before we look into the analysis of random waves we may recapitulate definitions of certain basic statistical parameters as under:

3.0. Basic definitions in random data analysis

Prob (A) = Probability of occurrence of an event 'A'
= number of times 'A' occurred / total number of all events

Ensemble = Collection of all events of a random process

Sample = Part of the ensemble selected for analysis

Statistics = various statistical parameters (apart from the branch of mathematics)

Probability distribution function (/Cumulative distribution function/Probability of non-exceedence) of a random variable: P(x) = Prob. (the variable ≤ a numerical value)

$$P\ (x = x_n) = P\ (x \le x_n)$$

Probability Density Function (pdf) of a random variable

P(x) = $\dfrac{d}{dx}P(x)$

$\int p(x) = P\ (x)$

$$\int_{-\infty}^{x_n} p(x)dx = [P(x)]_{-\infty}^{x_n}$$

$$= P\ (x = x_n)$$
$$= P\ (x_n)$$

Generalizing for all $x = x_n$; $P(x) = \int_{-\infty}^{x} p(x)dx$

$$\int_{-\infty}^{x_n} p(x)dx = [P(x)]_{-\infty}^{x_n}$$

$= P(x_n) - (-\infty)$
$= P(x_n)$

P (x) = $\int_{-\infty}^{x} p(x)dx$ for all x = x_n

Note: $\int_{-\infty}^{\infty} p(x)dx$ = P (∞) – P (-∞) = 1 – 0 = 1

A common pdf is Gaussian normal given by:

$$p(x)\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Where σ = standard deviation of x; μ = average of x

Expectations

If f (x) = any function of x; p (x) = pdf of x,

E{ f(x)} = $\int_{-\infty}^{\infty} f(x)p(x)dx$ = mean of x = $\overline{f(x)}$

Moment of x :

These are useful in getting shapes of pdf's and they reprewent the moment of the curve p (x) around any vertical axis.

Moments about the origin:

m_n = E { x^n } = $\int_{-\infty}^{\infty} x^n p(x)dx$

n = 0; m_0 = $\int_{-\infty}^{\infty} p(x)dx$ = 1

n = 1; m_1 = $\int_{-\infty}^{\infty} xp(x)dx$ = $\overline{x}$ = E { x }

n = 2; m_2 = $\int_{-\infty}^{\infty} x^2 p(x)dx$ = $\overline{x^2}$ = E { x^2 } = mean of square of x

Note: root mean square of x = $\sqrt{\overline{x^2}}$

Moments about mean (Central moments)

μ_n = E{(x – $\overline{x}$)^n} = $\int_{-\infty}^{\infty} (x-\overline{x})^n p(x)dx$

n = 0; μ_0 = $\int_{-\infty}^{\infty} p(x)dx$ = 1

n = 1; $\mu_1 = \int_{-\infty}^{\infty} (x - \bar{x})\, p(x) dx$

n = 2; $\mu_2 = \int_{-\infty}^{\infty} (x - \bar{x})^2\, p(x) dx$

$= \int_{-\infty}^{\infty} x^2 - 2x\bar{x} + \bar{x}^2\ )\, p(x) dx$

$= \overline{x^2} - 2\bar{x}\bar{x} + \bar{x}^2$

$= \overline{x^2} - \bar{x}^2$

= variance, σ_x^2

$\sqrt{\sigma_x^{\ 2}} = \sigma_x =$ standard deviation of x

$\mu_3 = \int_{-\infty}^{\infty} (x - \bar{x})^3\, p(x) dx$

$\mu_4 = \int_{-\infty}^{\infty} (x - \bar{x})^4\, p(x) dx$

Joint Probabilities

Let x and y be any two random variables and x_n and y_n be any numerical values of these variables. Then:

Joint probability density function of x and y is:

$$p(x,y) = [\lim asx \to \infty, y \to \infty] \frac{P\{.(x_n \le x \le (x_n + \Delta x); y_n \le y \le (y_n + \Delta y)}{\Delta x \Delta y}$$

Individual or Marginal pdf:

$$p(x) = \int_{-\infty}^{\infty} p(x,y) dy$$

$$p(y) = \int_{-\infty}^{\infty} p(x,y) dx$$

Conditional pdf:

$p(x,y) = p(x).p(y)$ if x and y are independent

$= p(\frac{x}{y = const}) p(y)$ otherwise

$$p(\frac{x}{y = const}) = \lim(\Delta x \to \infty) \frac{P(\frac{x_n \le x \le (x_n + \Delta x)}{y = y_n})}{\Delta x}$$

Expectation of a function: f(x, y)

$$E\{f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y).p(x,y)dxdy$$

Time Series Analysis of a Single Random Variable

Time series indicates a series with chronologically arranged values. In wave analysis we analyze such series by assuming that the wave process in the short term (typically over 3 hr) is both stationary (statistics like mean and variance are same over the short term) and ergodic (averaging along the ensemble (say means over time t_1, over t_2,…) same as the one over time axis of a given sample). With this assumption we redefine the earlier statistics as follows:

Let $x_1(t)$ = value of the random variable x_1 at time instant t
$x_1(t+\tau)$ = value of the random variable x_1 at time instant $t + \tau$
T = total duration of observations.

For the variable x_1 then:

Mean $\bar{x} = \frac{1}{T}\int_0^T x_1(t)dt$

Let $x(t) = x_1(t) - \bar{x}$

Variance: $\sigma_x^2 = \frac{1}{T}\int_0^T x^2(t)dt$

Standard deviation: $\sigma_x = \sqrt{\sigma_x^2}$

Second Central Moment: $\mu_2 = \sigma_x^2$

Third Central Moment : $\mu_3 = \frac{1}{T}\int_0^T x^3(t)dt$

Fourth Central Moment : $\mu_4 = \frac{1}{T}\int_0^T x^4(t)dt$

Auto-correlation function:

It is the average lagged product of neighboring values. For any time lag ‘τ’,

$R(\tau) = \frac{1}{T}\int_0^T x_1(t).x_1(t+\tau)dt$

Auto-covariance Function: $R'(\tau) = \frac{1}{T}\int_0^T x(t).x(t+\tau)dt$

Note: $R(\tau=0) = \frac{1}{T}\int_0^T x(t).x(t)dt$ = variance of x_1

Power or Energy Spectral Density Function: This is the Fourier Transform of R(τ). For any wave frequency 'f', $S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau$

A graph of S(f) versus f is called the wave spectrum. Note by the property of the Fourier Transform: $R(\tau) = \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} df$

Physical Significance of wave surface spectral density function: S(f)

Let η_1 (t) = instantaneous sea surface elevation

T = total duration of the wave record (say 3 hr)

$\bar{\eta}$ = mean of all η values = $\frac{1}{T}\int_0^T \eta_1$ (t)dt

Let us define η (t) = η_1 (t) - $\bar{\eta}$

Then $R_\eta(\tau) = \frac{1}{T}\int_0^T \eta(t).\eta(t+\tau)dt$

Hence $R_\eta(\tau=0) = \frac{1}{T}\int_0^T \eta^2(t).dt$

= variance of the sea surface elevation, σ_η^2

But since R (τ) and S (f) are Fourier transforms of each other,

$R_\eta(\tau=0) = \int_{-\infty}^{\infty} S(f)df$ = area under the spectrum = σ_η^2 (3.1)

Thus the area under a wave spectrum will represent the variance of the sea surface elevation.
Note: for a deterministic wave (linear) :

Surface elevation: $\eta(x, t) = a \cos(kx - \omega t)$ (3.2)

(where, x = horizontal coordinate, t = time instant, a = wave amplitude, k = wave number' ω = circular wave frequency = 2πf'.)

Energy/plan area, $E = (1/2)\, \gamma\, a^2$ (3.3)

For a random wave, we assume that the surface is formed by superposition of many linear waves, each having a different frequency, height and further that such a combination is formed by adding them randomly or by selecting their phases in a random manner within the interval (0, 2π).

Hence $\eta(x, t) = \sum_{allj} a_j \cos(k_j - \varpi_j + \phi_j)$ where ϕ_j = random within (0, 2π).

Comparing this equation with (3.1) it follows from (3.2) that for a random wave, the energy per unit plan area is:

$$E = \frac{1}{2}\gamma \sum_{allj} a_j^{\ 2}$$

According to Parseval's theorem in classical spectral analysis:

$$\frac{1}{2}\sum_{allj} a_j^{\ 2} = \sigma_\eta^2$$

Hence $E = \gamma\, \sigma_\eta^2$

$$= \gamma \int_{-\infty}^{\infty} S(f)df$$

Thus the area under the spectrum represents the total energy of the random wave in a given record per unit plan area. Similarly the wave energy, ΔE, within the band of df = S(f)df.

3.1 Wave Spectrum Analysis

Introduction

The short term wave analysis is restricted to a single wave record observed for a short interval of time (say twenty minutes or half an hour) for which the sea conditions are assumed to be stationary (not much change in mean, variance) and are such that the wave properties can be studied around mean values. As stated earlier the simplified method of spectral analysis consists of characterizing the sea state by superposing a large number of linear progressive waves each with different height, period, length and random phase difference (See Figure 3.1).

Mathematically, this can be expressed as follows:

$$\eta(x,t) = \sum_{n=1}^{M} a_j \cos(k_j x - \omega_j t + \theta_j) \qquad 3.2$$

Where

$\eta(x,t)$ = Sea surface elevation being considered at a point which is at a horizontal distance 'x' from any chosen origin and at time instant 't'.

M = Number of linear waves being added together.

a_j = Amplitude of the j^{th} wave.

k_j = Wave number of the j^{th} wave = (2π)/Length of the j^{th} wave

ω_j = Angular wave frequency = (2π)/Period of the j^{th} wave

= (2π)x frequency in cycles per second of the j^{th} wave

θ_j = Phase of the j^{th} wave assumed to be uniformly distributed over the interval $(0,2\pi)$

As seen earlier the amplitude of the component wave is related to an important statistical function called the Spectral Density Function by the relationship.

$$a_j = \sqrt{2S_\eta(\omega_j)\Delta\omega} \qquad 3.3$$

where

$S_{\eta}(\omega_j)$ = Spectral density function corresponding to the frequency ω_j for sea surface (η).

$\Delta\omega$ = frequency step or interval used in calculating above mentioned function.
(Note: since area under the spectrum has to be the same S(ω) dω = S(f) df).

From Equations (3.2) and (3.3), it may be clear that a wave spectrum can be derived from a given time history of past observations of sea surface elevations. The spectrum of waves so established is actually a simplified model of the generalized three dimensional representation of the sea surface. The wave spectrum has a number of practical applications. Once a wave spectrum is known, a variety of information can be deduced from it.

The significant information obtained is that of the wave frequency composition (sea or swell components) in a given wave sample.

Further the area under the wave spectrum gives the total energy of the irregular wave system per plan area and also the variance value of the water surface fluctuations.

The wave spectra when multiplied by suitable transfer functions yield the response spectra that are useful in structural design.

The wave spectrum is also used in generating the random sea in a laboratory.
The integration of the wave spectrum involving different powers of wave frequencies yields important design statistics like significant wave height and average zero cross period.

Let $m_n = \int_0^\infty f^n S_\eta(f)df$

Hence $m_0 = \int_0^\infty S_\eta(f)df = \sigma_\eta^{\ 2}$

$$m_2 = \int_0^\infty f^2 S_\eta(f)df$$

According to Cartwright and Longuett Higgins, the significant wave height or the average height of the highest one third is given by:

$$H_s = H_{1/3} = 4\sqrt{m_0}$$

While the average zero-cross wave period is:

$$T_z = \sqrt{\frac{m_0}{m_2}}$$

For the sake of economy as well as convenience in data collection and handling the wave records of instantaneous sea surface elevations are collected for about 10 to 30 minutes only, within each 3 hours' duration. The spectral density function for such a short-term record can be calculated by two different methods. (Note: Average zero cross period refers to an average of all periods defined by up-crosses of the zeroth level or Still Water Level as explained in Figure 3.5.

Sometimes a parameter called Spectral Width Parameter (ε) that shows whether a spectrum is narrow or broad banded is necessary to obtain, which is given by:

$$\varepsilon = \sqrt{1 - \frac{m_0^2}{m_0 m_4}}$$

The value of $\varepsilon < 0.75$ usually means a narrow banded spectrum.

Covariance Method

The surface (η) spectral density function $S_\eta(f)$ for wave frequency of *f* is obtained by taking the Fourier Transform of auto-correlation function $R_\eta(\tau)$ for all time lag values τ, i.e.

$$R_\eta(\tau) = \int_0^\infty \eta(t)\eta(t+\tau)dt \qquad 3.4a$$

$$S_\eta(f) = 4\int_0^\infty R_\eta(\tau)\cos(2\pi f\tau)d\tau \qquad 3.4b$$

where, the factor '4' results from the changed limits of integration [($-\infty, \infty$) to (0, ∞)] for avoiding negative frequency considerations and the fact that the area under the spectrum must remain same before and after changing the limits,

$\eta(t)$ = Sea surface elevation at time t

$\eta(t+\tau)$ = Sea surface elevation at time $t+\tau$

A part of an actual wave record is typically shown in Figure 3.3. An example of variation of $R_\eta(\tau)$ against various lag or τ values is given in Figure 3.4 from which it is evident that the 'auto-correlogram' shows an oscillatory decay for random ocean waves. As can be seen from this figure there is less correlation among the surface elevation values separated by larger time lags. The examples of wave spectral plots showing $S_\eta(f)$ versus *f* are given in earlier referred Figure 3.6.

Fast Fourier Transform

This is a faster method (FFT) to arrive at $S_\eta(f)$ values and is very useful when large data are required to be handled. This technique however is relatively complex and reference could be made to Bendat and Piersol (1986) and Newland (1975). However the principle involved in it is given below:

In the covariance method of obtaining $S_\eta(f)$ values, the same exponentials appear several times in the calculations. This can be avoided by taking total number of observations (N), say N = 2^m, or, 3^m or 5^m, where m is usually 10 or 11.

The actual formulae involved in the use of FFT are different than those used in the covariance method. In the FFT, the spectral density function for circular frequency ω (=2πf; f is frequency in Hertz) or S_η *(ω)* is calculated directly from the observed $\eta(t)$ value, as follows:

$$\eta_k = \frac{1}{N}\sum_{j=0}^{N-1}\eta_j e^{-i\left(\frac{2\pi jk}{N}\right)} \qquad 3.5$$

where,

η_k = Discrete Fourier Transform of N values

N = Total number of observed η_j values

η_j = j[th] value of the sea surface elevation

j = 0,.........,N-1

k = 0,.........,N-1

I = $(-1)^{1/2}$

Then, $$S_\eta(\omega_k) = \frac{2\pi}{T}\eta_k^{\,*}\eta_k \qquad 3.6$$

If $$\omega_k = \frac{2\pi k}{T} = \Delta f_k \qquad 3.7$$

where

ω_k = Circular wave frequency

T = Total duration of observations

Δ f = Frequency width = 2π/T

$\eta_k^{\,*}$ = Complex conjugate of η_k

***f-w* Conversions**

The wave spectrum can be plotted either as a graph of $S_\eta(f)$ versus f, (where f is wave frequency in Hz), or that of $S_\eta(\omega)$ versus ω (where ω is circular frequency in radians/sec). In any case, the energy in interval $\Delta\,\omega$ = Energy in interval df. Hence,

$$S_\eta(\omega)d\omega = S_\eta(f)df \qquad 3.8$$

Theoretical Spectra

When actual measurements of waves and their analysis as above are not intended, theoretical wave spectra would provide an approximate alternative.

There are several forms of such wave spectra proposed by different authors. Some of the important ones are given below.

Pierson-Muskowitz Spectrum

Kitaigorodskii had proposed a similarity hypothesis that the plots of the observed spectra have similar shapes if plotted in non-dimensional forms. Pierson-Muskowitz (1964) developed their spectrum based on this concept. They assumed that

$$S_\eta(\omega) = \phi(U, g, f) \qquad 3.13$$

where

U = Wind speed

f = Wave frequency,

$\omega = 2\pi f$

and further carried out dimensional analysis to arrive at a functional relationship for S(f). This involved constants that were determined by analysis of the data of North Atlantic Sea using curve fitting techniques. The resulting spectrum is

$$S_\eta(\varpi) = \frac{ag^2}{\varpi^5} e^{\left[-\beta\left(\frac{\varpi_0}{\varpi}\right)^4\right]} \qquad 3.14$$

where

α = Philip constant = 0.0081 (This is independent of U and wind fetch F)

β = 0.74

ϖ_0 = Frequency corresponding to the peak value of the energy spectrum

= $2\pi f_0$ = g/U_w

$$\text{Characteristics wind speed } U_w = \left\{\frac{H_s g\left(\beta/\alpha\right)^{1/2}}{2}\right\}^{1/2} \qquad 3.15$$

Equation (3.14) depends only on U_w and not on wind fetch, F (the distance over which the wind remains the same) or duration θ, Hence it is valid for fully developed sea that is produced when wind of unlimited fetch and duration blows, in which case the resulting wave height are not restricted by F or θ and all further input of energy from the wind is dissipated in breakers and not in wave growth.

Several alternative forms of equation (3.14) are available.

We have $S_\eta(\omega) = (1/2\pi)S_\eta(f); \omega = 2\pi f; \omega_0 = g/U_w$

Thus equation (3.14) becomes:

$$S_\eta(f) = \frac{\alpha g^2}{(2\pi)^4 f^5} e^{\left\{-\frac{\beta'}{f^4}\right\}} \qquad 3.16$$

where,

$$\beta' = 0.74\left(\frac{g}{2\pi U_w}\right)^4 \quad ; \alpha = 0.0081 \qquad 3.17$$

Bretschneider Spectrum

Bretschneider (1963) had earlier developed a similar form of spectrum but he had given $S_\eta(f)$ as a function of significant wave height H_s and T_s (which was empirically related to peak-energy frequency f_0) that are obtained from the SMB curves. The spectrum is described as:

$$S_\eta(f) = \frac{5H_s^2}{16 f_0 (f/f_0)^5} e^{\left\{-\frac{5}{4}\left(\frac{f}{f_0}\right)^{-4}\right\}} \qquad 3.18$$

or,

$$S_\eta(f) = \frac{\alpha'}{f^5} e^{\left[-\frac{\beta''}{f^4}\right]} \qquad 3.19$$

where

$$\alpha' = \frac{5H_s^2 f_0^4}{16} \quad \text{and} \quad \beta'' = \frac{5 f_0^4}{4} \qquad 3.20$$

Bretschneider spectrum is useful for undeveloped or developing sea, which are more generally met with.

JONSWAP Spectrum

A group led by Hasselmann et al. (1973) conducted wave observations under the Joint North Sea Wave Project (JONSWAP). They analyzed data collected in the North Sea and found out that the PM spectra underestimate the spectral peaks, which could be due to the assumption of fully developed sea conditions. Hence Hasselmann et al. suggested a new form of spectrum shown below that incorporates a peak enhancement factor (γ).

$$S(\omega) = \frac{\bar{\alpha} g^2}{\omega^5} e^{\left(-\bar{\beta}\frac{\omega_0^4}{\omega^4}\right)} \gamma^{e^{\left[-\frac{(\omega-\omega_0)^2}{2\omega_0\sigma^2}\right]}} \qquad 3.21$$

or

$$S(f) = \frac{\bar{\alpha} g^2}{2\pi^4 f^5} e^{\left(-\bar{\beta}\frac{f_0^4}{f^4}\right)} \gamma^{e^{\left[-\frac{(f-f_0)^2}{2 f_0 \sigma^2}\right]}} \qquad 3.22$$

where in

$\bar{\beta} = 1.25$ 3.22(a)

$\bar{\alpha} = 0.066\left(\frac{gF}{U^2}\right)^{-0.22}$ 3.22(b)

$\sigma = 0.07$ if $\omega \le \omega_0$ 3.22(c)

$= 0.09$ if $\omega > \omega_0$ 3.22(d)

ω_0 = peak-energy frequency = $2.84\left(\frac{gF}{U^2}\right)^{-0.33}$

γ = 3.3 average 3.23

Scott Spectrum

This is a modified form of the Darbyshire spectrum and it is based on data at a different site (Scotts 1965). This could be expressed in terms of H_s.

$$S(\omega) = 0.214 H_s^2 e^{\left[\frac{(\omega-\omega_0)^2}{0.065(\omega-\omega_0+0.26)}\right]^{1/2}} \qquad 3.24$$

for $\quad 0.26 < \omega - \omega_0 < 1.65$

$\quad = 0 \qquad$ otherwise

ω_0 is obtained by $\frac{d}{d\omega} S(\omega) = 0$

This is found to be good for Indian Conditions (Dattatri (1978) and Narasimhan and Deo (1979)) along with the Scotts Wiegel Spectrum explained subsequently. See Figure 3.6 as an example.

Scott-Wiegel Spectrum

Wiegel (1980) replaced the two constants (0.214 and 0.065) involved in the Scott spectrum by variables A' and B'.

$$S(\omega) = A' H_s^2 e^{-\left[\frac{(\omega-\omega_0)^2}{B'(\omega-\omega_0+0.26)}\right]^{1/2}} \qquad 3.25$$

A' and B' are given in tabular form as a function of H_s.

3.2 Wave Statistics

The objective of the above mentioned spectral analysis was to derive wave spectrum for every short term record and obtain a variety of information out of it. Complimentary to this we carry out statistical analysis of short term records to get various types of probability distributions of wave heights and wave periods and know design and operational values of the same from them.

Short term Wave Statistics

The Gaussian normal probability function describes the probability structures of naturally occurring processes well. Thus if η represents the sea surface elevation at any time instant 't' the its probability density function p (η) and distribution function P (η) can be respectively give as:

$$p(\eta) \frac{1}{\sigma_\eta \sqrt{2\pi}} e^{\frac{-(\eta-\bar{\eta})^2}{2\sigma_\eta^2}} \; ; \quad \text{and} \quad P(\eta) = \int_{-\infty}^{\infty} p(\eta) d\eta$$

Where, $\sigma_\eta^2 = \frac{1}{T}\int_0^T (\eta - \bar{\eta}^2)^2 dt$ and $\bar{\eta} = \frac{1}{T}\int_1^T \eta(t) dt$

Where, T = record duration

Assuming that (i) probability distribution of instantaneous water surface fluctuations is Gaussian normal and (ii) the wave energy is confined to a narrow range of frequencies, the probability density function of individual (or successively occurring) wave heights (H), which is two times the wave amplitude, is given by a typical 'Rayleigh distribution'

$$p(H) = \frac{H}{4\sigma_\eta^{\ 2}} e^{-\frac{H^2}{8\sigma_\eta^{\ 2}}} \qquad 3.26$$

where $\sigma_\eta^{\ 2} =$ variance of the sea surface elevation.

Hence $P(H) = \int_{-\infty}^{H} p(H)dH = 1 - e^{-\frac{H^2}{8\sigma_\eta^{\ 2}}}$

Based on this assumption the height H of all waves exceeding a numerical value H' can be derived and thus it can be shown that:

$\overline{H_{1/3}} = Hs = 4\sigma_\eta = 4\sqrt{m_0}$ (average height of the highest 1/3 of all waves)

$\overline{H}_{1/10} = 5.08\sigma_\eta$ (average height of the highest 10% of all waves)

$\overline{H}_{1/100} = 6.67\sigma_\eta$ (average height of the highest 1 % of all waves)

The expected value of the maximum wave (or the most probable maximum wave) height in a given duration can also be shown to be derived as:

$$E\{H_{\max}\} = \overline{H}_{\max} = 0.705 Hs\sqrt{\log_n N}$$

Where N = total number of waves in the given duration of time.
Based on the assumption of the Rayleigh distribution function Hs can be related to the root-mean square (rms) wave height as under:

$H_{1/3} = 1.416\ H_{rm}$; $H_{1/10} = 1.8\ H_{rms}$; $H_{max} = 2.172\ H_{rms}$ 3.30

Knowing any one wave height, we can compute other wave heights by the preceding relations.
An example of a typical short-term wave data analysis at an Indian location can be seen from Narasimhan and Deo (1979, 1980, 1981). Figure 3.7 shows how the observed data at Bombay High satisfactorily matches with the theoretical distribution of Rayleigh.

The statistical distribution of wave periods can also be described by theoretical distributions; though its use is very much restricted in practice. Typically if T_z is average zero cross period, the probability density function of individual wave period (T_z) is given by (Bretschneider, 1977).

$$p(T) = 2.7\left(T^3/T_z^4\right)\exp\left(-0.675\left(T^4/T_z^4\right)\right) \qquad 3.31$$

There is generally a lack of strong perceptible correlation in the joint occurrence of wave height and wave period values. This can be seen from Figure 3.8 which shows that a given wave height can occur along with a range of values of wave periods and further that the largest waves are rarely associated with the longest periods.

Tucker's Method

Tucker's method is a quick and simple method for analyzing wave data and is based on observing only a few larger surface fluctuations of the sea state along with the total number of waves in that sea state. Tucker (1963) assumed the nature of wave spectra as narrow banded and gave the following expression to determine the root-mean-square wave height of the record.

$$H_{rms} = (2)^{1/2} H_1 (2\theta)^{-1/2} \left(1 + 0.289\theta^{-1} - 0.247\theta^{-2}\right)^{-1} \quad 3.31a$$

$$= (2)^{1/2} H_2 (2\theta)^{-1/2} \left(1 - 0.289\theta^{-1} - 0.103\theta^{-2}\right)^{-1} \quad 3.31b$$

H_1 = A+C ; H_2 = B+D ; $\theta = \log N_z$

where ,

A = Height of the highest crest in the given record above SWL
B = Height of the second highest crest above SWL
C = Depth of the lowest trough below SWL
D = Depth of the second lowest trough below SWL
N_z = Total number of zero up-crosses in the record.

For typical wave records collected at Bombay High, Figure 3.9 shows the extent of agreement between the H_{rms} values calculated by using Tucker method and those obtained directly from the entire record. It may be seen that despite the fact that the Tucker's method relies only on few observations of the highest and the second highest waves instead of the entire record, it gives satisfactory estimation of H_{rms} value.

Long-term Wave Height Statistics

Most of the structures are designed to withstand the design significant wave height having a return period of 100 years or so. Such a design wave can be derived from the long-term statistical distribution of H_s values.

The pre-requisite for the long term description of the wave heights is that of collection of short term (or 3 hourly) wave records over duration of at least one year and preferably more. From each short-term wave record, a pair of significant wave height (H_s) and average zero cross period (T_z) is derived. These data are often summarized in the form of a scatter diagram shown in Figure 3.10.

The mean, variance and other higher distribution moments of H_s are then calculated. These are used to establish one of the few theoretical long-term distributions of H_s as given below:

The data of significant wave height can be managed and used in three different ways for doing extreme value analysis (Goda, 2000).

a) *Total sample method/initial distribution method/Cumulative Distribution Function method:* This method utilizes the entire observed data to fit to some distribution function to obtain the Cumulative Distribution Function (CDF). The best fitting CDF is identified and extrapolated to a given period of years. According to Herbich (1990) the total sample method has drawbacks including lack of independence and deviation of observed distribution from the fitted one at the upper tail. However, according to U K Department of Energy (1987) the total distribution method is justified because when a large number of regularly measured wave height values are used, lack of independence between neighbouring values is offset by the huge volume of data and further because of negligible difference between the N-year return period wave height values derived from both total population and annual maxima at higher values of N.

b) *Annual maxima method:* In this method the highest wave in each year is considered for the analysis.

c) *Peak Over Threshold (POT) method:* The wave heights above a certain arbitrarily introduced threshold value are considered for further analysis in this method based on selecting a population of stroms. A storm is defined as the time when the wave height exceeds this threshold. The threshold ensuring independece could theoretically be obtained by correlation analysis.

Data used in the analysis should ideally be statistically independent, with least correlation between the data and homogeneous, with the sample having common parent distribution. Annual maxima method and POT method both satisfy the requisite of independency. However, the length of data in such cases many time is many times too less in this case.

The mean rate of extreme event is denoted by λ. It is defined with the number of events N_T during the period K years as

$$\lambda = \frac{N_T}{K}$$

The Probability Distributions

Several probability distributions have been used or proposed to describe extreme wave statistics. These include the log-normal and Extremal Type I, II and III probability distributions.
Gumbel and Weibull distributions have been the most commonly used distribution functions used for fitting wave data (Forristall, 1978; Longuet, 1980; Vikebø et al., 2003; Kumar and Deo 2004; Panchang and Li, 2006; Muraleedharan et al., 2007; Neelamani et al, 2007a, **2007b**; Persson and Rydén, 2010). These two methods are used in the current study.

Gumbel Distribution

Gumbel was the first to develop a statistical method for predicting the extreme values of natural random events. This method has been later adopted by ocean engineers to predict extreme wave events. The Cumulative Distribution Function (CDF) for the significant wave height, Hs in Gumbel distribution is given by Eq (3.2)

$$P(Hs) = e^{-e^{-\left(\frac{Hs-\gamma}{\beta}\right)}}$$

Where
γ : location parameter and β : scale parameter

$$\frac{1}{\beta} = \theta = \frac{\pi}{\sqrt{6\sigma^2}}$$

$$\gamma = \overline{Hs}\,\frac{0.5772}{\theta}$$

$\overline{Hs}$: mean of all Hs values

σ^2 : variance of all Hs values

Weibull Distribution

Weibull (1951) introduced a distribution function which got wide acceptance from ocean engineers for fitting wave data. The CDF for the significant wave height, Hs in Weibull distribution is given by Eq (3.5)

$$P(Hs) = 1 - e^{\left(-\left\{\frac{Hs-\gamma}{\beta}\right\}^{\alpha}\right)}$$

where

γ : location parameter, β : scale parameter and α : shape parameter

There are two versions of this distribution: two parameters and three parameters Weibull distribution. In the two parameter Weibull distribution, the location parameter (γ) is taken as zero.

Parameter Estimation

Parameters of the distribution need to be estimated to find the CDF of the selected distribution. Methods commonly employed for this purpose include plotting on probability paper, method of moments, method of least square and method of maximum likelihood. Details of these techniques are given in Appendix 3.1.

The parameters of the Gumbel distribution are straight forward to determine from Eq (3.3) and (3.4) using sample mean and variance. However, Weibull distribution parameters need to be estimated by adopting a specific method like plotting points in a probability paper, Method of Moments(MOM), Method of Least Squares (MLS) and Method of Maximum Likelihood (MML). MML is a commonly used procedure in estimating distribution parameters (Al-Fawzan, 2000). It has been reported to be used in parameter estimation for distribution fitted to waves (Panchang and Li, 2006). This method can be better than the other methods (such as the plotting position method) since it provides estimates that are consistent and asymptotically efficient and no other estimator has a smaller variance (Panchang and Li, 2006).

Design Wave Height Estimation

Once the distribution is selected and the parameters estimated, based on the return period required for the design, the design wave height can be estimated. The return period, T_r, is defined as the average time interval between successive events of the design wave being equaled or exceeded. The return value is the threshold value which defines a given return period.

Then return period and return value are given by Eq (3.6) and (3.7)

$$T_r = \frac{1}{\lambda(1 - P(Hs))}$$

$$Hs_r = P^{-1}\left(1 - \frac{1}{\lambda T_r}\right)$$

Various recommended practices for extreme wave analysis are discussed by Mathiesen et al., (1994).
To summarise, the standard procedure in the analysis of extreme statistics of significant wave height data is:

- Select data for analysis
- Fit a candidate distribution to the observed data
- Calculate the parameters
- Check the goodness of fit of the selected distribution
- Compute (extreme) return values from the fitted distribution

Detailed explanation of the statistical distributions, fitting procedures and goodness of fits and other relevant procedures can be found in standard books and reports such as Sarpkaya and Issacson (1981), World Meteorological Organization (1988), Chakrabarti (1987), Goda (2000), Thompson (2002), Massel (2005), Kamphuis (2006).

Appendix 3.1

Extreme Wave Analysis

The extreme wave height values are generally determined by two basic methods

a) from grouped data from a complete long-term data set and

b) from ordered data derived using a limited number of extreme values(Kamphuis, 2006**)**

Statistical Analysis of grouped wave data

In this method various probability distributions are fitted to the available. These include the log-normal and Extremal Type I, II and III probability distributions.

<u>Log-normal Distribution</u>

If random variable $Y = \log(X)$ is normally distributed, *then* X is a random variable with a log normal distribution. The distribution has two parameters μ (mean, β) and σ (standard deviation, α). To estimate the parameter by regression, the equation can be rearranged as

$$Y = Z = \Phi^{-1}(\mathrm{P}) = \frac{\ln\mathrm{Hs} - \overline{\ln\mathrm{Hs}}}{S_{\ln\mathrm{Hs}}} = \frac{1}{S_{\ln\mathrm{Hs}}}\ln\mathrm{Hs} - \frac{\overline{\ln\mathrm{Hs}}}{S_{\ln\mathrm{Hs}}}$$

where $\overline{\ln\mathrm{Hs}}$: mean and $S_{\ln\mathrm{Hs}}$: standard deviation

$$Y = \Phi^{-1}(\mathrm{P})\,;\ X = \ln\mathrm{Hs};\ \ A = \frac{1}{S_{\ln\mathrm{Hs}}};\ \ B = \frac{\overline{\ln\mathrm{Hs}}}{S_{\ln\mathrm{Hs}}} \qquad \ldots$$

Plot of lnHs vs Z (reduced variate) gives a straight line if the data is log-normally distributed with the equation of the best fit line as Z=A (lnHs)+B. From this the parameters of the distribution can be estimated.

Fisher-Tippett I distribution (Double exponential/FT-I/Gumbel)

CDF for the significant wave height, Hs in Gumbel distribution is given by

$$P(Hs) = e^{-e^{-\left(\frac{H-\gamma}{\beta}\right)}}$$

where

$$\frac{1}{\beta} = \theta = \frac{\pi}{\sqrt{6\sigma^2}}$$

$$\gamma = \overline{Hs}\frac{0.5772}{\theta}$$

$\overline{Hs}$: mean of all Hs values

σ^2 : variance of all Hs values

This may be linearized by taking the logs of both sides

$$-\ln\left(ln\frac{1}{P}\right) = \left(\frac{Hs-\gamma}{\beta}\right) = \left(\frac{1}{\beta}\right)Hs - \frac{\gamma}{\beta}$$

The reduced variate Z can be plotted against Hs to obtain a straight line with equation of the best fit line as $Y = A(Hs) + B$ provided the data follows this distribution.

$Y = -\ln\left(ln\frac{1}{P}\right) = Z$; $X = Hs$; $A = \frac{1}{\beta}$; $B = -\frac{\gamma}{\beta}$

Fisher-Tippett II distribution (FT-II, Frechet distribution)

FT II type of distribution is widely used in ocean engineering for predicting extreme events associated with winds and waves.

PDF of FT-II is given by

$$P = e^{-\left(\frac{Hs}{\beta}\right)^{-\gamma}}$$

It can be linearised by taking log on both sides to obtain

$$-lnP = \left(\frac{Hs}{\beta}\right)^{-\gamma}$$

Taking log again

$$-\ln\left(ln\frac{1}{P}\right) = \gamma\ ln\left(\frac{Hs}{\beta}\right)$$

$$-\ln\left(ln\frac{1}{P}\right) = \gamma lnHs - \gamma ln\beta$$

$$Y = -\ln\left(ln\frac{1}{P}\right); X = \ln Hs;\ \ A = \gamma;\ B = -\gamma ln\beta$$

In the linear expression $Y = AX + B$

Plotting $\ln Hs$ vs $-\ln\left(ln\frac{1}{P}\right)$ will yield a straight line if the data set follows FT-II distribution, from which slope, A, and intercept, B, can be estimated to calculate the parameters.

Fisher-Tippett III distribution (FT-III/ Weibull Distribution)

The above distributions have two parameters. A more versatile extreme value distribution is the three-parameter Weibull distribution. The cumulative distribution function (CDF) for the significant wave height, Hs in Weibull distribution is given by

$$P = 1 - e^{\left(-\left\{\frac{Hs-\gamma}{\beta}\right\}^{\alpha}\right)}$$

For particular parameter values, this distribution reduces to Rayleight distribution (when α = 2, ε = 0) or to exponential distribution (when α =1). There are two versions of this distribution: two parameter and three parameter Weibull distribution. In the two parameter Weibull distribution, the location parameter (γ) is taken as zero.

If $Q = 1 - P$

$$Q = e^{\left(-\left\{\frac{Hs-\gamma}{\beta}\right\}^{\alpha}\right)}$$

By taking the logs of both sides:

$$-lnQ = \left(\frac{Hs-\gamma}{\beta}\right)^{\alpha}$$

This can be rearranged as

$$\left(ln\frac{1}{Q}\right)^{1/\alpha} = \frac{Hs-\gamma}{\beta} = \mathrm{Z}$$

A plot of reduced variate Z with Hs will yield a straight line with best fit equation Z=A(Hs)+B

$$Y = \left(ln\frac{1}{Q}\right)^{1/\alpha} = Z; X = Hs; A = \frac{1}{\beta};\ \ B = -\frac{\gamma}{\beta}$$

Weibull distribution has three parameters (α, β and γ). Linear regression provides only two constants (A and B). To use linear regression, trial and error is done to determination of third coefficient (α). Assuming different values of α will change the curvature of the points. The parameter γ in the Weibull distribution is a lower limit of H (when H= γ, Q=1 or P=0). Thus γ is theoretically equal to the threshold value in a POT data set.

Estimation of distribution parameters

Distribution properties depend upon the parameter values assigned to them. The parameter values should be determined to provide the best empirical fit between the distribution and the data. Simplest approach is to plot the individual data points on the selected probability paper and then draw a straight line through these by visual observation (Type III, α chosen in advance). The slope and intercept can be estimated and parameter calculated as explained in the previous section. Other techniques employed in determining the parameters are

a) Method of Least Square (MLS)

The parameters obtained using this method corresponds to the minimum quadratic difference between the data points and a theoretical straight line. It is the most basic form- directly applicable to log-normal, TypeI, Type II distributions. Method gives slope A and intercept B of best fit line Z=AX+B in terms of coordinates of all data points.

b) Method of Moments (MOM)

In this method first two or three moments of the distribution are equated to those of the data, establishing a relationship between the parameters to be estimated and the sample mean, variance and skewness.

c) Method of Maximum Likelihood (MML)

MML attempts to estimate parameter which would give the data sample the highest probability of being observed in its particular form. The method results in estimating parameter which are unbiased and have a relatively small variance.

Let the random observations be x_1, x_2,…, x_n and the unknown parameters be Θ_1, Θ_2,…, Θ_m. Their joint probability probability distribution is $p_x(x_1, x_2,\ldots, x_n; \Theta_1, \Theta_2,\ldots, \Theta_m)$. Since for a random sample the x_i's are independent their joint distribution can be written $p_x(x_1; \Theta_1, \Theta_2,\ldots, \Theta_m)\, p_x(x_2; \Theta_1, \Theta_2,\ldots, \Theta_m)\ldots p_x(x_n; \Theta_1, \Theta_2,\ldots, \Theta_m)$. This is proportional to the probability that the particular random sample would be obtained from the population and is known as likelihood function.

$$L(\Theta_1, \Theta_2, \ldots, \Theta_m) = \prod_{i=1}^{n} p_x(x_i;\ \Theta_1, \Theta_2, \ldots, \Theta_m)$$

The values of these m unknown parameters that maximize the likelihood that the particular sample in hand is the one that would be obtained if n random observations were selected from $p_x(x; \Theta_1, \Theta_2,\ldots, \Theta_m)$ are known as the maximum likelihood estimators (Haan, 1977). The parameter estimation procedure becomes one of the finding the values of Θ_1, Θ_2,…, Θ_m that. Maximization of the likelihood function can be done by taking the partial derivative of $L(\Theta_1, \Theta_2,\ldots, \Theta_m)$ with respect to each of the Θ_i's and setting the resulting expression equal to zero. These m equations in m unknowns are solved for the m unknown parameters. Since most of the common density functions have an exponential form, the maximum likelihood estimator is obtained by maximizing the logarithm of L.

The values of distribution parameters, resulting from these methods, will be different for the same sample of data.

Goodness of fit

Distribution which best fit to the observed data need to be selected as the most probable parent distribution. Goodness of fit test is done to reject or accept the distribution and to choose between various fitted distributions. Various available tests include Kolmogorov-Smirnov, Chi-square and correlation coefficient.

a) Correlation coefficient

When the parameter estimation is done by the least square method, the degree of goodness of fit is simply represented with the value of correlation coefficient between the ordered data x_m and its reduced variate y_m; nearer the coefficient towards 1, better is the fit.

b) Chi-Square Test

In this test a comparison between the actual number of observations and the expected number of observations (according to the distribution under test) that fall in the class intervals is done. The test statistics is calculated from the relationship

$$\chi_c^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Where k is the number of class intervals, O_i is the observed and E_i the expected number of observations in the i^{th} class interval. The distribution χ_c^2 is a chi-square distribution with k-p-1 degree of freedom where p is the number of parameters estimated from the data (eg., p=3 for Weibull distribution) and α, the significance level. The hypothesis that the data are from the specified distribution is rejected if

$$\chi_c^2 > \chi_{1-\alpha,k-p-1}^2$$

c) Kolmogorov-Smirnov Test

In this test, a distance statistics, D, which gives the largest vertical distance between observed and theoretical CDF is evaluated.

1. Let $P_x(x)$ be the completely specified theoretical CDF under null hypothesis
2. Let $S_n(x)$ be the sample CDF based on observed x
3. Determine the maximum deviation, D

$$D = max\ |P_x(x) - S_x(x)|$$

4. If for the chosen significance level, the observed values of D are greater than or equal to the critical tabulated value of Kolmogorov-Smirnov statistics, the hypothesis is rejected (Haan, 1977).

Use of chi-square and Kolmogorov-Smirnov test is discouraged by hydrologists when testing hydrologic frequency distributions as the hydrologic frequency distributions are important at the tails, but these statistical tests are insensitive in the tails of the distributions.

d) Confidence Interval

Once a distribution has been fitted to a set of data by one or more methods, it becomes desirable to appraise the closeness of fit of the data points to the fitted distribution. The scatter data may best be described in terms of confidence limits on either side of the fitted line. The curves on either side of best fit line provides a series of confidence bands which indicate the confidence attached to any particular data point.

Design Wave Height Estimation

The probability that the extreme variate Hs does not exceed a given value Hs_u in one year is $P(\mathrm{Hs}_u)$, the CDF, by definition. If event of $\mathrm{Hs} \geq \mathrm{Hs}_u$ occurred in one year - Hs did not exceed Hs_u during the other n-1 years and exceeded Hs_u in n^{th} year. Probability of non exceedence for n-1 years is given by $P^{n-1}(\mathrm{Hs}_u)$and that of exceedence in one year is 1-F(Hs $_u$), the probability of the above event is

$$P_n = P^{n-1}(\mathrm{Hs}_u)[1 - P(\mathrm{Hs}_u)]$$

The expected value of n is the return period by definition

$$T_r = E[n] = \sum_{n=1}^{\infty} n\,P_n = [1 - P(Hs_u)]\sum_{n=1}^{\infty} n\,P^{n-1}(Hs_u) = \frac{1}{1 - P(Hs_u)}$$

Return value

$$Hs_r = P^{-1}\left(1 - \frac{1}{T_r}\right)$$

In case of POT with mean rate λ, each year is divided into segments of 1/λ year by assuming that each time segment has the same probability of extreme events. Then return period, return value and exceedence probability are given by

$$T_r = \frac{1}{\lambda(1 - P(Hs))}$$

$$Hs_r = P^{-1}\left(1 - \frac{1}{\lambda T_r}\right)$$

$$Q = \frac{1}{\lambda T_r}$$

For Log-normal distribution Eq D.1 and D.2 yields

$$lnHs_{T_r} = \overline{\ln \mathrm{Hs}} + S_{\ln \mathrm{Hs}}\Phi^{-1}(\mathrm{P}) = \overline{\ln \mathrm{Hs}} + S_{\ln \mathrm{Hs}}\Phi^{-1}\left(1 - \frac{1}{\lambda T_r}\right)$$

Or

$$Hs_{T_r} = e^{\left(\overline{\ln \mathrm{Hs}} + S_{\ln \mathrm{Hs}}\Phi^{-1}(\mathrm{P}) = \overline{\ln \mathrm{Hs}} + S_{\ln \mathrm{Hs}}\Phi^{-1}\left(1 - \frac{1}{\lambda T_r}\right)\right)}$$

For Gumbel distribution Eq D.4 and D.5 gives

$$Hs_{T_r} = \gamma - \beta \ln\left(ln\frac{1}{P}\right) = \gamma - \beta \ln\left(ln\left\{\frac{\lambda T_r}{\lambda T_r - 1}\right\}\right)$$

For Weibul distribution Eq D.14 and D. 15 produces

$$Hs_{T_r} = \gamma + \beta\left(ln\frac{1}{Q}\right)^{1/\alpha} = \gamma + \beta(ln\{\lambda T_r\})^{1/\alpha}$$

Using Eq D.28 and D.29, for a given return period, value of Hs can be calculated for Gumbel and Weibul distributions.

Encounter probability, E, is another quantity used in selecting design wave. This is the probability that the design wave is equaled or exceeded during a prescribed period, L, say the design life of a structure. The relationship between these quantities along with return period, T_r , and recording interval r is

$$E = 1 - \left(1 - {}^{r}/_{T_r}\right)^{L/r}$$

When r is associated with the largest wave height during a year

$$E = 1 - (1 - \frac{1}{T_r})^L$$

eg: Return Period (T_r) giving rise to an encounter probability (E) of 0.1 for a design lifetime L=100 years is 949 years.

Statistical Analysis of ordered data

Statistical analysis of ordered data is carried out in the following stages:

- Data of wave height collected over a long time, or hindcasted is arranged in descending order with largest wave having rank m=1 and the smallest wave with rank m=N, where N is the number of observed data points.
- The data is assumed to follow a randomly chosen distribution (based on prior knowledge/literature)
- A plotting formula is used to reduce the data to a set of points describing the probability distribution of the wave heights.
- These points are plotted on an extreme value probability paper corresponding to the chosen distribution
- A straight line is fitted through the points to represent the trend and if the points are not fitting into a straight line a different distribution is selected and the process repeated
- The line is then extrapolated to locate a design value corresponding to a chosen return period T_r, or a chosen encounter probability E.

Plotting Formula

In order to plot data, a value of P(H) is assigned to each value in sample. Data ordered and assigned a rank, m=1 for largest wave till m=N for smallest wave.
A simple estimate of the exceedence Q(H)=1-p(H) for each of the N heights is then given as

$$Q(H_m) = 1 - P(H_m) = \frac{m}{N+1}$$

This plotting formula has been demonstrated to introduce a bias peculiar to the distribution being estimated. A more general plotting formula may be

$$Q(H_m) = 1 - P(H_m) = \frac{i - c_1}{N + c_2}$$

3.1 gives the coefficients for so-called unbiased plotting position for each distribution. As α influence both the plotting position and the curvature of the weibull graph, some trial and error is necessary. The line of best fit for these points can be determined using the method of moments, the method of maximum likelihood or the least square analysis.

Table 3.1: Coefficient for unbiased plotting position for various distributions (Source: Kamphius (2006) and Goda(2000))

Distribution	c_1	c_2
Log-normal	0.25	0.125
Gumbel	0.44	0.12
FT-II	$0.44 + {0.52}/{\alpha}$	$0.12 - {0.11}/{\alpha}$
Weibull	$0.20 + {0.27}/{\alpha}$	$0.20 + {0.23}/{\alpha}$

Summary

Examples of derivation of long term distribution of wave height, (H_s) and design wave height can be seen in Deo and Burrows (1986), Deo and Venugopal (1991), Goswami et al. (1991), Kirankumar et al. (1989), Pagrut and Deo (1992), Soni et al. (1989), Baba and Shahul Hameed (1989) and Baba and Kurian (1988).

Derivation of the distribution parameters like α and u using Equations (3.32b) and (3.33) involves use of what is called the method of moments to fit the data where the underlying equations are worked out by equating the sample moments to the population moments. Alternatively, a least squares approach as well as the method of maximum likelihood functions can also be employed to obtain the distribution parameters like α ,u, A, B, C (Sarpkaya and Issacson, 1981). However, these techniques are more laborious and need not necessarily mean a better accuracy in the resulting estimates.

For better accuracy in the estimation of design H_s values, some times, the derived distribution of observed H_s values (i.e. $P(H_s)$ versus H_s) is fitted to all the four theoretical distributions as mentioned above. Then the theoretical goodness of fit criteria, like the Chi-square test, Kolmogorov-Smirnov test or Confidence Bands are applied to choose one particular distribution that most closely fits the observed H_s distribution. The prediction of 100 years H_s value is thereafter made based on this 'best-fit' distribution. (See Soni et al. 1989 for more details). Following example will illustrate the simple method to get the design H_s having a return period of say 100 years.

Example: Annual data of significant wave heights collected for a site along the East coast of India is given below:

H_s (in m)	0	1	2	3	4	5
No. of observations	1198	999	322	112	15	2

Obtain the design H_s value corresponding to 100 years return using the Gumbel distribution.

Solution:

Gumbel distribution is

$$P(H_s) = e^{\left\{-e^{[-\alpha(H_s-u)]}\right\}}$$

where, $\alpha = \pi / \left(6\sigma^2 H_s\right)^{1/2}$ and $u = \overline{H}_s - (0.5772)/\alpha$

$\overline{H}_s = \left(\sum H_s\right)/N$

= [0.5(1198)+1.5(999)+2.5(322)+3.5(112)+4.5(15)+5.5(2)]/2648

= 1.27 m

$\sigma^2_{H_s} = [\sum\left(H_s - \overline{H}_s\right)^2]/N$

= $[1198(0.5\text{-}1.27)^2+999(1.5\text{-}1.27)^2+322(2.5\text{-}1.27)^2+112(3.5\text{-}1.27)^2+15(4.5\text{-}1.27)^2+2(5.5\text{-}1.27)^2]/2648$

= 0.76

$\alpha = \pi / \left(6\sigma^2 H_s\right)^{1/2}$

= π /[6(0.76)]$^{1/2}$

= 1.47

$u = \overline{H}_s - (0.5772)/\alpha$

= 1.27-(0.5772/1.47)

= 0.877

For 100 years, $P(H_s)$ = 1-1(8*365*100) = 0.999 99 66

Hence from $P(H_s) = e^{\left\{-e^{[-\alpha(H_s-u)]}\right\}}$, H_s = u + [-ln-ln(0.999 99 66)]/α

= 0.877 + [-ln-ln(0.999 99 66)]/1.47

= 9.44 m

Long Term Distribution of Individual Wave Heights:

The long term distribution of individual wave heights was initially derived by Battjes and subsequently modified by Burrows as below:

$$\left[P(H)\right]_{LT} = 1.0 - \sum_{i=1}^{M^*} \exp\left(-2H^2 / H_{si}^2\right)\overline{T}_{zi}^{-1} W_i \Big/ \overline{T}_z^{-1}$$ where,

$P(H)_{LT}$ = Long term distribution of individual wave heights
H = Individual wave heights
M^* = Corresponds to limiting value of H_s at the site say due to water depth
H_{si} = Midpoint Hs value corresponding to i^{th} row of (H_s, T_z) scatter diagram (See Figure 3.11)

$$\overline{T}_{zi} = \sum_{allj} T_{zj} \left(W_{ij} \Big/ \sum_{allj} W_{ij} \right)$$

where,
T_{zj} = T_z value corresponding to the j_{th} column of (H_s, T_z) scatter diagram
W_{ij} = Total number of occurrence of H_s values in the (i, j) interval
W_i = $P(H_{is} + 0.5\Delta H_s) - P(H_{is} - 0.5\Delta H_s)$ obtained from the underlying observed and fitted distribution of H_s.

$$W = \sum_{alli} \sum_{allj} W_{ij}$$

P() = Cumulative distribution function of ()
ΔH_s = Class width of H_s in the (H_s, T_z) scatter diagram

$$\overline{T}_z = \sum_{i=1}^{M^*} T_{zi} W_i$$

A design individual wave height having a return period of 100 years is derived by reading the value of H from such a distribution curve that corresponds to a cumulative probability, $P(H) = 1 - 1/\left(\check{T}_z 3600 * 24 * 365 * 100)\right)$ since $\check{T}_z$ represents average number of waves per second in the long term. Figure 3.11 shows an example of long-term distribution of H values using the above equations.

TABLE 1

A Part Of The Typical Computer Printout For Spectral Analysis (DATE 2.7.78)

LAG NO.	AUTO CORRELATION	FREQUENCY	SPECTRAL DENSITY
0	1.000000	0.0	0.091563
1	0.668255	0.010	0.099117
2	0.127182	0.020	0.061174
3	-0.345670	0.030	0.055347
4	-0.609706	0.040	0.065691
5	-0.605127	0.050	0.084348
6	-0.412819	0.060	0.223711
7	-0.125942	0.070	1.463952
8	0.116177	0.080	5.341101
9	0.282754	0.090	9.498674
10	0.377365	0.100	8.477833
11	0.373429	0.110	5.138504
12	0.253866	0.120	3.327433
13	0.076564	0.130	2.585230
14	-0.116333	0.140	2.156492
15	-0.258884	0.150	2.038771
16	-0.318416	0.160	1.873202
17	-0.271008	0.170	1.090199
18	-0.128157	0.180	0.598183
19	0.033781	0.190	0.640248
20	0.163707	0.200	0.566144
21	0.203942	0.210	0.299311
22	0.182920	0.220	0.221035
23	0.129335	0.230	0.245811
24	0.049667	0.240	0.219826
25	-0.014684	0.250	0.237477
26	-0.090852	0.260	0.289445
27	-0.155121	0.270	0.274819
28	-0.165966	0.280	0.238758
29	-0.097720	0.290	0.179577
30	0.005123	0.300	0.176351

TABLE II

SOME OF TH IMPORTANT WAVE STATISTICS OBTAINED FROM THE SPECTRAL ANALYSIS

No.	Date	Variance m^2	Standard Deviatio m^2	Third Central Moment m^2	Fourth Central Moment m^2	Zeroth spectral moment m^2	Second spectral moment m^2	Fourth spectral moment m^2	Significant Wave Height m	Average Zero cross period Sec	Spectral width parameter
1	2	3	4	5	6	7	8	9	10	11	12
1	17.6.78	0.998	0.999	-0.030	2.940	0.997	0.020	0.0013	3.99	7.02	0.83
2	18.6	0.645	0.803	-0.153	3.426	0.641	0.015	0.0012	3.20	6.55	0.84
3	19.6	0.530	0.728	-0.028	3.024	0.528	0.010	0.0006	2.91	7.23	0.84
4	20.6	0.652	0.80	0.057	3.345	0.651	0.016	0.0011	3.23	6.36	0.79
5	21.6	0.827	0.909	0.044	2.869	0.822	0.021	0.0015	3.63	6.25	0.79
6	22.6	0.977	0.989	0.130	2.771	0.980	0.024	0.0015	3.96	6.36	0.78
7	23.6	1.069	1.034	0.034	2.969	1.066	0.024	0.0014	4.13	6.66	0.78
8	24.6	0.793	0.891	0.101	2.684	0.789	0.023	0.0016	3.55	5.84	0.76
9	25.6	0.732	0.856	-0.078	2.608	0.728	0.019	0.0013	3.41	6.17	0.78
10	26.5	0.514	0.717	-0.034	2.643	0.512	0.014	0.0009	2.86	5.96	0.75
11	27.6	0.391	0.626	-0.028	3.169	0.388	0.009	0.0006	2.49	6.63	0.81

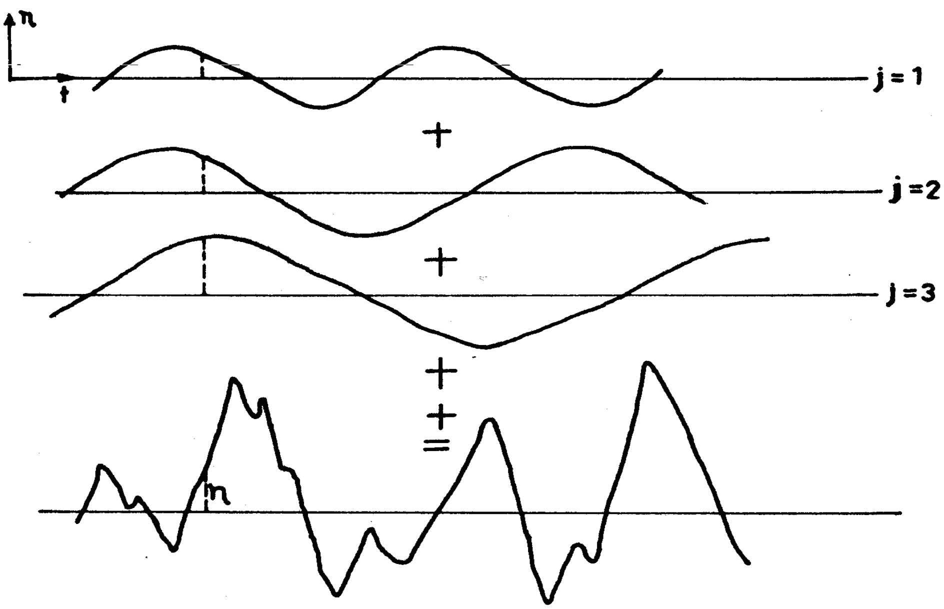


Fig 3.1 Superposition of Linear Waves

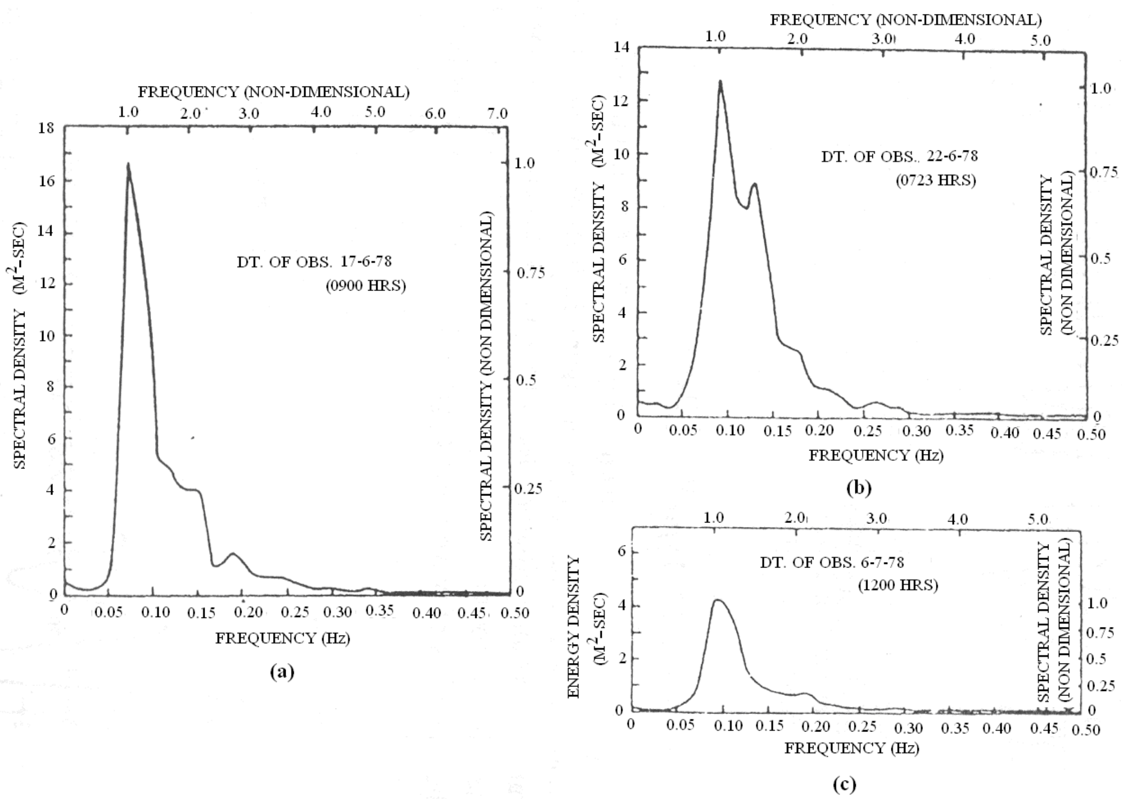


Fig.: 3.2 Examples of Wave Spectra

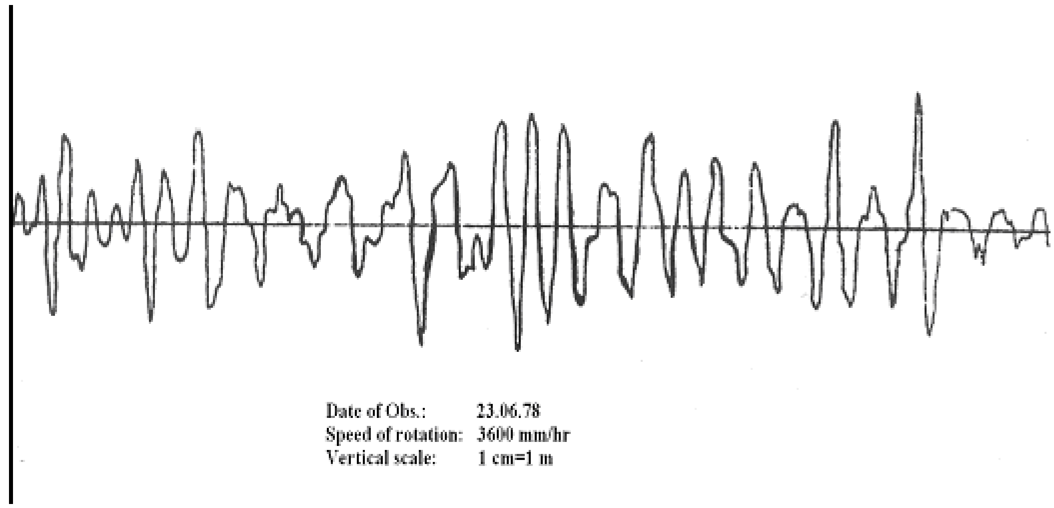


Fig 3.3 Typical Wave Record

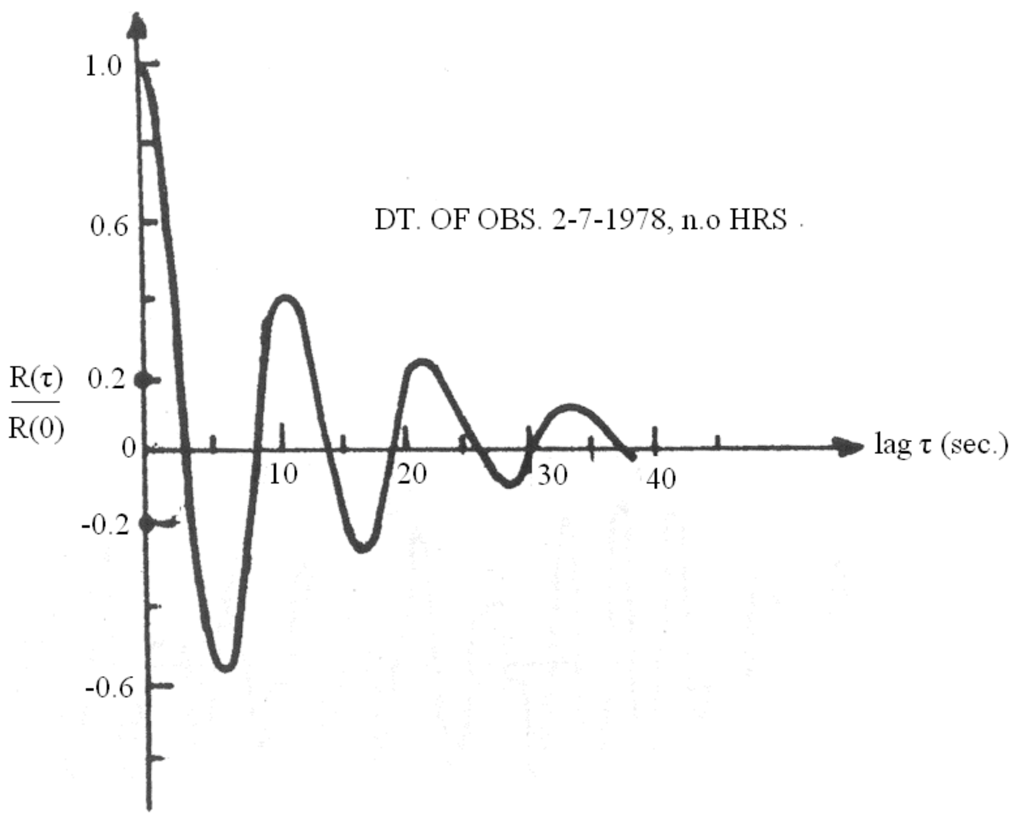


Fig 3.4 Correlogram

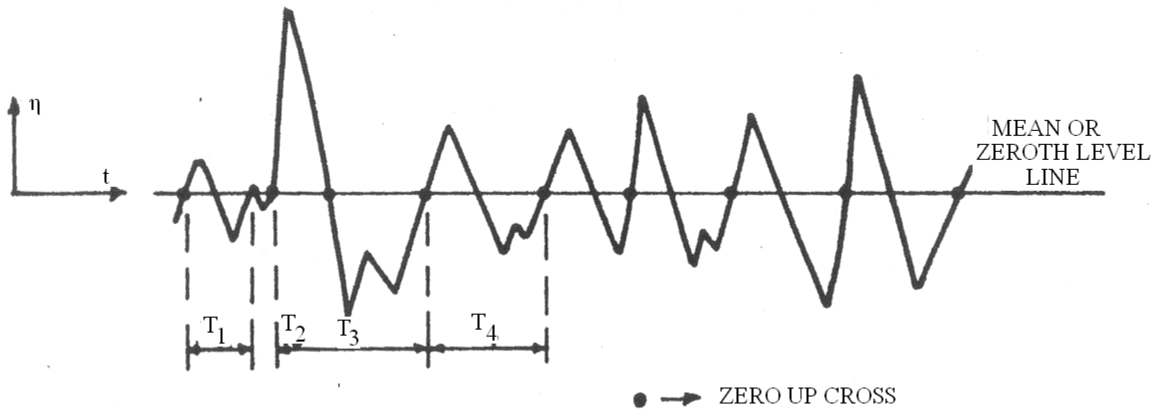


Fig 3.5 Individual Wave Periods

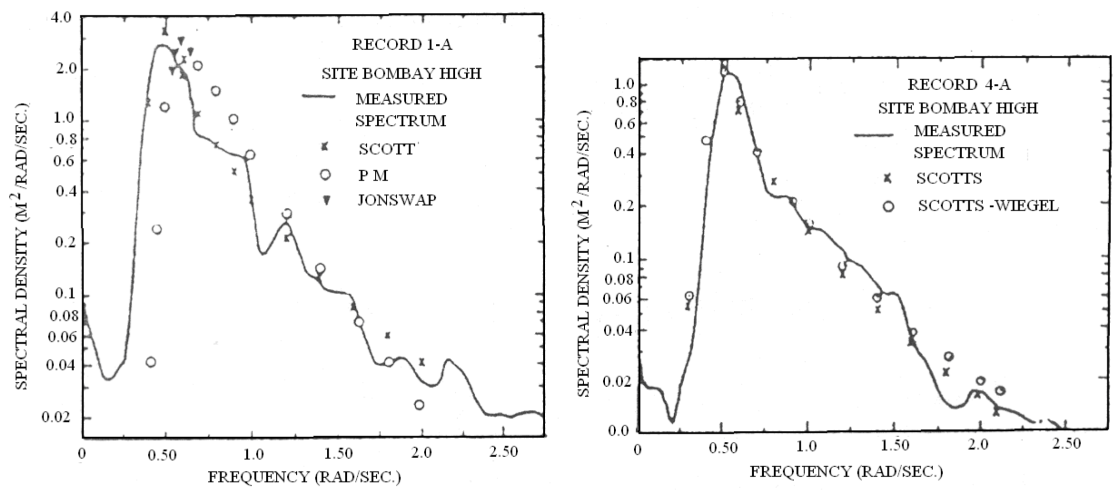


Fig 3.6 Comparison of Wave Spectra

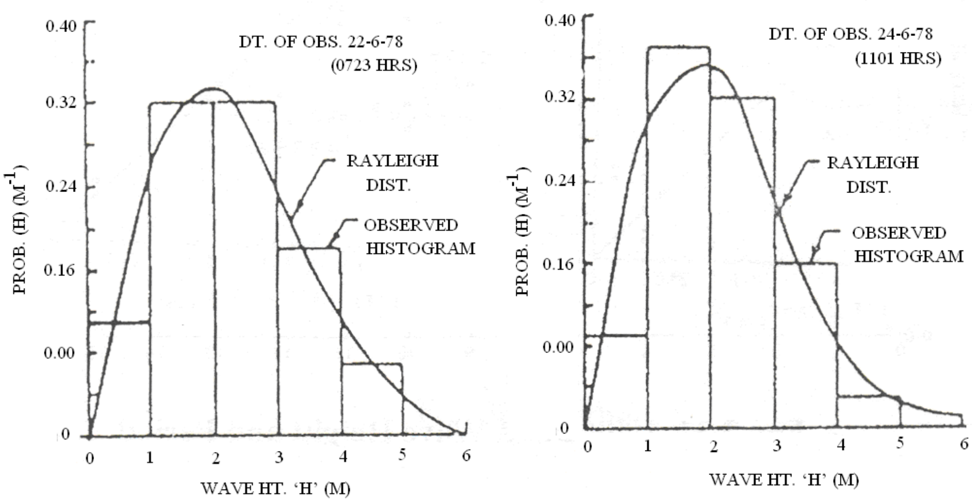


Fig 3.7 Wave height Distribution

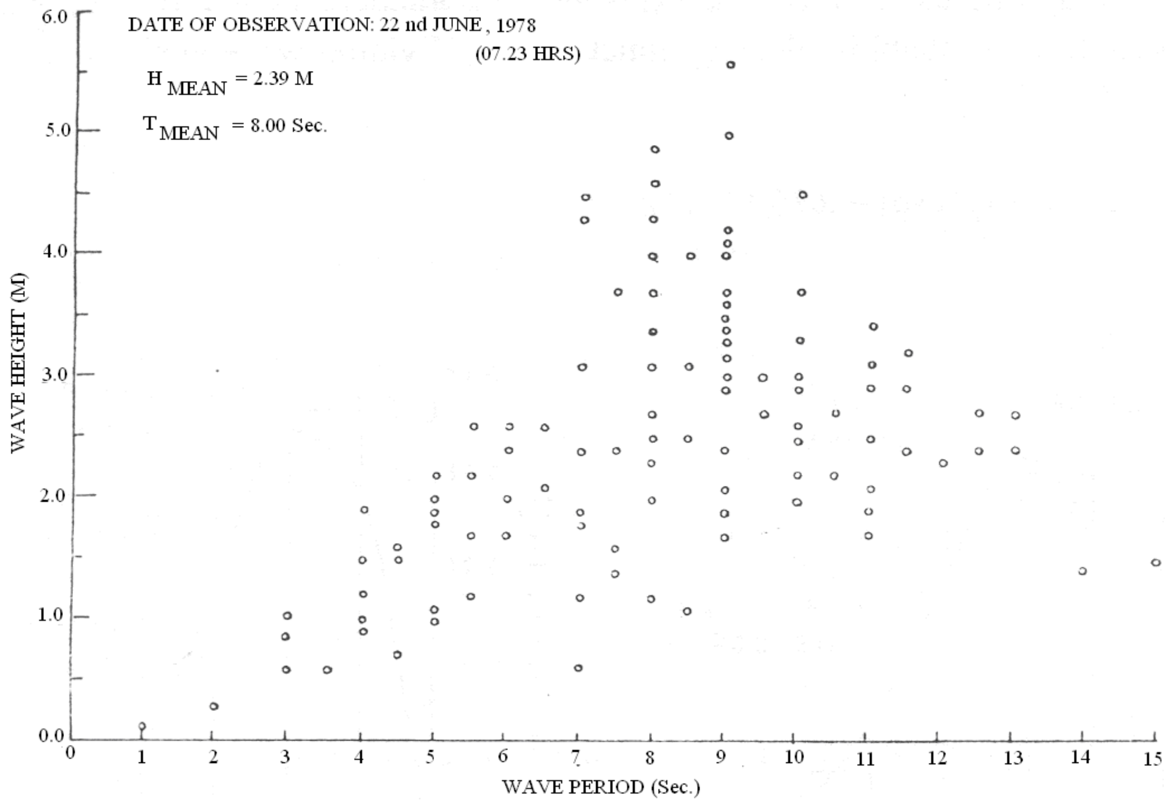


Fig 3.8 Joint Distribution of Wave Height and Period

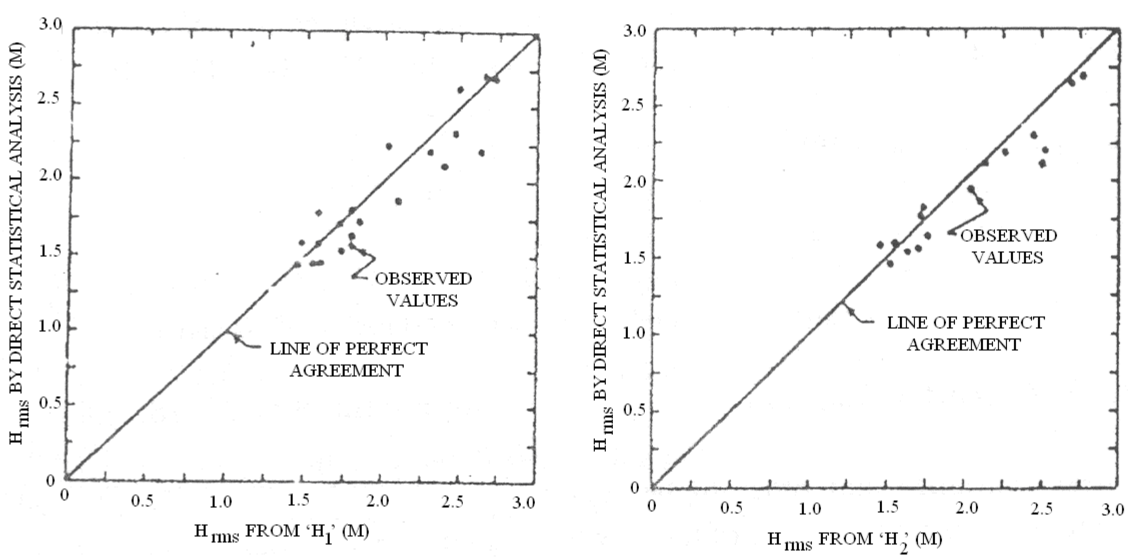


Fig 3.9 Tucker Method

T_z (sec)

H_s(m)	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17
0.0–0.5				3	4	1	1	2		2			1	
0.5–1.0	2	56	220	191	136	103	59	40	18	9	9	1	1	2
1.0–1.5		74	235	259	150	53	14	2	1	1				
1.5–2.0		4	33	34	36	8	2							
2.0–2.5			2		2									

Fig 3.10 Scatter Diagram (Wave data off Machilipatnam-From 9.5.83 – 18.12.83)

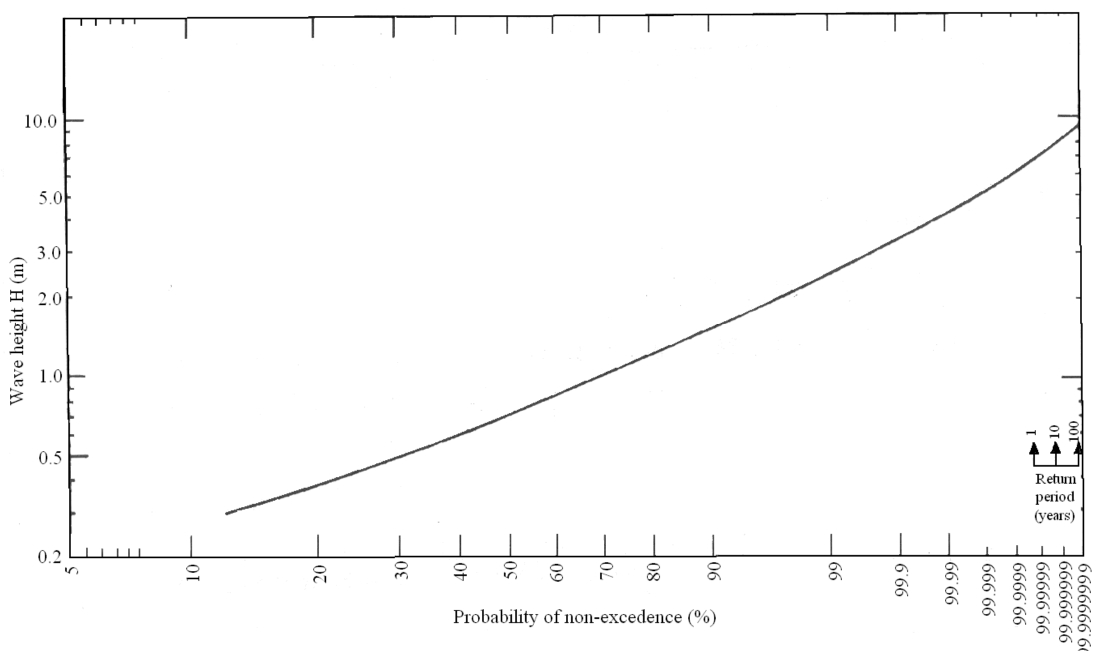


Fig. 3.11 Long Term Distribution of ‘H’
(Underlying H_s distribution – Weibull)

CHAPTER 4

WAVE PROPAGATION

When waves move from deep to shallow water their heights and angles of propagation change due to the change in water depth. This is caused by the effect of wave shoaling, refraction and breaking. Further, if waves meet a barrier, like breakwater, they get diffracted or reflected or both and this additionally causes change in wave heights and direction of their propagation. There is also a possibility of bottom induced diffraction and run up over a barrier structure.

4.1 Wave Shoaling

The change in water depth (d) produces corresponding change in the wave speed (C), which in turn modifies the wave group velocity (C_g). However, as the energy flux P (=E. C_g) always remains constant, the energy (E) also changes. This gives rise to change in wave height (H) as E α H^2. Equating the flux in deep water to the one in intermediate water, we get from linear theory,

$$H = K_s H_0 \tag{4.1}$$

where,

H = Wave height at any given water depth

H_0 = Wave height in deep water

K_s = Shoaling coefficient $K_s = \sqrt{\dfrac{C_0}{2nC}}$

C_0 = Deep water wave speed

C = Wave speed in intermediate water

N = f (kd) = $\dfrac{1}{2}\left[\dfrac{4\pi d / L}{\sinh(4\pi d / L)}\right]$

4.2 Wave Refraction

Referring to Figure 4.1, when deep water wave crest line strikes the sea bed contours at some non-zero angle, it tends to change its direction and align its wave crest with the sea bed contours. This is called the wave refraction. It occurs due to the fact that since the wave speed is proportional to the water depth, the part of the crest line on the landward side of contours moves slowly than the one in the seaward side. Wave refraction causes change in the wave height and direction (or pattern) of wave approach. Using the linear wave theory and assuming no lateral transfer of wave energy, it is possible to show by equating flux before and after refraction that:

$$H = K_s K_r H_0 \qquad 4.2$$

$$K_r = \sqrt{\frac{b_0}{b}}$$

b_0 is a chosen distance between any two orthogonals (direction of wave travel) before refraction and b is that after refraction. (See Figure 4.2)

If the contours are roughly straight and parallel,

$$\frac{b_0}{b} = \frac{\cos(\alpha_0)}{\cos(\alpha)} \qquad 4.3$$

where,

α_0 = Angle made by the incident crest line with the bottom contour over which it is passing

α = Angle made by the refracted crest line with the next bottom contour.

The value of b for given b_0 can be determined by following a graphical procedure (SPM 1984) which is called ray or orthogonal method. This method is briefly outlined in Appendix 4.1. By analogy with refraction of an optical ray the value of refracted angle α can be determined using:

$$\frac{C_0}{C} = \frac{\sin(\alpha_0)}{\sin(\alpha)} \qquad 4.4$$

Where C_0 and C are the wave speeds before and after refraction, respectively.

Example:

A wave has 3 m height and 7 seconds period in deep water. It travels towards shore over parallel bed contours. If its crest line makes and angle of 30^0 with the bed contour of 10 m before refraction, calculate the wave height after crossing this contour line.

Data:	d/L_0	d/L_0	n
	0.1300	0.1655	0.7621
	0.1310	0.1674	0.7606

Solution:

$$L_0 = \frac{gT^2}{2\pi} = 1.56(7)^2 = 76.5 \text{ m}$$

d/L_0 = 10/76.5 = 0.1307. This gives d/L = 0.1671 and n = 0.7616 from data.

Hence, L=59.84 m ; C= L/T = 59.84 / 7= 8.55 m/sec

$C_0 = L_0/T$ = 76.5 / 7 =10.93 m/sec

Now, $\frac{C_0}{C} = \frac{\sin(\alpha_0)}{\sin(\alpha)}$ 10.93/8.55 = sin 30^0/sin α and hence α =23.02^0

$$K_r = \sqrt{\frac{b_0}{b}} = (\cos\alpha_0/\cos\alpha)^{1/2} = (\cos 30/\cos 23.02)^{1/2} = 0.97$$

$$H_r = K_s K_r H_0 = \sqrt{\frac{C_0}{2nC}}\ K_r H_0 = \sqrt{10.93/[2(0.7616)(8.55)]}(0.97)(3) = 2.76 \text{ m.}$$

4.3 Wave Diffraction

In the harbour area, waves get diffracted or scattered when they strike a barrier like the tip of a breakwater (Figure 4.3). Unlike refraction, diffraction of waves involves energy transfer laterally along the crest line. Height of the incident wave as well as the pattern of its direction changes following diffraction. Based on analogy with optical diffraction, contours of equal diffraction coefficient (i.e. the ratio of diffracted to incident wave heights) have been presented (SPM 1984) for a single as well as a pair of breakwaters under regular wave attack. Figure 4.4 and 4.5 indicates two such typical cases involving incident wave attacks at angles of 30 and 120 with the length of a single breakwater, while Figure 4.6 and 4.7 show the isolines of diffraction coefficients in case of a pair of breakwaters spaced with two different gap lengths in between. If the gap length is more than two times the incident wavelength then each breakwater is assumed to diffract waves independent of each other. Before using these diffraction diagrams care has to be taken to reduce or enlarge them so that the linear scale of the diagram is same as the one of the given hydrographic chart showing the breakwater location. An irregular wave attack will involve a combination of many wave frequencies, each with an associated wave height (or energy density function). Diffraction for each constituent wave can be obtained separately and combined as per the directional spreading of wave energy. This has given rise to diffraction diagrams to obtain significant wave height and period of diffraction of an irregular wave from a breakwater or through a gap of two breakwaters (SPM 1985). (See Figure 4.8 as an example). Generally irregular waves indicate higher diffracted heights than the regular ones and further, a larger energy spread of the irregular incident waves is found to produce still larger wave heights.

Example: Confused sea waves are striking approximately normally against a semi-infinite breakwater in 6 m deep water with H_s = 2.5 m and T_s = 10 sec.
What are the values of H_s and T_s at a location 400 m behind and 400 m on the lee of the breakwater?

Solution:

$L_0 = \frac{gT^2}{2\pi} = 1.56(10)^2 = 156$ m

d/L_0 = 6/156 = 0.03846. This gives from the Tables given by SPM (1984),

d/L = 0.0816. Hence L = 75.53 m

x/L = 400/75.53 = 5.3 = y/L

S_{max} = 10

For the above values, height ratio = 0.31 and period ratio = 0.87 from Figure 4.8.

Hence, H_s = (0.31) 2.5 = 0.78 m and T_s = 0.87 (10) = 8.7 sec.

4.4 Wave Reflection

After striking a barrier the waves may have their energies swept back. Such a reflection of waves is undesirable for coastal or harbour structures. The amount of reflection depends on the barrier characteristics as well as properties of the incident waves. Slope of the obstruction, its permeability and roughness together with steepness of the incident wave and its angle of approach determine the amount of reflection. Reflected wave heights will be higher if barrier slope is steep, its permeability is low and if its face is smooth. Steeper waves will undergo more dissipation and less reflection. Wave attack that is normal to barrier length will produce more reflection than the obliquely striking waves.

Based upon the work of Seeling and Aherens (1981), SPM (1984) recommends graphs to determine the coefficient of reflection (K_r), defined as the ratio of reflected to incident wave heights), for beaches, plane slopes and rubble mound breakwaters and other protected slopes as a function of the Surf Similarity parameter, which is defined as below:

$$\xi = \frac{\tan\theta}{\sqrt{H_i/L_0}} \qquad 4.5$$

where,

θ = Reflecting slope angle

H_i = Incident wave height

L_0 = Deep water wave length

SPM (1984) also presents further graphical relationships to obtain the amount of reflection due to a sudden change in water depth and because of wave movement over bed ripples.

When an incident wave gets reflected from a smooth, vertical and an impermeable wall (Figure 4.9) it undergoes a pure reflection. Let the profile of the incident linear wave be given by:

$$\eta_r = \frac{H_i}{2}\cos\left(2\pi\left(\frac{X}{L} - \frac{t}{T}\right)\right) \qquad 4.6$$

where,

η_i = Sea surface elevation above SWL

H_i = Height of the incident wave

X = x co-ordinate where η_i is considered

t = time instant

L = Wave length

T = Wave period

The reflected wave will propagate in opposite direction and will have the profile:

$$\eta_r = \frac{H_i}{2}\cos\left(2\pi\left(\frac{X}{L} + \frac{t}{T}\right)\right) \qquad 4.7$$

The resultant profile then becomes

$$\eta = \eta_i + \eta_r = H_i \cos\left(\frac{2\pi X}{L}\right) + \cos\left(\frac{2\pi t}{T}\right) \qquad 4.8$$

This indicates that the resultant wave will have an amplitude equal to height of the incident wave (**Figure 4.9)**. The water particles at the nodes will undergo horizontal motion while those at the antinodes will follow a vertical one.

4.5 Combined Effects Using Numerical Solutions

When the sea bed contours are complex, rather than straight and parallel, the graphical ray orthogonal method fails as the neighbouring orthogonals cross each other forming what is called a caustic. In the vicinity of caustics bottom diffraction or scattering effects also become important and it becomes necessary to take into account combined effects of refraction and diffraction. Numerical solution of a differential equation describing the wave propagation is normally adopted in the situations (Kanetkar, 1996). Such a governing equation may be Laplace continuity Equation (2.11). However, the 3-D nature of this equation makes it difficult to solve.

Assuming that the sea bed slope in mild (in practice upto 1:3) integration of equation (2.11) over a vertical yields the following 2-D mild slope equation:

$$\nabla.\left(CC_g \nabla \phi\right) + \frac{C_g}{C}\omega^2 \phi = 0 \qquad 4.9$$

This equation can be solved numerically to get wave height and phase values over a space grid. It caters to both refraction and internal diffraction; but due to its elliptic nature requires solution formulation over the entire region. Hyperbolic and parabolic solutions are therefore better alternatives. Based on the time dependent form of the mild slope equation, Copland (1985) derived the following first order hyperbolic equation,

$$\left.\begin{aligned} &\frac{\partial \phi'}{\partial t} + \frac{C_g}{C}\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) = 0 \\ &\frac{\partial p}{\partial t} + CC_g \frac{\partial \phi'}{\partial x} = 0 \\ &\frac{\partial q}{\partial t} + CC_g \frac{\partial \phi'}{\partial y} = 0 \end{aligned}\right\}$$

$$\phi' = \phi(x, y)e^{-i\omega t} \qquad 4.10$$

x, y = Horizontal co-ordinate

p, q = pseudo fluxes that are dependent on vertical integration of x and y particle velocities.

The equations are solved in space and time using a Finite Difference scheme. They have the advantage of considering reflection effect also. But they involve more computational effort than the parabolic type due to time steeping in iterations.

A set of non-linear equations of hyperbolic form, derived by depth averaging of the Euler's equations forms an alternative to above equation (4.10). They are solved numerically to obtain the sea surface elevations at different times at various grid points. These can cover non-linear wave profiles as well as range of wave frequencies. However, converged solutions are possible where water depths are less than around 12% of the wave length in deep water.

When the boundary of wave propagation domain is a beach involving smaller reflection effects, the boundary value problem of the elliptic and hyperbolic equations can be converted into an initial value problem using a parabolic type of governing equation.

Ebersole (1985), using the wave irrotationality condition, separated the mild slope equation into real and imaginary parts. Then neglecting the reflecting component, he solved the resulting three equations in space domain using the Finite Difference scheme. The wave irrotationality condition restricts use of this equation where wave orthogonals cross each other.

Radder (1979) obtained the parabolic form by splitting the wave field into transmitted and reflected components and then neglecting the later. This equation is:

$$\frac{\partial \overline{\phi}}{\partial x} = \left(iK' - \frac{1}{2K'}\frac{\partial K'}{\partial x} + \frac{i}{2K'}\frac{\partial^2}{\partial y^2} \right)\overline{\phi} \qquad 4.11$$

$\overline{\phi}$ = Complex velocity potential of transmitted wave

K' = Modified wave number

One of the consequences of the parabolic approximations is that if the wave directions vary fro more than $\pm\ 30^0$ from the principal one then large errors are experienced. Kirby (1986) and Dalrymple and Kirby (1988) provided a remedy to this. However, generally computation with parabolic equations demand a rectangular domain which may not be possible in all of the fields situations.

4.6 Effect of Currents

When waves traveling with certain speed and direction meet a current flow their heights and lengths change. Superposition of the current speed on the wave speed and determination of wave energy thereafter gives the expressions to obtain the changes in wave heights and lengths caused by the current. Figure 4.10 can be used as an aid to determine wave height and length after it meets either following or opposing current. It may be noted that the opposing current reduces wavelength and increases wave height while the opposite is true for the current flowing in the same direction as that of the wave.

4.7 Wave Breaking

As mentioned in Section 1.3,when the steepness of the wave, i.e. the ratio of its height to its length increases beyond the theoretical value of 1/7 (which occurs when the angle subtended at the wave crest exceeds 120^0) the wave form becomes so unstable that it breaks. This is however true when the water depth is large enough not to interfere with wave particle motions. When this is not the case, the depth of water along with the seabed slope affects wave breaking. Empirical relationships in the form of a set of graphs are available to obtain the height of a wave at the time of breaking viz H_b (Figure 4.11) from its value in the deep water, viz., $H^{0'}$. The period of the wave (T) along with the slope of the seabed (m) are other input parameters that are needed to obtain H_b. A wave having unrefracted deep water height ($H^{0'}$) period (T) and propagating over shoaling bottom with bed slope (m) will break when it comes to depth (d_b). Figure 4.12 is then used to obtain this unknown value d_b once we find H_b from the previous Figure 4.11.

Waves are found to break in four different ways, viz. Spilling, Plunging, Collapsing and Surging. As waves approach the beach from deeper water, these four forms can be seen one after the other. The breaking involved in each case is schematically shown in Figure 4.13, which also gives the sequence of breaking in different cases.

The spilling type of breaking occurs in deep water or over gentle bed slope (m = 1:50). This involves gradual release of energy and is characterized by appearance of foam on forward side of the crest. Plunging and collapsing breakers occur on moderately steep slopes. In plunging, the water mass plunges and falls on forward face of the crest while in collapsing type the breaking is sudden, the crest form is steep and the foam is seen on lower side of the forward face. The last type of breaking i.e. surging, takes place when the seabed slope is steep with m = 1:10 or so. In this case the entire water mass gets piled up and the foam is seen on beach face.

Two parameters, viz. Galvin's and Irribarrens's (or Surf Similarity) are often used to distinguish between different types of breaking. If H_0 is the deep water height, L_0 is corresponding wavelength and m is the seabed slope then,

$$\beta = \left(H_0 / L_0\right) / m^2$$

$$N_1 = (\beta)^{-1/2}$$

Figure 4.13 shows different values of β and N_I exhibited in various breaking modes.

4.8 Wave Set up and Set down

Wave set up indicates the rise in the Mean Water Level (MWL) due to wave generated onshore transport of water mass. When the waves break near the coast in the surf zone, the broken water mass gets piled up against the bench slope following conversion of kinetic energy to potential energy. As a result the water level for a considerable distance offshore rises (See Figure 4.14). For this to happen a sustained wave attack of an hour or so is necessary so that the equilibrium surface gets formed. Calculation of wave set up is necessary to know whether low lying coastal area would be flooded or not in stormy waves.

Referring to Figure 4.14 at the point of wave breaking there is a lowering of water level. This is called Set Down at the breaking point. Quantitatively, it is the depression of MWL (the level of symmetry for wave oscillations) below the Still Water Level or SWL (the water level in absence of any waves) at the point of wave breaking and is denoted by S_b. From this point onwards there is an increase in MWL. The maximum MWL occurs when it touches the beach slope. This value measured from the SWL is called the net wave setup (S_w), while the same measured from the lowest MWL is called the total wave setup, (ΔS).

The set down at the breaking zone, (S_b), can be given by,

$$S_b = \frac{(g)^{1/2} H_0^2 T}{64 \pi d_b^{3/2}} \qquad 4.12$$

where,

H_0 = Unrefracted deep water significant wave height

T = Wave period

d_b = Depth of water at the breaking point.

Experimental investigations have shown that

$$\Delta S = 0.15\, d_b \qquad 4.13$$

$$\Delta S = S_w + S_b \qquad 4.14$$

This indicates that given the values of H_0, T and d_b the total set up at the shore (ΔS) can be calculated from equations (4.12), (4.13) and (4.14).

The value of d_b involved in above equations can be evaluated from Figure 4.16 referred to earlier or analytically by using,

$$d_b = \frac{H_b}{b - \frac{a H_b}{g T^2}} \qquad 4.15$$

$$a = 43.75\left(1 - e^{-19m}\right) \qquad 4.16$$

$$b = \frac{1.56}{\left(1 + e^{-19.5m}\right)} \qquad 4.17$$

m = Seabed slope

The set up produced by a group of irregular or random waves has different and complex features than the one generated by regular waves as described above. Some of the complications that arise in this case are that the random wave may strike the beach in groups with a small calm period in between any two groups. During this calm period some water will be pushed back to the sea reducing the effective set up.

If there is a wide flat bed, called a berm or a reef, present in between the surf zone and the beach slope as shown in Figure 4.15 the larger waves in a sea wave spectrum will break at the seaward end of the flat bed producing a set up which can sustain smaller waves within the flat portion. These waves will now break on the beach slope causing additional set up effects.

Laboratory as well as numerical experiments have been conducted to determine the wave set up for the random waves. Figure 4.16 shows the laboratory study based graphs to obtain $S/H_0^{'}$ values from $d/H_0^{'}$ values for different wave steepness or $H_0^{'}/L_0$ values for a 1:30 beach slope while Figure 4.17 shows numerical results for obtaining $S/H_0^{'}$ for a fixed magnitude of $d/H_0^{'}$ for different sea bed slopes.

Wave Runup

The height above the SWL up to which the incident wave rises on the face of the barrier is known as the wave run up. In Figure 4.18, R is the run up produced along a structure (of height h above its toe). Wave run up indicates a complex process that is dependent on a number of wave characteristics, structure conditions and local effects. These include regularity of randomness of the wave as well as its broken or unbroken state, wave steepness, slope of the structure, its roughness and permeability, water depth and sea bed slope.

Laboratory experiments have given rise to a set of guiding curves to obtain the run up. Figure 4.18 indicates an example of the same that gives the relative run up (R/H_0) (where $H_0^{'}$ is unrefracted wave height in deep water) against the structure slope (expressed as cot θ) for various values of deep water steepness, $H_0^{'}/gT^2$). This Figure 4.18 typically applicable when the ratio of water depth at depth (d_s) to unrefracted deep water wave height (H_0) is 0.8. Similar graphs are specified for few other values of the ratio $d_s/H_0^{'}$ (SPM 1984). Graphs like these suffer from small scale effects due to their derivation from laboratory works and hence the resulting run up value is required to be increased by applying a correction factor determined from structure slope and intensity of wave attack. SPM (1984) also gives graphs (Figure 4.18) to determine the wave run up in a similar way for vertical, stepped and curved cross section walls that are smooth and impermeable. If the wall face is rough and permeable then a roughness and porosity correction factor becomes applicable which is also tabulated in this reference. If the barrier has a composite slope, e.g. a beach with berms, then it is assumed to be replaced by a hypothetical uniform slope and the same design curves as referred to earlier are applied to evaluate the run up.

If the waves at a location can no longer be assumed regular then above mentioned design graphs are not sufficient. In such case the individual run ups produced by a train of random waves is assumed to follow Rayleigh distribution, like that of the incident wave heights, and run up (R_p) having the probability exceedence (P) is calculated as:

$$R_p = R_s \sqrt{\frac{-\ln P}{2}} \qquad 4.18$$

where,

$R_s =$ Run up produced by a regular significant wave height (H_s)

(H_s) = Significant Wave Height determined from the above referred design curves.

Above equation (4.18) assumes that the wave larger than the significant one break on the structure. If this is less likely in practice and if they break on the seabed slope away from the structure then the actual run up value will be smaller than the one given by above equation (4.18).

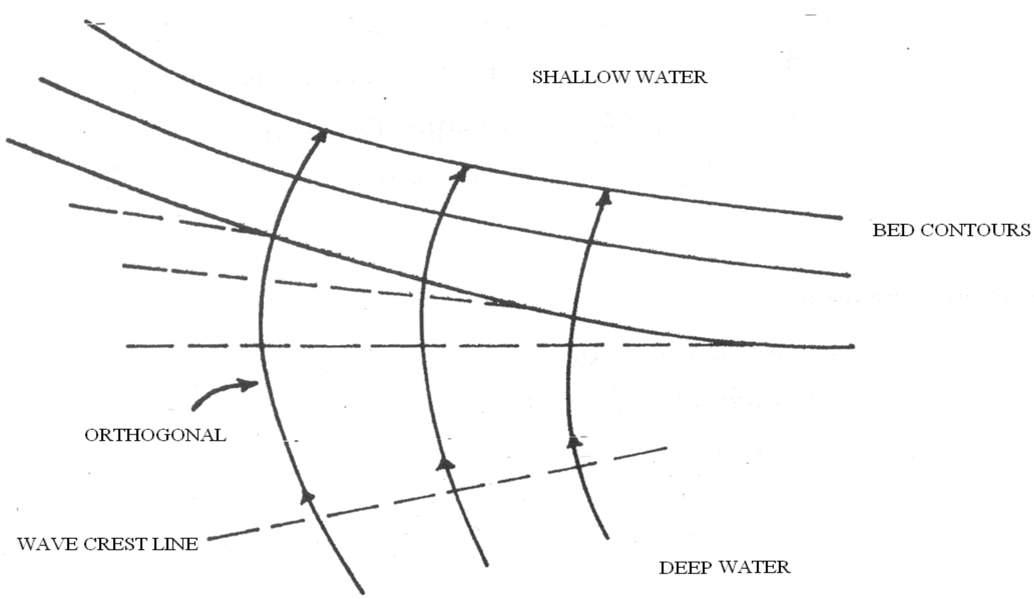


Fig 4.1 Wave Refraction

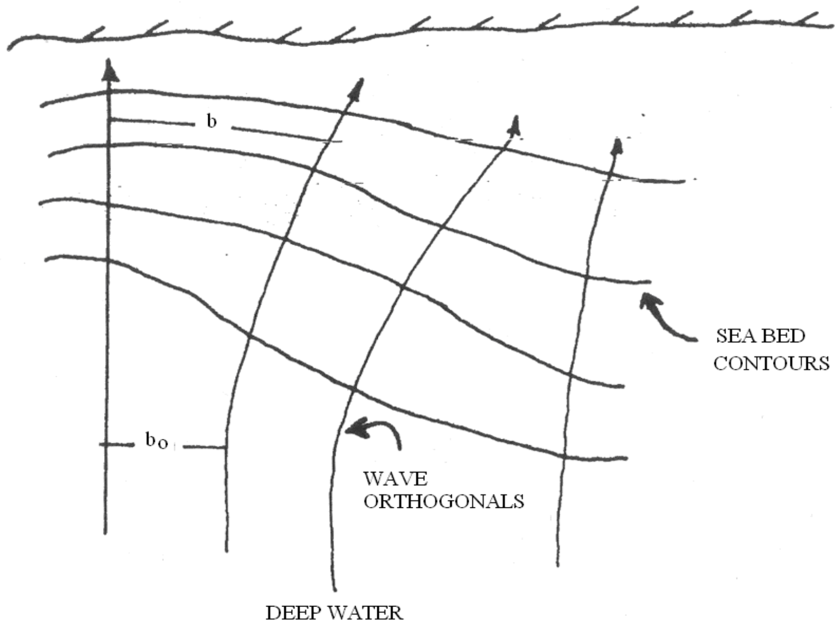


Fig 4.2   Wave Refraction

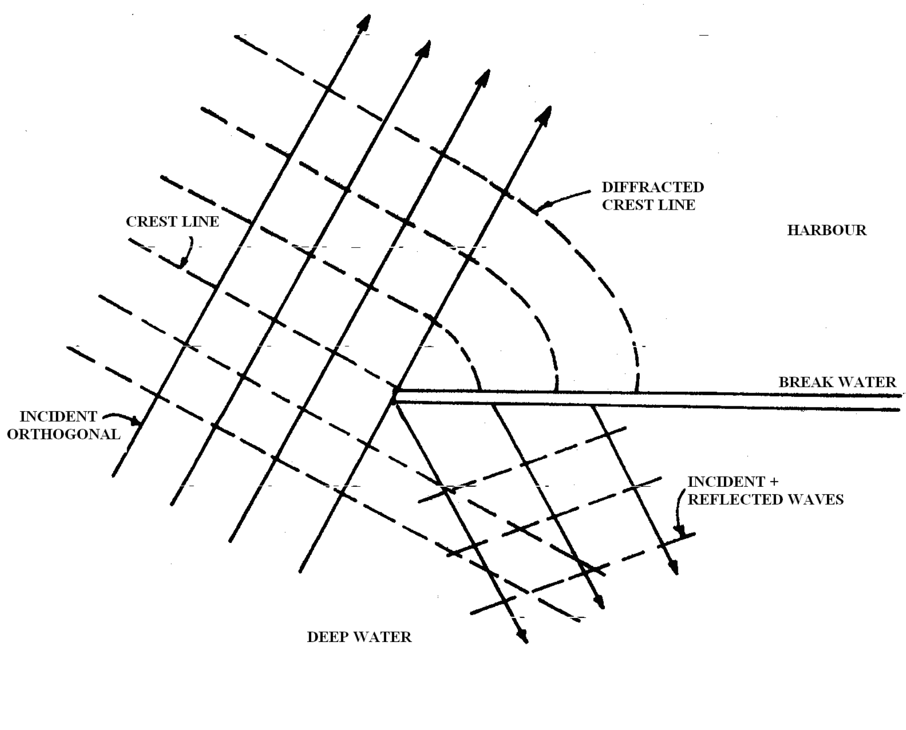


Fig 4.3 Wave Diffraction

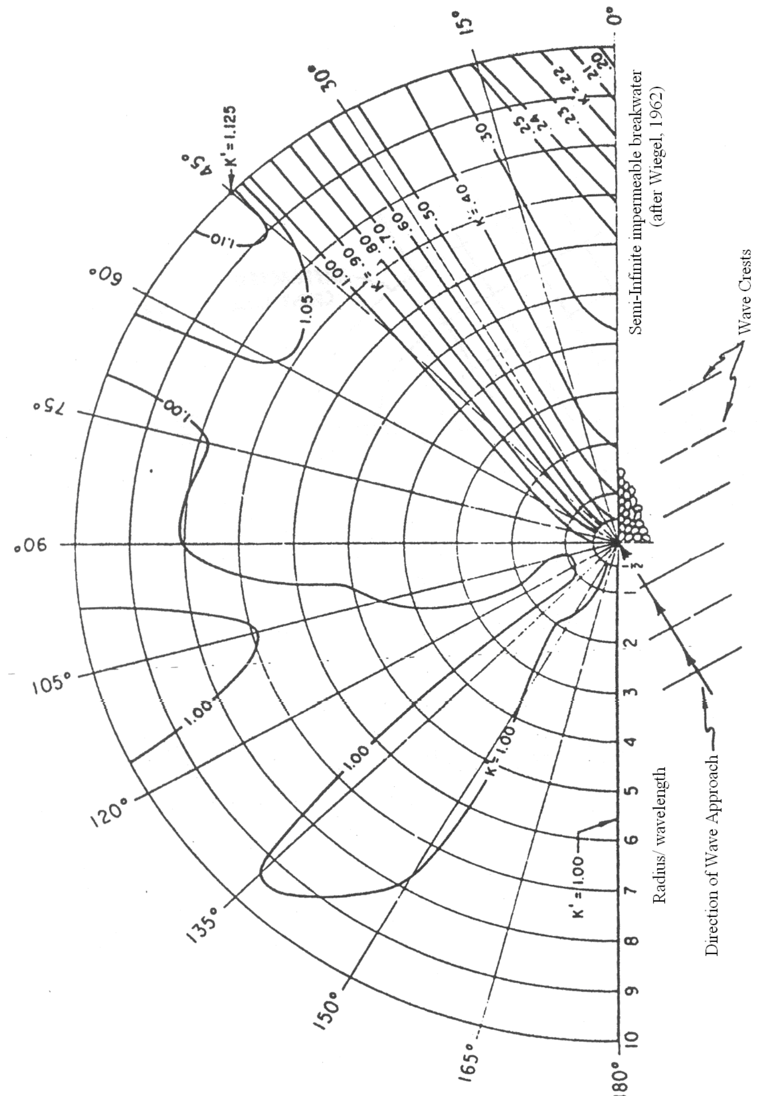


Fig 4.4 Wave Diffraction Diagram (Angle of Attack – 30^0) **(Ref.: SPM, 1984)**

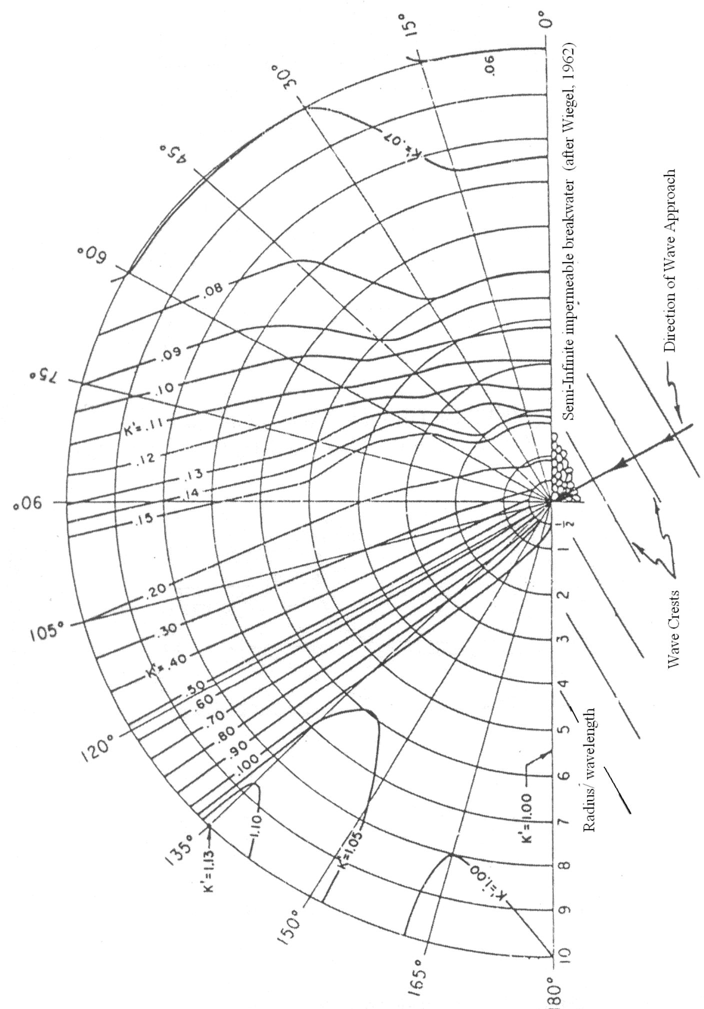


Fig 4.5 Wave Diffraction Diagram (Angle of Attack – 120^0) **(Ref.: SPM, 1984)**

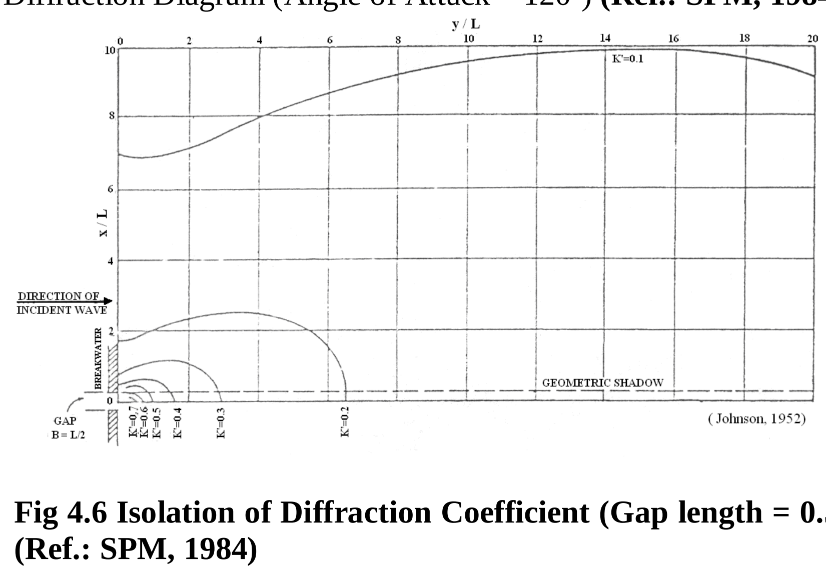


Fig 4.6 Isolation of Diffraction Coefficient (Gap length = 0.5 * Wave Length) (Ref.: SPM, 1984)

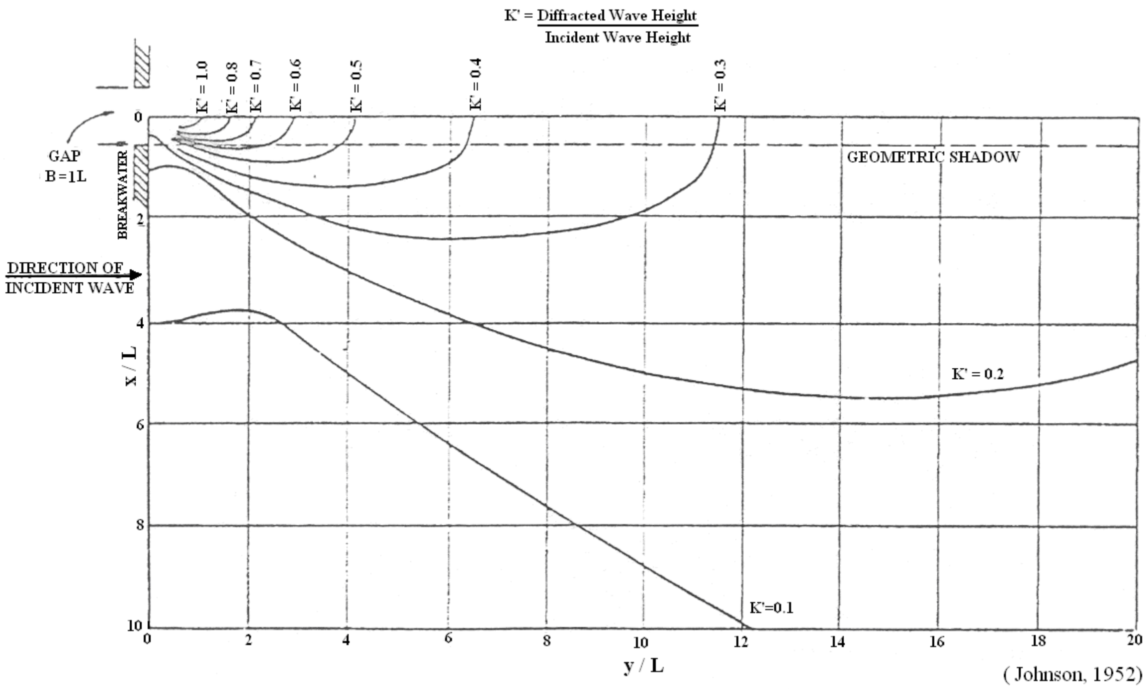


Fig 4.7 Isolation of Diffraction Coefficient (Gap length = Wave Length) (Ref.: SPM, 1984)

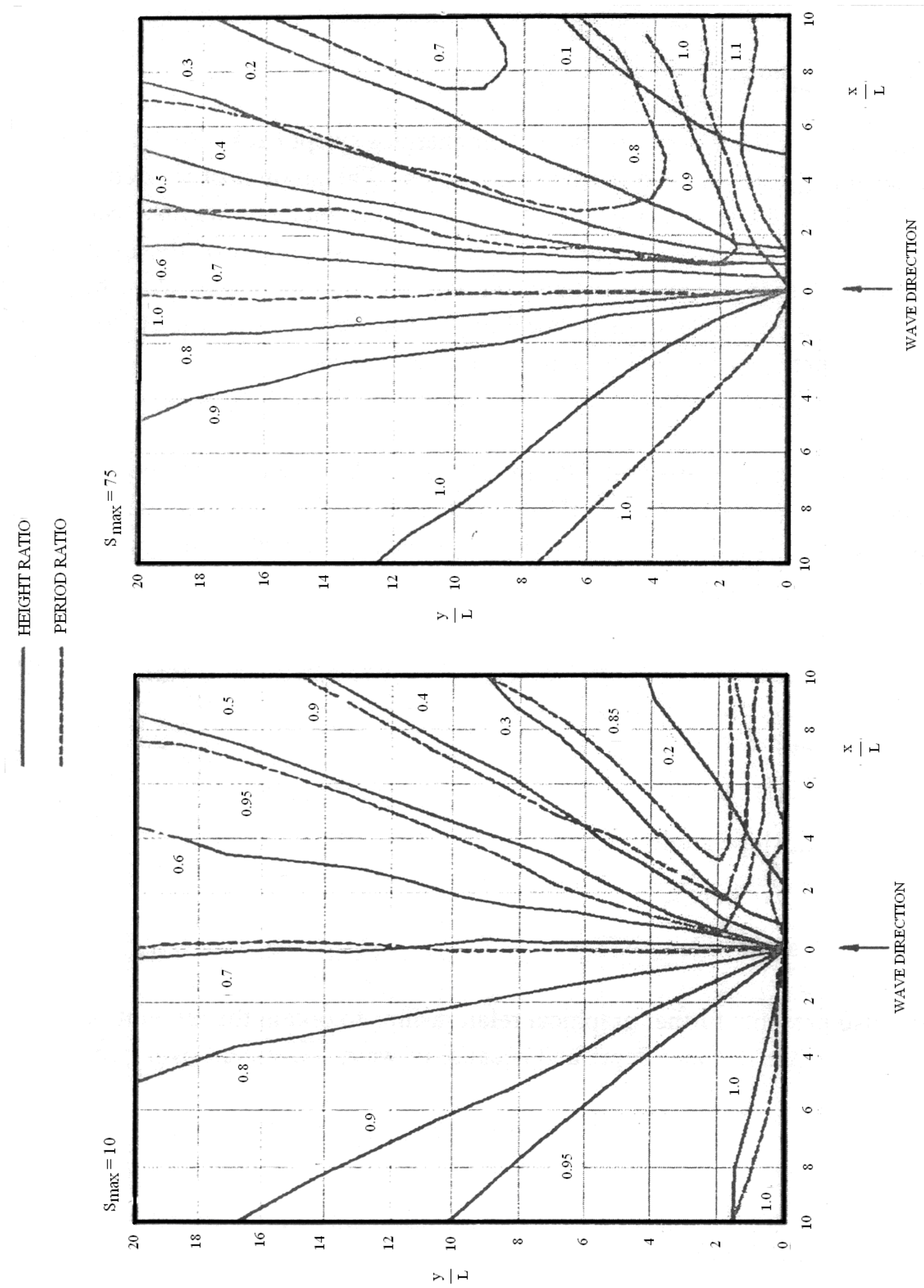


Fig 4.8 Random Waves Diffraction (Single Breakwater) **(Ref.: SPM, 1984)**

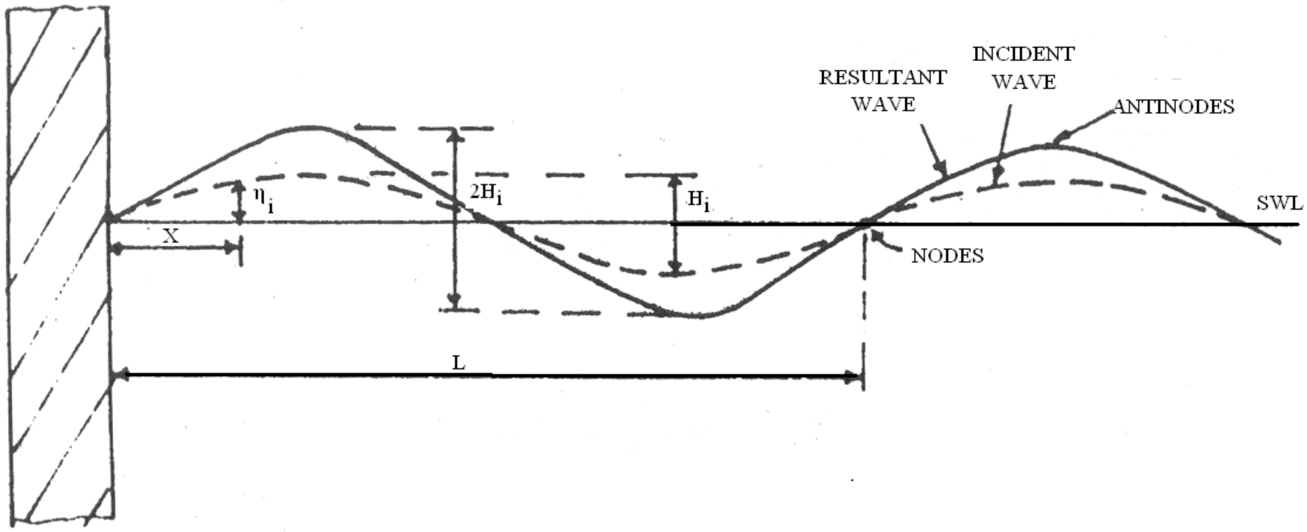


Fig 4.9 Pure Reflection

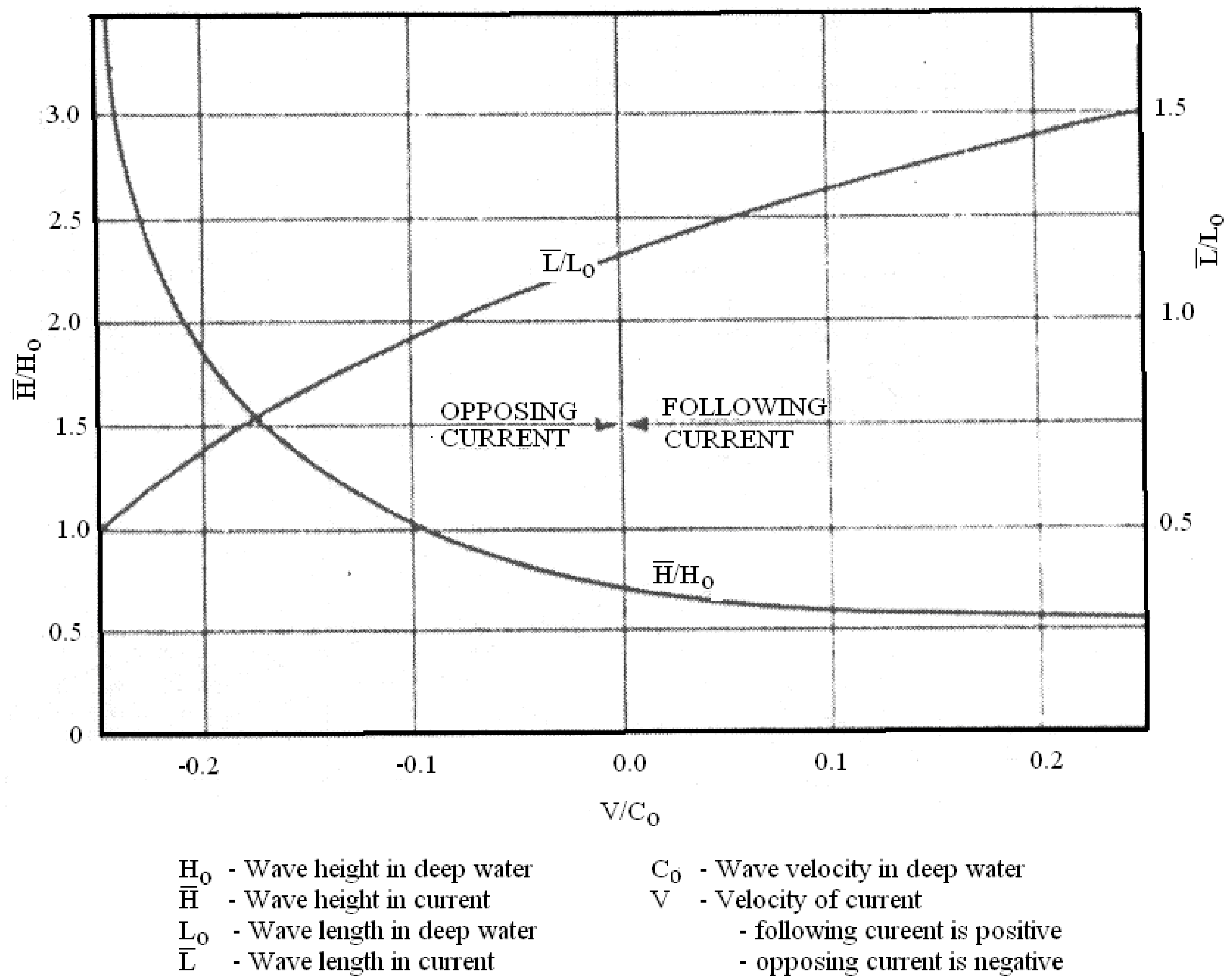


Fig 4.10 Effect of Current on Waves **(Ref.: Gerwick, 1986)**

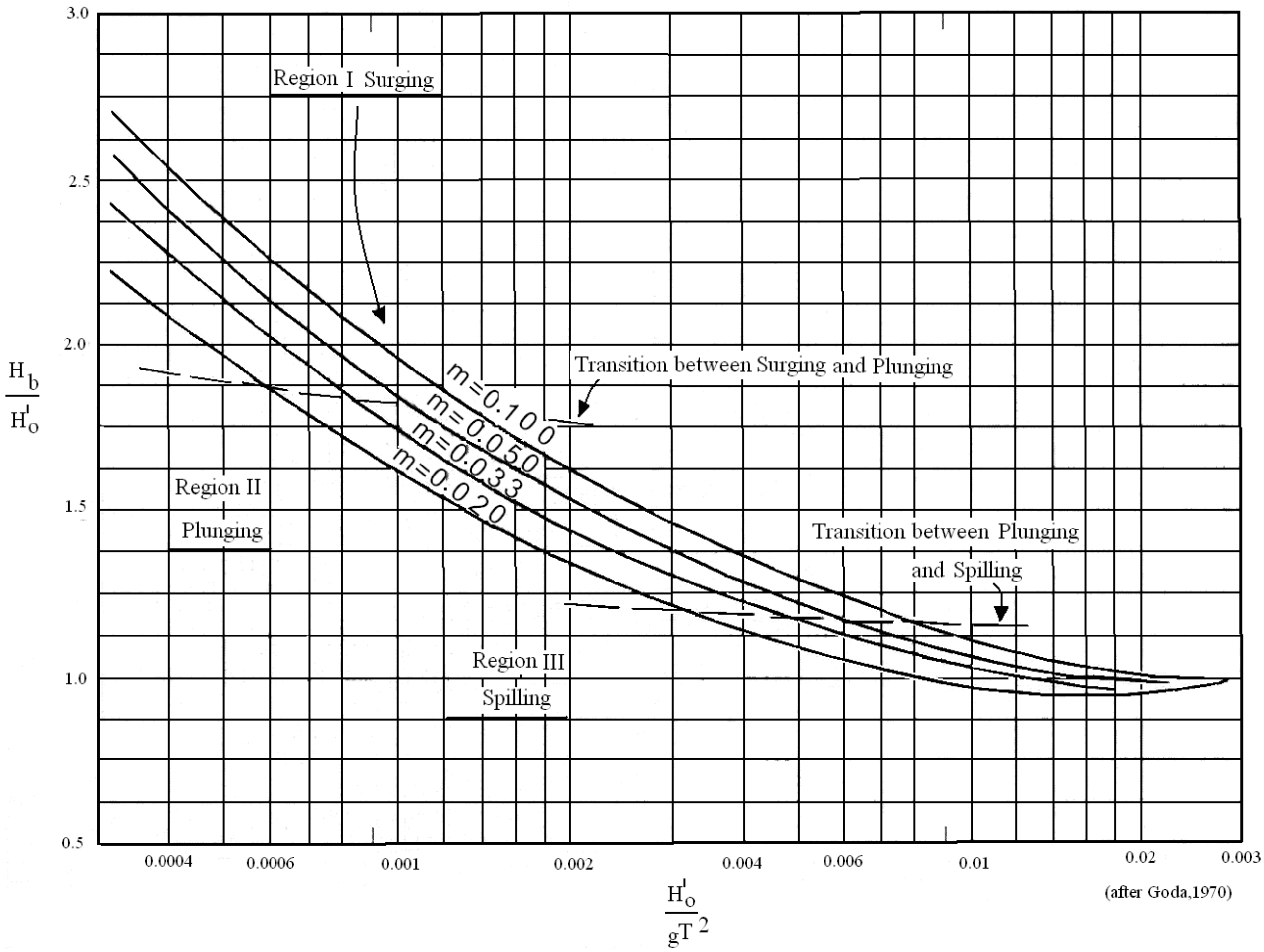


Fig 4.11 Determination of Breaking Wave Height **(Ref. SPM, 1984)**

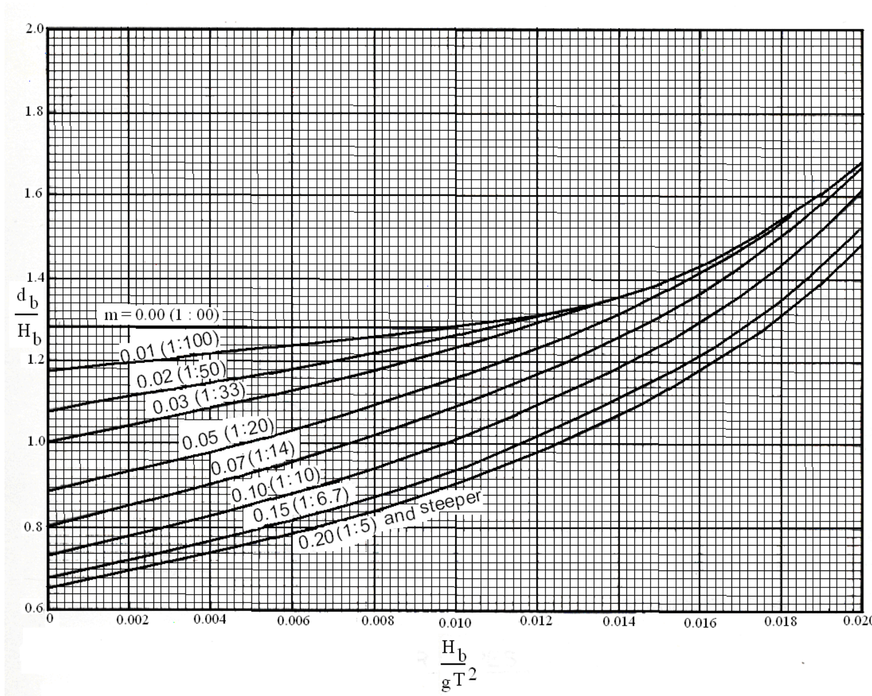


Fig 4.12 Determination of Breaking Water Depth **(Ref. SPM, 1984)**

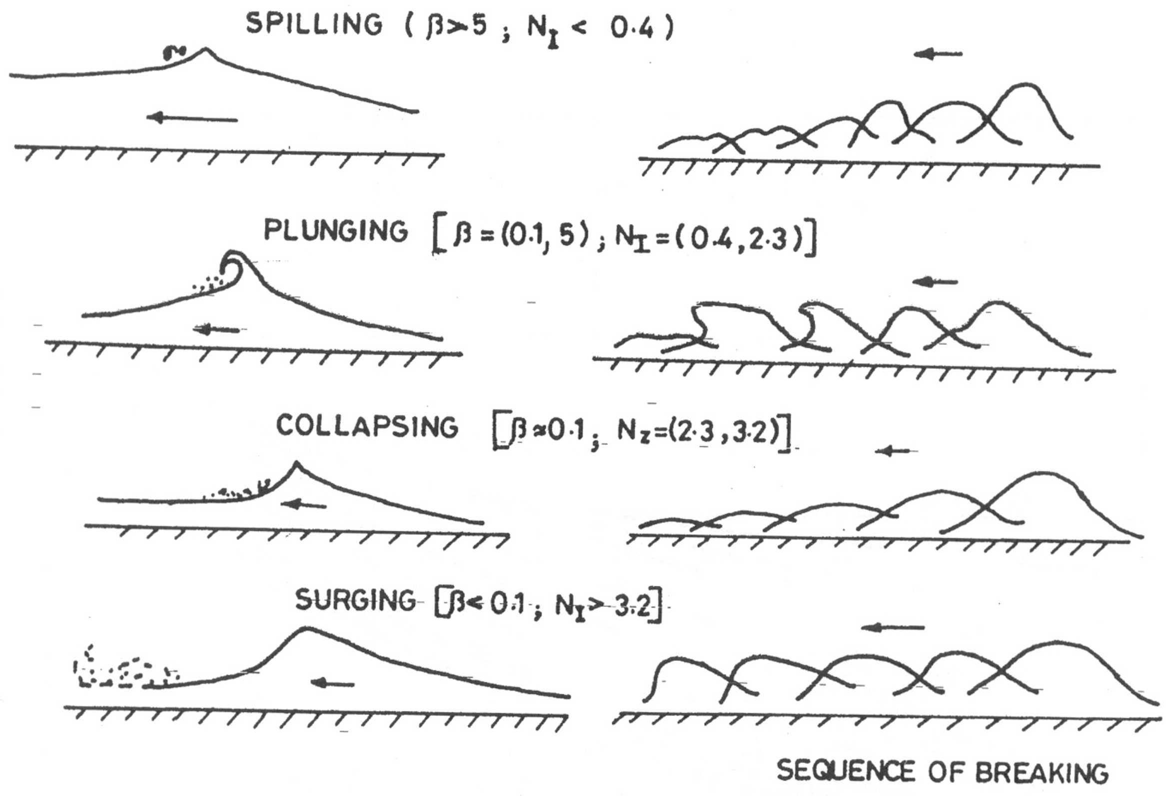


Fig 4.13 Breaker Types

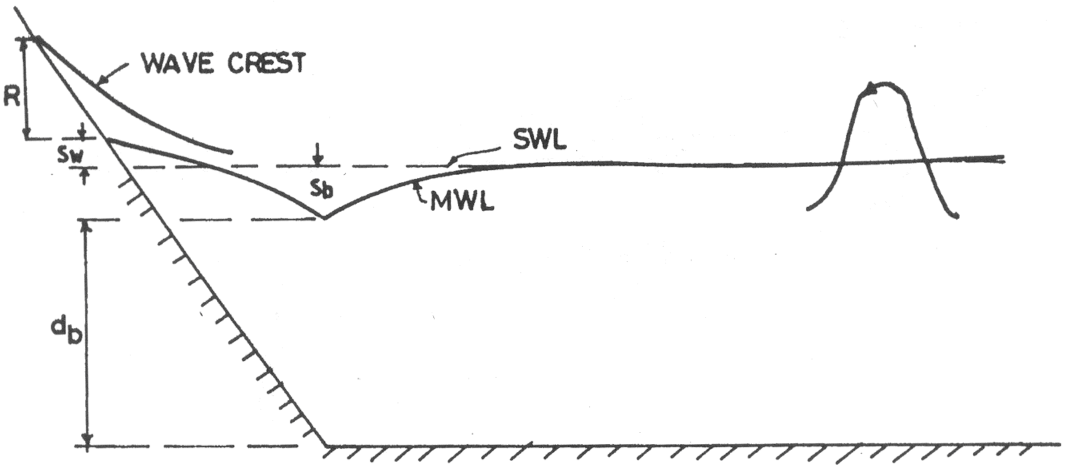


Fig 4.14 Wave Set Up, Down, Run-Up on a Beach Slope

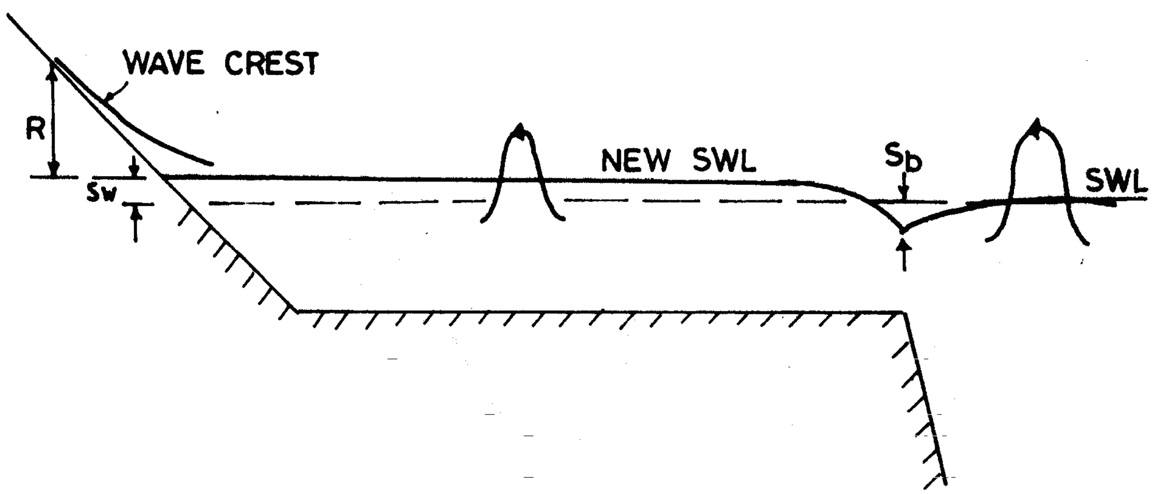


Fig 4.15 Wave Set Up, Down, Run-Up over Flat Bed

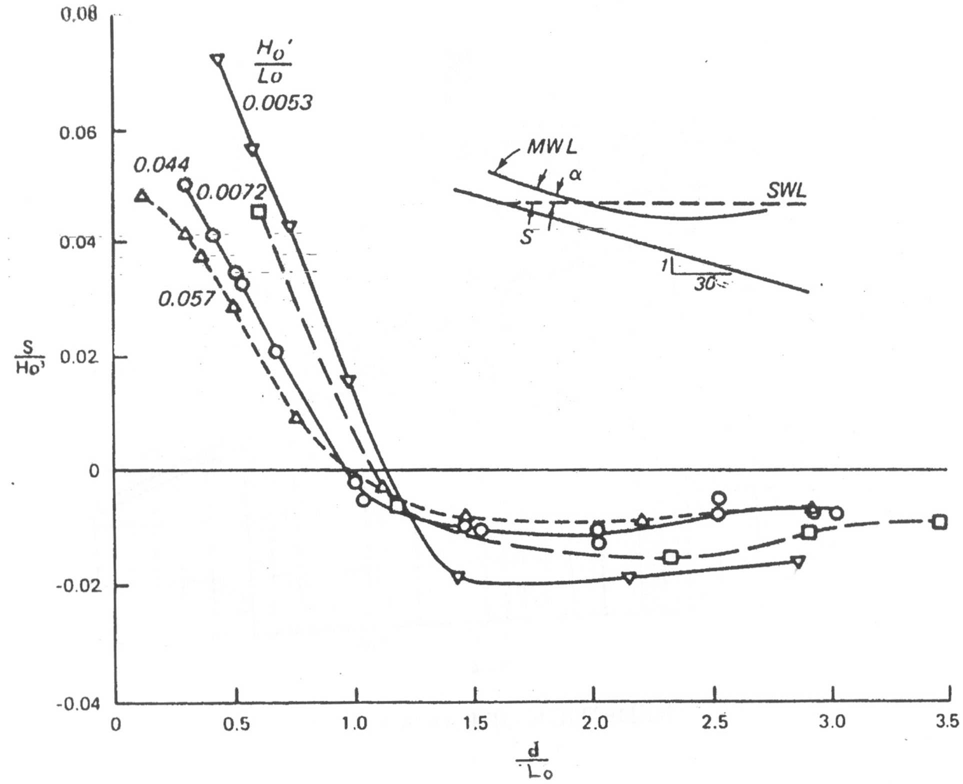


Fig 4.16 Measurements of Set-Up Due To Random Waves
(Ref.: SPM, 1984)

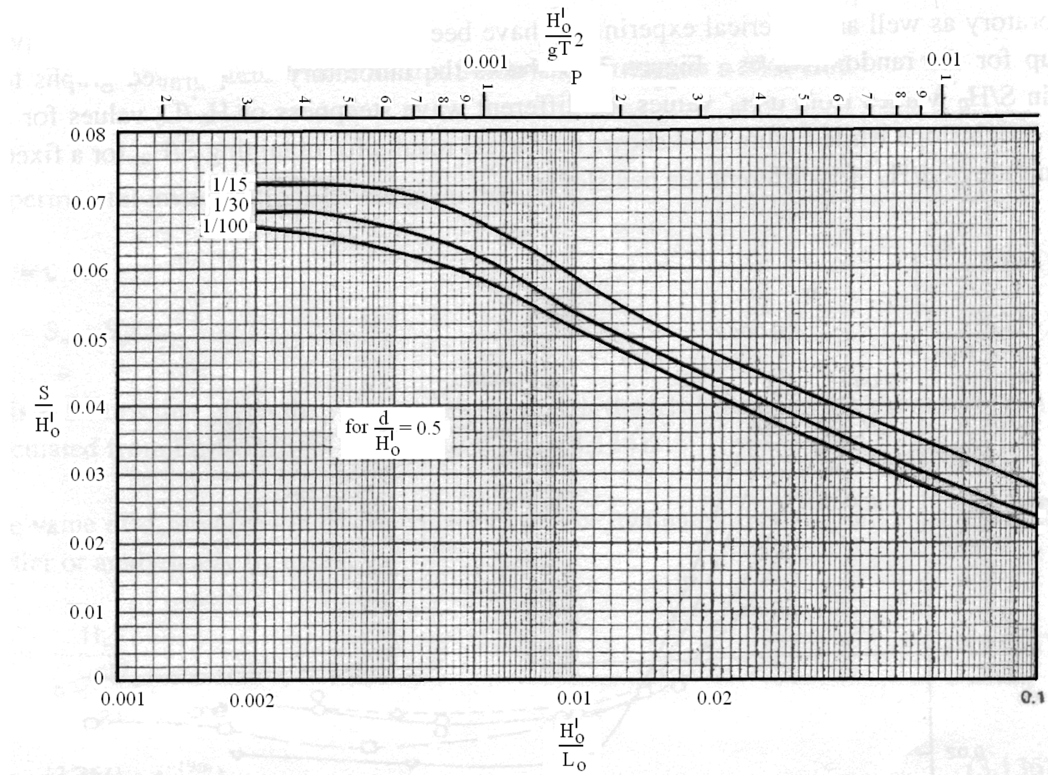


Fig 4.17 Prediction of Random Wave set-up **(Ref.: SPM, 1984)**

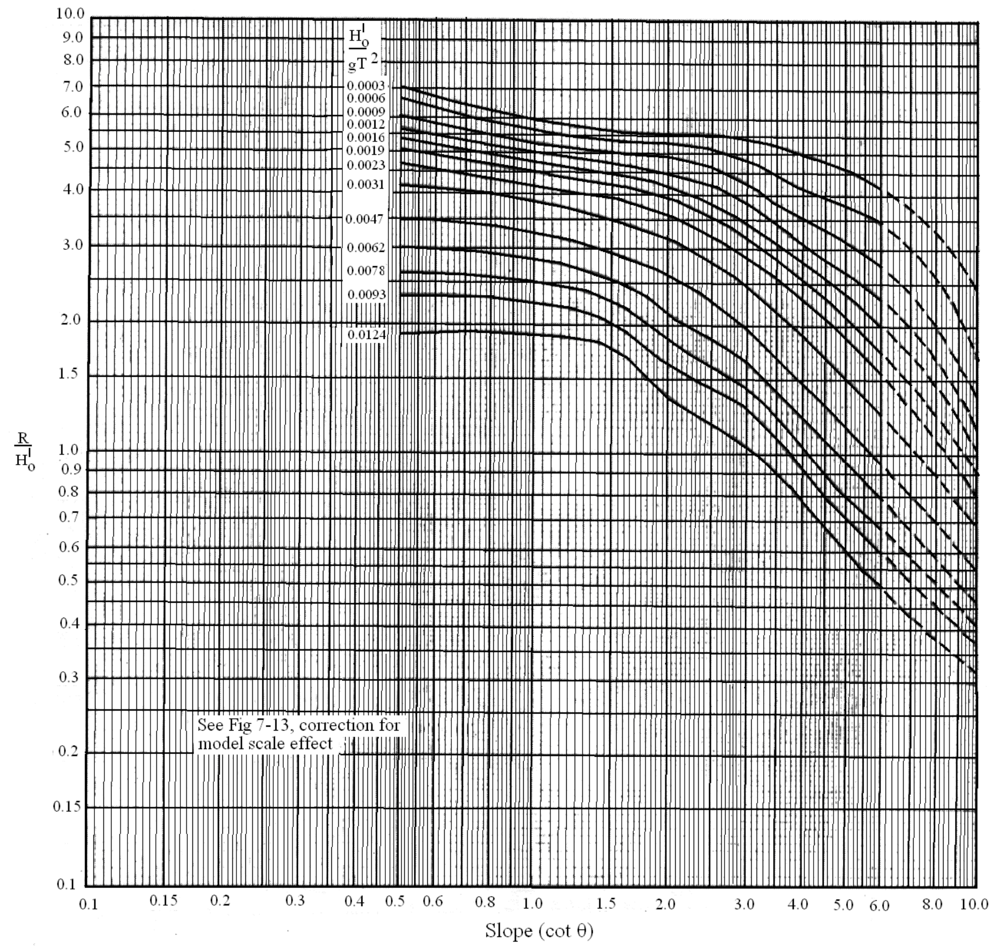


Fig 4.18 Determination of Wave Run-up (for smooth, impermeable slope, $d_s / H_0 = 0.8$, bed Slope=1:10) **(Ref.: SPM, 1984)**

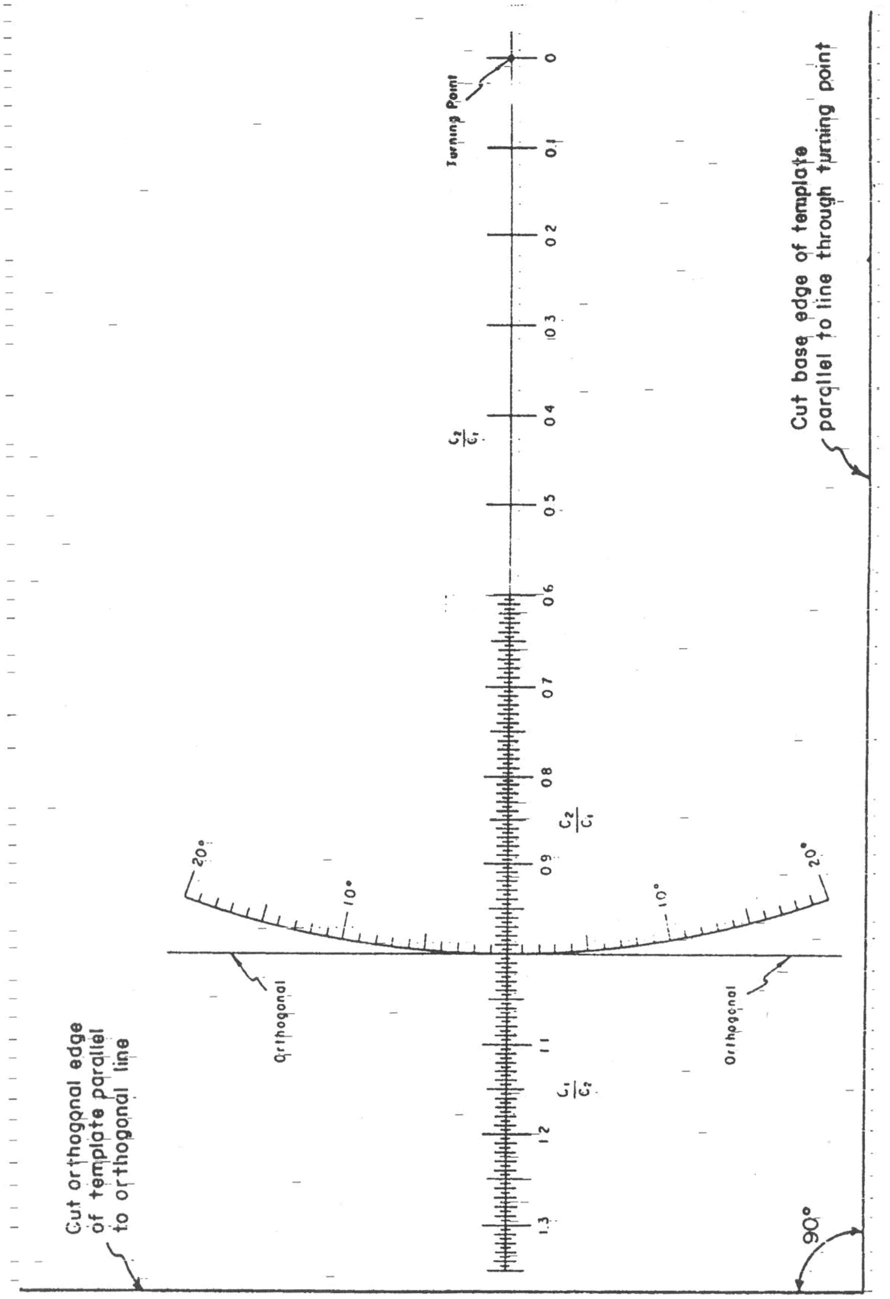


Fig 4.19 Refraction Template (Ref.: SPM, 1984)

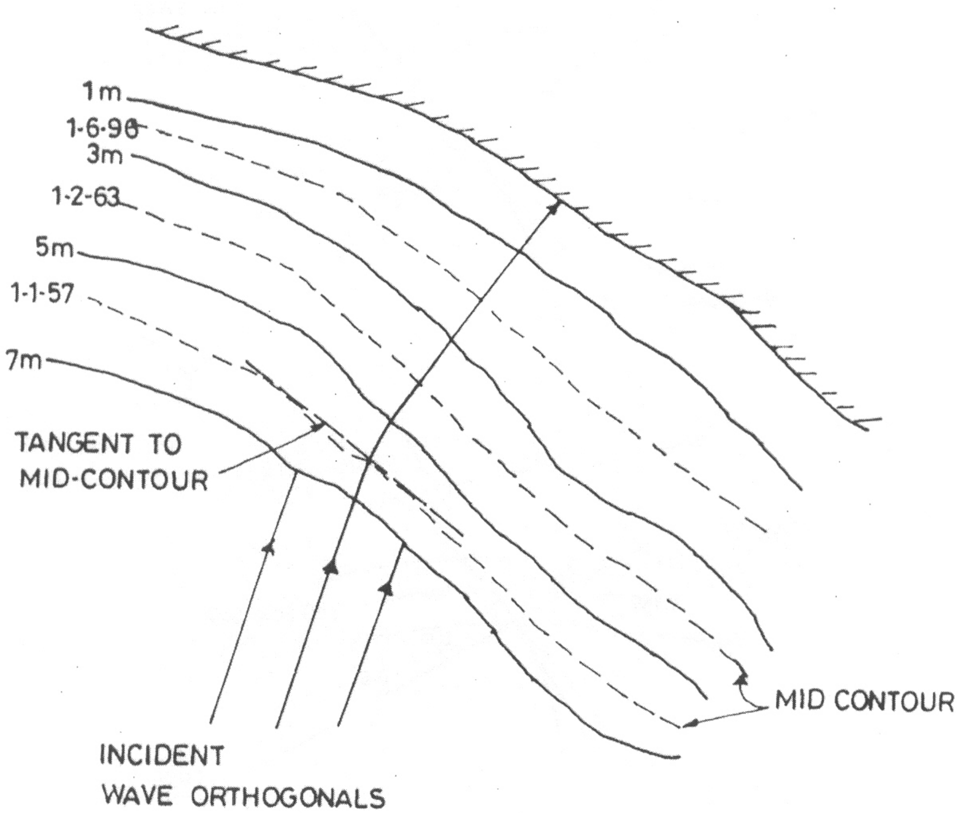


Fig 4.20 Ray Methods

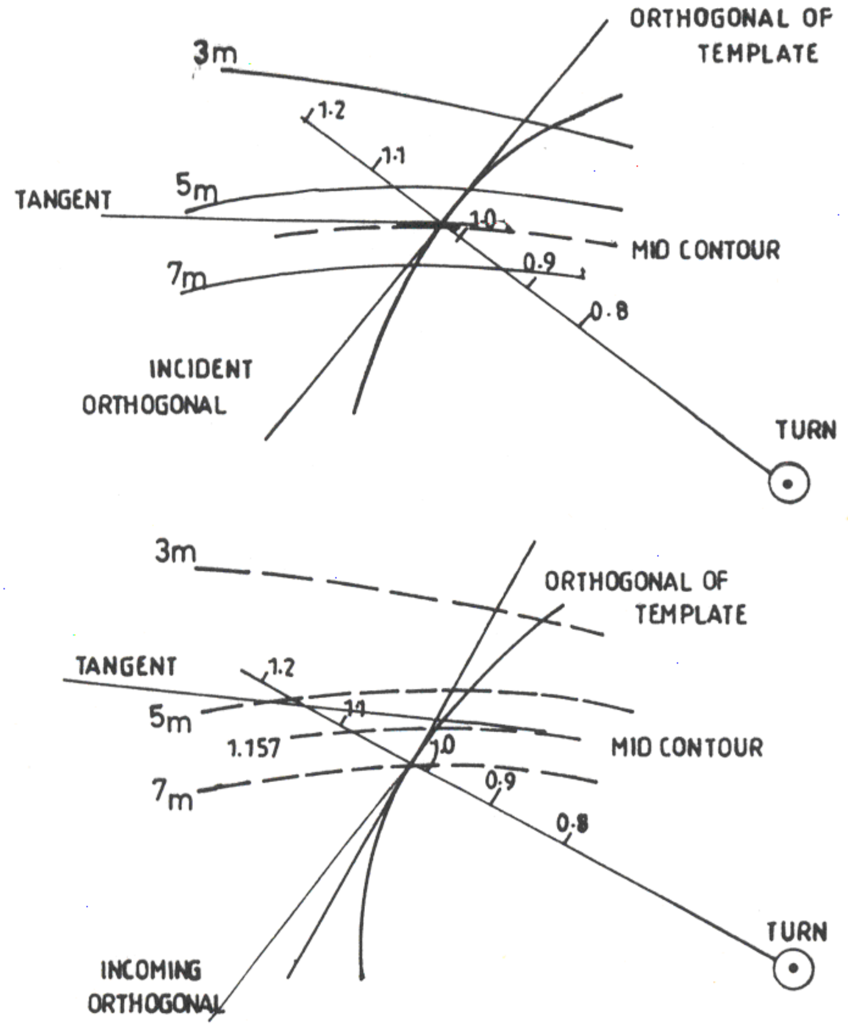


Fig 4.21 Ray Method

4.10 APPENDIX 4.1

GRAPHICAL PROCEDURE TO DRAW REFRACTION DIAGRAMS

A refraction template shown in Figure 4.19 made up of a transparent material, like a Perspex sheet, necessary. The procedure to dram the refraction diagrams with its help is as under:

(i) Obtain the hydrographic chart of the area of interest.

(ii) Draw smooth bed contours at suitable intervals, like 2 m or so.

(iii) Select a range of wave periods (T) and wave directions and draw separate diagrams for each T and θ as below:

Obtain C_0/C_1 values, where C_0 is wave speed before refraction and C_1 is the same after refraction over bed contour. In open areas rays are drawn from deep to shallow water and the C_0 belongs to deep water and C_1 to shallow water case. In highly sheltered locations, it may become more useful to follow a reversed procedure to draw rays from the shallow to deep water. A tabular representation shown below is beneficial. Assume that the wave period is 8 seconds. This gives $L_0 = gT^2/(2\pi) = 99.84$.

Water depth,d (m)	d/L_0	$\tanh 2\pi\, d / L$	$\frac{C_0}{C_1} = \frac{\tanh 2\pi . d_1 / L}{\tanh 2\pi . d_0 / L}$
1	0.01	0.2480	
			1.696
3	0.03	0.4205	
			1.263
5	0.05	0.5310	
			1.157
7	0.07	0.6144	

As shown in Figure 4.20 draw orthogonals (in deep water) to some spacing and write the C_0/C_1 values obtained from the above table on the hydrographic chart at appropriate places within the region of the two underlying contours.

If the angle between the underlying contour line and the crest line is less than 90^0, follow the procedure given below (Figure 4.21)

(A) Referring to Figure 4.20 and 4.21 construct a mid-contour. Extend the incident orthogonal upon it and draw tangent to the mid-contour at the intersection point.

(B) Place the refraction template (Figure 4.19) on the top of the hydrographic chart such that the template orthogonal coincides with the intersection point # at mark '1.0' on the template.

(C) Turn the template around its turning point until C_0/C_1 value of the contour interval read on the template crosses the tangent to the mid-contour. The template orthogonal is now orthogonal to the changed direction of the incident orthogonal.

(D) Move the template orthogonal parallel to itself so that incident and changed or turned orthogonals have same lengths within the portion of the contour intervals.

(E) Repeat the procedure (A) to (D) for all contour intervals and for all incident orthogonals.

If the angle between the crest line and the contour line exceeds 80^0, then the orthogonals appear almost parallel to the contours. In such case the region between two contour lines is to be divided into few blocks and the orthogonals turning is required to be made within each block (SPM, 1984).

CHAPTER 5

NUMERICAL MODELING OF WAVES

5.1 Introduction

Numerical modeling of waves is used to predict waves in the open ocean as well as to obtain distribution of waves in harbor and coastal areas. It is adopted when wind has a large temporal or spatial variation, and also in cases involving superposition of sea and swell, irregular bathymetry and complex coastline geometry. There are two methods of the modeling:

1) Phase Resolving models
2) Phase Averaging models

1) Phase Resolving Models:

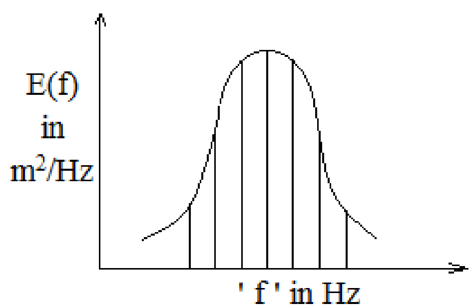


Fig. 5.1 A spectrum showing variation of wave energy against frequency

In this type of model individual waves in a spectrum are resolved as per their phases and phase, amplitude and surface elevations are predicted. They are used when average properties of waves change rapidly (over a distance of few 'L') and when there is rapid variations in depth and shoreline. Wave propagation in harbors, wave-structure interaction are typical problems handled conveniently by these models, despite the fact that they involve high computational complexity. The phase resolving models can be further classified as

1) Boundary Integral Models
2) Mild Slope Equation.
2) Boussinesq Equation Models.

5.2 Boundary Integral Models:

These models do not involve any assumptions for wave conditions or site conditions. They solve the Laplace equation given as:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad (5.1)$$

The solution involved is complex as both horizontal (x and y direction) variations are required to be determined. This model is good for irrotationality dominance (breaking) cases but bad for viscosity dominance situations as well as for wave-structure interaction.

5.3 Mild Slope Equation Models:

The assumption in these models is that: the sea bed slope (m) is very much less than kd, and, the waves are weakly nonlinear. (ak<<1). The unknown velocity potential ϕ is expanded as Taylor's series in the formulation. Berkhoff in 1972 originally gave the mild slope equation as:

$$\nabla\left(CC_g\nabla\phi\right)+K^2CC_g\phi=0 \qquad (5.2)$$

This formulation of the model can cater to the effects of wave shoaling, refraction, diffraction, reflection. For computational efficiency, a parabolic version of the original elliptic equation is often used. Boundary conditions over full domain are required. Using this model a solution at all the points is obtained. In modified forms, the parabolic equation is used to include current, wider approach angles, and non-linear dispersion, dissipation, wind input.

5.4 Boussinesq Equation Models:

These formulations assume that the bed slope, m, and 'kd' are very small compared to unity and further that the waves are weakly non-linear. ($\frac{a}{d}<<1$). Peregrine (1967) gave the original 2-D form as follows.

$$\frac{\partial\eta}{\partial t}+\nabla\left[(d+\eta)\bar{u}\right]=0 \qquad (5.3)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\nabla\bar{u} + g\nabla\eta = \frac{d}{2}\nabla\left[\nabla\left(d\frac{\partial \bar{u}}{\partial t}\right)\right] - \frac{d^2}{6}\nabla\left[\nabla\left(\frac{\partial \bar{u}}{\partial t}\right)\right] \qquad (5.4)$$

where $\bar{u}$ =depth averaged horizontal velocity. This model includes refraction, diffraction, shoaling, reflection, wave-current interaction effects. This also includes dissipation and wind input.

5.5 Phase Averaging Models: (Spectral Models)

By using these models, averaged information over a spectrum is obtained. These models consider evolution of a directional spectrum. They predict averaged or integral properties like significant wave height, average wave period etc. These are used when average properties change slowly (over few 'L'). The basic equation in deep water (advective transport equation) is:

$$\frac{\partial F}{\partial t} + C_g\nabla f = S_{total} \qquad (5.5)$$

where $F \equiv F(f,\theta,x,y,t)$ is the directional Spectrum (f = wave frequency, θ = angle of wave approach, x and y are horizontal coordinates, t = time instant),

$$S_{total} = S_{in} + S_{nl} + S_{ds} \qquad (5.6)$$

S_{in} = source term representing wind energy input; S_{nl} = source term representing input energy from non-linear interaction between different wave frequencies in a spectrum resulting in energy transfer to another component. There can be two components in a spectrum which resonate. If they transfer energy to third one then it is known as Triad. If four components interact, then it is known as Quadruplate. S_{ds} = source term representing energy loss in dissipation like breaking.

The solution to (5.5) involves specification of the initial values, boundary conditions along with an appropriate advection scheme to represent energy transport within the computational domain. As per the source term representation, phase averaging models are classified as follows.

a) First Generation:

These were developed after 1957 and assumed:

$$S_{total} = S_{in} + S_{ds} \qquad (5.7)$$

These were site specific and proved to be inaccurate in storms.

b) Second Generation:

These were presented around 1973 and they specify:

$$S_{total} = S_{in} + S_{nl} + S_{ds} \tag{5.8}$$

where S_{ds} indicates 'white-cap' dissipation (deep water, H / L dependent)

S_{nl} specification is based on a parametric form (JONSWAP spectral)

For hybrid models instead of parametric-fixed, parametric-evolving forms were used.

c) Third Generation:

These are developed around 1985 and they specify:

$$S_{total} = S_{in} + S_{nl} + S_{ds}$$

where, S_{nl} is represented in a different way than the second generation models mentioned above. The Third generation model use Discrete Interaction approximation to model S_{nl}, which retains the basic physics of wave - wave interaction.

The basic equation in phase averaging models is as given in (5.5), i.e, :

$$\frac{\partial F}{\partial t} + \overline{C}_g \nabla \bar{f} = S_{tot}$$

where $\overline{C}_g \nabla \bar{f}$ is advection of wave energy at $\overline{C}_g$. For this purpose, discrete bins $(\Delta f \Delta \theta)$ of directional spectrum are formed. (Fig. 5.2)

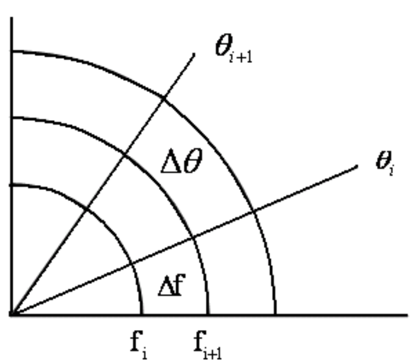


Fig. 5.2 The discrete bins of frequency and approach angle intervals

Each bin is advected at its C_g. The equation mentioned above is solved using Finite Difference or Finite Element or Full Ray (Line to advect) or Piecewise Ray method.

Typically the WAM model considers of 25 frequencies and 12 directions. Recently efforts were made to collect data from buoys or satellite and by using error back propagation the model was fine tuned; the resulting procedure is model refinement being called the Data Assimilation.

CHAPTER 6

DESIGN WATER DEPTH

Still water level (SWL) may be defined as the level in absence of gravity waves. There are many factors contributing to changes in such a level at a specified location and a knowledge of all of them is necessary for many purposes, like, obtaining the extent of damage due to flooding, determining safe elevations for sensitive structures (like nuclear plants) and knowing the design water depth at a given location.

Astronomical tide is a major factor in calculation of the mean water depth. Statistical analysis of tidal levels at a given location is carried out and the mean value is derived. All other factors, namely, tsunamis, wave set up, wind set up and pressure set up are studied as allowances to this value.

Following are the major causes of change in SWL.

6.1 Astronomical tides

This is a major cause of change in the SWL. Basically the Mean Tide Level is obtained by carrying out a statistical analysis of all tidal levels observed at the given location and then variations in this level created by all other factors described below are studied.

Tides represent periodic rise and fall of still water level due to differential attraction of sun and moon, influenced further by gravity and centrifugal forces. The moon has more influence in tide generation because of its proximity with the earth compared to the sun. The period of tides could be 12 hours (semi-diurnal Tide) or 24 hours (diurnal tide). (See Fig. 6.1). Vertical difference between the high water (the highest elevation in one tidal cycle) and the low water (the lowest elevation in one tidal cycle) is the tidal range.

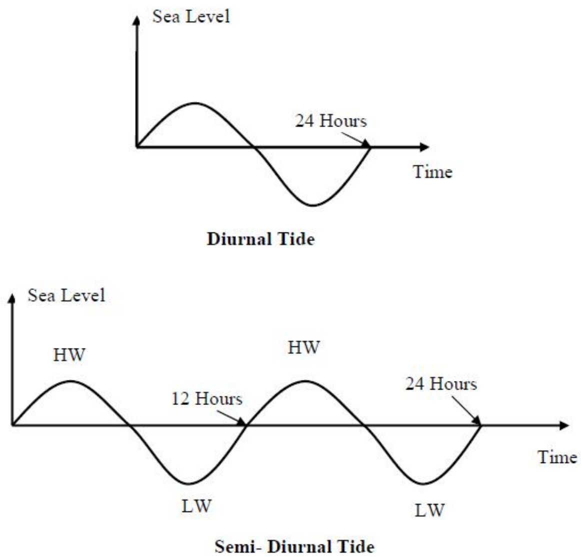


Fig. 6.1. The types of a tide

The tidal range changes from cycle to cycle because of continuous shifts in the positions of the earth, sun and moon. (Fig. 6.2)

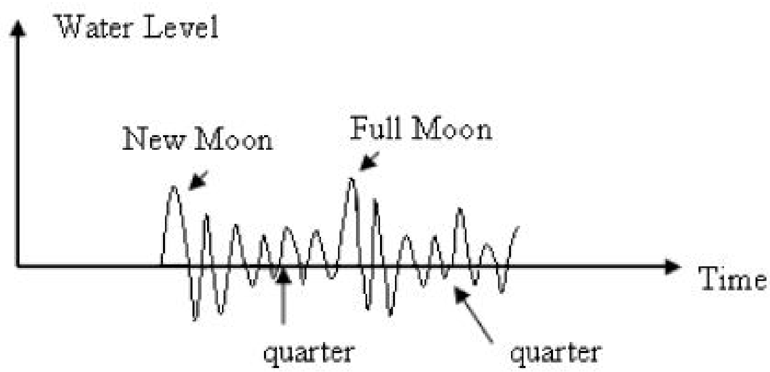
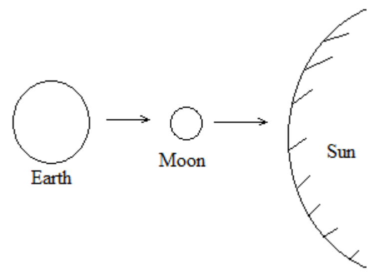


Fig. 6.2. Time history of tidal elevations

It is small in open sea. The range attains maximum value (in a fortnight) during new or full moon days. (Figure 6.3). This is called spring tide. Similarly it takes minimum value (in a fortnight) during quarters, which is called neap tide.

Spring Tide:



Neap Tide:

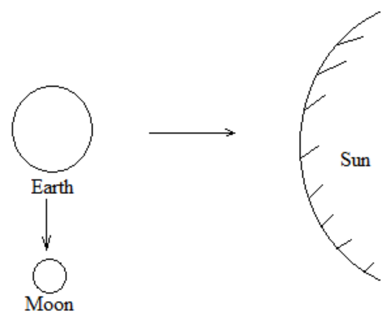


Fig. 6.3. The spring and neap tide

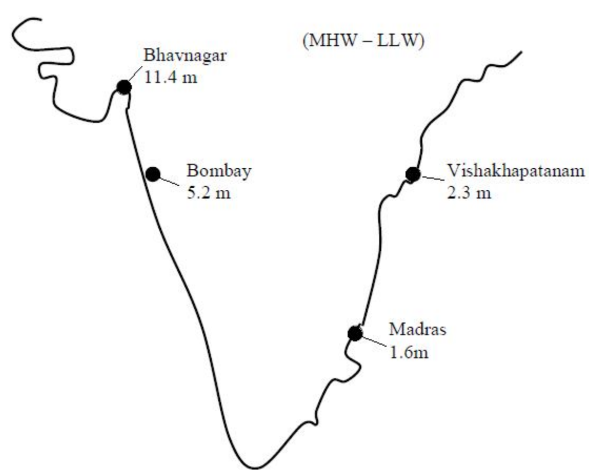


Fig. 6.4 Tide levels (highest HW – lowest LW) at some Indian locations

Figure 6.4 shows the highest tidal ranges at some important locations around our country. Figure 6.5 shows relative levels, averaged over all occurrences, of Highest High Water(HHW), Mean High Water Springs(MHWS), Mean High Water Neaps(MHWN), Mean Low Water

Neaps(MLWN), Mean Low Water Springs(MLWS), Lowest Low Water(LLW). The HHW and LLW occur when the centers of the three celestial bodies involved passes through a common straight line, a phenomenon that takes place in every 11 years and 11 days.

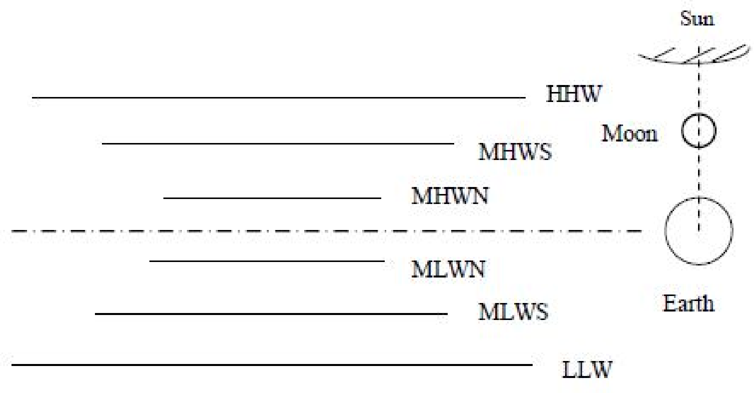


Fig. 6.5. Relative water levels

Mean Tide Level: It is average of all tidal heights (Astronomical).

Mean Sea Level: It is actual average of sea levels (Astronomical or Meteorological)

For structural design, Highest Astronomical Tide as well as Lowest Astronomical Tide are considered.
Highest Astronomical Tide (HAT) = Mean Sea Level (MSL) +1.2 (Sum of certain location dependent tidal constituents like $M_2, S_2,\ K_1, O_1$). (6.1)

The data of $M_2, S_2,\ K_1, O_1$ are available with Director, Geodetic and Research, Survey of India.

Lowest Astronomical Tide (LAT) = Mean Sea Level (MSL) -1.2 (Sum of certain location dependent tidal constituents like $M_2, S_2,\ K_1, O_1$). (6.2)

Tidal levels are specified above 'Chart Datum', which is equal to the Indian Spring Low Water.
Indian Spring Low Water = Mean Sea Level (MSL) - (Sum of certain location dependent tidal constituents like $M_2, S_2,\ K_1, O_1$). (6.3)

Chart Datum varies with location as shown in Fig. 6.6.

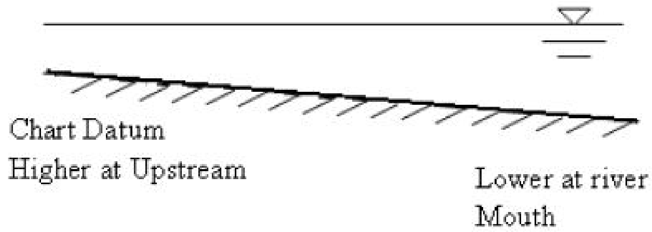


Fig. 6.6. Chart datum

Tide Prediction

This is done using harmonic analysis, which is based on the assumption that because the tide is effected by motions of sun, moon, earth which are periodic, the tide levels can be resolved into periodic components, i.e.,

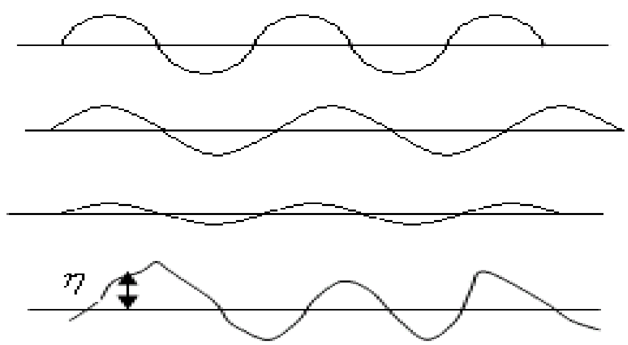


Fig. 6.7. Tide superposition

$$\eta = A_1 \cos(a_1 t + M_1) + A_2 \cos(a_2 t + M_2) + \ldots \tag{6.4}$$

(See Fig. 6.7).

Where η = tidal elevation above or below the mean level, A_1 = amplitude, a_1 = rate of change of phase. About 69 such harmonic components have been identified.

At any important location, Tide Tables give predictions of High Water, Low Water, as well as their timings based on harmonic analysis. (**Note**: A Tide Chart gives tidal levels at a time at several locations in a bay). Tide tables are available only at few ‘reference’ stations. Tide at any ‘subordinate’ station is estimated from these reference station data.

Actual water level is a function of coastline configuration, local water depth, sea bed topography, wind and weather (rain, runoff, etc.). Hence predicted level differs from actual level. For low barometric pressure and onshore wind, actual water level is higher while for high barometric pressure and offshore wind, actual water level is lower than predicted. Prediction in the presence of high wind, rain, runoff, enclosed locations is bad while the one in open location and normal condition is always good.

National Oceanographic & Atmospheric Administration (NOAA), USA maintains its website www.noaa.org.gov, which gives online tide prediction at some 3000 subordinate stations on weekly, monthly, yearly basis. Most of the information is available free of charge. Many other companies and their software also provide tide prediction.

Tide tables repeat exactly after 18 years 11 days because earth, moon, sun occupy exactly same positions after every such period.

Tidal theories explain tidal mechanics. One of them is based on Newton's law, as indicated below, which considers equilibrium of any particle on earth assuming statistical condition. (F = force of attraction between two bodies, ν = Universal constant, m_1 = mass of one body, m_2 = mass of another body, r = distance between them.

$$F = v\frac{m_1 m_2}{r^2} \tag{6.5}$$

Another theory due to Laplace, considers rotational effect of the earth.

6.2 Tsunamis:

Introduction

Tsunamis represent a sequence of ocean waves generated by an underwater impulsive action primarily caused by earthquakes and secondarily created by landslides, volcanic activities and meteorite-strike as well as an earthquake induced submarine slump. In tsunami formation huge water mass gets thrown vertically upwards due to the underwater impulse. This mass falls down due to gravity, transferring its momentum and energy to remaining water. Such action is comparable to that of a stone falling on a pond and creating dispersing ripples. Continuous action of gravity as a restoring force ensures formation of a series of waves, which may typically be five to seven in number. It is seen that the second or the third wave is most formidable. At the source the wave height is only a few cm, but when it comes towards the shore the wave gets compressed with heights rising even up to 20 m or more. The time required by a tsunami to complete one oscillation is of the order of 30 to 60 minutes while the same for wind-generated gravity waves it is 3 to 20 seconds and for the tides (tidal wave) created by sun and moon's attraction it is 12 or 24 hours.

Time of travel

Because tsunamis have large lengths (700 km or more) the so-called long wave or shallow water wave theory can be applied to study their behavior and so also the shallow water wave speed equation, namely,

$$C = \sqrt{g \times d} \tag{6.6}$$

Where, C = speed of propagation of tsunami, g = acceleration due to gravity and d = water depth. Hence the time of travel of tsunami can be given by:

$$t = \sum \frac{\Delta s}{\sqrt{g \times d_s}} \tag{6.7}$$

where, t = total travel time from the epicenter to the desired location, Δs = incremental distances over which the depth, d_s, can be assumed to be the same

Depending on the time of travel from the source to land it could be either a distant tsunami or a local tsunami. Distant tsunamis travel long distances of the order of thousands of km with relatively same speed and can be easily warned against while local tsunamis resulting from submarine landslides propagate over a small distance of a hundred km or less and are more dangerous because they leave no time to warn for evacuation.

Properties of tsunamis

Properties of tsunamis like their shape, phase velocity, group velocity, flow kinematics; induced pressures can be studied with the aid of wave theories. All wave theories presume that the flow is potential and 2-Dimensional (in the vertical plane). They initially assume that the velocity potential is some unknown function of wave height, period and water depth. The unknown function is obtained by making such potential to satisfy the Laplace continuity equation as well as various boundary conditions including those at the free surface and at the seabed. Once the function is determined its derivatives yield wave profile, flow kinematics, pressure and other desirable parameters. Out of several wave theories the first order Airy's theory as well as higher order Stokes and Laitone solitary theory describe the tsunami properties well and hence they are used in deep, intermediate and shallow waters, respectively.

The Airy's first order theory assumes that ratios of wave height to wavelength, and to water depth are small and so also the ratio of water depth to wavelength. It gives sinusoidal profile and yields simplified equations but it is valid in deeper water away from the shore.

The cnoidal wave theory of Laitone or solitary theory is more rigorous. At a first order of approximation the ratio of wave height to depth is assumed to be small, the vertical distribution of horizontal velocity is uniform and there is absence of transport of water mass. The second order approximation gives a realistic nonuniform velocity distribution and can predict the mass transport. In general this solitary theory is appropriate when the tsunami height is less than 30 percent of depth.

Many tsunami researchers use the theory of Stokes at a second order of approximation in intermediate water. The wave profile in it is obtained by superposing two sinusoidal components of period T and T/2, respectively. As a result the wave crest becomes peaked and the trough becomes flatter. From this theory the water mass transport resulting from irrotationality and nonlinearity can be estimated.

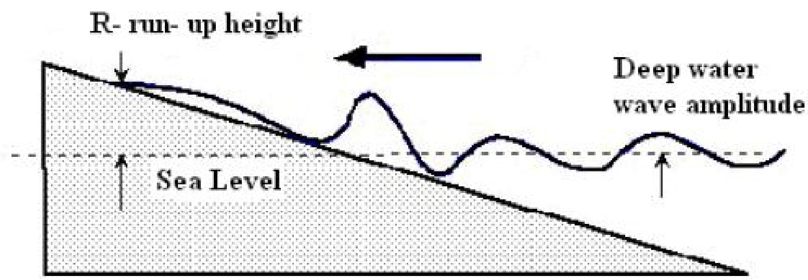


Fig. 6.8. Tsunami run-up

Inundation levels

Damage caused by tsunamis is due to factors like associated hydrostatic as well as hydrodynamic forces, impact of objects being carried by the attacking water mass, high speed currents, overtopping, and, resulting flooding, and current induced erosion. Additionally, a large wall of water advancing in the form of a bore may get developed if the forward speed of the tsunami front exceeds its phase speed (eq. 1) and this may result in flooding large areas.

As tsunamis leave the generating area and disperse they undergo shoaling (change due to the bottom effect), refraction and diffraction before finally reaching the coast. Research on inundation in coastal areas due to tsunamis has been going on since last four to five decades. A good account of the earlier works can be seen in Murty (1977).

Run-up or vertical elevation above the meal sea level (Fig. 1) reached by a tsunami in coastal areas can be found by empirical formulae and curves and more accurately by a numerical solution to governing differential equations. Laboratory experiments conducted by Kaplan (1955) first resulted in a simple empirical formula for run-up over a beach or a structure slope. It is as below:

$$\frac{R}{H} = a(\frac{H}{L})^b \qquad (6.8)$$

where R = run-up above the undisturbed water level, H = incident tsunami height at the start of the slope, L = length of the tsunami at the start of slope; a and b are two constants whose values depend on the beach slope. Typically a = 0.381 and 0.206 for slopes of 1:30 and 1:60

respectively while b = 0.336 and –0.315 respectively for the same slopes. There are many such empirical studies available in the literature including the one by Bretschneider and Wibro (1976) who considered the effect of an important parameter, namely, the seabed roughness.

From the studies reported it is seen that the tsunami run-up increases with wave height, sand specific gravity and flatness of the slope and decreases with slope roughness and permeability. Also, the largest run-up occurs normal to the major axis of the seabed displacement.

The above empirical studies have limitations due to underlying simplified assumptions and idealizations. In recent years significant advances have been made in developing mathematical numerical models to describe the entire process of generation, propagation and run-up of tsunami events. (Yeh, 1996). The most common numerical modeling of tsunamis is based on two sets of governing differential equations called shallow water equations and Navier-Stokes equations in 2 and 3 dimensions, with and without consideration of water compressibility. The differential equations represent conservation of mass, momentum, and energy as indicated below (Mader, 2004):

Conservation of mass equation:

$$\left(\frac{\partial}{\partial t} + U \bullet \nabla\right)\rho = -\rho\nabla \bullet U \qquad (6.9)$$

where, t = time instant, U = particle velocity vector in 3-D, $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$

ρ = water density.

Conservation of momentum equation:

$$\rho(\frac{\partial}{\partial t} + U \bullet \nabla)U = -\nabla \bullet \sigma + \rho g \qquad (6.10)$$

where, $\sigma = \delta_{ij}P - S_{ij}$; δ = Dirac delta function, P = pressure, S = viscosity with no shearing forces

Conservation of energy equation:

$$\rho(\frac{\partial}{\partial t} + U \bullet \nabla)I = -\sigma : \nabla U + \lambda\nabla^2 T \qquad (6.11)$$

where, I = internal energy, $\sigma : \nabla U = \sigma_{ji}\frac{\partial U_i}{\partial X_j}$; X = {x, y, z}; λ = real viscosity coefficient, T = wave period.

The shallow water or long wave models are applicable when the water depth is small (less than 5 percent) compared to the wavelength. It assumes that the wave motion is essentially horizontal, i.e., the vertical component of the motion is negligible. Hence when the tsunami propagates over underwater shoals the shallow water model becomes less accurate, but it could be good for large wave length (exceeding 10 times the depth), long area-generated and long period (more than 15 minutes) tsunamis propagating over bed slopes steeper than 2 percent.

Long distance or deep water or small width (less than 40 times the depth) propagations can be better simulated either by the Navier-Stokes equation or by the linear wave model.

In above modeling water is assumed to be incompressible. But large earthquake-generated tsunamis or those created by impact landslides have to be simulated by the compressibility-incorporated Navier-Stokes equation.

The differential equations as above are conveniently solved using finite difference methods. Sophistication in considering different terms of governing equations coupled with advances in numerical procedures have yielded the latest models that can work out water levels in as small a space grid as 10 m x 10 m and as small a time step as 0.2 seconds. They can yield flooding around different buildings in a residential colony.

Closure

Occurrence of the tsunami-generating earthquake can be recorded on seismographs and communicated via satellites to the warning center, where computer-based models as described above calculate tsunami heights and travel times. This could be verified by a series of wave rider buoys and accordingly tsunami warning can be issued. Such warning systems exist in Pacific countries including the U.S. and Canada since 1964.

Research on tsunami hydraulics in this country in future may involve following studies (NIO, 2005): (a) Development of numerical models for tsunami in the Indian ocean, (b) simulation of past tsunami events identified in the tide-gauge records, (c) reconciliation of source parameters with arrival times using inverse methodology (d) construction of Green's function for the Indian Ocean, (e) development of finite-element and fractal models for tsunami run-up simulations, and soft computing tools (ANN, fuzzy systems, hybrid approaches) to evaluate water levels due to tsunamis

6.3 Pressure set-up: (Atmospheric or Barometric)

It is the rise or fall in the SWL produced by changes in the atmospheric pressure as in storms or cyclones.

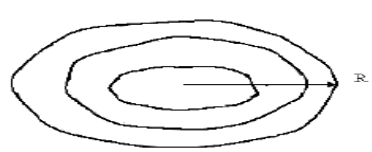


Fig. 6.9. Isobars

At distance R, (Fig. 6.9) where the maximum wind speed occurs from eye of the storm, the pressure set up is given by:

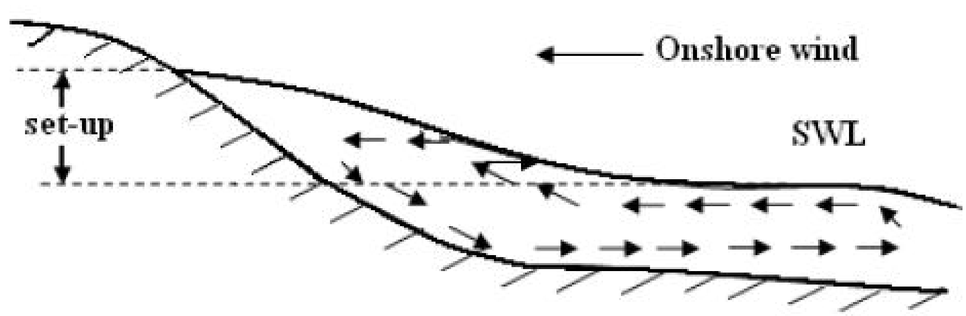
$$(P_s)_R = (const)\Delta P \qquad (6.12)$$

where ΔP = actual pressure with respect to normal atmosphere measured in inches of mercury. At any radial distance r from the storm eye, the set up is:

$$(P_s)_R = \left(1 - e^{\frac{R}{r}}\right) P_s \qquad (6.13)$$

6.4 Wind set-up: (also called storm surge)

This indicates the change in the SWL produced by wind accompanying a storm. As shown in the following figure (6.10) an onshore wind can cause inland flooding due to the circulation pattern generated.



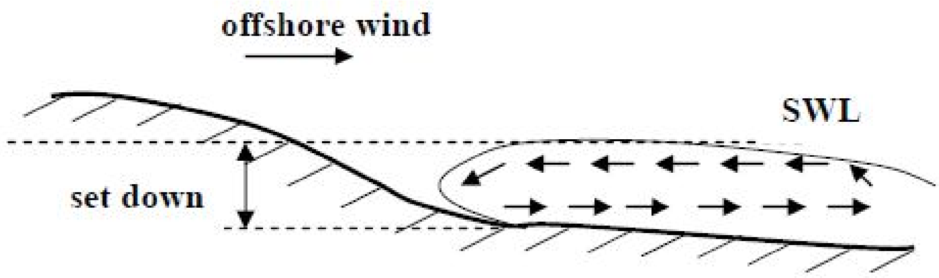


Fig. 6.10. Set up and down due to wind

An offshore wind creates a circulation pattern in the opposite direction as shown in the figure, which may results in navigational draft problems as well as head problems for pumps.

Prediction of storm surge

Change in SWL because of wind set-up depends up on storm characteristics, sea bed features, and interaction with tide and gravity waves. The storm characteristics are wind speeds, duration of storm, path and pressure pattern. The seabed features include roughness, size and shape of basin. Storm surge can also be studied by physical scale modeling or by numerical models based on solution of governing differential equations describing the water motion. The underlying equations involve following assumptions:

a) Vertical acceleration of water particles is negligible.

b) Sea water is inviscid.

c) Sea bed is impermeable.

d) Effect of earth's curvature is small.

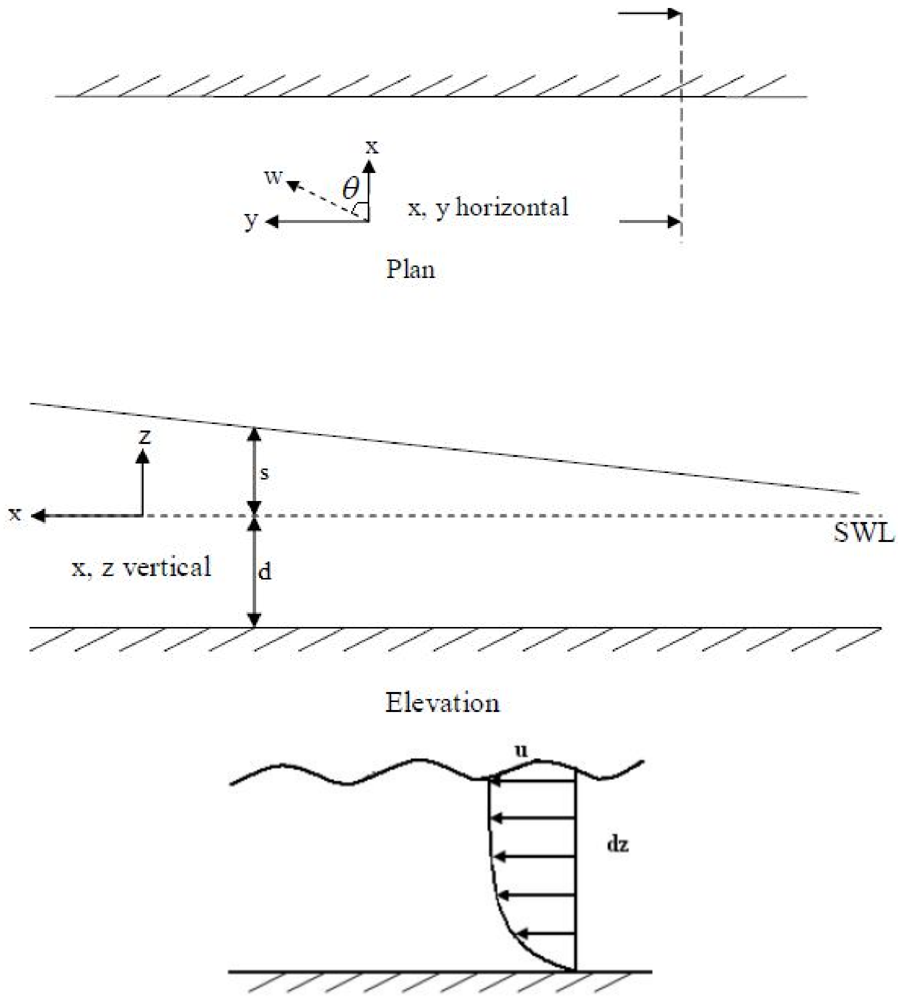


Fig. 6.11 The definition sketch for storm surge

For any horizontal position (x, y) and time't': (Fig. 6.11), let d = water depth, s = surge height, D = d + s; u = current speed along x direction, v = current speed along y direction, t = time.

$$U = \int_{-d}^{s} udz = \text{Volume of water transported per unit width along x direction.} \quad (6.14)$$

$$V = \int_{-d}^{s} vdz = \text{Volume of water transported per unit width along y direction} \quad (6.15)$$

P = precipitation rate.

Two sets of equations, continuity and motion, are involved.

Equation of continuity:

$$\frac{\partial s}{\partial t}+\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=P \tag{6.16}$$

Equation of Motion: (based on the equation of force = mass x acceleration)

$$\frac{\partial U}{\partial t}+\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}= fV-gD\frac{\partial S}{\partial x}+gD\frac{\partial \xi}{\partial x}+gD\frac{\partial \varsigma}{\partial x}+\frac{\tau_{Sx}}{\rho}-\frac{\tau_{bx}}{\rho}+W_x P$$

$$\frac{\partial V}{\partial t}+\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}=-fU-gD\frac{\partial S}{\partial y}+gD\frac{\partial \xi}{\partial y}+gD\frac{\partial \varsigma}{\partial y}+\frac{\tau_{Sy}}{\rho}-\frac{\tau_{by}}{\rho}+W_y P$$

(6.17 and 6.18)

where, M_{xx} = transport of x-direction momentum along x direction; M_{xy} = transport of x direction momentum along y direction; M_{yy} = transport of y-direction momentum along y direction; f = Coriolis parameter = $2\omega\sin\phi$, where ω is angular speed of earth and ϕ is latitude; g = acceleration due to gravity; D = d + S, where d = average water depth and S = storm surge; ξ =Pressure difference (with respect to atmosphere) in water head; ς =tide potential in water head equivalent; τ_{Sx}, τ_{Sy} = x,y components of surface wind stress; W_x, W_y = x, y components of wind speed; τ_{bx}, τ_{by} = x,y components of bottom wind stress; ρ =mass density of sea water;

$$M_{xy}=\int_{-d}^{S}U^2dz \tag{6.19}$$

$$M_{xy}=\int_{-d}^{S}UVdz \tag{6.20}$$

$$M_{yy}=\int_{-d}^{S}V^2dz \tag{6.21}$$

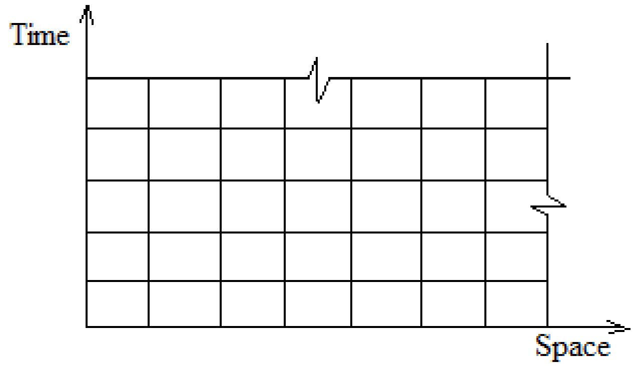


Fig. 6.12. The solution domain

The above equations are solved numerically at several points simultaneously in the time–space grid (Fig. 6.12) using the Finite Difference scheme or alternatively the Finite Element method.

6.5 Wave set-up:

Breaking waves produces it. After breaking, waves travel to shore and during this journey the kinetic energy is converted to potential energy, generating the wave set up. (Fig. 6.13.)

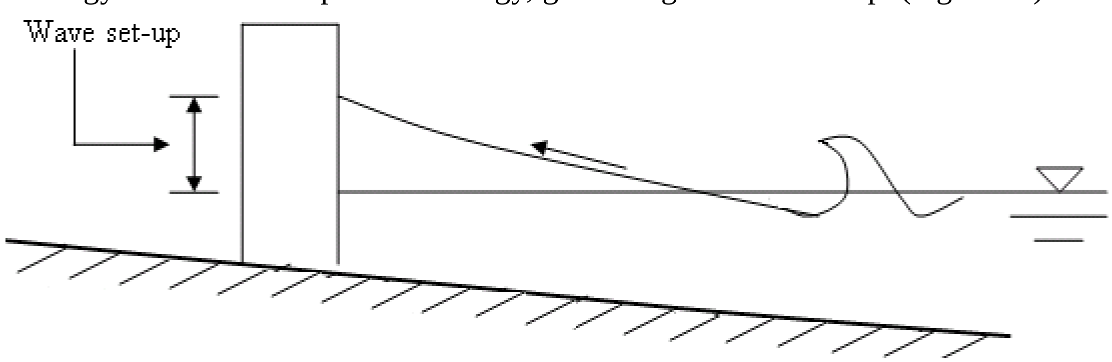


Fig. 6.13. Wave set-up in general

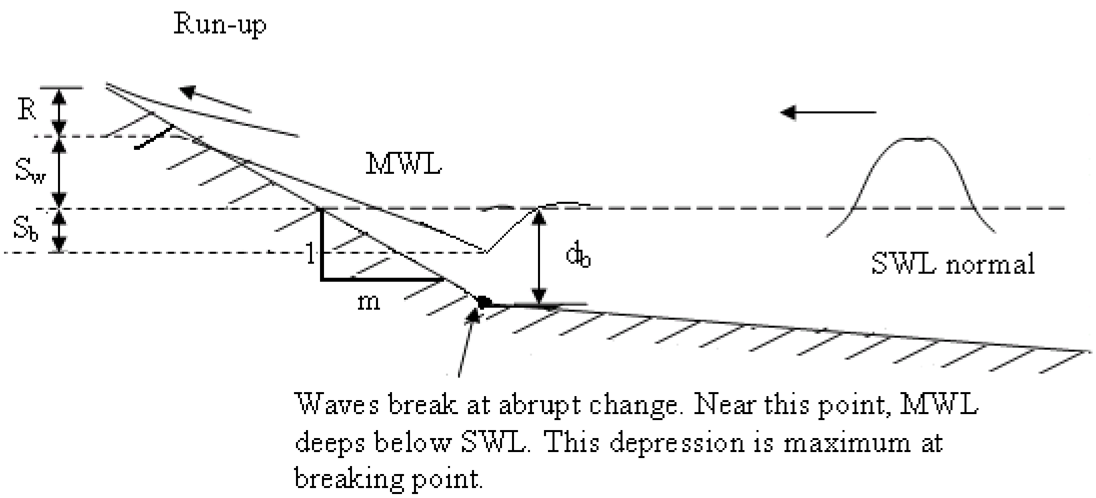


Fig. 6.14. Wave set up over sloping beach

Wave set-up on a Sloping Beach: It is as shown in the fig. 6.14.

$$S_w + S_b = \Delta S \qquad (6.22)$$

Wave set-up = ΔS = max. MWL – min. MWL

S_w = max. MWL – normal SWL

S_b = set down = normal SWL – min. MWL

Wave set-up on a berm (or reef): (Fig. 6.15).

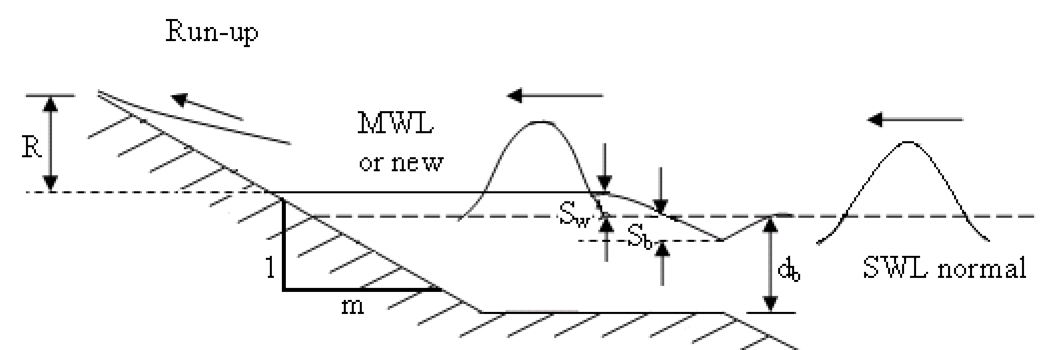


Fig. 6.15. Wave set up over berm

Consider a spectrum of larger or smaller wave heights. Larger waves break in depth d_b producing set-up. This increases depth upstream. Hence smaller waves still exist over the berm and do not break. They break on the shore and this additional set-up has to be considered in calculations. Thus, very high waves contribute less to wave set-up.

Set-up for regular waves:

Based on the solitary wave theory Reid (1972) provided useful expressions as below:
The set-down is given by:

$$S_b = -\frac{\sqrt{g}H_0^2 T}{64\pi d_b^{\frac{3}{2}}} \tag{6.23}$$

While the set-up is obtained by:

$$\Delta S = 0.15 d_b \tag{6.24}$$

Where $d_b = \dfrac{H_b}{b - \left(a\dfrac{H_b}{gT^2}\right)}$; $a = 43.75\left(1 - e^{-19m}\right)$ and $b = \dfrac{1.56}{\left(1 + e^{-19.5m}\right)}$

CHAPTER 7

WAVE FORCES ON SHORE-BASED STRUCTURES

7.1 Introduction

The method to calculate wave-induced forces on shore-based structures depends primarily on its type, namely, vertical faced or sloping faced.

Vertical faced structures: A common example is a sea wall, of concrete or steel, constructed parallel to the shore in order to protect it against erosion
Sloping faced structures: It mainly includes a rubble mound breakwater formed by dumping rubble one over the other and constructed normally at an angle to the shoreline with a view to create calm water conditions.

7.2 Forces on Vertical Faced Structures:

The first step to make the force calculations is to choose an appropriate design wave height and also to understand the relative location of the structure with respect to the wave breaking zone.
The choice of the wave height out of alternative definitions likes $H_{1/100}$ or $H_{1/10}$, or $H_{1/3}$ (denoting average value of the top 1, 10 and 33.33 percent of all waves in a collection respectively) is guided by the structure type and the required harbor protection. The type of the structure indicates rigid, semi-rigid or flexible. Rigid structures like concrete walls do not involve any absorption of the incident wave energy, offer maximum protection, but can fail against a single large wave and hence are designed for the highest value, $H_{1/100}$. Semi-rigid structures include steel sheet pile walls and can absorb incident energy to some extent and hence could be designed for less than maximum wave, falling between $H_{1/100}$ to $H_{1/10}$. Finally flexible structures like rubble mound breakwater, designed to absorb large amount of incident wave energy and easy to repair if damaged, can be economically designed with the help of $H_{1/3}$.

Waves break in a depth ranging from about 0.8 to 1.4 times their height, depending on their steepness and seabed slope. If the structure is located in this range of water depth then it would be subjected to the action of breaking wave forces. On the contrary if it is installed in depth deeper than this range, it would be subjected to non-breaking wave forces. Finally structures in depths smaller of this range would be influenced by broken water action. While non-breaking wave forces are static in nature the remaining two are dynamic or time varying.

Non-breaking wave forces:

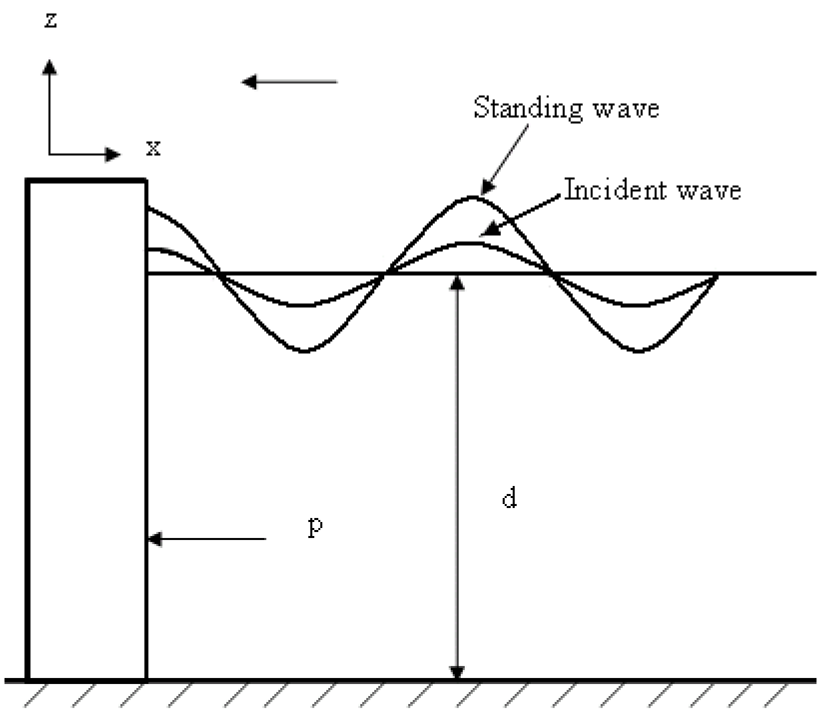


Fig. 7.1 Standing wave

Assume a smooth faced vertical wall. Then the incident wave would undergo pure reflection and standing waves will be formed. Assuming linear theory to be valid the subsurface pressure at depth 'z' is given by (as per SAINFLOU in 1928)

$$p = -\gamma z + \gamma\eta \frac{\cosh k(d+z)}{\cosh kd} \tag{7.1}$$

where γz is static and $\gamma\eta \frac{\cosh k(d+z)}{\cosh kd}$ is dynamic part.

For a standing wave

$$\eta = H \cos kx \cos \omega t \qquad (7.2)$$

Hence pressure for a standing wave at any depth 'z' is:

$$p = \gamma H \cos kx \cos \omega t \frac{\cosh k(d+z)}{\cosh kd} - \gamma z \qquad (7.3)$$

Choosing x = 0, cos (kx) =1, the maximum pressure will occur when cos wt =1 or physically when the crest appears on wall. (Fig. 7.2)

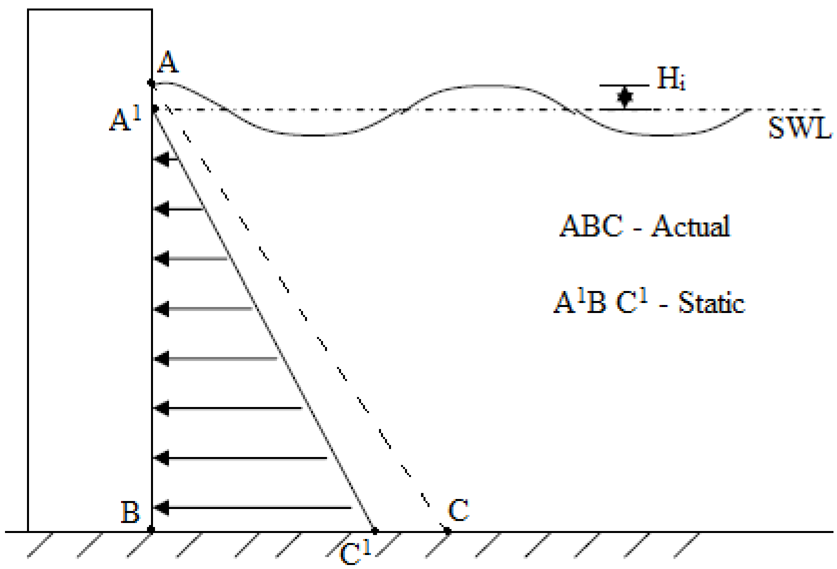


Fig. 7.2 The wave pressure diagram when crest appears on the wall

Hence

$$P_c = \gamma H \frac{\cosh k(d+z)}{\cosh kd} - \gamma z \qquad \text{--- at } z = z \qquad (7.3)$$

and

$$P_c = \frac{\gamma H}{\cosh kd} + \gamma d \qquad \text{---at } z = -d \qquad (7.4)$$

Similarly the minimum pressure will occur when cos wt = -1, or physically when the trough appears on wall. (Fig. 7.3)

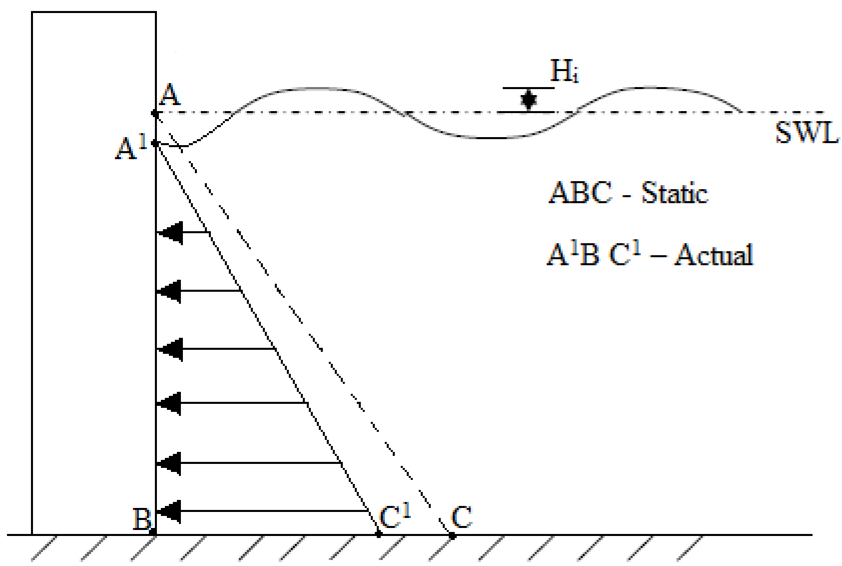


Fig. 7.3 Wave force when trough appears on the wall

$$P_t = -\gamma H \frac{\cosh k(d+z)}{\cosh kd} - \gamma z \qquad \text{------ at } z = z \qquad (7.4)$$

$$P_t = \frac{-\gamma H}{\cosh kd} + \gamma d \qquad \text{--- at } z = -d \qquad (7.5)$$

However in practical calculations some non-linear effects are assumed, because the above expression was found to lead to overestimation. Miche (1994) & Rundgren(1958) accordingly gave the following corrected procedure:

CREST ON WALL case:

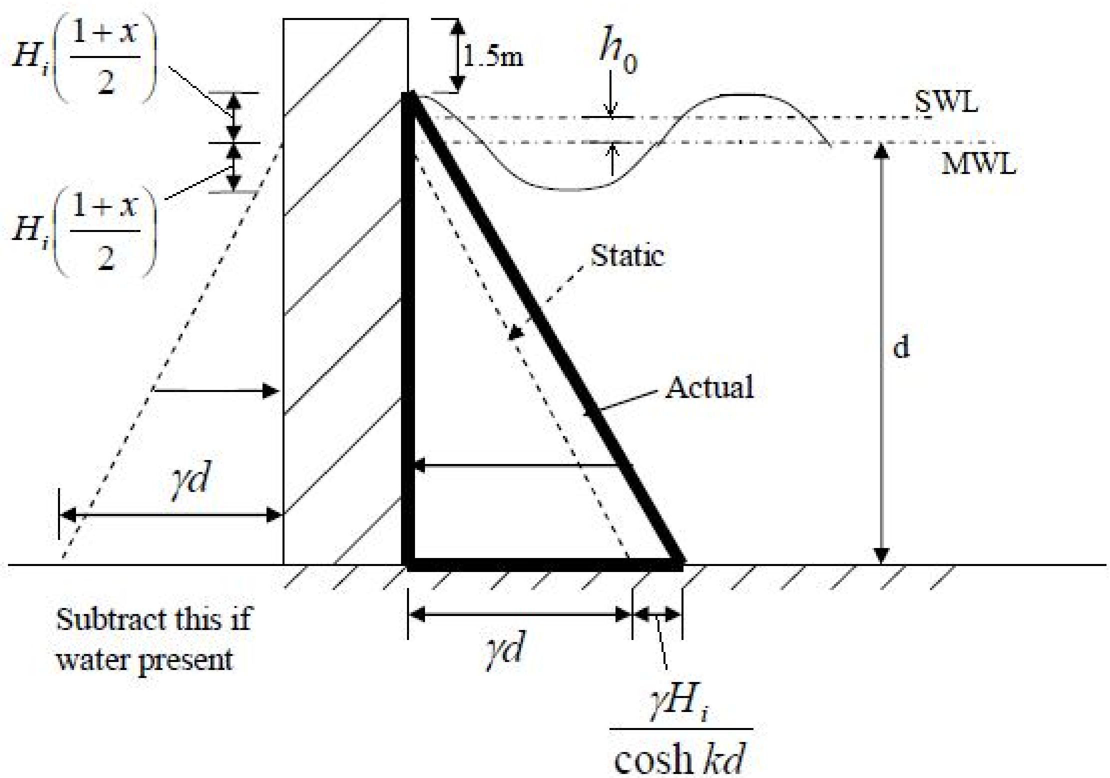


Fig.7.4 Wave pressure diagram when crest appears on the wall.

$$h_0 = \frac{\pi H_i^2}{L} \coth \frac{2\pi d}{L} \tag{7.6}$$

Wave height at wall $= H_i + H_r$

$= \left(1 + \frac{H_r}{H_i}\right) H_i$

$=(1+ X)\ H_i$ (7.7)

where X is reflection coefficient in range (0.9 , 1). For rip-rap and rough wall the reflection coefficient is 0.9.

TROUGH ON WALL case:

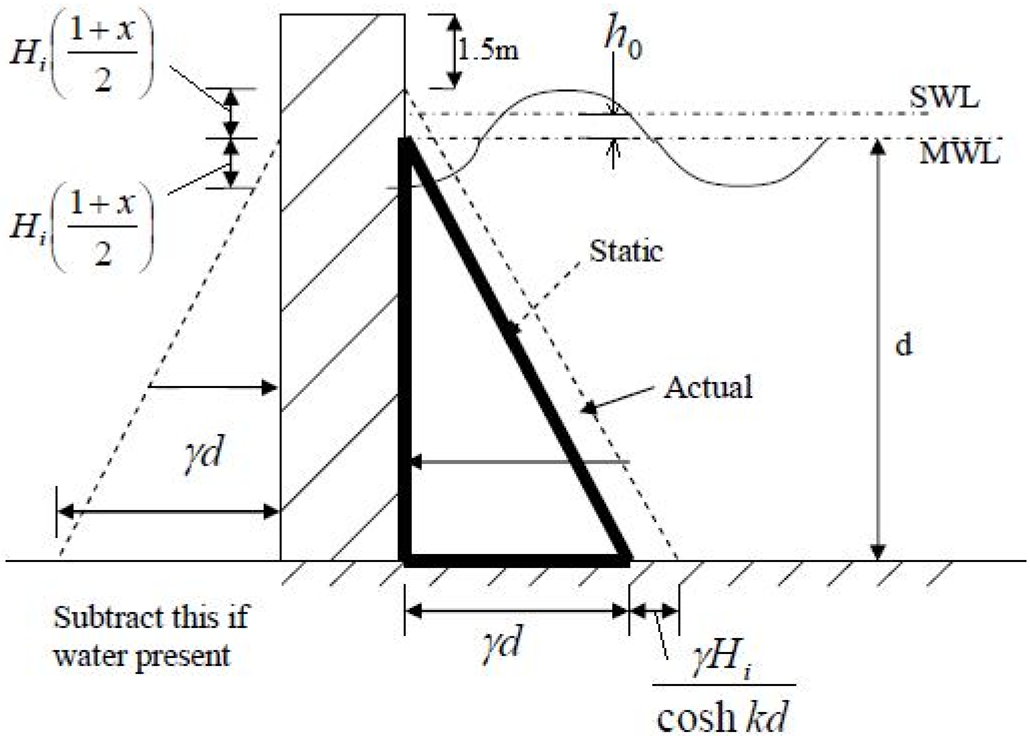


Fig. 7.5 Wave pressure diagram when trough appears on the wall

To find total force acting on the wall (F) and total overturning moment at base (M):

The overtopping conditions:

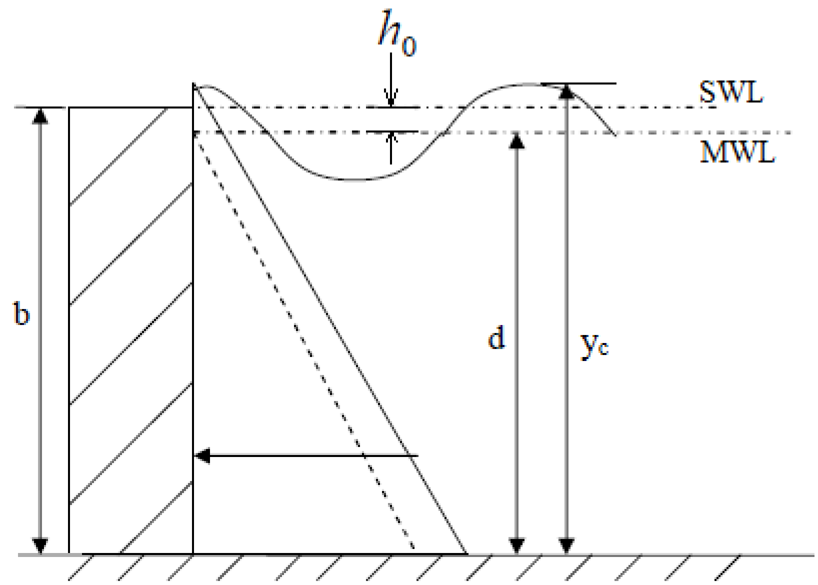


Fig. 7.6 The overtopping condition

Total force $F^1 = r_F F$

$$r_F = \frac{b}{y_c}\left(2 - \frac{b}{y_c}\right) \tag{7.8}$$

Total Moment $M^1 = r_M M$

$$r_M = \left(\frac{b}{y_c}\right)^2 \left(3 - 2\frac{b}{y_c}\right) \tag{7.9}$$

Replace y_c by y_t for trough on wall

The composite wall:

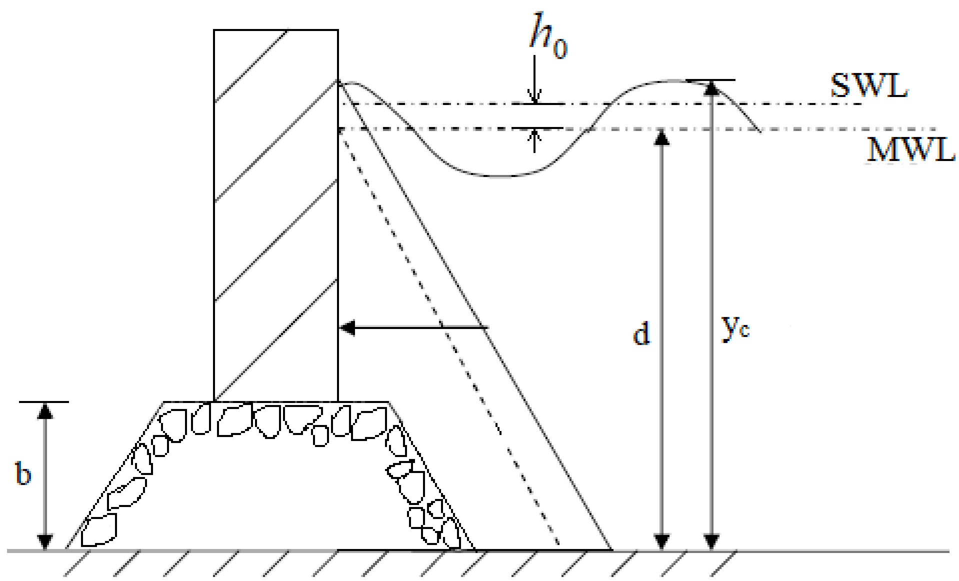


Fig. 7.8. The composite wall

Total Force $F^{11} = F - r_F F$

$$= (1 - r_F)F \tag{7.10}$$

Total Moment at the rubble bottom

$$M_1^{11} = (1 - r_M)M \tag{7.11}$$

Total Moment at the rubble top

$$M_2^{11} = M_1^{11} - b(1 - r_F)F \tag{7.12}$$

BREAKING WAVE FORCES

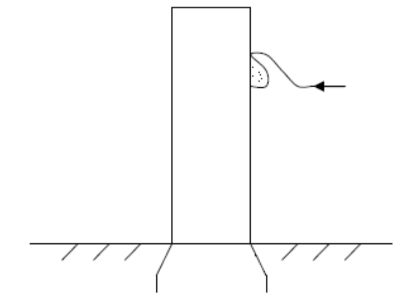


Fig. 7.9 Wave breaking on the wall

Bagnold's experimental analysis in 1939 indicated that the large pressure is exerted only for 1/100 seconds duration when air is entrapped by plunger breakers. Minikin in 1955-63 combined this work with proto measurements and suggested following pressure diagram:

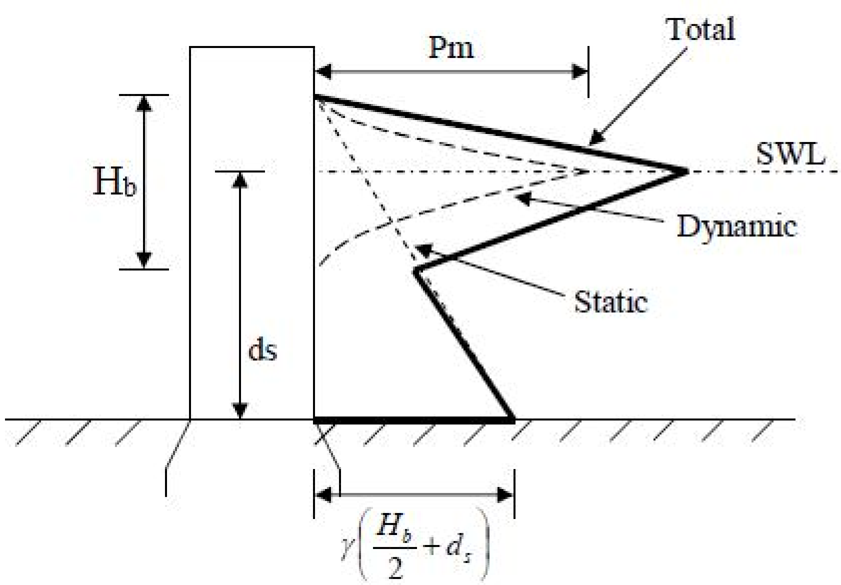


Fig. 7.10 The wave pressure diagram for breaking waves

The maximum dynamic pressure due to wave breaking is given by:

$$P_m = 101\gamma \frac{H_b}{L_D}\frac{d_s}{D}(D+d_s) \qquad (7.13)$$

which has a parabolic variation over the wall. Here L_D = wave length in depth 'D', D= depth at one wave length away from the structure.

$$D = d_s + mL_{d_s} \qquad (7.14)$$

$$P_{Total} = P_{st} + P_{Dynamic} \qquad (7.15)$$

Hence $$F_{Total} = \frac{1}{2}\left[\gamma\left(\frac{H_b}{2}+d_s\right)\right]\left(\frac{H_b}{2}+d_s\right)+\frac{1}{3}P_m H_b \qquad (7.16)$$

$$M_{Total} = \frac{1}{6}\gamma\left(\frac{H_b}{2}+d_s\right)^3+\frac{1}{3}P_m H_b d_s \qquad (7.17)$$

FORCES BY BROKEN WAVES

After breaking, the broken water mass travels towards the shore with velocities same as those before breaking. Depending on whether the structure is located landward or seaward the shoreline we have two types of forces as below:

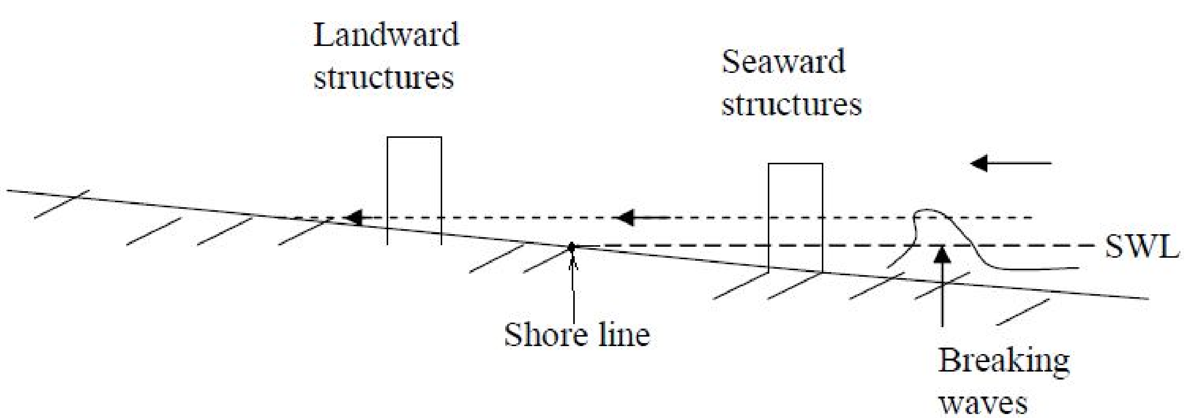


Fig. 7.11 Landward and shoreward structures

FORCES ON SEAWARD STRUCTURES:

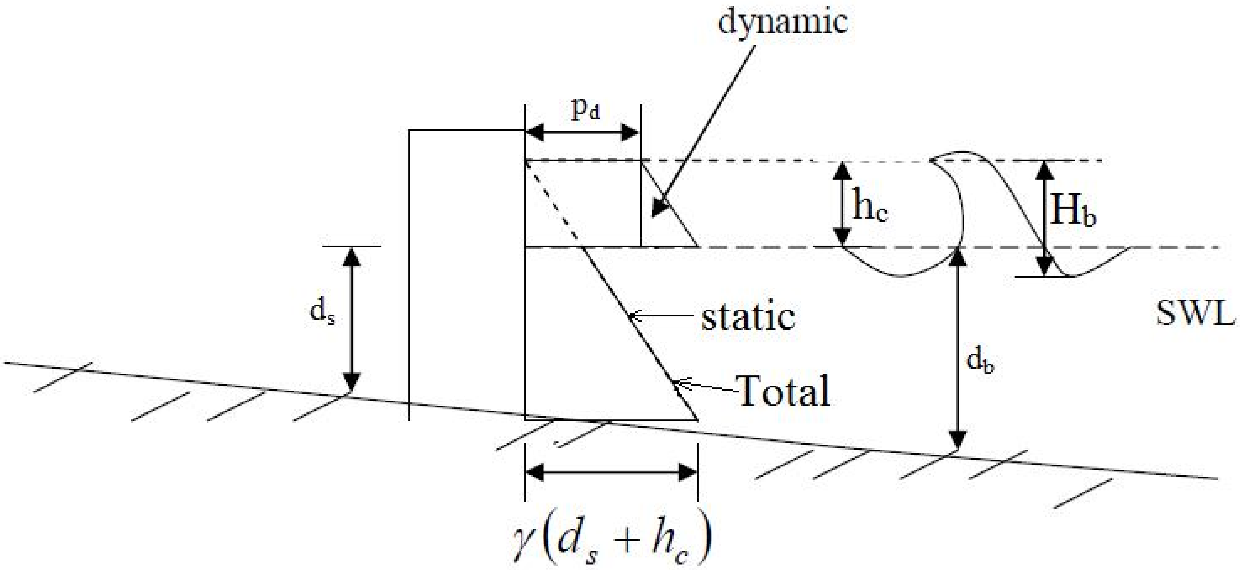


Fig. 7.12 Forces on seaward structures

Take $h_c = 0.78 H_b$ (7.18)

$$P_d = \frac{\gamma c^2}{2g}$$

$$= \frac{\gamma\left(\sqrt{g d_b}\right)^2}{2g}$$

$$= \frac{\gamma}{2} d_b$$

$$F_{Total} = F_d + F_{Static}$$

$$= \frac{\gamma}{2} d_b h_c + \frac{1}{2}\gamma(d_s + h_c)[(d_s + h_c)] \qquad (7.19)$$

$$M_{Total} = \frac{\gamma}{2} d_b h_c\left(d_s + \frac{h_c}{2}\right) + \frac{1}{6}\gamma(d_s + h_c)^3 \qquad (7.20)$$

FORCES ON LANDWARD STRUCTURES:

Assume: 1) Waves break at d_b and travel up to shoreline with speed c becoming a translatory wave.

2) Then they up rush up to a height = $2H_b$ above SWL over the beach

3) Velocity decreases from ‘c’ at shoreline to ‘0’ at the point of maximum up rush.

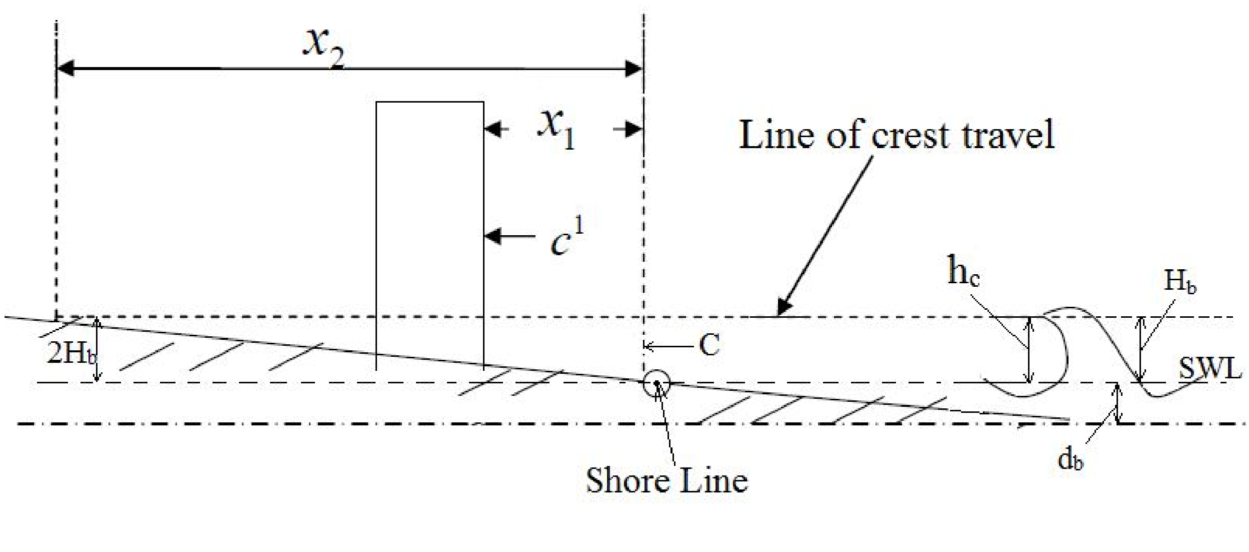


Fig. 7.13 Forces on landward structures

x_1 =Horizontal distance from shoreline to structure

x_2 = Horizontal distance from shoreline to limit up to up rush

α =Beach slope

$$\tan\alpha = \frac{2H_b}{x_2}$$

C^1 =celerity of broken wave at structure toe

h^1 = Height of structure under influence of broken wave

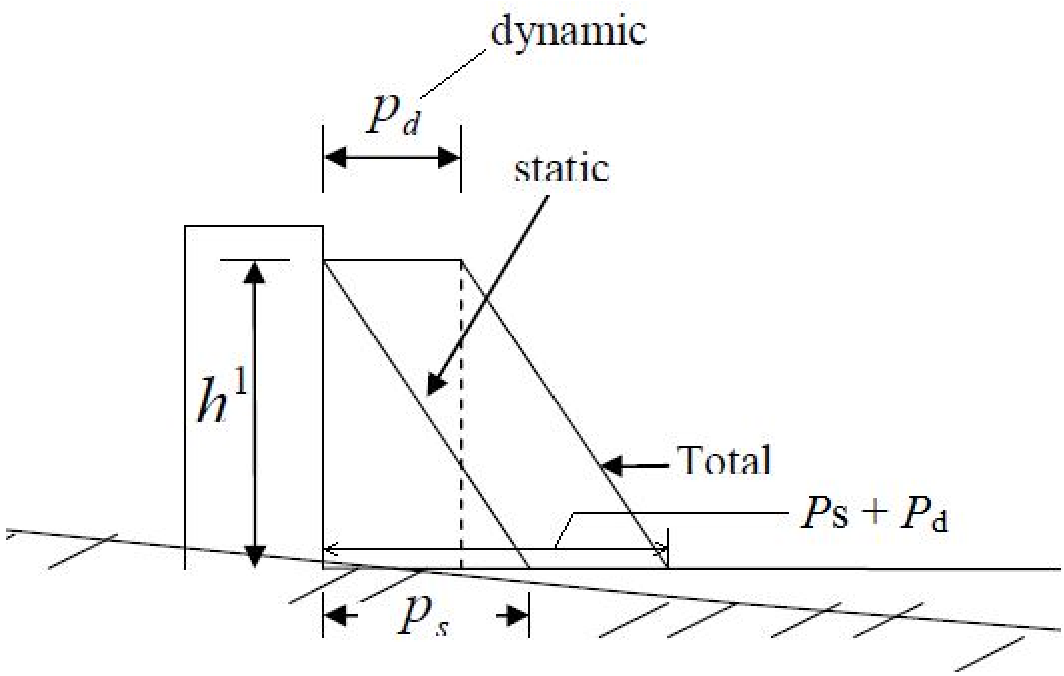


Fig. 7.14 Forces on landward structures

$P_d = \frac{1}{2g}\gamma c^{1^2}$ ------ uniform along height h^1 (7.22)

P_s =0 ----- at top of height h^1

=γ h^1 ----- at bottom of wall

But $c^1 = c\left(1 - \frac{x_1}{x_2}\right)$ ------- Since within distance x_2 change in velocity =c-0 , hence within distance x_1 change in velocity=$\frac{x_1}{x_2}c$

Similarly $h^1 = h_c\left(1 - \frac{x_1}{x_2}\right)$

Hence $P_d = \frac{1}{2g}\gamma c^2\left(1 - \frac{x_1}{x_2}\right)^2$

But $c=\sqrt{gd_b}$

Hence $P_d=\frac{\gamma}{2}d_b\left(1-\frac{x_1}{x_2}\right)^2$

$F_{Dynamic}=P_d h^1$; $F_{static}=\frac{\gamma}{2}h_c{}^2\left(1-\frac{x_1}{x_2}\right)^2$

$$\text{Hence}\quad \text{total Force } =\frac{\gamma}{2}d_b h_c\left(1-\frac{x_1}{x_2}\right)^3+\frac{1}{2}\gamma h_c{}^2\left(1-\frac{x_1}{x_2}\right)^2 \qquad (7.23)$$

$$\text{Total Moment}=\frac{\gamma}{4}d_b h_c{}^2\left(1-\frac{x_1}{x_2}\right)^4+\frac{1}{6}\gamma h_c{}^3\left(1-\frac{x_1}{x_2}\right)^3 \qquad (7.24)$$

OBLIQUE WAVE ATTACK:

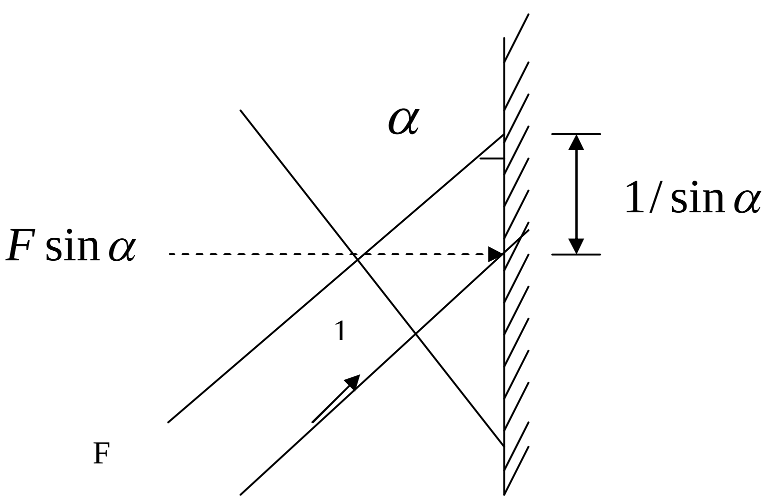


Fig. 7.15 Oblique wave attack

In this case reduce dynamic force only.

$$F_{net}=\frac{F\sin\alpha}{1/\sin\alpha}$$

$$=F\sin^2\alpha \qquad (7.25)$$

STRUCTURE SLOPE: (CONCRETE WALLS ONLY)

$$F^{1}{}_{net} = F_{net} \sin^{2}\theta$$

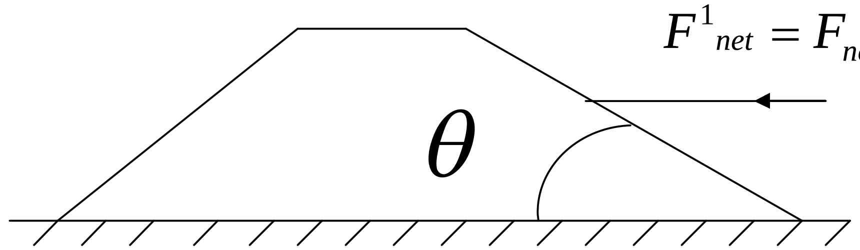


Fig. 7.16 Forces on landward structures

7.3 FORCES ON SLOPING FACE STRUCTURES:

The most common example of such structures is a rubble mound breakwater.

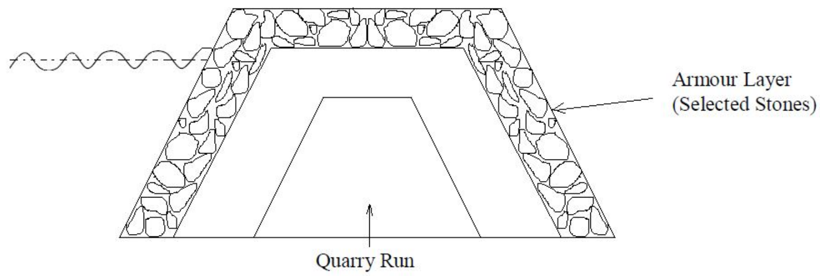


Fig.7.17 Forces on landward structures

Its outer layer, which is subjected to direct wave action, is known as armor layer. The wave action on this structure is complex. There are no explicit formulae to determine it. Only empirical expressions describing armor stability are popular. Rubble mound can be either single or Composite.

SINGLE RUBBLE MOUND

Following Irribarren (1938)'s work, Hudson (1953) did model testing, which resulted in the Vicksburg formula given below:

The weight of the armor unit stone:

$$W = \frac{\gamma_r H^3}{K_D (S_r - 1)^3 \cot\alpha} \qquad (7.26)$$

---- For ungraded stones or when H>5 feet.

where W=Weight of armour unit (N or Kg)

γ_r =Specific wt. of armour unit – Saturated , surface dry (N or kg/m^3)

H=Design wave height (m)

S_r = Specific gravity of armour unit = $\frac{\gamma_r}{\gamma_w}$

α = angle – side slope horizontal in degrees.

K_D = damage coefficient (2,15) which is function of shape , roughness, degree of locking , damage , etc.

If graded stones are used or if H< 5 feet:

$$W_{50} = \frac{\gamma_r H^3}{K_{RR}(S_r - 1)^3 \cot\alpha} \tag{7.26}$$

Where W_{50} = Weight of the stone of 50% size in the gradation

$W_{\max} = 3.6W_{50}$

$W_{\min} = 0.22W_{50}$

$K_{RR} = 1.3$ if d<20 feet

= 1.7 if d>20 feet --------- Assuming 5% damage.

COMPOSITE BREAKWATER

Rubble as foundation :

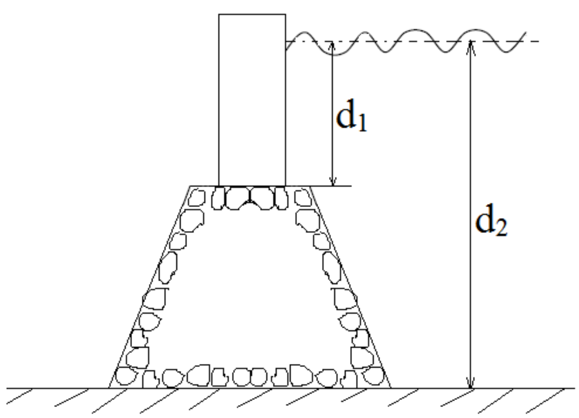


Fig. 7.18 Rubble as foundation

Rubble as toe protection:

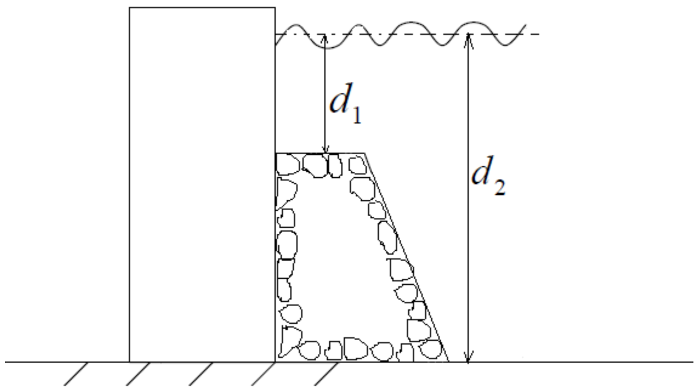


Fig. 7.19 Rubble as toe protection

$$W = \frac{\gamma_r H^3}{N_s^3 (S_r - 1)^3} \qquad (7.27)$$

Where W=Mean weight of rubble

γ_r =Specific weight of rubble

S_r = Specific gravity of rubble= $\frac{\gamma_r}{\gamma_w}$

N_s =Stability Number obtained from the following Figure 7.20.

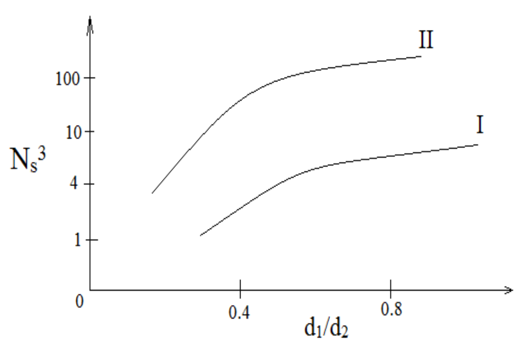


Fig. 7.20 Variation of stability number

CHAPTER 8

WAVE FORCE ON SMALL DIAMETER MEMBERS

8.1 The Morison's equation

A structural member is considered to be of 'small diameter' when its diameter is less that about 0.15 times the wave length; for example, members of Jacket structures and piled jetties.

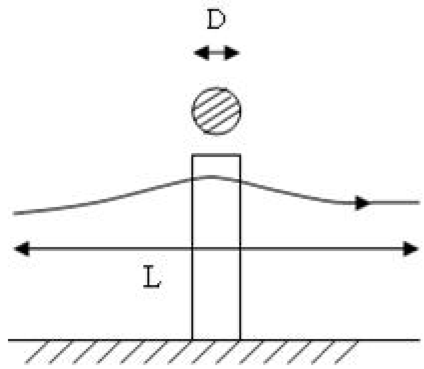


Fig.8.1 Definition sketch

When member diameter is small incident waves do not get much scattered by the obstruction and in that case the equation given by Morison et al. (1950) becomes applicable.
Morison et al. (1950)'s equation:

It states that the total force, F_I, in-line with the wave direction can be obtained by addition of the drag, F_D, and the inertia, F_I components, i. e.,

$$F_T = F_D + F_I \tag{8.1}$$

The force due to drag is proportional to kinetic head, i. e.,

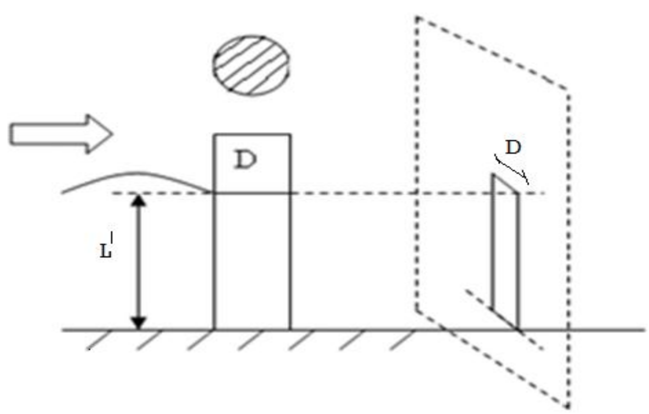


Fig. 8.2 Area projection on a vertical plane

$$F_D \propto \frac{1}{2}\rho A u^2 \qquad (8.2)$$

Where ρ =mass density of fluid

A= area of object projected on a plane held normal to flow direction

u = flow velocity

Introducing the constant of proportionality, C_D, and assuming a steady, uniform flow in a viscous fluid, we have

$$F_D = C_D \frac{1}{2}\rho A u^2 \qquad (8.3)$$

where C_D is coefficient of drag. Its value depends on body shape, roughness, flow viscosity and several other parameters.

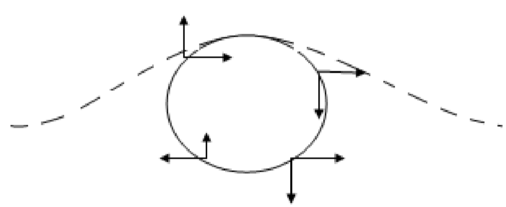


Fig. 8.3 Particle velocities

Because the direction of wave induced water particle velocity reverses after every half cycle, we write,

$$F_D = \frac{1}{2} C_D \rho A u|u| \qquad (8.4)$$

The force of inertia is proportional to mass times the fluid acceleration:

$$F_I \alpha \rho v \dot{u}$$

$$F_I .\alpha. \rho V\dot{u}$$

where V = volume of fluid displaces by the object.

$\dot{u}$ = acceleration of fluid

Hence,

$$F_I = C_m \rho v \dot{u} \tag{8.5}$$

Where C_m =Coefficient of Inertia. It depends on shape of the body, its surface roughness and other parameters.

Most of the structural members are circular in cross section. Hence,

$$F_D = \frac{1}{2} C_D \rho D L' u |u|$$

$$F_I = C_m \rho \frac{\pi d^2}{4} L' \dot{u}$$

Because u and $\dot{u}$ vary along L' and further considering unit pile length i.e. L'=1. Hence,

$$F_T = \frac{1}{2} C_D \rho D u |u| + C_m \rho \frac{\pi d^2}{4} \dot{u} \tag{8.6}$$

where, F_T = in-line (horizontal) force per meter length at member axis at given time at given location.

$\frac{1}{2} C_D \rho D u |u|$ = in-line (horizontal) water particle velocity at the same time at the same location.

$C_m \rho \frac{\pi d^2}{4} \dot{u}$ is in-line (horizontal) water particle acceleration at the same time at the same location.

Note that $u = f(\cos\theta)$ and $\dot{u} = f(\sin\theta)$. Hence u and $\dot{u}$ are out of phase by 90^0 and are not maximum at the same time.

Basically C_D and C_m are functions of size and shape of the object. If that is fixed then they depend on Keulegan-Carpenter number, Reynold's number as well as roughness factor.

Keulegan-Carpenter number: K_C

It is basically a ratio of maximum drag to maximum inertia. We have,

$$\left(F_D\right)_{\max} = \frac{1}{2} C_D \rho D u_{\max}^2$$

where, $$u_{\max}^2 = \frac{\pi^2 H^2 \cosh^2 k\left(d+z\right)}{T^2 \sinh^2 kd} \cos^2\theta$$

$$\left(F_I\right)_{\max} = C_m \rho \frac{\pi d^2}{4} \dot{u}_{\max}$$

where $$\dot{u}_{\max} = \frac{2\pi^2 H \cosh k\left(d+z\right)}{T^2 \sinh kd} \sin\theta$$

At z=0,

$$\frac{\left(F_D\right)_{\max}}{\left(F_I\right)_{\max}} = \frac{C_D}{C_m}\frac{H}{D}\frac{1}{\pi}\frac{\cosh kd}{\sinh kd} = \frac{C_D}{C_m}\frac{1}{\pi^2}\frac{u_{\max}T}{D} \qquad (8.7)$$

The ratio of maximum drag to maximum inertia can thus be taken as proportional to

$= \frac{u_{\max}T}{D}$ Where $u_{\max}$ =Maximum velocity in the wave cycle

T= wave period

D= Diameter

The above ratio also stands for (Total horizontal motion of the particle / Diameter).

If $K_C < 5$ then inertia is dominant,

If $K_C > 15$ then drag is dominant and regular eddies are shed at downstream section.

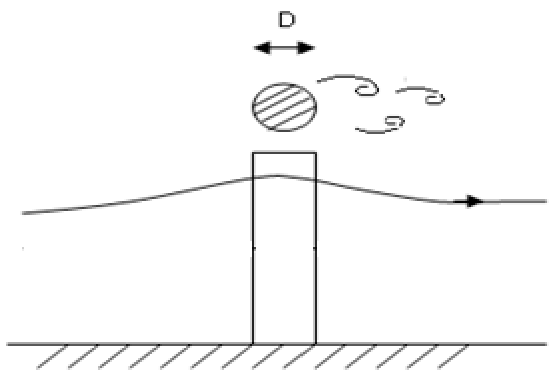


Fig. 8.4 Eddy shedding

at frequency of $fe = \frac{Sv}{D}$ where S = Strouhal No. $\approx$ 0.2.

Alternate eddy shedding gives rise to alternate lift forces due to pressure gradient across the wake.

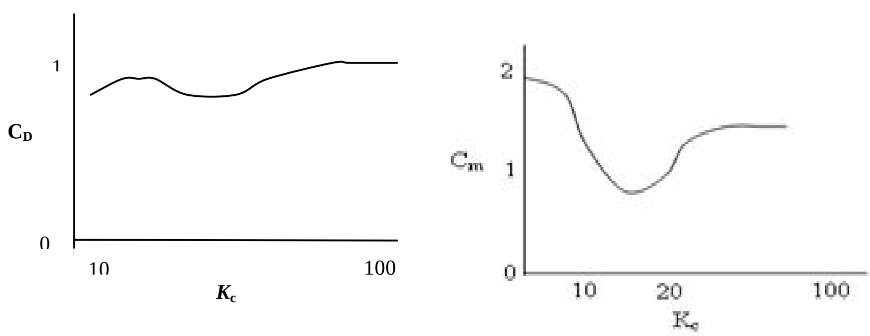


Fig. 8.5 Variation of C_D and C_M against K_C

Reynold's Number, R_e:

It is the ratio of the inertia force to the viscous force, i. e., R_e:

$$= \frac{u_{max} D}{v} \tag{8.8}$$

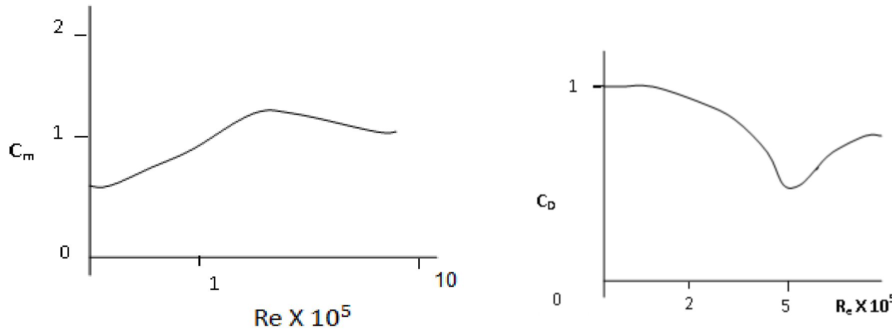


Fig. 8.6 Variation of C_D and C_M against Re

Roughness Factor:

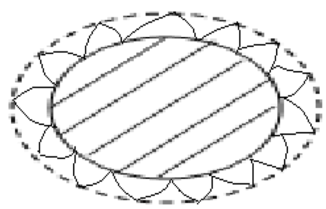


Fig. 8.7 Encrustation around cylindrical members

Structural members are in course of time covered by sea weeds, barnacles, shell fish etc. Due to this, effective diameter changes, effective mass increases, flow pattern, eddy structure changes . Finally the wave force also changes. Lab studies have shown that C_m does not change much. C_D changes appreciably and can become 2 to 3 times more than the initial value.

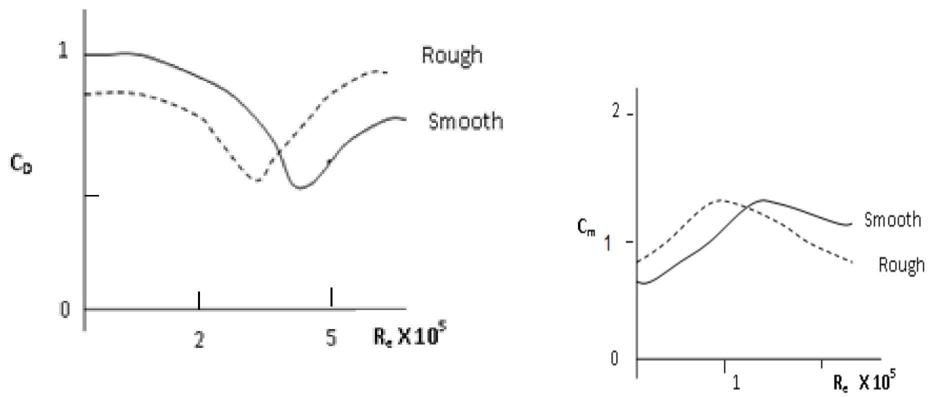


Fig. 8.8 Effect of roughness on C_D and C_M

Scatter in C_D, C_m values: Many laboratory and field studies have been made to assess the effects of all unaccounted factors like eddy shedding , past flow history, initial turbulence , wave irregularity directionality, local conditions , data reduction techniques.

But experiments are inconclusive.

Experiments to evaluate C_D, C_m are performed in the following way.

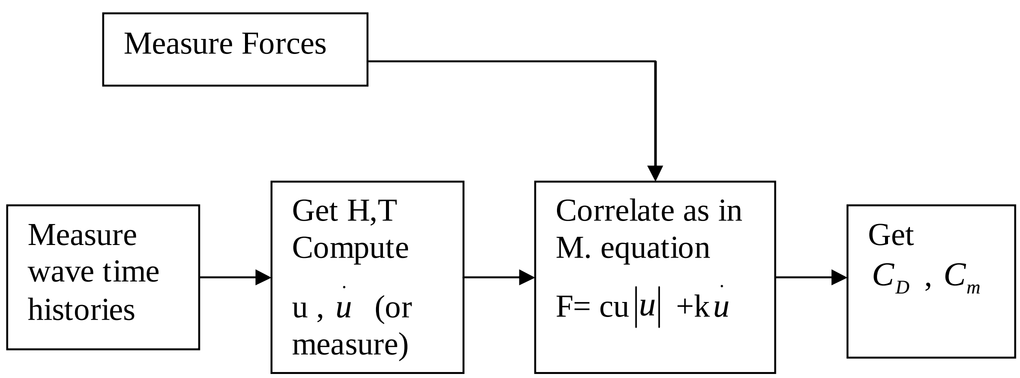


Fig. 8.9 Flow chart to obtain C_D and C_M through lab measurement

Almost all experiments suffer from widely scattered values. Major reasons of the scatter are: (1) use of either steady/ oscillatory / wavy flow, (2) difficulty in achieving high Re (10^7), (3) wave theories over predict velocity, (4) definition of Re is arbitrary, (5) waves are irregular, hence C_D,

C_m are large, (6) use of $\frac{\partial u}{\partial t}$ (not $\frac{du}{dt}$) overestimate forces, (7) no accounting for directionality, current, 3-D flow.

Recommendations:

1) For Indian conditions C_D =0.7 ; Cm=2 are generally used.
2) DnV : C_D =0.7-1.2 ; Cm=2
3) A.P.I. : C_D =0.6-1.0 ; Cm=1.5-2
4) Shore Protection Manual: C_D -Refer Fig. ; Cm=1.5 if Re>5 x 10^5

   =2 if Re<2.5 x 10^5

   $= 2.5 - \frac{\text{Re}}{5x10^5}$, otherwise

EXAMPLE:

A one m diameter jacket leg is subjected to an attack of waves which are 5m high, 80 m long and 10 seconds in period.

Determine maximum Drag Force , maximum Inertia Force ,Total Force @ $\theta = \frac{\pi}{4}$, at a location 10m below SWL . The water depth is 60m.

Take C_D =1; Cm=2; Use linear theory.

ρ =1030 $\frac{kg}{m^3}$

Solution :

$$F_{D\max} = \frac{1}{2} C_D \rho D |u_{\max}| u_{\max}$$

$$u = \frac{\pi H \cosh k(d+z)}{T \sinh kd} \cos\theta$$

$$= \frac{\pi(5)\cosh\frac{2\pi}{80}(50)}{(10)\sinh\frac{2\pi}{80}(60)}.1$$

=0.717 m/s

$$F_{D\max} = \frac{1}{2}(1)1030(1)\left|0.717\right|0.717$$

=264.76 N/m.

$$\left(F_I\right)_{\max} = C_m \rho \frac{\pi d^2}{4} \dot{u}_{\max}$$

$$\dot{u}_{\max} = \frac{2\pi^2 H \cosh k\left(d+z\right)}{T^2 \sinh kd}\sin\theta$$

$$= \frac{2\pi^2(5)\cosh\frac{2\pi}{80}(50)}{(10^2)\sinh\frac{2\pi}{80}(60)}.1$$

$$=0.45 \ \frac{m}{s^2}$$

$$F_{I_{\max}} = 2(1030)\frac{\pi(1^2)}{4}(0.45)$$

=728 N/m.

$$u = \frac{\pi H \cosh k\left(d+z\right)}{T\sinh kd}\cos\theta$$

$$= \frac{\pi(5)(25.3869)}{(10)(55.1544)}.\cos\frac{\pi}{4}$$

=0.5067 m/s

$$\dot{u}_{\max} = \frac{2\pi^2 H \cosh k(d+z)}{T^2 \sinh kd} \sin\theta$$

$$= \frac{2\pi^2 (5)(25.3869)}{(10^2)(55.1544)} .\sin\frac{\pi}{4}$$

$$=0.318 \ \frac{m}{s^2}$$

$$\text{F} = \frac{1}{2} C_D \rho D u|u| + C_m \rho \frac{\pi d^2}{4} \dot{u}$$

$$= \frac{1}{2}(1)1030(1)(0.5067)^2 + 2(1030)\frac{\pi(1)^2}{4}(0.318)$$

=646.72 N/m

8.2 Total Wave Force on the Entire Member Length

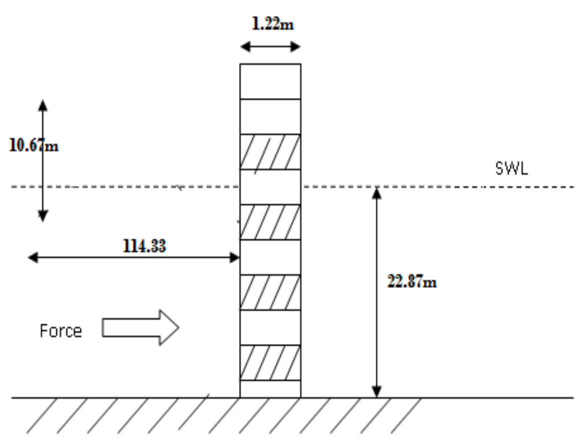


Fig. 8.10 Variation of drag and inertia over a vertical

Consider a vertical located at x=0 as shown above.

Consider Linear Theory

$$\eta = \frac{H}{2}\cos\frac{2\pi t}{T} \quad \text{as x=0}$$

$$u = \frac{\pi H \cosh k\left(d+z\right)}{T \sinh kd}\cos\left(\frac{2\pi t}{T}\right)$$

$$\left(\frac{\partial u}{\partial t} \approx \right)\dot{u} = \frac{2\pi^2 H \cosh k\left(d+z\right)}{T^2 \sinh kd}\sin\left(-\frac{2\pi t}{T}\right)$$

$$F_D = \frac{1}{2}C_D \rho D u|u| = \frac{1}{2}C_D \rho D \frac{\pi^2 H^2}{T^2}\frac{\cosh^2 k(d+z)}{\sinh^2 hd}\left|\cos\tfrac{2\pi}{T}t\right|\cos\tfrac{2\pi}{T}t$$

When t=0, $F_D \rightarrow (F_D)_{\max}$

$$F_I = C_m \rho \frac{\pi d^2}{4}\dot{u} = C_m \rho \frac{\pi d^2}{4}\frac{2\pi^2 H}{T^2}\frac{\cosh k(d+z)}{\sinh hd}\sin\left(-\tfrac{2\pi}{T}t\right)$$

When t=0, $F_I \rightarrow (F_I)_{\max}$

But

$(F_I)_{\max}$ When $\sin\left(-\tfrac{2\pi}{T}t\right)$=1 or when $-\tfrac{2\pi}{T}t = \frac{\pi}{2}$

Or when t=$-\frac{T}{4}$

At this time $\cos\tfrac{2\pi}{T}t = \cos\left(-\frac{\pi}{2}\right)$=0 Hence F_D =0

Note: When $(F_D)_{\max}$ occurs F_I =0

When $(F_I)_{\max}$ occurs F_D=0

$(F_D)_{\max}$ occurs after time $\frac{T}{4}$ when $(F_I)_{\max}$ occurs.

If η is small, $\eta=0$, if η is not small , $\eta = \frac{H}{2}\cos\frac{2\pi t}{T}$

$$F_T = \int_{-d}^{\eta} F_D dz + \int_{-d}^{\eta} F_I dz$$

$$M = \int_{-d}^{\eta} F_D (d+z) dz + \int_{-d}^{\eta} F_I (d+z) dz$$

Hence total horizontal force on entire member length at any time ‘t’ : $F_T = F_{TD} + F_{TI}$

$$F_{TD} = \int_{-d}^{\eta} F_D dz = \int_{-d}^{\eta} \frac{1}{2} C_D \rho D \frac{\pi^2 H^2 \cosh^2 k(d+z)}{T^2 \sinh^2 kd} \left|\cos\left(\frac{2\pi t}{T}\right)\right| \cos\left(\frac{2\pi t}{T}\right) dz$$

$$= \frac{1}{2} C_D \rho D \frac{\pi^2 H^2 |\cos\omega t| \cos(\omega t)}{T^2 \sinh^2 kd} \int_{-d}^{\eta} \cosh^2 k(d+z) dz$$

$$= \frac{1}{2} C_D \rho D \frac{\pi^2 H^2 |\cos\omega t| \cos(\omega t)}{T^2 \sinh^2 kd} \left[\frac{k(d+z)}{2k} + \frac{\sinh 2k(d+z)}{4k} \right]_{-d}^{\eta}$$

[Using $\int \cosh^2 x = \frac{x}{2} + \frac{\sinh 2x}{4}$]

$$= \frac{1}{2} C_D \rho D \frac{\pi^2 H^2 |\cos\omega t| \cos(\omega t)}{T^2 \sinh^2 kd} \left[\frac{1}{4k} \left\{ 2k(d+z) + \sinh 2k(d+z) \right\} \right]_{-d}^{\eta}$$

$$= \frac{1}{2} C_D \rho D \frac{\omega^2 H^2}{4} \frac{1}{4k} \left[\frac{\left\{ 2k(d+z) + \sinh 2k(d+z) \right\}}{\sinh^2 kd} \right]_{-d}^{\eta} |\cos\omega t| \cos(\omega t)$$

$$F_{TD} = \frac{C_D \rho D}{32k} (\omega H)^2 \left[\frac{\left\{ 2k(d+z) + \sinh 2k(d+z) \right\}}{\sinh^2 kd} \right]_{z=\eta} |\cos\omega t| \cos(\omega t) \tag{8.9}$$

$$F_{TI}=\int_{-d}^{\eta}F_I dz=\int_{-d}^{\eta}C_m\rho\frac{\pi d^2}{4}\left(\frac{2\pi^2 H\cosh k(d+z)}{T^2\sinh kd}\sin(-\omega t)\right)dz$$

$$=-C_m\rho\frac{\pi d^2}{4}\left(\frac{2\pi^2 H}{4}\frac{\sin(\omega t)}{\sinh kd}\right)\left[\frac{\sinh k(d+z)}{k}\right]_{-d}^{\eta}$$

$$\text{Hence}\quad F_{TI}=-C_m\rho\frac{\pi d^2}{4}(\omega H)^2\left(\frac{1}{2k}\frac{\sinh k(d+z)}{\sinh kd}\right)_{z=\eta}\sin(\omega t)$$

(8.10)

Similarly, $M_T=M_{DT}+M_{IT}$

$$M_{DT}=\frac{C_D\rho D}{64k^2}\frac{(\omega H)^2}{\sinh^2 kd}\left\{2k(d+z)\sinh 2k(d+z)-\cosh 2k(d+z)+2\left[k(d+z)\right]^2+1\right\}\left|\cos\omega t\right|\cos(\omega t)$$

$$M_{IT}=\frac{-C_m\rho}{2k^2}\frac{\pi d^2}{4}\frac{(\omega H)^2}{\sinh kd}\left(\frac{k(d+z)\sinh k(d+z)-\cosh k(d+z)+1}{\sinh kd}\right)\sin(\omega t)$$

(8.11)

EXAMPLE

Obtain variation of total horizontal force and moment at the sea bed with time for a circular vertical pile of diameter 1.22 m extending into a water depth of 22.9 m. The wave height is 10.67m and the wavelength is 114.3m. Take C_D =1 and C_m =2 .

γ =10.06 $\frac{KN}{m^3}$.

What are the maximum force and moment values?

Use Linear Theory.

Consider two cases (a) Integration up to SWL.

(b) Integration up to free surface.

SOLUTION:

$$K=\frac{2\pi}{L}=\frac{2\pi}{114.3}=0.05497\text{cycles/m}$$

$$\omega=\left\{gk\tanh kd\right\}^{\frac{1}{2}}=\left\{9.81(0.05497)\tanh\left[0.05497\left(22.9\right)\right]\right\}^{\frac{1}{2}}=0.6773\text{ rad/s}$$

$$F_{TD}=\frac{C_D\rho D}{32k}\left(\omega H\right)^2\left[\frac{\left\{2k\left(d+z\right)+\sinh 2k\left(d+z\right)\right\}}{\sinh^2 kd}\right]_{z=\eta}\left|\cos\omega t\right|\cos\left(\omega t\right)$$

$$=\frac{C_D\rho D}{32k}\left(\omega H\right)^2\left[\frac{\left\{2k\left(d\right)+\sinh 2k\left(d\right)\right\}}{\sinh^2 kd}\right]\left|\cos\omega t\right|\cos\left(\omega t\right)$$

$$=\frac{1(10.06)1.22}{32(9.81)(0.05497)}\left(0.6773\left(10.67\right)\right)^2\left[\frac{\left\{2(0.05497)(22.9)+\sinh 2(0.05497)\left(22.9\right)\right\}}{\sinh^2(0.05497)\left(22.9\right)}\right]\left|\cos\omega t\right|\cos\left(\omega t\right)$$

$$=123.022\left|\cos\omega t\right|\cos\left(\omega t\right)\text{ KN}$$

$$F_{TI}=-C_m\rho\frac{\pi d^2}{4}\left(\omega H\right)^2\left(\frac{1}{2k}\frac{\sinh k\left(d+z\right)}{\sinh kd}\right)_{z=\eta}\sin\left(\omega t\right)$$

$$=-C_m\rho\frac{\pi d^2}{4}\left(\omega H\right)^2\left(\frac{1}{2k}\frac{\sinh k\left(d\right)}{\sinh kd}\right)\sin\left(\omega t\right)$$

$$=-2\left(\frac{10.06}{9.81}\right)\frac{\pi\left(1.33\right)^2}{4}\left(10.67\left(0.6773\right)\right)^2\left(\frac{1}{2\left(0.05497\right)}\right)\sin\left(\omega t\right)$$

$$=-106.74\ \sin\left(\omega t\right)\text{ KN}$$

Hence

$$F_T=123.022\left|\cos\omega t\right|\cos\left(\omega t\right)-106.74\ \sin\left(\omega t\right)$$

Vary t=0, T

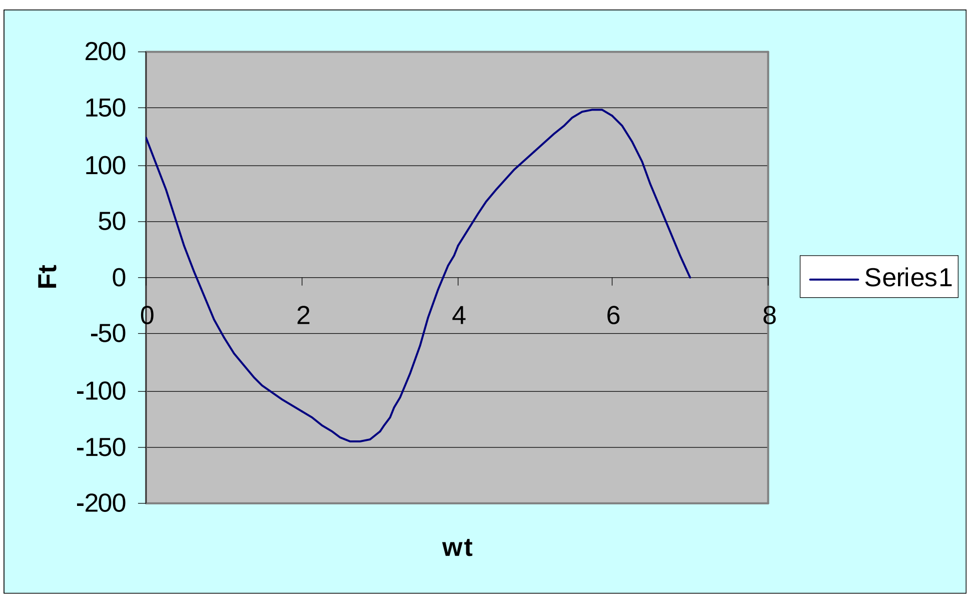
$\frac{2\pi t}{T}$=0, $\frac{2\pi}{T}.T$

ω t=0, 2π

=0, 6.284

=0, 1,2, …,7

ω t	F_{TD} (KN)	F_{TI} (KN)	F_T (KN)
0	123.02	0	123.02
1	35.01	-89.82	-53.91
2	-21.31	-97.06	-118.37
3	-120.57	-15.06	-135.64
4	-52.56	80.78	28.22
5	9.9	102.86	112.26
6	113.42	29.83	143.24
7	69.92	-70.13	-0.21



We can express: $F = \overline{C}\left|\cos\theta\right|\cos\theta + \overline{K}\sin\theta$. For maximum conditions this equation can be worked out using θ as: $\dfrac{\partial F}{\partial\theta} = 0 \Rightarrow \theta = \sin^{-1}\dfrac{\overline{K}}{2\overline{c}}$.

8.3 Wave Forces Using Stokes (V) Theory

Water particle kinematics are calculated at every m length of the vertical structural member (at its center along the immersed length of the member axis) using the Stokes Fifth Order theory. Corresponding forces are worked out using the Morrison's equation at every such segment and then they are added up to cover the full member length. For a typical case of wave attack shown below, the results are further indicated in the following figure:

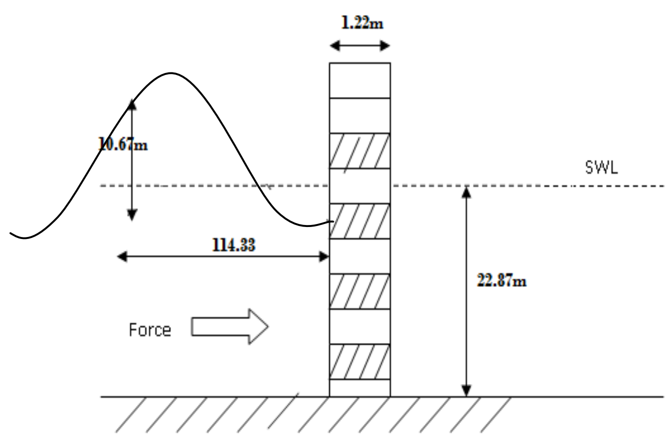


Fig. 8.11 Calculation of total wave force

C_D =1 and C_m =2

8.4 Calculation Of Wave Forces Using Dean's Theory

For circular vertical piles, based on Dean's theory and Morison's equation, it is possible to express approximately the total maximum force within the wave cycle as:

$$F_m = \Phi_m \rho g C_D H^2 D \tag{8.12}$$

Where, Φ_m = Coefficient to be read from curves plotted for various values of

$$W = \frac{C_m}{C_D}\frac{D}{H} \tag{8.13}$$

Similarly the total maximum moment at the base is:

$$M_m = \alpha_m \rho g C_D H^2 Dd \tag{8.14}$$

Where α_m =Coefficient to be read from curves

LIFT FORCES:

For high drag ($K_c > 15$) there is regular and alternate eddy shedding on the downstream side on both sides of cylinder at a frequency.

$$f_{eddyshedding} = \frac{sv}{D} \tag{8.15}$$

where s = Strouhal No. $\approx$ 0.2, ν = kinematic viscosity of sea water, D = diameter.
This gives rise to lift force given by:

$$F_L = \frac{1}{2} C_L \rho D u |u| \tag{8.16}$$

Where C_L is Lift coefficient = $f(K_c)$

$\approx C_D$ If K_c>20

$\approx$ If K_c<3

If the frequency of eddy shedding goes close to the natural frequency of the structural member then resonance occurs and high structural vibrations result.

8.5 WAVE FORCE ON INCLINED MEMBERS

Let V_n, a_n = normal components of total particle velocity and accelerations, respectively. Then the normal (to member axis) wave force at any time 't' is given by:

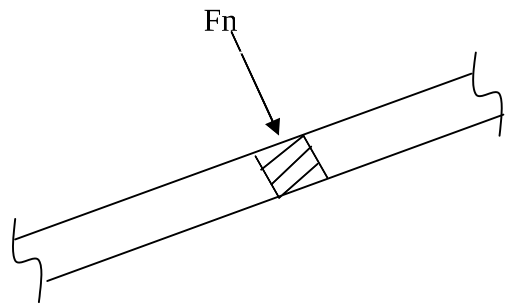


Fig. 8.12 Normal to axis force

$$F_n = C|V_n|V_n + Ka_n \tag{8.17}$$

Where $C = C_D \rho \dfrac{D}{2}$

$$K = C_m \rho \frac{\pi d^2}{4}$$

Where V_n and a_n are normal components of total velocity (V') and acceleration (a')

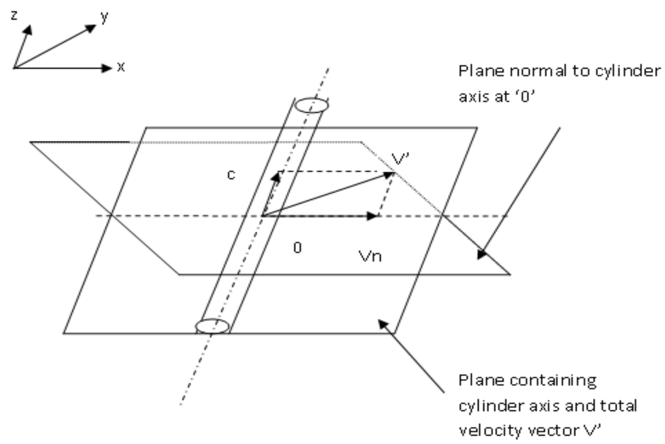


Fig. 8.13 Normal to axis force

$\overline{V_n}$ (or $\overline{a_n}$) lies along the line of intersection of the two planes

$$\overline{V_n} = \bar{c}x\left(\overline{V'}x\bar{c}\right)$$

$$\overline{a_n} = \bar{c}x\left(\overline{a'}x\bar{c}\right)$$

If $\bar{c}$ is unit vector along axis and $\overline{c_x}$, $\overline{c_y}$, $\overline{c_z}$ are its direction cosines and if,

$$\overline{V_n} = V_{nx}i + V_{ny}j + V_{nz}k$$

$$\overline{a_n} = a_{nx}i + a_{ny}j + a_{nz}k$$

$$\bar{c} = c_x i + c_y j + c_z k$$

then evaluating the products,

$$\begin{Bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \end{Bmatrix} = \begin{bmatrix} 1-c_x^{\,2} & -c_x c_y & -c_x c_z \\ -c_x c_y & 1-c_y^{\,2} & -c_y c_z \\ -c_x c_z & -c_y c_z & 1-c_z^{\,2} \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}$$

and $|V_n| = \sqrt{V_{nx}^{\,2} + V_{ny}^{\,2} + V_{nz}^{\,2}} = \left\{V_x^{\,2} + V_y^{\,2} + V_z^{\,2} - \left[c_x c_x + c_y c_y + c_z c_z\right]^2\right\}^{\frac{1}{2}}$

Thus we get,

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = c|V_n| \begin{Bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \end{Bmatrix} + K \begin{Bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{Bmatrix} \qquad (8.18)$$

Where $$\begin{Bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{Bmatrix} = \begin{bmatrix} 1-c_x^{\,2} & -c_x c_y & -c_x c_z \\ -c_x c_y & 1-c_y^{\,2} & -c_y c_z \\ -c_x c_z & -c_y c_z & 1-c_z^{\,2} \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

Note: If wave theory is used then $V_y = a_y = 0$.

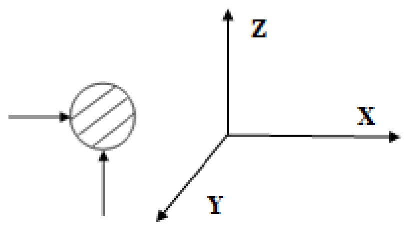


Fig. 8.14 Special case of a pipeline

Further for a horizontal member, $c_x = c_z = 0$ and $c_y = 1$

Note on calculation of direction cosines,

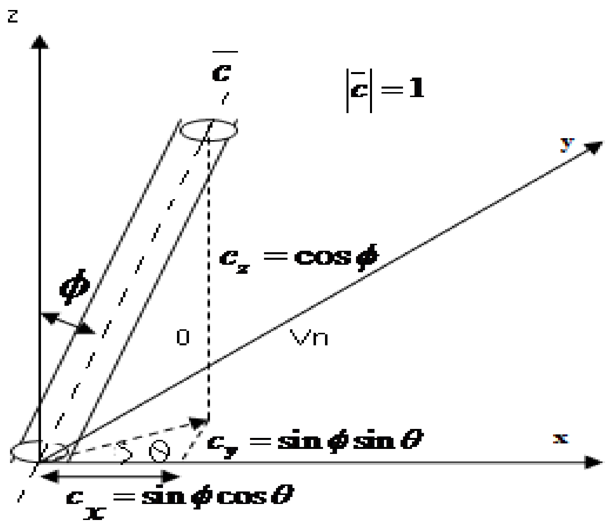


Fig. 8.15 Direction cosines

$\bar{c} = c_x i + c_y j + c_z k$; where,

$c_x = \sin\phi \cos\theta$

$c_y = \sin\phi \cos\theta$

$c_z = \cos\phi$

A note on flexible cylinders

The previous discussion was based on the assumption that the cylinder on which the force was exerted was rigidly held at its bottom. On the contrary if it is free to move appreciably with waves, not only the exact volume of water displaced by the cylinder contributes to the inertia force but also some volume surrounding it behaves as one with the cylinder and contributes to the force due to inertia. This volume is some fraction of the displaced volume V. The resulting inertia force is thus:

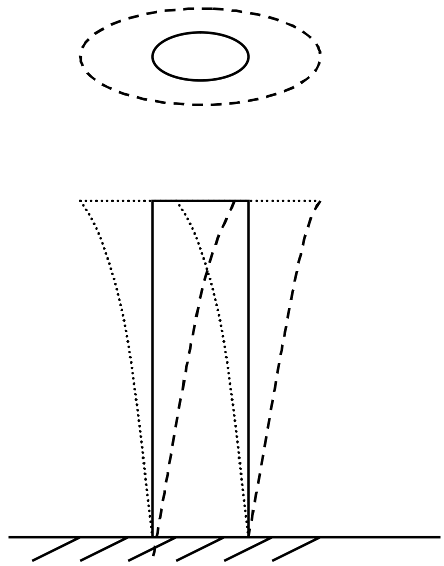


Fig. 8.16 Added mass effect

$$F_I = (C_F + C_a)\rho V \dot{u} \tag{8.19}$$

Where C_F = Froude-Crylov coefficient and

C_a = Coefficient of added mass =1 (theoretically)

Hence Total F_I = Froude-Crylov Force + Added mass force

The Froude-Crylov Force is the force required to accelerate the fluid particles within the volume of cylinder in its absence, whereas, the added mass force is the force due to acceleration of water surrounding the cylinder and oscillating with it.

8.6 Wave Slam:

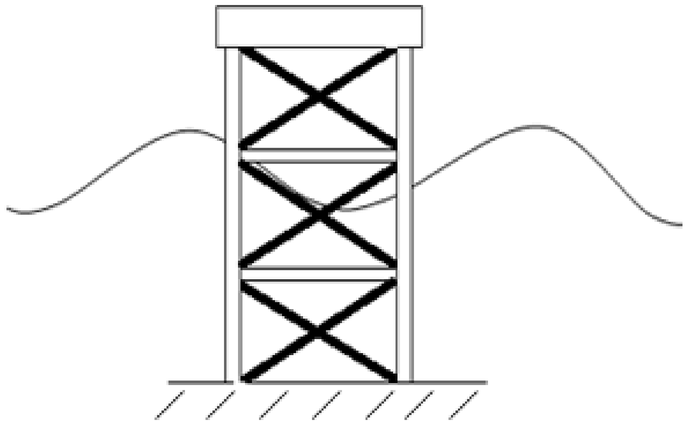


Fig. 8.17 Wave attack on a Jacket

When wave surface rises, it slams underneath horizontal members near the SWL and then passes by them. The resulting slamming force (nearly vertical) due to sudden buoyancy application is given as follows

$$F_z = C_s \frac{1}{2} \rho D u_z^2 \tag{8.20}$$

Where $C_s \approx \pi$ (theoretically for circular cylinder)

The American Petroleum Institute (API) suggests that it should be taken into consideration to calculate total individual member loads and not to get the global horizontal base shear and overturning moments. Impulsive nature of this force however can excite natural frequency of the members creating resonant condition and large dynamic stresses.

8.7 Limitations of the Morrison's Equation:

1) Physics of wave phenomenon is not well represented in it.
2) The drag force formula and the inertia force formula involve opposite assumptions. The former assume that the flow is steady while the latter implies that the flow is unsteady

3) Real sea effects like ‘transverse forces’, ‘energy spreading (directionality)’ are unaccounted for.

4) There is a high amount of scattering in values of C_D and C_m.

5) Inaccuracies in the wave theory based values of water particle kinematics get reflected in the resulting force estimates.

CHAPTER 9

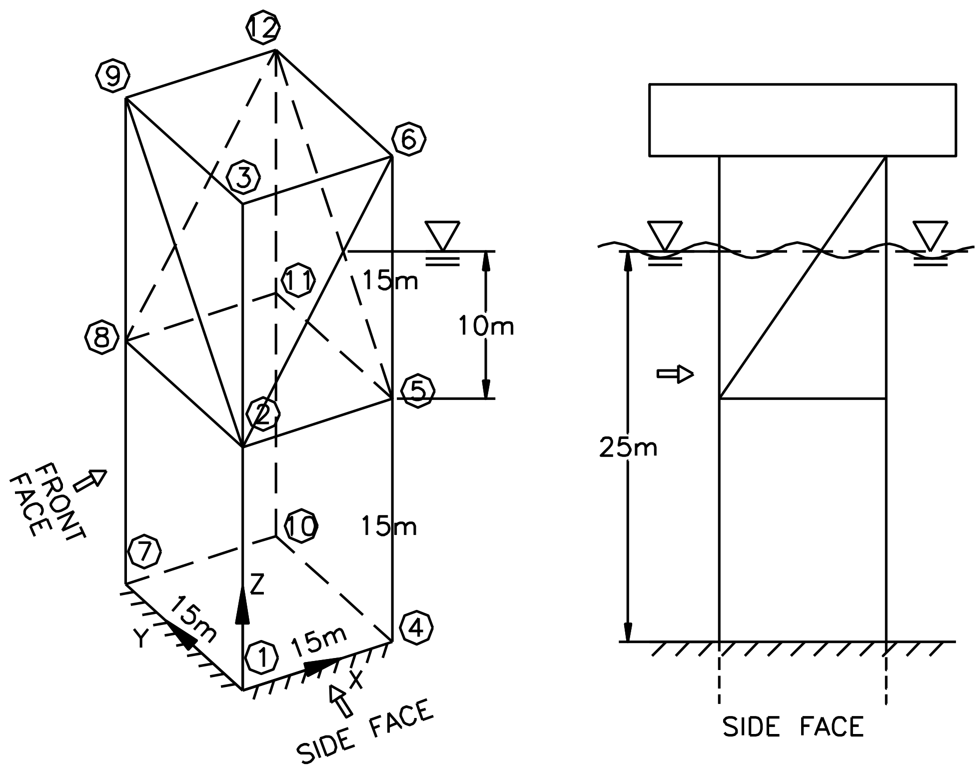
MAXIMUM WAVE FORCE ON THE ENTIRE STRUCTURE

9.1 Example

Obtain the variation with time of total horizontal wave force and moment for the entire offshore structure as shown below:

Also calculate maximum force and moments. (Horizontal) Use linear theory (In-line)

Given



H = 6m L = 90m d = 25m

Pile dia. = 1.2m bracing dia. = 0.6m

$C_D = 1$ $C_M = 2$ $\gamma = 10.06 \dfrac{kN}{m^3}$

Solution

The members on which wave forces should be considered are:

Group A:	Vertical Piles	:	1-3/ 7-9 /4-6/ 10-12
Group B:	Horizontal Bracings	:	2-8/ 5-11
Group C:	Diagonal Front Face Bracings:		2-9/ 5-12
Group D:	Diagonal Side Face Bracings	:	2-6/ 8-12

Calculate F_T for each member at same time instant wt = 0, 1, 2, ……7

Example for wt = 6

[A] F_T for pile 1-3: $k = \frac{2\pi}{L} = 0.0698\frac{c}{m}$

$$w = \{gk \tan h\, kd\}^{\frac{1}{2}}$$

$$= \{9.81(0.0698)\tan h\, 0.0698(25)\}^{\frac{1}{2}}$$

$$= 0.8026\frac{r}{s}$$

$$F_{TD} = \frac{C_D SD}{32k}\frac{(WH)^2}{\sin h^2\, kd}\{2k(d+z)+\sin h\, 2k\,(d+z)\}_{z=\eta} / \cos\theta / \cos\theta$$

θ = kx – wt $\qquad = \frac{H}{2}\cos\theta$

= -6

=3 cos (-6)

x = 0

= 2.881m

$$= 1\frac{(10.06)}{9.81}\frac{1.2}{32(0.0698)}\frac{\left[\frac{0.8026}{6}\right]^2}{\sin h^2\, 0.0698(25)}\{2(0.0698)(27.881)+\sin h 2(0.0698)(27.881)\}/\cos(-6)/\cos(-6)$$

= 43.4 kN

$$F_{TI} = C_M S\frac{\pi D^2}{4}(HW^2)\frac{\{\sin h\, k(d+z)\}_{z=\eta}}{2k\sin h\, kd}\sin\theta$$

$$= 2\frac{10.16}{9.81}\frac{\pi(1.2)^2}{4}\left(6(0.8026)^2\right)\frac{\{\sin h[0.0698(27.881)]\}\sin\theta}{2(0.0698)\sin h[(0.0698)25]}$$

= 22.17 kN

$\therefore F_T$ = 43.4 + 22.17 = 65.57 kN

IDENTICALLY:

F_T for pile 7-9 = 65.57 kN

F_T for pile 4-6: Here θ = kx – wt

= 0.0698(15) – 6

= -4.953 rad

and $z = \eta = \frac{H}{2}\cos(-4.953) = 0.7149m$

$$F_{TD} = \frac{C_D SD}{32k}\frac{(WH)^2}{\sin h^2\, kd}\{2k(d+z)+\sin h\, 2k\,(d+z)\}_{z=0.7149} / \cos\theta / \cos\theta$$

$$= 1.6583\{2(0.0698)(25.715)+\sin h 2(0.0698)(25.715)\} / \cos\theta / \cos\theta$$

= 2.04 kN

$$F_{TI} = C_M S\frac{\pi D^2}{4}\left(HW^2\right)\frac{\{\sin h\, k(d+z)\}_{z=0.715}}{2k\sin h\, kd}\sin\theta$$

= $8.9652\{\sin h\, 0.0698(25.715)\}\sin\theta$ θ = - 4.953

= 65.75 kN

$\therefore$ Total F_T on pile 4-6 = 2.04 + 65.75 = 67.8 kN

Identically F_T on pile 10-12 = 67.8 kN

[B] F_T on horizontal member 2-8: special case of inclined members

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = C|V_n|\begin{Bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \end{Bmatrix} + K\begin{Bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{Bmatrix}$$

$$\therefore F_x = C|V_n|V_{nx} + Ka_{nx}$$

$$\begin{Bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \end{Bmatrix} = \begin{bmatrix} 1-C_x^2 & -C_xC_y & -C_xC_z \\ -C_xC_y & 1-C_y^2 & -C_yC_z \\ -C_xC_z & -C_yC_z & 1-C_z^2 \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}$$

(Here $C_x = 0;\ C_y = 1;\ C_z = 0$)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix} = \begin{Bmatrix} V_x \\ 0 \\ V_z \end{Bmatrix}$$

Similarly, $$\begin{Bmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{Bmatrix} = \begin{Bmatrix} a_x \\ 0 \\ a_z \end{Bmatrix}$$

$$\therefore |V_n| = \sqrt{V_x^2 + V_z^2} \qquad \& \qquad F_x = C\sqrt{V_x^2 + V_z^2}^{V_x + Ka_x}$$

[<u>Note</u>: For calculating $u, w, \dot{u}, \dot{w}$ $\longrightarrow$ ORIGIN is AT SWL]

$$V_x = u = \frac{\pi H}{T}\frac{\cos h\,K(d+z)}{\sin h\,kd}\cos(kx - wt) \quad ; \qquad T = \frac{2\pi}{w}$$

$$= \frac{2\pi}{0.8026} = 7.829^{11}$$

$$= \frac{\pi(6)}{7.829}\frac{\cos h[(0.0698)(15)]}{\sin h[0.0698(25)]}\cos(-6)$$

$$= 1.333\ \frac{m}{s}$$

$$V_z = w = \frac{\pi H}{T}\frac{\sin h\,k(d+z)}{\sin h\,kd}\sin(kx - wt)$$

$$= 0.303\frac{m}{s}$$

$$a_x = \dot{u} = \frac{2\pi^2 H}{T^2}\frac{\cos h\,k(d+z)}{\sin h\,kd}\sin(kx - wt)$$

$$= 0.311\frac{m}{s^2}$$

$$C = C_D S\frac{D}{2} = 1\ \frac{10.06}{9.81}\frac{0.6}{2} = 0.3076$$

$$K = C_M S\frac{\pi D}{4} = 2\ \frac{10.06}{9.81}\pi\frac{(0.6)}{4} = 0.58$$

$$\therefore F_X = 0.3076\sqrt{1.333^2 + 0.303^2}1.333 + 0.58(0.311)$$

$$= 0.741\frac{kN}{m}$$

$\therefore F_T$ for entire member length 2-8 = 0.741(15) = 11.16 kN

F_T for horizontal member 5-11:

Here x = 15*m*; $\eta\frac{H}{2}\cos(kx - wt)$

= 3 cos (0.0698(15)-6)

= 0.715*m*

$$V_x = \frac{\pi H}{T}\frac{\cos h K(d+z)}{\sin h kd}\cos(kx - wt)$$

$$= \frac{\pi 6}{7.829}\frac{\cos h[0.0698(15)]}{\sin h[0.0698(25)]}\cos(0.0698(15) - 6) = 0.331\frac{m}{s}$$

$$V_z = \frac{\pi H}{T}\frac{\sin h K(d+z)}{\sin h kd}\sin(kx - wt) = 1.052\frac{m}{s}$$

$$a_x = \frac{2\pi^2 H}{T^2}\frac{\cos h K(d+z)}{\sin h kd}\sin(kx - wt) = 1.082\frac{m}{s}$$

$$\therefore F_X = 0.3076\sqrt{0.331^2 + 1.052^2}(0.331) + 0.58(1.082) = 0.74kN$$

Therefore, Total F_T on entire member 5-11 = 15(0.74) = 11.1kN

[C] Diagonal Front Face Member:

F_T on member 2-9:

$$F_X = C|V_n|V_{nx} + Ka_{nx}$$

where; $$\begin{Bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \end{Bmatrix} = \begin{bmatrix} 1-C_x^2 & -C_xC_y & -C_xC_z \\ -C_xC_y & 1-C_y^2 & -C_yC_z \\ -C_xC_z & -C_yC_z & 1-C_z^2 \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}$$

Here $C_x = 0$; $C_y = 0.707$; $C_z = 0.707$

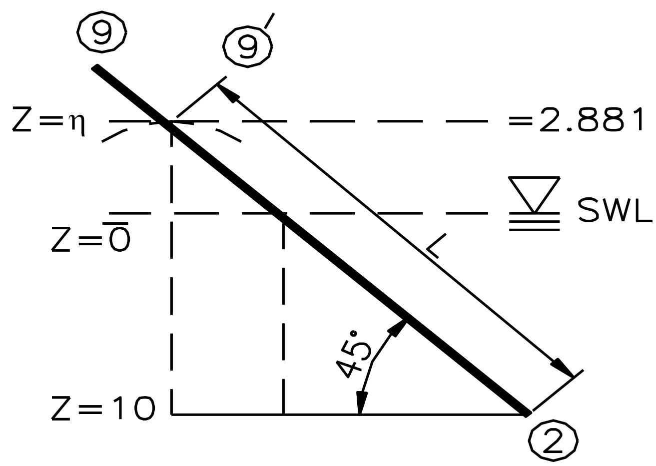
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} V_x \\ 0 \\ V_z \end{Bmatrix}$$

$\therefore V_{nx} = V_x$

$V_{ny} = -0.5V_z$ $\qquad\qquad$ $\therefore |V_n| = \sqrt{V_x^2 + V_z^2}$

$V_{nz} = 0.5V_z$ $\qquad$ & $\qquad$ $a_{nx} = a_x$

$$\therefore F_X = C\sqrt{V_x^2 + \frac{1}{2}V_z^2}V_x + Ka_x$$



Here, x = o $\quad$ & $\quad$ SWL is 10 *m* about “2”

$z = \eta = \frac{H}{2}\cos(kx - wt)$

$= \frac{6}{2}\cos(-6)$

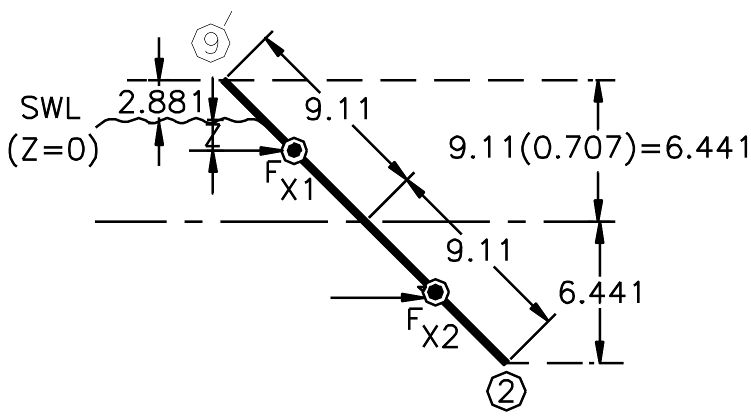
$= 2.881m$

$\therefore$ Immersed member length = $\frac{10 + 2.881}{0.707}$

= <u>18.22*m* = *l*</u>

Actually we should calculate Force per each *m* of this *l* as z varies.

But approximately, divide *l* into $\frac{l}{2}$ each (= 9.11*m*) and calculate F_X at each midlength, multiply this by $\frac{1}{2}$ and add up.



$$Z = \frac{6.441}{2} - 2.881$$

$$= 0.34m$$

F_{X1}: z = -0.34*m* ; x = 0 ; wt = 6

$$V_X = u = \frac{\pi H}{T}\frac{\cos h(d+z)}{\sin h\, kd}\cos(kx - wt)$$

$$= \frac{2.408}{2.7756}(2.8852)(0.9602) = 2.4\frac{m}{s}$$

$$V_Z = w = \frac{\pi H}{T}\frac{1}{\sin h\,kd} * \sin h\ k(d+z)\sin(kx - wt)$$

$$= \frac{2.408}{2.7756} 2.706(0.2794) = 0.656\frac{m}{s}$$

$$\dot{u} = a_x = \frac{2\pi^2 H}{T^2}\frac{1}{\sin h\,kd}\cos h(d+z)\sin(kx - wt)$$

$$= \frac{1.9323}{2.7756}(2.885)(0.2794) = 0.5612\frac{m}{s^2}$$

$$F_X = c\sqrt{V_x^2 + \frac{1}{2}V_z^2}\ V_x + Ka_x$$

$$= 0.3076\sqrt{2.4^2 + \frac{1}{2}(0.656)^2}(2.4) + 0.58(0.5612)$$

$$= 2.13\frac{kN}{m}$$

$\therefore F_X$ for $\frac{l}{2} = 9.11m$ is 9.11(2.13)=19.4kN

F_{X2}: X = 0 ; Z = -0.34 – 6.441 = -6.781m

$$V_X = \frac{2.408}{2.7756} 1.9237(0.9602) = 1.603\frac{m}{s}$$

$$V_Z = \frac{2.408}{2.7756}(1.643)(0.2794) = 0.398\frac{m}{s}$$

$$a_x = \frac{1.9323}{2.7756}(1.9237)(0.2794) = 0.374\frac{m}{s}$$

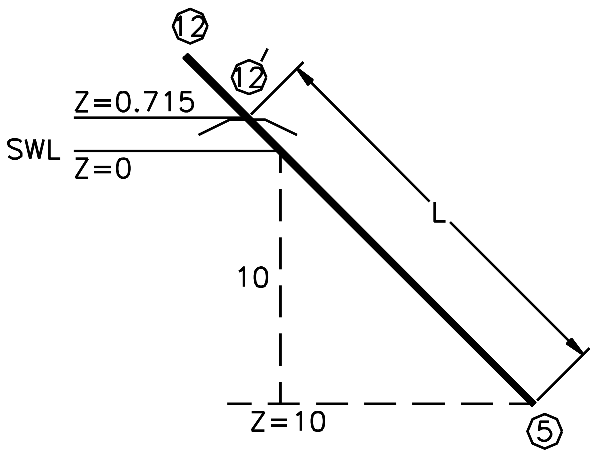
$$\therefore F_X = 0.3076\sqrt{1.603^2 + \frac{1}{2}0.398^2}(1.603) + 0.58(0.374)$$

$$= 1.02\frac{kN}{m}$$

$\therefore F_X$ for $\frac{l}{2} = 9.11m$ is 9.11 (1.02) =9.21kN

$\therefore$ Total F_X for member 2 – 9 is 19.4 + 9.21 28.61kN

Force on member 5-12:



$$F_X = c\sqrt{V_x^2 + \frac{1}{2}V_z^2}\ V_x + Ka_x$$

Here, x = 15m

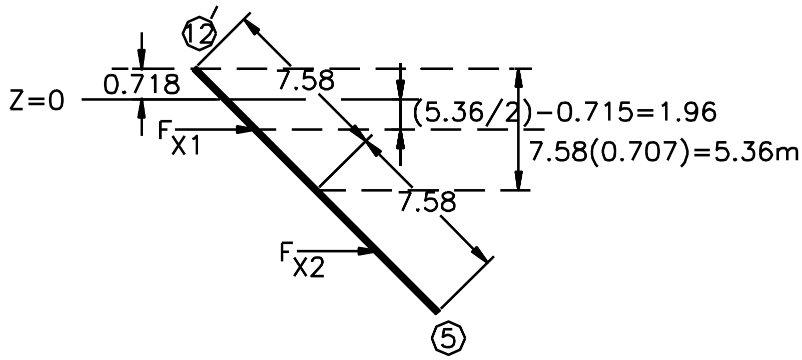
$$z = \eta = \frac{H}{2}\cos(kx - wt)$$

$$= \frac{6}{2}\cos(0.0698(15) - 6)$$

$$= 0.715m$$

$\therefore$ Immersed length $l = \frac{10 + 0.7156}{0.707} = 15.16m$

Divide l into 2 segments & calculate F_X at center of each half-length $\frac{l}{2} = 7.58m$



$\therefore F_{X1}$: x = 15m ; z = 1.96m

$$\therefore \quad V_X = \frac{2.408}{2.7756}(2.597)(0.2383) = 0.537\frac{m}{s}$$

$$V_Z = 0.8676(2.3968)(0.9712) = 2.02\frac{m}{s}$$

$$a_x = 0.6962(2.597)(0.9712) = 1.756\frac{m}{s^2}$$

$$\therefore \quad a_x = 0.6962(2.597)(0.9712) = 1.756\frac{m}{s^2}$$

$$\therefore F_{X1} = 0.3076\sqrt{(0.537)^2 + \frac{1}{2}(2.02)^2}\,0.537 + 0.58(1.756)$$

$$= 1.27\frac{kN}{m} \qquad \therefore F_{X1} = 7.58(1.27) = 9.63kN$$

$\therefore F_{X2}$: x = 15m ; z = -1.96-5.36 = -7.32m

$$\therefore \qquad V_X = 0.8676(1.8631)(0.2383) = 0.385\frac{m}{s}$$

$$V_Z = 0.8676(1.572)(0.9712) = 1.325\frac{m}{s}$$

$$a_x = 0.6962(1.8631)(0.9712) = 1.26\frac{m}{s^2}$$

$$\therefore F_{X2} = 0.3076\sqrt{0.385^2 + \frac{1}{2}(1.325)^2}\,(0.385) + 0.58(1.26)$$

$$= 0.851\frac{kN}{m} \qquad \therefore F_{X2} = 7.58(0.851) = 6.45kN$$

$\therefore$ Total F_X = 9.63 + 6.45 = 16.08 kN

[D] Diagonal Side Face Members:

F_T for 2-6: $\quad F_X = c|V_n|V_{nx} + Ka_{nx}$

$$\begin{Bmatrix} V_{nx} \\ V_{ny} \\ V_{nz} \end{Bmatrix} = \begin{bmatrix} 1-C_x^2 & -C_xC_y & -C_xC_z \\ -C_xC_y & 1-C_y^2 & -C_yC_z \\ -C_xC_z & -C_yC_z & 1-C_z^2 \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}$$

putting $C_x = 0.707$

$C_y = 0$

$C_z = 0.707$

$$= \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}$$

$V_y = 0$

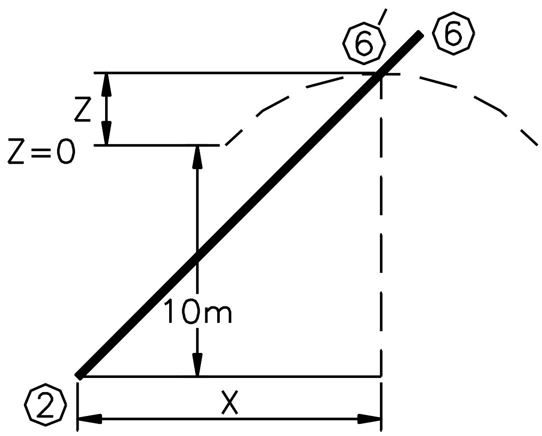
$$\therefore \quad V_{nx} = 0.5V_x - 0.5V_z = \frac{1}{2}(V_x - V_z)$$

$$V_{ny} = V_y = 0$$

$$V_{nz} = -0.5V_x + 0.5V_z = \frac{1}{2}(V_z - V_x)$$

$$|V_n| = \frac{1}{\sqrt{2}}(V_x - V_z)\ ;\ V_{nx} = \frac{1}{2}(V_x - V_z)\ ;\ a_{nx} = \frac{1}{2}(a_x - a_z)$$

$$\therefore F_X = 0.1088(V_x - V_z)^2 + 0.29(a_x - a_z)$$



Note: Here x & z both vary:

$$\therefore z = \eta = \frac{H}{2}\cos(kx - wt)$$

$$= \therefore 3\cos(kx - wt)$$

Also: $\frac{z+10}{x} = \tan 45 = 1$

$\therefore$ z = x – 10 $\quad \therefore$ x = z + 10

$$x = 3\cos(kx - wt) + 10$$

$$= 3\cos(kx - 6) + 10$$

Solving by trial, $x \approx 12m$

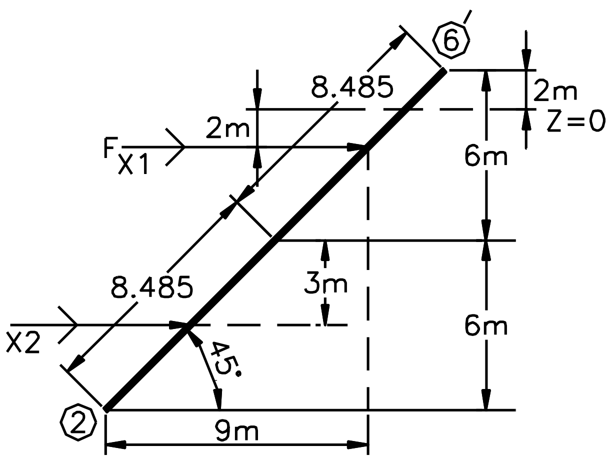
$\therefore \quad z = 2m$

$\therefore$ Immersed length of member at wt = 6 is:

$$l = \frac{12}{0.707} = 16.97m$$

Divide the member length l into half $\left(= \frac{l}{2} = 8.485m \right)$

Calculate F_X at the centre of each & add up.



F_{X1}: $x = 9m$; $z = -1m$

$$V_x = 0.8676(2.7636)(0.6127) = 1.469\frac{m}{s}$$

$$V_z = 0.8676(2.576)(0.7964) = 1.766\frac{m}{s}$$

$$a_x = 0.6962(=.7636)(0.7904) = 1.521\frac{m}{s^2}$$

$$a_z = -0.6962(2.576)(0.6127) = -1.1\frac{m}{s^2}$$

$$\therefore F_{X1} = 0.1088(1.469 - 1.766)^2 + 0.29(1.521 + 1.1) = 0.7697\frac{kN}{m}$$

$$\therefore F_{X1}\, on \frac{l}{2} (= 8.485) = 6.531kN$$

F_{X2}: x = 3m ; z = -7m

$$\therefore V_x = 0.8676(1.8987)(0.8811) = 1.4515\frac{m}{s}$$

$$V_z = 0.8676(1.614)(0.4729) = 0.662\frac{m}{s}$$

$$a_x = 0.6962(1.8987)(0.4729) = 0.625\frac{m}{s^2}$$

$$a_z = -0.6962(1.614)(0.8811) = -0.99\frac{m}{s^2}$$

$$\therefore F_{X2} = 0.1088(1.4515 - 0.662)^2 + 0.29(0.625 + 0.99) = 0.5362\frac{kN}{m}$$

$$\therefore F_{X2}\ on\ \frac{l}{2}(= 8.485) = 4.549kN$$

$\therefore$ Total F_X on member 2 - 6 is 6.531 + 4.549 = 11.08 kN

IDENTICALLY F_X on member 8 - 12 = 11.08 kN

$\therefore$ TOTAL WAVE FORCE ON ENTIRE STRUCTURE AT wt = 6 is:

Group	[A]	:	65.57 + 65.57 + 67.8 +67.8
	[B]	:	11.16 + 11.1
	[C]	:	28.61 + 16.08
	[D]	:	11.08 + 11.08

= 355.85kN

≈ 356kN

CHAPTER 10

WAVE FORCES ON LARGE DIAMETER MEMBERS

10.1 Introduction

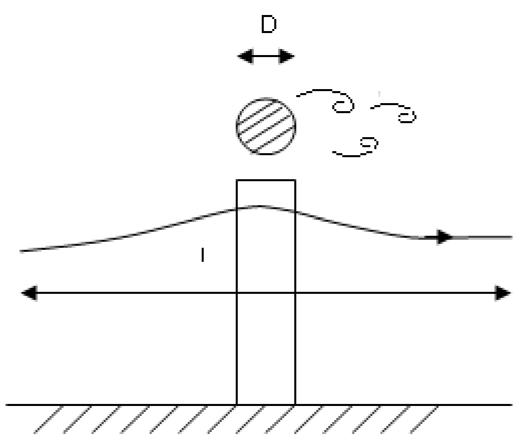


Fig. 10.1 Small diameter member case

If the member diameter D is less than 15 percent of the incident wave length, L, flow separation takes place and a wake region is formed at the downstream of the flow direction. (Fig.10.1). This increases the drag force. Hence the Morrison's equation becomes valid, which among other facts assumes that the particle kinematics u and $\dot{u}$ remain constant along the length 'D'.

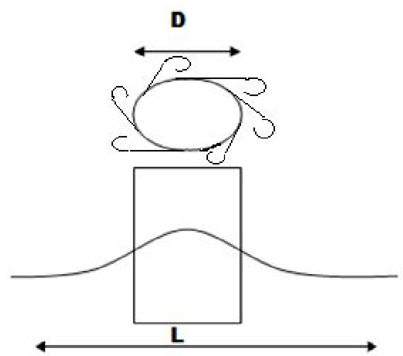


Fig. 10.2 Large diameter member case

But if the diameter of member D exceeds 15 % of the wave length L then the flow separation is localized and confined to the small region of boundary layer around member surface. (Fig.10.2). Hence the resultant drag force (viscous effects) takes a small value. Then the wave force can be calculated by the potential flow theory. Further, the incident wave is scattered or diffracted and the effect of scattered wave potential is required to be considered. Also the variation of particle kinematics along 'D' becomes considerable. However, if D is not very large compared to L (not of the order of L) then the scattered wave height H remains small compared to (not of the order of) incident 'H'. Then the following approximate theory of Froude-Krylov can be used.

10.2 Froude-Krylov Theory:

It involves evaluation of the force by calculating the wave induced pressure over an element of the body surface area, multiplying it with the elemental area and then integrating the product to cover the whole submerged surface. First we have to calculate pressure at the body surface points (by wave theory).

It is assumed that the actual force is directly proportional to the Froude-Krylov force, i.e.,

$$\text{Actual force } \alpha \text{ F.K.force}$$
$$= [\text{force coefficient}] \text{ x Froude-Krylov force}$$

This assumption is not valid when high diffraction is expected. Note that the Froude-Krylov theory is approximate, generally not used in rigorous design but found to be good for submerged objects or objects with small diffraction.

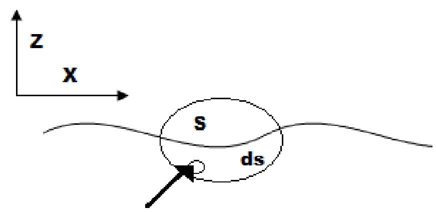


Fig. 10.3 Definition sketch

Let p = normal instantaneous pressure (given say by a wave theory) acting on 'ds' (elemental submerged area).

n_x and n_z = x and z components of unit vector normal to ds respectively.

S = total submerged area. (Fig. 10.3)

Then total horizontal force component $F_x = C_H \iint_S p.n_x ds$ (10.1)

Total vertical force component $F_z = C_V \iint_S p.n_z ds$ (10.2)

Where p = wave induced pressure, n_x, n_y = horizontal and vertical components of the unit axial vector; C_H, C_V = horizontal and vertical Froude-Krylov coefficients.

1) Submerged Horizontal Cylinder:

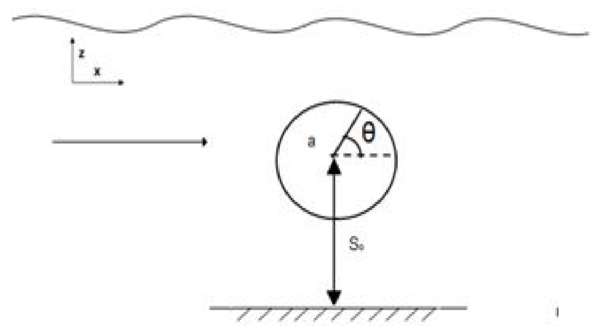


Fig. 10.4 Definition sketch

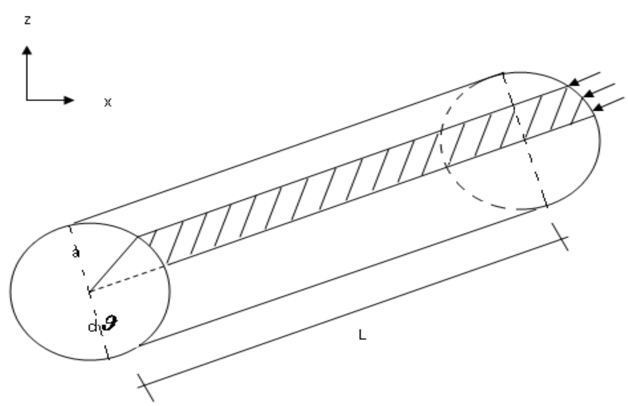


Fig. 10.5 Horizontal cylinder

$$F_x = C_H \iint p.n_x ds$$

$$p = r\frac{H}{2}\frac{\cosh k\left(d+z\right)}{\text{coshh}\, kd}\cos\left(kx-\omega t\right)$$

where, (d+z)= S_0 +asinθ (S_0 = distance of cylinder center from the sea bed; a = cylinder radius; θ = angle from the reference direction

$x = a\cos\theta$

$n_x = \cos\theta$; $n_y = \sin\theta$

$ds = a\,(d\theta)\,l$,   (where l = length of the cylinder)

Substituting and using series expansions of cosh , cos and simplifying and neglecting end effects,

$$F_x = C_H \rho \pi a^2 l \dot{u}_0 \qquad (10.3)$$

where, $\pi a^2 l$ = volume of cylinder, $\dot{u}_0$ = horizontal acceleration at '0'.

$$F_z = C_V \rho \pi a^2 l \dot{w}_0 \qquad (10.4)$$

where w_o = vertical water particle acceleration at 'o'. C_H and C_V are functions of Diffraction parameter = $\pi \frac{D}{L} = ka =$, wave characteristics, proximity with boundaries. Assuming small H and cylinder location away from bottom and top boundaries at least by 'D/2', $C_H = C_V = 2.0$ if $\pi \frac{D}{L} \Rightarrow (0,1)$

If $\pi \frac{D}{L}$ >1, high diffraction effects would be encountered.

2) Submerged Horizontal Half Cylinder:

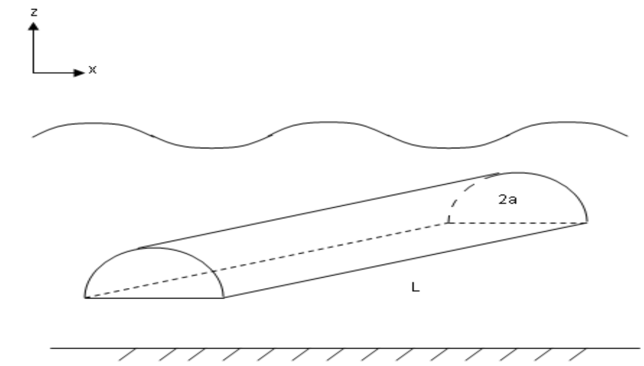


Fig. 10.6 Horizontal half cylinder

Let, a = cylinder radius.   Following a similar procedure:

$$F_x = C_H \rho \frac{\pi a^2}{2} l \left[\dot{u}_0 + c_1(ka)\omega w_0 \right] \tag{10.5}$$

$$F_z = C_V \rho \frac{\pi a^2}{2} l \left[\dot{w}_0 + c_2(ka)\omega u_0 \right] \tag{10.6}$$

Where $c_1(ka) = \frac{2}{\pi}\left[\frac{\cos ka}{ka} - \frac{\sin ka}{(ka)^2} + \int_0^{ka} \frac{\sin\alpha}{\alpha} d\alpha \right]$ (10.7)

$$c_2(ka) = \frac{2}{\pi}\left[\frac{\cos ka}{ka} + \frac{\sin ka}{(ka)^2} + \int_0^{ka} \frac{\sin\alpha}{\alpha} d\alpha \right] \tag{10.8}$$

ω = circular wave frequency; $C_H = 2.0$; $C_V = 1.1$ -------If ka $\Rightarrow (0,1)$ and if H is small and if the cylinder is away by distance D/2 from top and bottom boundaries. Table 10.1 gives values of the coefficients used in the Froud Krylov force calculations

Table 10.1. Values of the coefficients in the Froud Krylov force calculations

k_a	C_1	C_2	C_3	C_4
0.1	0.037	15.019	0.042	12.754
0.2	0.075	7.537	0.085	6.409
0.3	0.112	5.056	0.127	4.308
0.4	0.149	3.825	0.169	3.268
0.5	0.186	3.093	0.210	2.652
0.6	0.223	2.612	0.252	2.249
0.7	0.259	2.273	0.292	1.966
0.8	0.295	2.024	0.332	1.760
0.9	0.330	1.834	0.372	1.603
1.0	0.365	1.685	0.411	1.482
1.5	0.529	1.273	0.591	1.156
2.0	0.673	1.105	0.745	1.034
2.5	0.792	1.031	0.867	0.989
3.0	0.886	0.999	0.957	0.977
3.5	0.955	0.989	1.015	0.978
4.0	1.000	0.987	1.045	0.985
4.5	1.025	0.990	1.054	0.993
5.0	1.034	0.994	1.047	0.998

Note on the Bessel's equation of order 'v':

For any $y = f(t, v)$ and $t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + \left(t^2 - v^2\right) y = 0$

has solution $$y = \sum_{n=0}^{\infty} \frac{(-1)^m t^{v+2m}}{2^{v+2m} m! \left|(v+m-1)\right.} = J_V(t) \tag{10.9}$$

which is called the Bessel function of first kind of order 'v'.

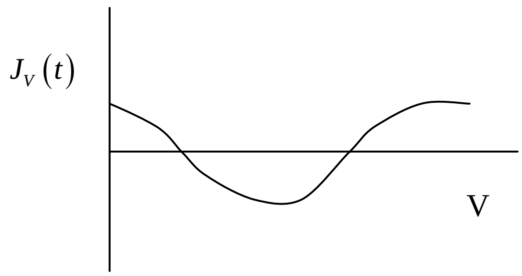


Fig. 10.7. Bessel function

The Bessel Function of <u>Second kind</u> of order v is:

$$Y_V(t) = \frac{\cos v\pi \, J_V(t) - J_{-V}(t)}{\sin v\pi} \tag{10.10}$$

Bessel Function of <u>third kind</u> of order v: (<u>Hankel Function</u>)

$$H_V(t) = J_V(t) \pm iY_V(t) \tag{10.11}$$

Positive sign indicates $H^1{}_V(t)$ of first kind and negative one means $H^2{}_V(t)$ of second kind.

3) Submerged Sphere:

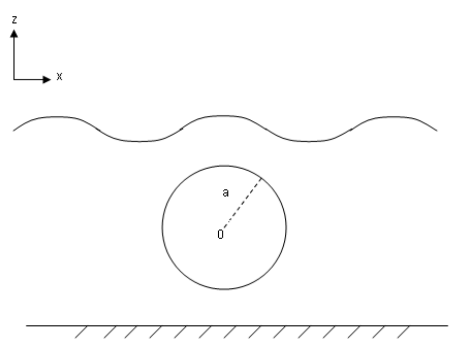


Fig. 10.8. Submerged sphere

$$F_x = C_H \rho \frac{4}{3} \pi a^3 \dot{u}_0 \tag{10.12}$$

$$F_z = C_V \rho \frac{4}{3} \pi a^3 \dot{w}_0 \tag{10.13}$$

Where a = radius, u_o and w_o = horizontal and vertical accelerations of water particles, $C_H = 1.5$, $C_V = 1.1$, if K_a varies from 0 to 1.75 and if H is small and if, the sphere is away by distance 'a' from boundaries.

4) Submerged Hemisphere

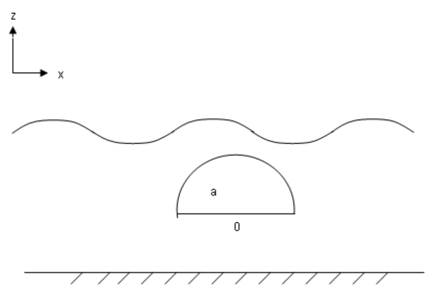


Fig. 10.9. Submerged half sphere

$$F_x = C_H \rho \frac{2}{3} \pi a^3 \left[\dot{u}_0 + c_3 (K_a) \omega w_0 \right] \tag{10.14}$$

$$F_z = C_V \rho \frac{2}{3} \pi a^3 \left[\dot{w}_0 + c_4 (K_a) \omega u_0 \right] \tag{10.15}$$

Where $c_3(K_a) = 3 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!} (K_a)^{n-1} J_{n+2}(K_a)$

$$c_4(K_a) = 3 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n)!} (K_a)^{n-2} J_{n-1}(K_a)$$

Note: $J_{n+2}(K_a)$ and $J_{n-1}(K_a)$ are Bessel functions.

w_o and u_0 are vertical and horizontal particle velocities, $C_H = 1.5$, $C_V = 1.1$ ------if K_a varies from 0 to 0.8, if H is small and if the sphere is away by distance 'a' from the boundaries.

5) Submerged Vertical Cylinder

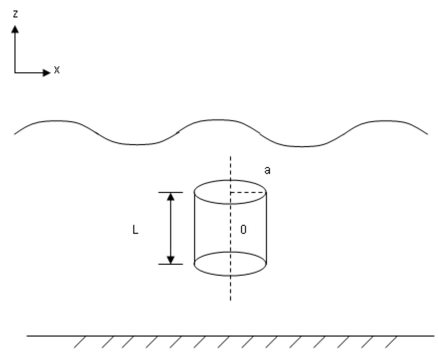


Fig. 10.10 Submerged vertical cylinder

$$F_x = C_H \rho \pi a^3 l \frac{2J_1(K_a)}{K_a} \frac{\sinh \frac{kl}{2}}{\frac{kl}{2}} \dot{u}_0 \qquad (10.16)$$

$$C_H = 1.5 \ ?$$

6) Submerged Rectangular Block:

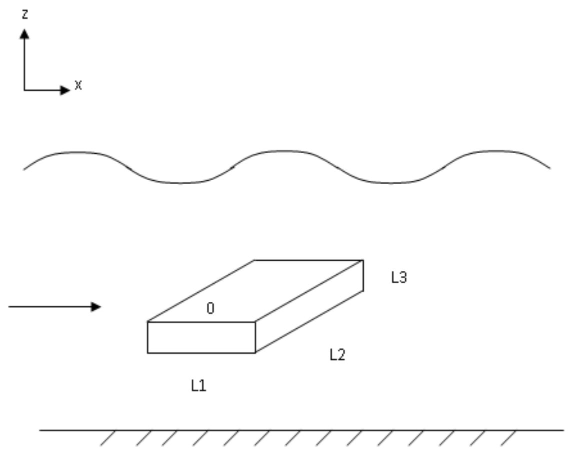


Fig. 10.11 Submerged rectangular block

$$F_x = C_H \rho l_1 l_2 l_3 \frac{\sinh \frac{kl_3}{2}}{\frac{kl_3}{2}} \frac{\sinh \frac{kl_1}{2}}{\frac{kl_1}{2}} \dot{u}_0 \qquad (10.17)$$

$$F_z = C_V \rho l_1 l_2 l_3 \frac{\sinh \frac{kl_3}{2}}{\frac{kl_3}{2}} \frac{\sinh \frac{kl_1}{2}}{\frac{kl_1}{2}} \dot{w}_0 \qquad (10.18)$$

l_1, l_2 *and* l_3 = in-line, transverse and vertical dimensions of the block, $C_H = 1.5$, $C_V = 6.0$ ---- If K_a varies from 0 to 5

7) Submerged Circular Disc:

An example of such a disc is the cap of a cylinder.

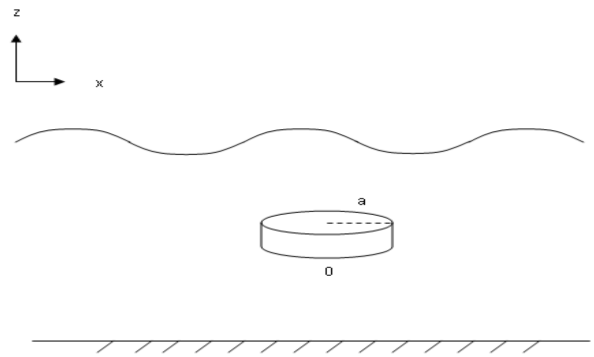


Fig. 10.12. Submerged disc

The design wave force on such a cap is calculated by considering it to act from one side only.

$$F_z = C_V \rho \frac{\pi a^2}{k} \frac{2J_1(K_a)}{K_a} \omega u_0 \qquad (10.19)$$

$$C_V \approx 1.5 \ ?$$

10.3 Diffraction Theory

Following figure explains regions of applicability of the Morison's equation and the diffraction theory.

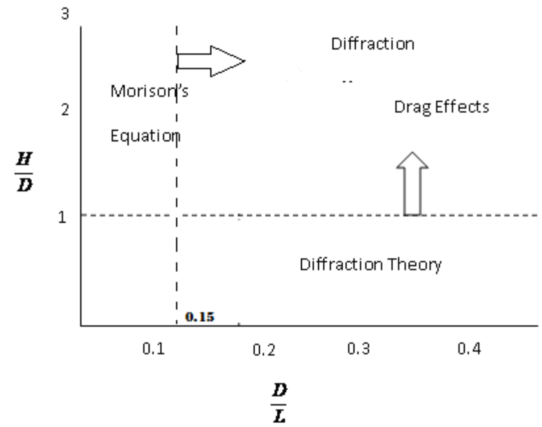


Fig.10.13 Regions of applicability

In diffraction theory the sea water is assumed to be irrotational, incompressible and inviscid.

$$\phi = \sum_{n=1}^{\infty} \varepsilon^n \phi_n \qquad \text{---- where } \varepsilon = \frac{KH}{2} \qquad (10.20)$$

$$= \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots$$

$\phi = \varepsilon^1\phi_1$ in linear (First Order) diffraction theory

$= \varepsilon^1\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots\ldots$ in non-linear (Higher Order) diffraction theory

Total Velocity Potential: $\phi = \phi_i + \phi_s$ (10.21)

Initially each velocity potential ϕ_1, ϕ_2,… is conceived as an unknown function of H,T,D and later these unknowns are found by making the potentials to satisfy the Laplace equation and various boundary conditions. Thereafter the total potential ϕ is put in the dynamic equation to get wave pressure and force values.

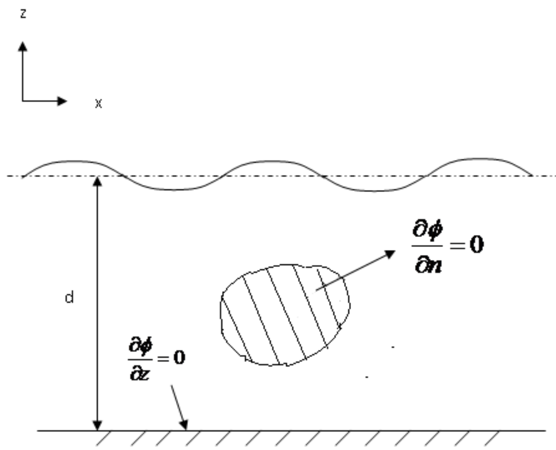


Fig. 10.14 Boundary conditions

The total potential is a summation of incident as well as scattered one, i.e.,

$$\phi = \phi_i + \phi_s$$

Each one of them has to satisfy:

Laplace Equation:

$$\nabla^2 \phi = 0 \text{, i.e.,} \tag{10.22}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

DFSBC: (Dynamic free surface boundary condition)

$$p + gz + \frac{1}{2} V^2 - \frac{\partial \phi}{\partial t} = 0$$

where, at the free surface , $z = \eta$, ϕ = 0.

$$\text{Hence } g\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = \frac{\partial \phi}{\partial t}$$

KFSBC: (Kinematic free surface boundary condition)

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial z} = 0 \text{ ....at } z = \eta$$

BBC: (Bed boundary condition)

$$\text{At } z = -d, \qquad \frac{\partial \phi}{\partial z} = 0$$

BSBC: (Body surface Boundary condition)

$$\text{At body surface,} \qquad \frac{\partial \phi_i}{\partial \eta} + \frac{\partial \phi_s}{\partial \eta} = 0$$

Radiation Condition:

$\phi_s \rightarrow 0$, at very large radial distance from the object.

Sommerfield R.C.: (at large radial distance R the scattering effect is zero)

$$\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial}{\partial R} \pm i\lambda\right)\phi_s = 0 \qquad (10.23)$$

where, λ = Eigen value.

Linear Diffraction Theory:

It assumes that the wave steepness is small. Hence the dynamic equation and boundary conditions can be linearalized as in the linear wave theory.

Application to the case of vertical circular cylinder: (as per McCamy and Fuchs 1954)

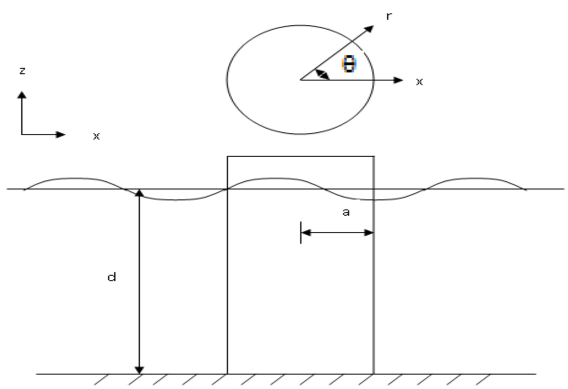


Fig.10.15 Large diameter member

It is convenient to use the cylindrical co-ordinates (r, θ, z), because then the Laplace equation takes the form of the Bessel's equation for which ready solutions (Bessel functions) are available.

Expressing ϕ in complex form and applying boundary conditions,

$$\phi = \frac{gH}{2\omega}\frac{\cosh k(d+z)}{\cosh kd}\left\{\sum_{m=0}^{\infty} i\beta_m\left[J_m(K_r) - \frac{J'_m(Ka)}{H^{(1)'}_m(Ka)}H_m^{(1)}(Kr)\right]\cos m\theta\right\}e^{-i\omega t}$$

(10.24)

where g = acceleration due to gravity

H = wave height

$\omega = 2\pi$ / wave period

K = 2π / wave length

d = water depth

z = vertical co-ordinate where ϕ is determined.

β_m = 1 if m=0

$= 2i^m$ if $m \geq 1$

$J_m(K_r)$ = Bessel Function of First kind of order 'm'

r = radial distance of the point

$J'_m(Ka)$ = derivative with respect to ka of $J_m(Ka)$

a = cylinder radius

$H^{(1)'}_m(Ka)$ = derivative of Hankel function of first kind of order m (of ka)

$H^{(1)}_m(Kr)$ = Hankel function of first kind of order m

θ = co-ordinate θ

t = time instant

Putting the above value of ϕ in the dynamic equation yields the wave induced pressure, which when integrated over the body surface gives the net wave force per unit axial length. Thus, the instantaneous (horizontal) wave force per unit axial length:

$$\Delta F = 2\rho g H a \frac{A(Ka)}{(Ka)} \frac{\cosh k(d+z)}{\cosh kd} \cos(\omega t - \delta) \tag{10.25}$$

where $A(Ka) = \left[J_1'^2(Ka) + Y_1'^2(Ka) \right]^{-\frac{1}{2}}$

$\delta = -\tan^{-1}\left[Y_1'(Ka) / J_1'(Ka) \right]$ = phase shift between max force w.r.t. $\eta = 0$

Horizontal Instantaneous Total wave force: (on entire cylinder)

$$F = 2\rho g H a d \frac{A(Ka)}{(Ka)} \frac{\tanh kd}{kd} \cos(\omega t - \delta) \tag{10.26}$$

Instantaneous overturning moment at the base:

$$M = 2\rho g H a d^2 \frac{A(Ka)}{(Ka)} \left[\frac{kd \sinh kd + 1 - \cosh kd}{(kd)^2 \cosh kd} \right] \cos(\omega t - \delta) \tag{10.27}$$

FORCE ON IRREGULAR SHAPED CYLINDERS:

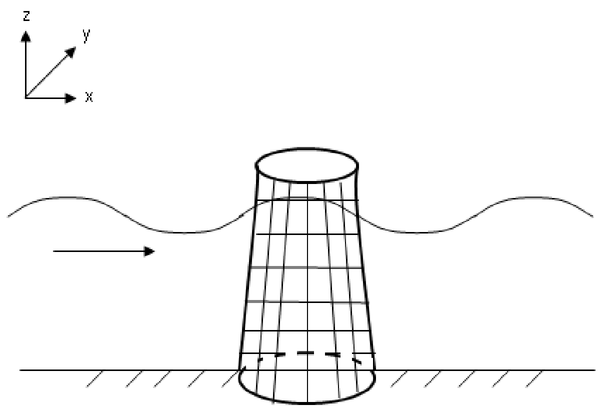


Fig. 10.15 Discretization for structures of arbitrary shapes

The problem is solved by the integral equation method. Typical of them is the wave source method

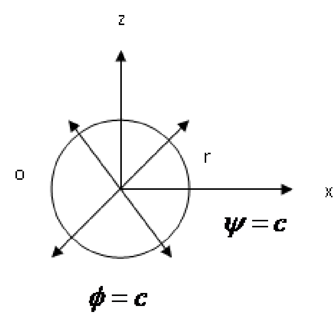


Fig. 10.16 A Source

Figure 10.16 shows the imaginary Source (2 Dimensional) point from which the fluid moves radially outwards in all directions The strength of the source is the total flow coming out per unit time.

The equation for ϕ on the cylinder surface is developed and solved by discretizing the surface into different elements and getting ϕ in each element.

Total $\phi = \phi\,(x,y,z,t)$

$$= \mathrm{Re}\left\{\phi'(x,y,z)e^{-i\omega t}\right\} \tag{10.28}$$

where $\phi' \Rightarrow \phi_i + \phi_s$, represented by continuous distribution of point sources over the entire body surface. This ϕ is made to satisfy various boundary conditions. It is obtained numerically and then the pressure, force,… are derived.

Many published results for specific object configurations are available. Some of them are as below.

Axisymmetric bodies, (conical, truncated)

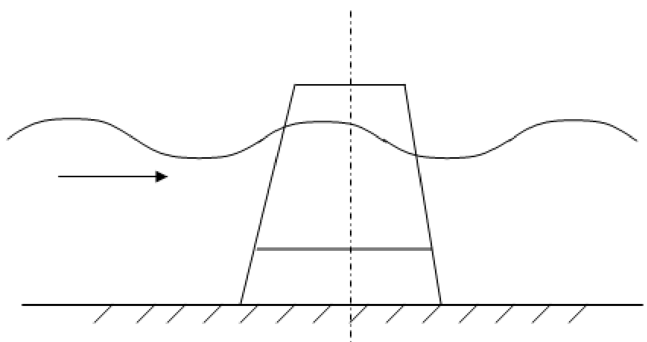
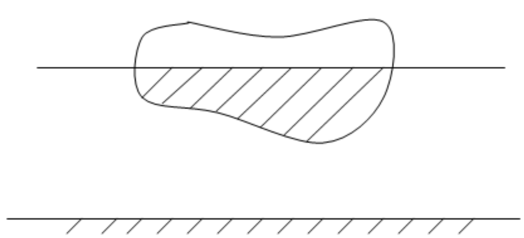


Fig. 10.17 Force on conical shaped structures

Horizontal cylinders:



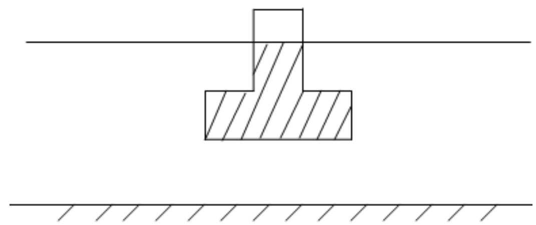


Fig. 10.18 Force on floating structures

Vertical cylinders with square, rectangular, random shapes

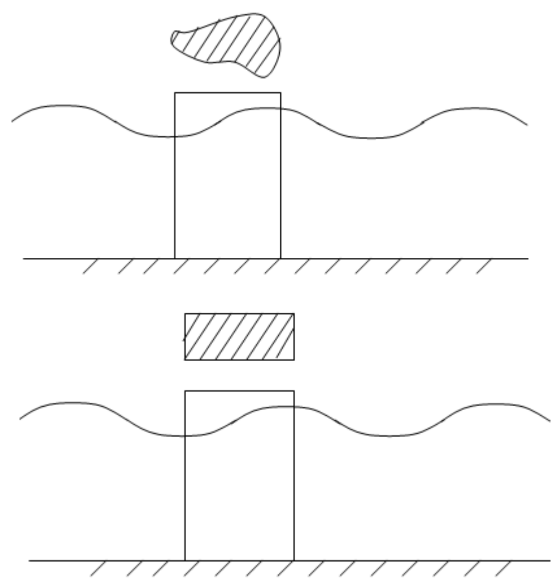


Fig.10.19 Force on arbitrary shapes structures

For such specific cases design curves are available to obtain the diffraction force, moments as functions of cross-sectional areas.

Floating Large Diameter Objects:

In this case the total potential is conceived as: $\phi = \phi_i + \phi_s + \phi_f$

where ϕ_f is velocity potential due to disturbance (waves) created by the movements of floating objects.

Each potential satisfies the dynamic equation,

$$\phi_f \rightarrow \frac{\partial \phi_f}{\partial n} = (vel)_n \qquad (10.29)$$

Drift force:

The total potential is given by: $\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots$

where $\varepsilon = \dfrac{KH}{2}$; $\varepsilon^2\phi_2$ and $\varepsilon^3\phi_3$ are higher order terms which give the higher order force, which is called the Drift Force

Unlike the first order force,

$$\int_{t=0}^{T} DriftForce \neq 0$$

(where T = wave period), i.e., the drift force is non-oscillatory. For regular waves, it is steady, while for random waves, it is oscillatory but the period of oscillation is very high. The drift forces are an order of magnitude smaller than the first order force but if frequency-coupled with disturbances, they could cause large structural oscillations.

There are two approximate approaches as alternatives to the above analytical one in arriving at the diffraction force as mentioned below:

Diffraction Coefficient method:

Approximately,

Total force on vertical cylinder = C_H x Froud Krylov force

C_H is the diffraction coefficient expressed as the ratio of the maximum diffraction force to the maximum FK force given by,

$$C_H = \frac{2A(Ka)}{KaJ_1(Ka)} \tag{10.30}$$

Effective Inertia Coefficient Technique:

Since the diffraction force is acceleration-dependent, the diffraction force per unit axial length at any depth 'z' can be given by

$$F_z = C_m \rho \frac{\pi d^2}{4} \dot{u}_m \cos(\omega t - \delta) \tag{10.31}$$

Where C_m is effective inertia coefficient $= \dfrac{4A(Ka)}{\pi (Ka)^2}$ (10.32)

Substituting for u_m the diffraction force per unit axial length (at z, t) can be given as

$$dF = C_m \rho \frac{\pi D^2}{8} gHK \frac{\cosh k(d+z)}{\cosh kd} \cos(\omega t - \delta)$$

Hence diffraction force for entire length at 't' is given by

$$F = C_m \rho \frac{\pi D^2}{8} gH \tanh kd \cos(\omega t - \delta) \tag{10.33}$$

Total moment at time 't' is:

$$M = C_m \rho \frac{\pi D^2}{8} \frac{gH}{K} \left\{ \frac{kd \sinh kd + 1 - \cosh kd}{\cosh kd} \right\} \cos(\omega t - \delta) \tag{10.34}$$

Also ,

$$F_{\max} = C_m \frac{\pi^2 D^2}{4} \frac{\rho HL}{T^2} \tag{10.35}$$

$$M_{\max} = C_m \rho gHLD^2 \left\{ \frac{kd \tanh kd + \sec hkd - 1}{16} \right\} \tag{10.36}$$

CHAPTER 11

SPECTRAL AND STATISTICAL ANALYSIS OF WAVE FORCES

11.1 SPECTRAL ANALYSIS OF WAVE FORCES

Time series analysis of two random variables:

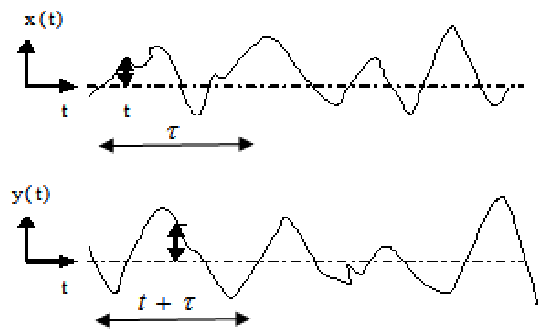


Fig. 11.1 Time series analysis

Let X(t) and Y(t) be stationary, ergodic and real valued processes. For example, X(t) is sea surface elevation, Y(t) is horizontal water particle velocity .

Cross Correlation Function:

$$R_{XY}(\tau) = E\{X(t)Y(t+\tau)\} = \frac{1}{T}\int_0^T X(t)Y(t+\tau)dt \qquad \text{for } \tau = 0,1,2,\ldots \quad (11.1)$$

Cross Spectral Density Function:

$$S_{XY}(f) = \text{Fourier Transform of Cross Correlation Function} = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i2\pi f\tau} d\tau \quad (11.2)$$

The real part is the co spectrum and the imaginary part is the quadrature spectrum.

If we consider only a single process instead of the two, then, we can define:

$$R_{XX}(\tau) = E\{X(t)Y(t+\tau)\} = R_X(\tau) = \text{ Auto Correlation Function} \qquad (11.3)$$

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i2\pi f\tau} d\tau = S_X(f) \quad = \text{Auto Spectral Density Function} \qquad (11.4)$$

Estimation of spectral density of the wave force spectrum at wave frequency, f: $S_F(f)$

This information is required in the spectral analysis of structural response. Its direct computation is obviously not possible in absence of any wave force time history at the design stage.

Application of the Morison's Equation:

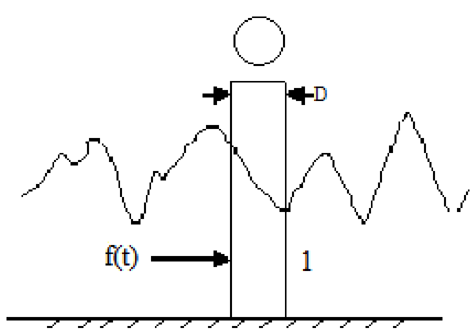


Fig. 11.2 Wave force on a small cylinder

Total horizontal force per unit length of vertical circular cylinder is given by,

$$F(t) = C|u(t)|u(t) + Ka(t) \qquad (11.5)$$

Where $C = C_D \rho \dfrac{D}{2}$ (11.6)

$u(t)$ = horizontal wave particle velocity

$K = C_M \rho \dfrac{\pi d^2}{4}$ (11.7)

a(t) = horizontal wave particle acceleration.

$$F(t) = C\left|u(t)\right|u(t) + Ka(t)$$

$$F(t+\tau) = C\left|u(t+\tau)\right|u(t+\tau) + Ka(t+\tau)$$

Wave force auto-correlation function:

$$R_{FF}\left(\tau\right) = E\left\{F(t)F(t+\tau)\right\} = \frac{1}{T}\int_0^\infty F(t)F(t+\tau)dt$$

(using over bar to denote the time averages)

$$= \overline{F(t)F(t+\tau)} =$$

$$C^2\overline{u(t)\left|u(t)\right|u(t+\tau)\left|u(t+\tau)\right|} + K^2\overline{a(t)a(t+\tau)} + CK\overline{u(t)\left|u(t)\right|a(t+\tau)} + CK\overline{u(t+\tau)\left|u(t+\tau)\right|a(t)}$$

Using classical theorems in time series analysis:

$$R_{FF}\left(\tau\right) = C^2\sigma_u^4 G\left(\frac{R_{uu}\left(\tau\right)}{\sigma_u^2}\right) + k^2 R_{aa}\left(\tau\right) \tag{11.8}$$

Where σ_u^2 =Variance of u

$C = C_D \rho \dfrac{D}{2}$; $K = C_m \rho \dfrac{\pi d^2}{4}$

$R_{uu}\left(\tau\right)$ = auto-correlation function of u

If $\dfrac{R_{uu}\left(\tau\right)}{\sigma_u^2}$ =r; $G(r) = \dfrac{(2+r^2)\sin^{-1} r + 6r(1-r^2)^{\frac{1}{2}}}{\pi} = \dfrac{1}{\pi}\left\{8r + \dfrac{4}{3}r^3 + \dfrac{r^5}{15} + \dfrac{r^7}{70} + \ldots\right\}$

(11.9)

$R_{aa}\left(\tau\right)$ =auto-correlation function of 'a'.

The above equation involving a non-linear velocity term is difficult to solve.

Noting that the series G(r) converges fast and that if we retain only first term of G(r), then the difference: [full series of G(r) - truncated series $G_1(r)$] could be low as 15%. Therefore we may write:

$$G_1(r) \approx \frac{8r}{\pi} \approx \frac{8}{\pi} \frac{R_{uu}(\tau)}{\sigma_u^2}$$

It is then possible to show that the auto-correlation function of the wave force can be given as:

$$R_{FF}(\tau) = c^2 \sigma_u^4 \left[\frac{1}{\pi} \left(\frac{8R_{uu}(\tau)}{\sigma_u^2} \right) \right] + K^2 R_{aa}(\tau)$$

where $c = C_D \rho D/2$; σ_u = standard deviation of velocity u; $R_{uu}(\tau)$ = auto-correlation function of velocity for time lag τ; $K = C_M \rho \pi D^2 / 4$; $R_{aa}(\tau)$ = auto-correlation function for acceleration for time lag τ.

$$R_{FF}(\tau) = \frac{8}{\pi} c^2 \sigma_u^2 R_{uu}(\tau) + K^2 R_{aa}(\tau) \qquad (11.10)$$

It can be proved that the same result can be obtained if,

$$u|u| = u \left\{ \sigma_u \sqrt{\frac{8}{\pi}} \right\} \qquad (11.11)$$

Let $F = C|u|u + Ka$ Hence

$$F(t) = C|u(t)|u(t) + Ka(t) \approx c\sigma_u \sqrt{\frac{8}{\pi}} u(t) + Ka(t)$$

$$F(t+\tau) = c\sigma_u \sqrt{\frac{8}{\pi}} u(t+\tau) + Ka(t+\tau)$$

Multiplying:

$$F(t)F(t+\tau) = c^2\sigma_u^2 \frac{8}{\pi} u(t)u(t+\tau) + K^2 a(t)a(t+\tau) + CK\sigma_u \sqrt{\frac{8}{\pi}} u(t)a(t+\tau) + CK\sigma_u \sqrt{\frac{8}{\pi}} a(t)u(t+\tau)$$

Time Averaging,

$$R_{FF}(\tau) = c^2\sigma_u^2 \frac{8}{\pi} R_{uu}(\tau) + K^2 R_{aa}(\tau) + CK\sigma_u \sqrt{\frac{8}{\pi}} R_{ua}(\tau) + CK\sigma_u \sqrt{\frac{8}{\pi}} R_{au}(\tau)$$

Spectral density function of force F:

Spectral density function of force F = Fourier Transform of auto correlation function.

Hence,

$$S_{FF}\left(f\right)=\int_{-\infty}^{\infty}R_{FF}(\tau)e^{-i2\pi f\tau}d\tau=\frac{8}{\pi}c^2\sigma_u^2\int R_{uu}\left(\tau\right)e^{-i2\pi f\tau}d\tau+K^2\int R_{aa}\left(\tau\right)e^{-i2\pi f\tau}d\tau$$

$$S_{FF}\left(f\right)=\frac{8}{\pi}c^2\sigma_u^2S_{uu}\left(f\right)+K^2S_{aa}\left(f\right) \qquad (11.12)$$

Where $S_{uu}(f)$ is Spectral Density Function of 'u' and $S_{aa}(f)$ is Spectral Density function of 'a'.

We have,

$$u(t)=2\pi f\,\frac{\cosh k\left(d+z\right)}{\sinh kd}\eta(t)$$

$$u(t+\tau)=2\pi f\,\frac{\cosh k\left(d+z\right)}{\sinh kd}\eta(t+\tau)$$

$$\overline{u(t)u(t+\tau)}=\left(2\pi f\right)^2\frac{\cosh^2 k\left(d+z\right)}{\sinh^2 kd}\overline{\eta(t)\eta(t+\tau)}$$

$$\text{Hence } R_{uu}\left(\tau\right)=\left(2\pi f\right)^2\frac{\cosh^2 k\left(d+z\right)}{\sinh^2 kd}R_{\eta\eta}\left(\tau\right)$$

Taking Fourier Transform,

$$S_{uu}\left(f\right)=\left\{4\pi^2f^2\frac{\cosh^2 k\left(d+z\right)}{\sinh^2 kd}\right\}S_{\eta\eta}\left(f\right) \qquad (11.13)$$

where $\left(2\pi f\right)^2=gk\tanh kd$

$$=\left\{T_u(f)\right\}S_{nn}\left(f\right)$$

$\left\{T_u(f)\right\}$ is the velocity transfer function

Similarly we get

$$S_{aa}\left(f\right)=\left\{\left(4\pi^2f^2\right)^2\frac{\cosh^2 k\left(d+z\right)}{\sinh^2 kd}\right\}S_{\eta\eta}\left(f\right) \qquad (11.14)$$

$$=\left(2\pi f\right)^2S_{uu}\left(f\right)$$

$$= \{T_a(f)\} S_{\eta\eta}(f)$$

$\{T_a(f)\}$ = acceleration transfer function

11.2 Statistical Analysis of Wave Forces

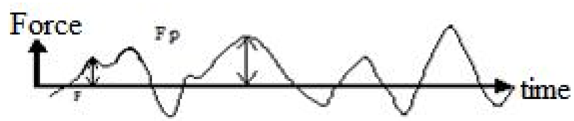


Fig.11.3. Force time history

This mainly involves deriving probability distribution functions of instantaneous force F, or peak force Fp whose knowledge could be used to obtain values of the design force with known probability of recurrence.

There are two main methods to derive this information.

1) Semi-deterministic
2) Probabilistic- a) Linearized and b) Non-linearized

Semi-deterministic:

This method is popularly used to obtain the probability distribution of peak force Fp.

The steps involved are as follows

1) Draw scatter diagram of (Hs ,T_z)
2) Obtain $P_{LT}(H)$ versus H
3) Associate appropriate 'T_z' to each 'H'
4) For each pair of (H,T), obtain u, $\dot{u}$ using a wave theory.
5) Obtain force using Morison's equation.
6) Assume:

P(Fp) = P(H).

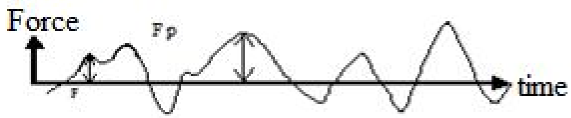


Fig. 11.4. Long term peak force

Probabilistic-Linearized:

$$F = C|u|u + Ka$$

If $|u|u \approx \sqrt{\frac{8}{\pi}}\sigma_u u$, then force will be a linear function of u and a and further because u and a are linear functions of sea surface elevation, which is Gaussian distributed, the wave force also can be described by the Gaussian distribution.

Hence in short term,

$$p(F) = \frac{1}{\sigma_f \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{F}{(\sigma_f)^2}\right)} \qquad (11.15)$$

$$P(F) = \int p(F)dF$$

and the probability distribution of the peak force is given by the Rayleigh Distribution Function.

The above equation of *p(F)* involves standard deviation of force, σ_F. Following procedure is stated to obtain its value.

Obtain $S_\eta(f)$ versus f, then derive $S_u(f)$, $S_a(f)$. Integrate these spectra and get σ_u^2, σ_a^2.

Finally estimate σ_F^2 using:

$$\sigma_f^2 = \frac{8}{\pi} c^2 \sigma_u^4 + K^2 \sigma_a^2 \qquad (11.15)$$

This equation can be easily derived by integrating:

$$S_{FF}(f) = \frac{8}{\pi} c^2 \sigma_u^2 S_{uu}(f) + K^2 S_{aa}(f)$$

Probabilistic-Nonlinear:

This is based on the full form of Morison's equation, which when used in this way makes the force variable non-Gaussian distributed as below:

Probability density function of instantaneous force F is:

$$p(F) = \frac{1}{2\pi K \sigma_u \sigma_a} \int_{\infty}^{-\infty} \exp - \left[\left(\frac{u^2}{(\sigma_u)^2} + \frac{a^2}{(\sigma_a)^2} \right) / 2 \right] du \qquad (11.16)$$

Where, σ_u = standard deviation of particle velocity

σ_a = standard deviation of particle acceleration

$$a = \frac{F - C|u|u}{K}$$

Long term force distributions

Long term probability distribution function of force F is obtained by multiplying the short term distributions as above by long term probability of occurrence of the given short term sea state and taking their summation as below:

$$P_{LT}(F) = \sum_{alli} p\left(\frac{F}{Hs_i}\right) prob(Hs_i) \qquad (11.17)$$

Where $p\left(\frac{F}{Hs_i}\right)$ = conditional probability density function of F for given wave Hs_i value in the short term

STATISTICS OF RESPONSE:

Structural response indicates its behavior against loading. In general response spectrum (of say member stresses, displacements) can be obtained by multiplying the force spectrum by appropriate transfer functions.

$$S_r(f) = |H(f)|^2 S_F(f) \qquad (11.18)$$

Where $S_r(f)$ = spectral density function of response 'r' at wave frequency 'f'.

$|H(f)|^2$ = transfer function obtained from theoretical / experimental considerations

$S_F(f)$ = spectral density function of force 'F' at frequency 'f'

If we assume that the response amplitude is linearly related to wave force amplitude, then all statistics of response will be similar to that of force. For example:

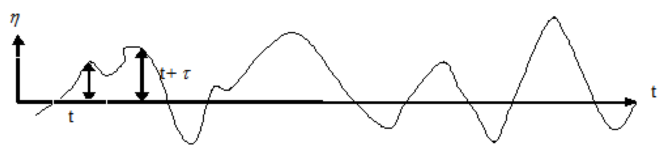


Fig. 11.5. Response time history.

$$p(r) = \frac{1}{\sigma_r \sqrt{2\pi}} e^{-\left(\frac{r^2}{2(\sigma_r)^2}\right)} \tag{11.19}$$

where, p(r) = probability density function of response 'r having a standard deviation of σ_r.

$$p(r_{peak}) = \frac{r_{peak}}{2\sigma_r^{\,2}} e^{-\left(\frac{r_{peak}^{\,2}}{2(\sigma_r)^2}\right)} \tag{11.20}$$

$p(r_{peak})$ = probability density function of peak of the responses belonging to different cycles having a standard deviation of σ_r.

$$E(r_{max}) = 0.705\, r_s\, (\ln N)^{1/2}$$

where $E(r_{max})$ = expected value of the maximum response within N cycles for a given significant response r_s.

Above equations for response evaluation pertain to short term sea states. For long term response, all long term theoretical distributions used for wave heights are valid. For example the long term peak response is given by,

$$p(r_{peak}) = 1 - e^{-\left[-\left(\frac{r_{peak} - A}{B}\right)^{c}\right]} \qquad (11.21)$$

where A, B = constants, dependent on response data.

A general flow chart depicting all major steps involved in the stochastic analysis involving waves along with typical variations in the values of functions with frequencies is as shown below:

11.3 FULL STOCHASTIC ANALYSIS OF WAVES

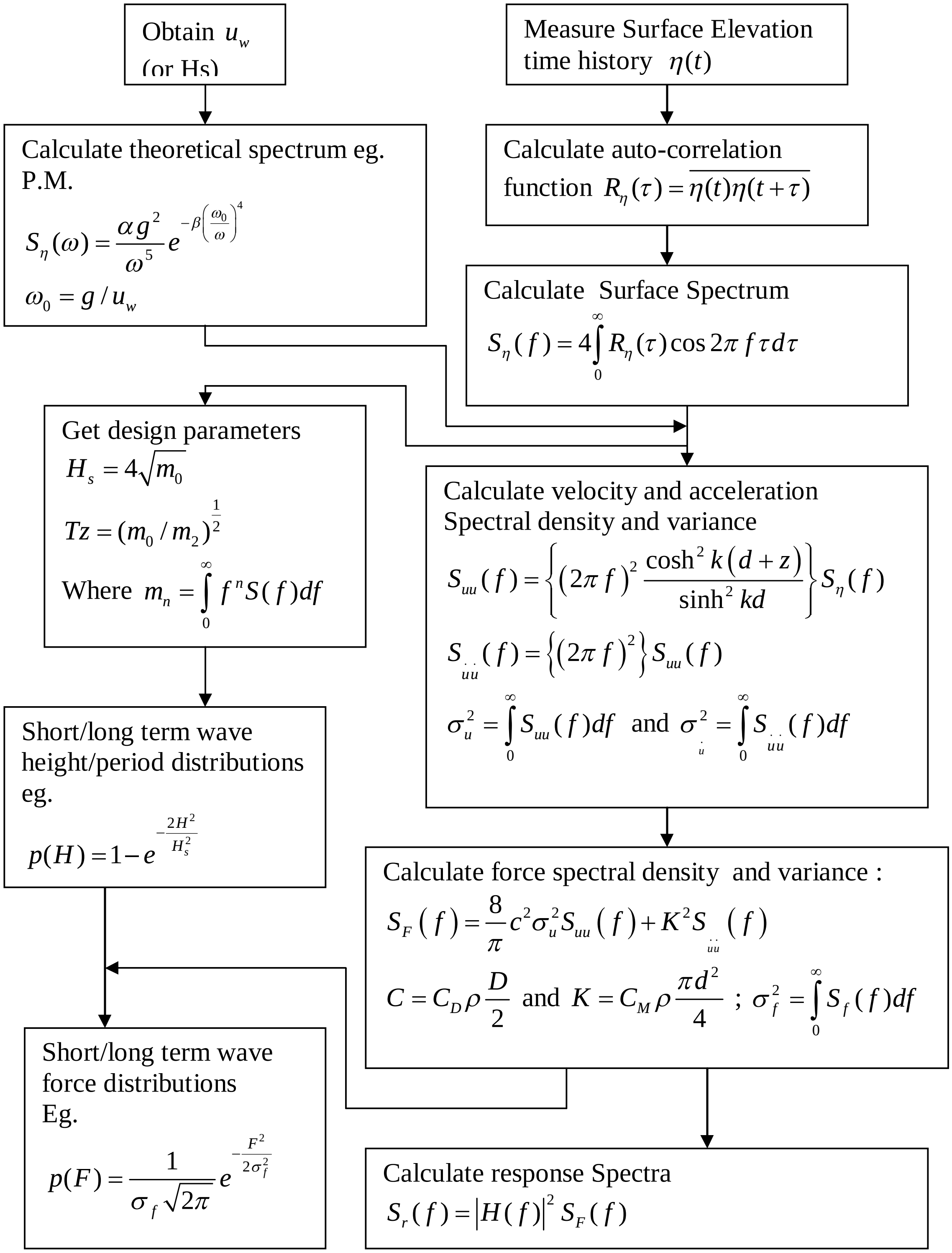


Fig. 11.6 Summary of wave and wave force analysis

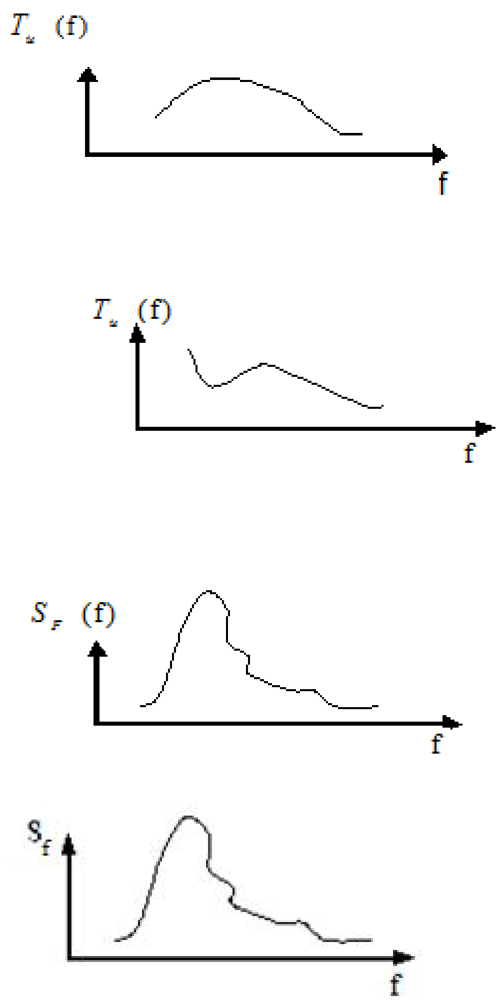
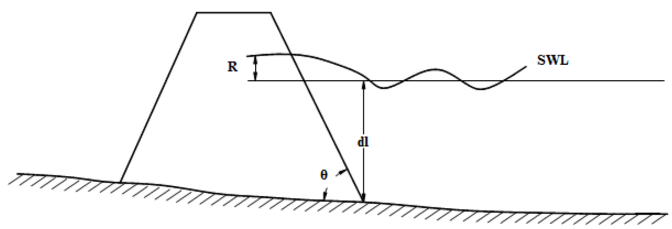


Fig. 11.7 Components of the wave and wave force analysis

CHAPTER 12

WAVE RUN UP

12.1 Introduction



When waves hit a barrier they move up its face with or without breaking. The vertical distance 'R' moved up by them over and above the Still Water Level (SWL) is called the wave run up. Knowledge of run up is important to know the height of structure. Run up is a complex process and depends upon the structural characteristics, (slope/roughness/ permeability) site conditions, (water depth/bed slope) and wave attack (steepness and other relationships between wave height, period and water depth).

12.2 RUN UP

RUN-UP FOR REGULAR WAVES

This is given as:

$$\frac{R}{H_i} = C_t \left[\sqrt{\frac{\pi}{2\theta}} + \frac{\pi H_i}{L_s} \cot \frac{2\pi d_s}{L_s} \right]$$

where θ is surface slope, L_s $_s$ is wave length, H_s is the height in deep water.

C_t depends on structure roughness and permeability. For smooth concrete slope its value is 0.9 ~ 1.0. For rip rap it is in between 0.5 to 0.7.

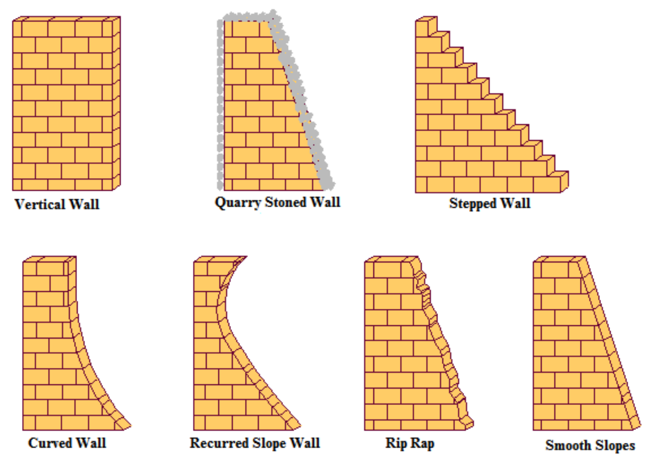
FOR BREAKING WAVES

$$\frac{R}{H} = C_t \left[\frac{\tan\theta}{H_i / L_o} \right]$$

In practice graphical relationships based on model tests are preferred. The values so obtained for smooth slopes need to be corrected for rough slopes by multiplying by certain constants. For rough slopes, use roughness, porosity correction factor,

$$\text{‘r’} = \frac{R\ rough\ slope}{R\ smooth\ slope}$$

Graphs are available for following type of walls as in Table 12.1:



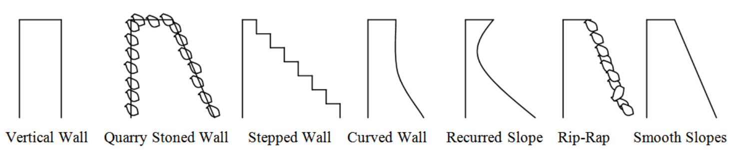
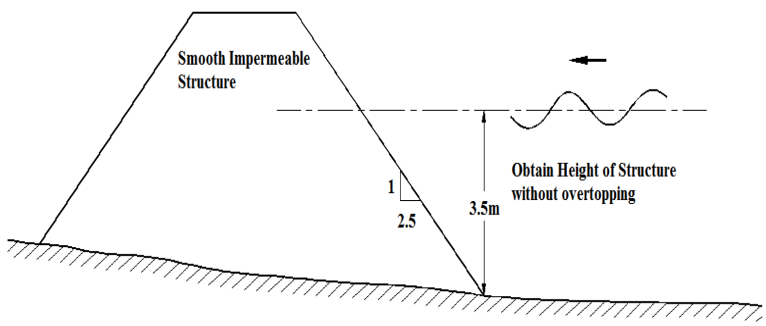


Table 12.1 Correction factor ‘r’:

Slope Surface Characteristics	**Placement**	**r**
Smooth, impermeable	---	1.00
Concrete blocks	Fitted	0.90
Basalt blocks	Fitted	0.85 to 0.90
Gobi blocks	Fitted	0.85 to 0.90
Grass	---	0.85 to 0.90
One layer of quarrystone (impermeable foundation)	Random	0.80

Quarrystone	Fitted	0.75 to 0.80
Rounded quarrystone	Random	0.60 to 0.65
Three layers of quarrystone (impermeable foundation)	Random	0.60 to 0.65
Quarrystone	Random	0.50 to 0.55
Concrete armor units (~ 50% void ratio)	Random	0.45 to 0.50

Example:



Solution:

H_o :

$$\frac{d}{L_o} = \frac{5.0}{1.56(9)^2} = 0.03957$$

$$\Longrightarrow \quad \frac{H}{H_o} = 1.067$$

$$\Longrightarrow \quad H_o = \frac{2.5}{1.067} = 2.343\,m$$

Therefore, $\dfrac{H_o}{gT^2} = \dfrac{2.343}{9.81(9)^2} = 0.00295$

$$\frac{d_s}{H_o} = \frac{3.5}{2.343} = 1.494$$

From the Figure: $\Longrightarrow$ for $\dfrac{d_s}{H_o} = 0.8 \quad \Longrightarrow \quad \dfrac{R}{H_o} = 2.8$

for $\dfrac{d_s}{H_o} = 2.0 \Rightarrow \dfrac{R}{H_o} = 2.7$

Therefore, R = 2.75 (2.343)

$= \boxed{6.44\ m}$ Uncorrected

From the figure: $\Rightarrow$ for $\tan\theta = \dfrac{1}{2.5} = 0.4$

k = 1.17

Therefore, R = 1.17 (6.44)

$= \boxed{7.53\ m}$

RUN-UP FOR IRREGULAR WAVES

The run up height varies with time regularly for a regular wave attack while it changes randomly for an irregular attack. The probability distribution of run up heights is given by the Rayleigh distribution.

Therefore, $P(H \rangle H') = e^{-(H'/Hrms)^2}$

$$= e^{-2(H'/H_s)^2}$$

$$\ln P(H \rangle H') = -2\left(\frac{H'}{H_s}\right)^2$$

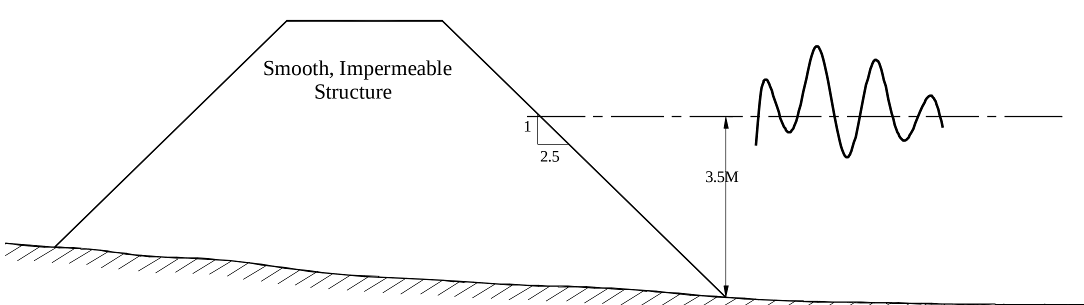
Therefore, $H' = \left[\dfrac{-\ln P(H \rangle H')}{2}\right]^{\frac{1}{2}} .H_s$

Where, H' =Height of run-up associated with problem of exceedence $P(H \rangle H')$

H_s =Height of run-up associated with significant wave height [given by previous graphs]

Example:

Data: Significant wave height = 2.5m measured in 5m water depth.



OBTAIN: (a) 'R' from $H_s, H_{\frac{1}{10}}, H_{\frac{1}{100}}$

(b) Probability of exceedance for 18 m run up.

Solution:

(a) from previous example, $H_s = 2.5m \Longrightarrow$ $\boxed{7.53\ m}$

for $H_{\frac{1}{10}}, = \left[\frac{-\ln 0.1}{2}\right]^{\frac{1}{2}}(7.53) = \boxed{8.08\ m}$

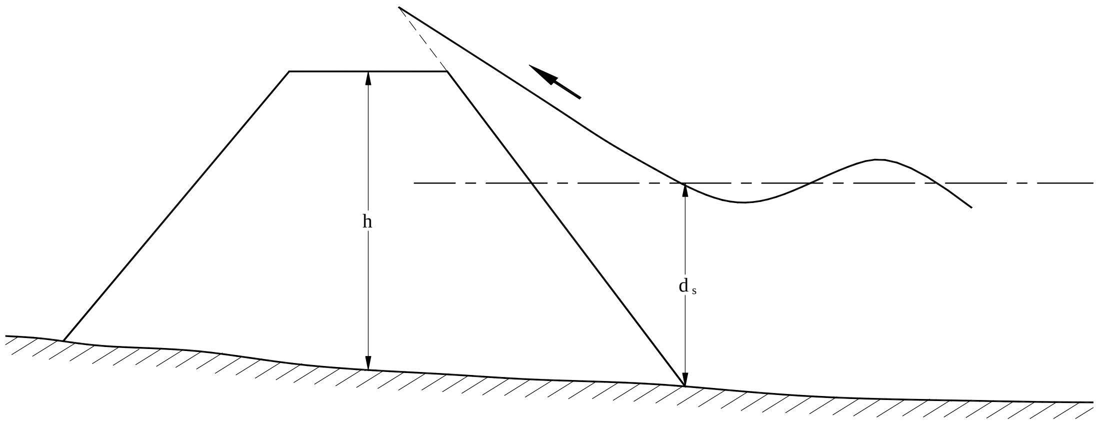
for $H_{\frac{1}{100}}, = \left[\frac{-\ln 0.01}{2}\right]^{\frac{1}{2}}(7.53) = \boxed{11.43\ m}$

(b) $12 = \left[\frac{-\ln P(H \rangle H')}{2}\right]7.53$

Therefore, $P(H \rangle H') = \boxed{0.6\%}$

12.3 Wave Overtopping

In regular waves:



From economical considerations many times height of the structure, h, is required to be kept less than d_s + R, where d_s is depth of water at the toe of the structure and R is the run up above the SWL. In these situations a few largest waves may overtop the structure. If the structure is in the form of an embankment (either a levee for retaining purpose or a dyke to prevent low areal flooding) it is necessary to know the volume of water overtopping for pumping out which is known through rate of overtopping in $m^3/\sec - m$ length of wall. Saville (1953 ~) has given following formula for this purpose:

$$\Longrightarrow \quad Q = K\left(g\overset{*}{Q_o} H_o^3\right)^{\frac{1}{2}} . e^{\left[\frac{0.217}{\alpha}\tan h^{-1}\left(\frac{h-d_s}{R}\right)\right]}$$

Where, Q is the rate of overtopping $m^3/\sec - m$ length.

K is onshore wing correction factor

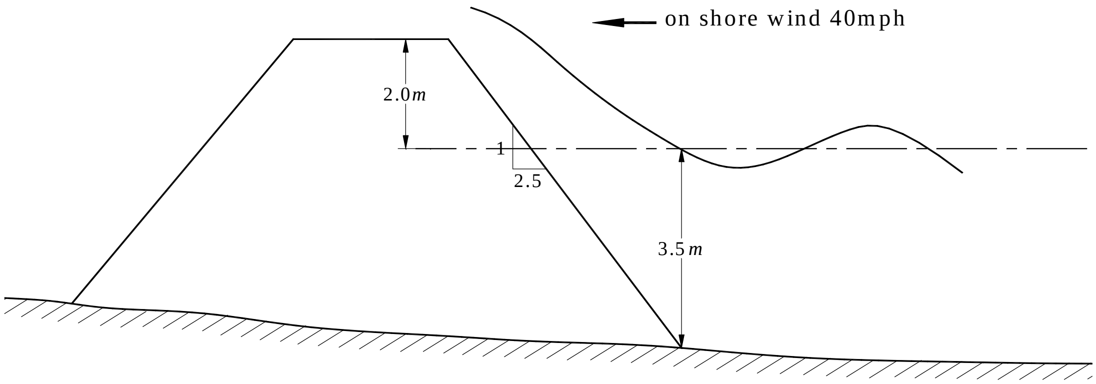
$\overset{*}{Q}$ & α are impirical constants of $\left(\frac{H_o}{gT^2}, \frac{d_s}{H_o}\right)$

$$K = 1 + K'\left(\frac{h-d_s}{R} + 0.1\right)\sin\theta$$

$$K' = \begin{cases} u \geq 60mph \leftarrow 2 \\ u \approx 30mph \leftarrow 5 \\ u \approx 0mph \leftarrow 0 \end{cases}$$

R = run-up for no overtopping

Example:



For the impermeable structure with smooth slope, obtain the volume of water overtopping over its 500*m* length in 3hrs. if $H_o = 1.75m$; T = 7 Sec.

Solution:

Run-up: $\frac{d_s}{H_o} = \frac{3.5}{1.75} = 2.0$; $\frac{H_o}{gT^2} = \frac{1.75}{(9.81)(7)^2} = 0.00364$

from figure $\frac{R}{H_o} = 2.6$ uncorrected for sale

from figure, for tanθ = 0.4 ⟹ k = 1.17

R=2.6 (10/75) (1.75) = 5.32 *m*

$\alpha : Q_o^* :$ for (nearby slope 1:3) Figure ⟹ $\alpha = 0.08; Q_o^* = 0.025$

$$Q = K\left(g\,\overset{*}{Q_o}\,H_o^3\right)^{\frac{1}{2}}.\,e^{\left[\frac{0.217}{\alpha}\tan h^{-1}\left(\frac{h-d_s}{R}\right)\right]}$$

$$= K\left[9.81(0.025)(1.75)^3\right]^{\frac{1}{2}}.e^{-\left[\frac{0.217}{0.08}\tan h^{-1}\left(\frac{2}{5.32}\right)\right]}$$

$$= 0.45K = \left\{1 + K'\left(\frac{h-d_s}{R} + 0.1\right)\sin\theta\right\} \qquad = 1 + 1\left(\frac{2}{5.32} + 0.1\right)0.3714$$

$$= 0.53\frac{m^3}{s-m} \qquad = 1.1765$$

$$= 0.53\,(3600)3*500 = 2.862*10^6\,m^3$$

IN IRREGULAR WAVES: Ahrens (1977 ~) had suggested as follows:

If R_{H_s} = wave run-up corresponding to H_s (obtain from previous procedure)

$R_{H'}$ = waver run-up corresponding to H' having problem of exceedence 'P',

$$\frac{R_{H'}}{R_H} = \sqrt{\frac{-\ln P(H \rangle H')}{2}}$$

Rate of overtopping corresponding to 'P':

$$Q_p = \sqrt{g\,\overset{*}{Q_o}(H_o)_s^3}\,.\exp^{-\left[\frac{0.217}{\alpha}\tan h^{-1}\left(\frac{h-d_s}{R_{H_s}}\right)\frac{R_{H_s}}{R_{H'}}\right]}$$

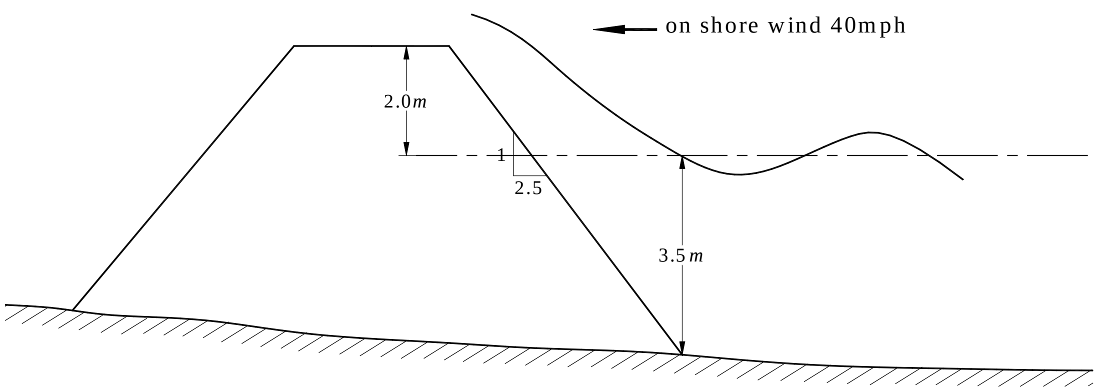
$\overset{*}{Q}$ & α are impirical constants = f $\left(\frac{H_o}{gT^2}, \frac{d_s}{H_o}\right)$

Normally, $Q_{0.5\%}$ $\Longrightarrow$ extreme overtopping rate,

$\overline{Q}$ $\Longrightarrow$ average overtopping rate

Desired, $= \dfrac{Q_{0.005} + Q_{0.010} + Q_{0.015} + - - - - + Q_{0.995}}{199}$

Example:



For the impermeable structure with smooth slope, $\lfloor H_{o_s} = 1.75m;\ T = 7\,Sec \rfloor$

Obtain: a) Rate of overtopping corresponding to H_s,

b) Extreme rate of overtopping for $Q_{0.5\%}$

c) Average rate of overtopping, for $\overline{Q}$

Solution:

(a) Run-up corresponding to H_s:

$$\frac{d_s}{H_o}=\frac{3.5}{1.75}=2.0;\frac{H_o}{gT^2}=\frac{1.75}{9.81(7)^2}=0.00364$$

⟹ from Figure, $\frac{R}{H_o}=2.6$ uncorrected for scale effects

⟹ from Figure, (for tanθ = 0.4), k = 1.17

Therefore, R = 2.6 (1.75) (1.17) = 5.32 *m*

From figure (for nearby slope 1:3), $\alpha=0.08;\overset{*}{Q}_o=0.025$

Therefore, $Q=K\left(g\overset{*}{Q}_o H_o^3\right).\, e^{-\left[\frac{0.217}{\alpha}\tan h^{-1}\left(\frac{h-d_s}{R}\right)\right]}$

$$=K\left[9.81(0.025)(1.75)^3\right]e^{-\left[\frac{0.217}{0.08}\tan h^{-1}\left(\frac{2}{5.32}\right)\right]}$$

= 0.45 K Where, $K=\left\{1+K'\left(\frac{h-d_s}{R}+0.1\right)\sin\theta\right\}$

$$=\left\{1+1\left(\frac{2}{5.32}+0.1\right)0.3714\right\}$$

$$=1.1765$$

$$=0.53\frac{m^3}{s-m}$$

(b) $\alpha=0.08;\frac{h-d_s}{R_{H_s}}=\frac{2}{5.32}=0.376$

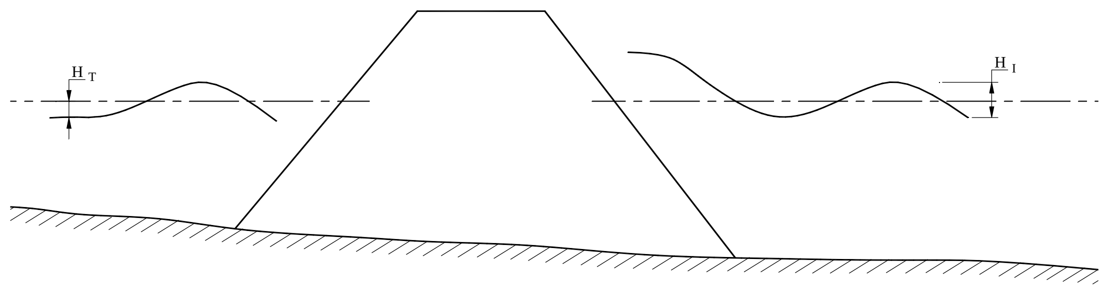
⟹ From Figure; $Q_{0.05\%}=1.5Q=1.5(0.53)$

$$=0.795\frac{m^3}{s-m}$$

(c) $\Longrightarrow$ From Figure; $\overline{Q} = 0.45Q = 0.45(0.53)$

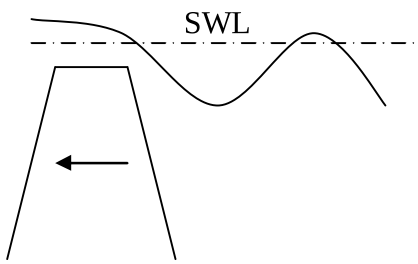
$$= 0.239 \frac{m^3}{s-m}$$

12.4 Transmission of waves

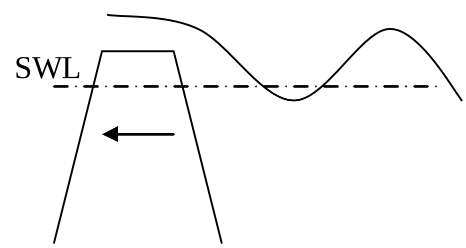


Transmission of waves is usually studied with respect to a structure submerged below SWL.

Sub-merged below water



Sub-aerial below water

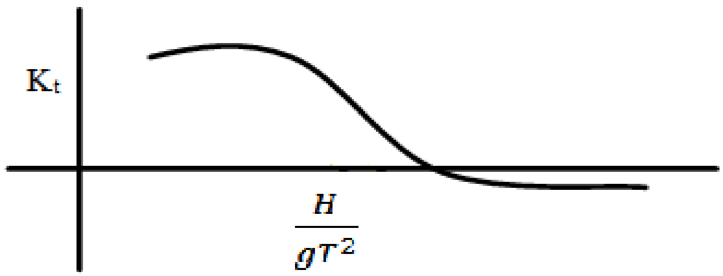


Transmission due to

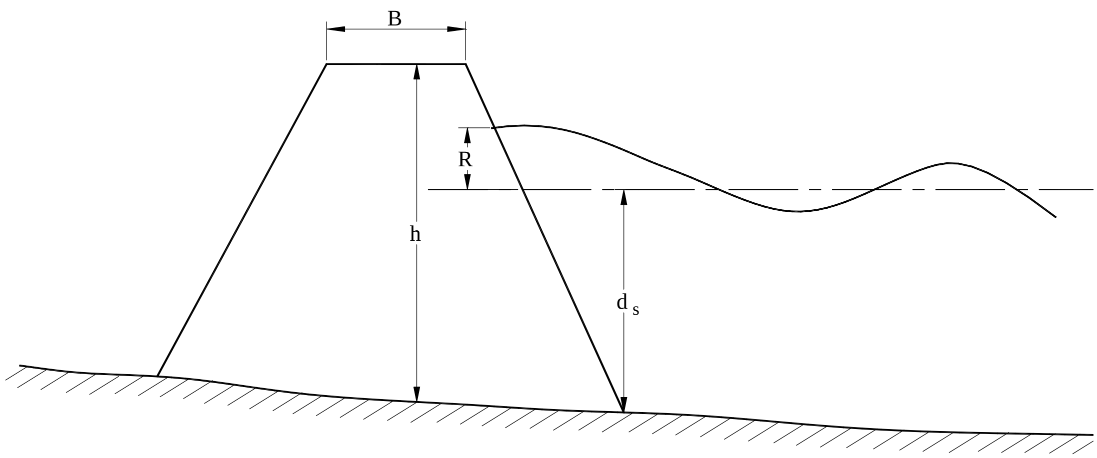
- Permeability of below water
- Overtopping by high run-ups

Submerged BW

Transmission Coefficient, $K_T = \frac{H_T}{H_I}$. It is high, more than 0.4 in this case. It is less if incident waves are high (as they may break). Empirical curves re available to know the transmission coefficient.



Subaerial BW
Hhere larger wave height means higher transmission coefficient.



Seeling (1980): has given the coefficient as: $K_T = C\left[1 - \frac{F}{R}\right]$

where 'C' = 0.51 - 0.11$\left(\frac{B}{h}\right)$; $if\ B \langle 3.2h$

'F' = free board = $h - d_s$

'R' = run-up for no overtopping

(K_T) irregular waves < (K_T) regular waves

since, it involves smaller waves. Therefore, less run-up. It requires less 'h' than regular wave concept.

K_T canbe obtained by empirical curves

$$K_T = f\left(\frac{B}{h}, \frac{F}{R_s}\right)$$

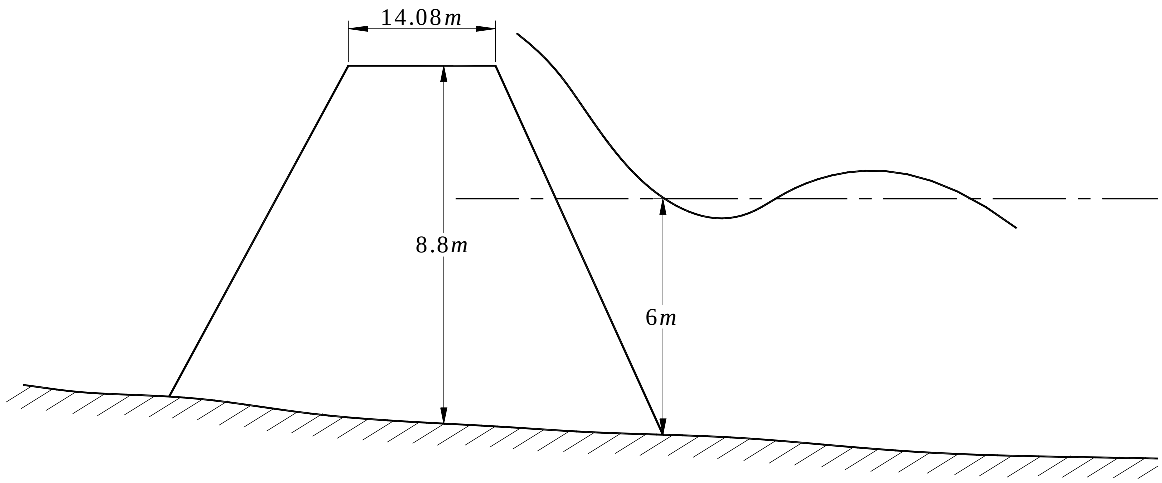
= where 'B' is (impermeable) BW's widt
= 'h' is height of BW
= 'F' = $(h - d_s)$

= ‘ R_s ’ is run-up corresponding to H_s .

Example

For the permeable BW with smooth slope, incident significant wave height is 5m with corresponding run-up of 4m:

(a) Obtain the transmitted significant wave height.



(b) Obtain the % of time the transmission by overtopping occurs.

(c) Obtain the transmitted wave height for 1% exceedence.

Solution:

(a) $\dfrac{F}{R_s} = \dfrac{8.8-6}{4} = 0.7;$

$\dfrac{B}{h} = \dfrac{14.08}{8.8} = 1.6$

from figure: $K_T = \dfrac{(H_s)_T}{(H_s)_I} = 0.07 \quad \Longrightarrow \quad (H_s)_T = 0.35m$

(b) from figure: $(H_T)_P \rangle 0 \quad \Longrightarrow$ P = 38%

from figure: $C_F = 0.65 \; for \dfrac{B}{h} = 1.6$

$\Longrightarrow$ 38 (0.65) = 24.7% of time

(c) from figure $\Longrightarrow$ for 1% P:

$(H_T)_P = 0.4 C_F (H_s)_I$

= 0.4 (0.65) 5

= 1.3m

CHAPTER 13

PIPELINE HYDRODYNAMICS

13.1 Introduction

The stability requires that the pipeline should not move from its installed position. The movement may be caused by forces, which could be natural or artificial, like environmental forces and man made changes in the environment.

The pipeline can be fixed at its position by, Anchors, Gravel sand bags / concrete bags, providing higher conc. coating or wall thickness. If the pipeline is subjected to high wave, current loads, lengthwise soil sliding, earthquakes and faults and if it is hazard to navigation then it is better that it is buried below the sea bed.

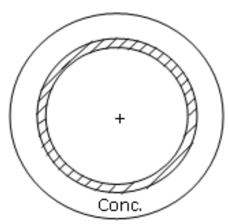


Fig.13.1 Pipe cross-section

.2 Hydrodynamic loading

One of the common exercises in pipeline design involves determination of minimum pipe weight for stability.

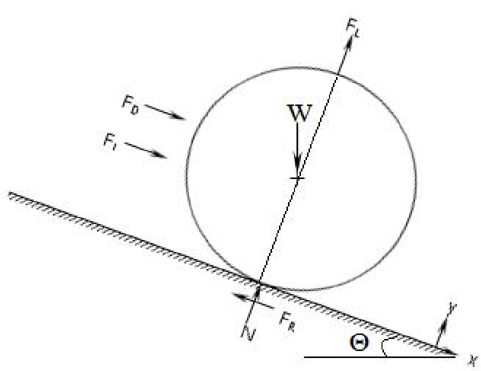


Fig. 13.2 Forces over a pipe cross section

$$\sum F_x = 0 \quad \Rightarrow F_D + F_I - F_r + W \sin\theta = 0 \qquad (12.1)$$

$$\sum F_y = 0 \quad \Rightarrow N + F_L - W \cos\theta = 0 \qquad (12.2)$$

Since $F_r = \mu N$, (μ = 0.3 for clay and 0.7 for gravel)

from (12.2) $\frac{F_r}{\mu} + F_L = W \cos\theta$

$\therefore \quad F_r + \mu F_L = \mu W \cos\theta$

from (12.1) $F_D + F_I + \mu F_L - \mu W \cos\theta + W \sin\theta = 0$

$\therefore F_D + F_I + \mu F_L = W(-\sin\theta + \mu \cos\theta)$

$$\therefore W = \frac{F_D + F_I + \mu F_L}{-\sin\theta + \mu\cos\theta} \qquad (12.3)$$

IF θ = 0; MINIMUM SUBMERGED WEIGHT OF PIPE FOR STABILITY,

$$W = \frac{F_D + F_I + \mu F_L}{\mu}$$

Or $$\boxed{W = F_L + \frac{1}{\mu}(F_D + F_I)} \qquad (12.4)$$

<u>note</u>: $F_I = 0$, if F_D is maximum.

Use of 'effective Velocity', u_e:

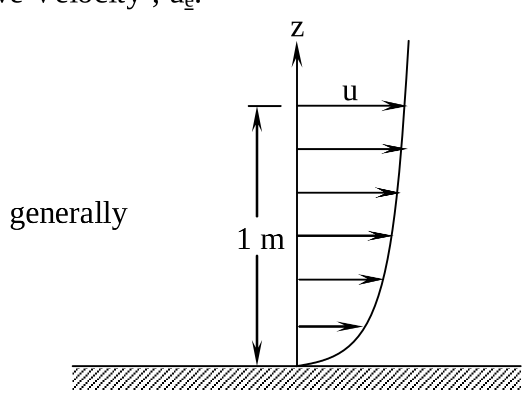


Fig.13.3 Variation of horizontal velocity

In actual,

$$F_D = \int_{z=0}^{Pipeheight} \frac{1}{2} C_D\, \varsigma\, D\, u^2(z)\, dz$$

In practice, $\boxed{F_D = \frac{1}{2} C_D\, \varsigma\;D\, u_e^2}$ (12.5)

To obtain <u>u_e</u>:

Within the boundary layer $\frac{u(z)}{u(1)} = \left(\frac{z}{1}\right)^{\frac{1}{7}}$

(Assuming 1 m boundary layer)

Approximately, $u_e^2 = \frac{1}{D} \int_{z=0}^{D} u^2(z)\, dz$

(over pipe height)

Substituting u(z) from (1),

$$\boxed{u_e^2 = 0.778\, u^2(1)\;\left(\frac{D}{1}\right)^{0.286}} \qquad (12.6)$$

Where u(1) = maximum water particle velocity given by wave theory at 1 m above sea bed

Similarly,

$$F_I = C_M\;\varsigma\;\frac{\pi D^2}{4}\,\dot{u}_e$$

Where $\dot{u}_e$ = horizontal particle acceleration at 1 m above sea bed

<u>AND ,</u>

$$\boxed{F_L = \frac{1}{2}\,\varsigma\;C_2\, u_e^2}$$

(12.7)

Recommendation to obtain C_D and C_M from given R_e values:

R_e	C_D	C_L	C_M
< 50,000	1.3	1.5	2.0
50,000 – 100,000	1.2	1.0	2.0
100,000 – 250,000	$1.53 - \frac{R_e}{300,000}$	$1.2 - \frac{R_e}{500,000}$	2.0
250,000 – 500,000	0.7	0.7	$2.5 - \frac{R_e}{500,000}$
500,000 >	0.7	0.7	1.5

Example:

Deep water Hs = 3.315 m $\qquad \mu = 0.5$

T_s = 9.8 sec $\qquad \gamma = 10.06 \ KN/m^3$

d = 30.49 m $\qquad \nu = 0.33 \times 10^{-5} \ m^2/\text{sec}$

D = 1 m

$\theta = 0$ $\qquad$ Determine submerged wt of the pipe

Assume no refraction

Sol^n: $L_o = gT^2/(2\pi)$

$= 1.5613\ (9.8) \qquad = 149.95$ m

$d/L_o = 30.49/149.95 \quad = 0.2033 \Rightarrow d/L = 0.225$

$$\frac{H}{H_o} = 0.92$$

$L = 30.49/0.225 = 135.51$ m

H = 3.315 (0.92) = 3.05 m

$$u = \frac{H_g T}{2L} \frac{\cos h\ k(d+z)}{\cos h\ kd} \ . \ \cos(-\omega t)$$

@ 1 m above sea bed

$$u_{max} = \frac{3.05\,(9.81)\ (9.8)}{2\ (135.51)} \quad \frac{\cos h \frac{2\pi}{135.51}(1)}{\cos h \frac{2\pi}{135.51} 30.49}$$

$= 0.4975$ m/s

$\therefore\ u_e^2 = 0.778\ u^2(1)\ (D/1)^{0.286}$

$u_e = \left[0.778(0.4975)^2\right]^{\frac{1}{2}} = 0.4388$ m/s

$R_e = u_e\ {D}/{\vartheta}$

$= 0.4388\ (1) / 0.33 \times 10^{-5} = 133000$

$\therefore\ C_M = 2\ ;\ C_D = 1.53 - [133000 / 300000] = 1.09$

$C_L = 1.2 - [133000 / 500000] = 0.934$

$$F_D = \frac{1}{2}\ C_D\ \varsigma\ D\ u_e^2$$

$$= \frac{1}{2}\ (1.09)\ \frac{10.06}{9}\ (1)\ 0.4388^2 = 0.1076\ {kN}/{m}$$

$$F_L = \frac{1}{2}\ C_L\ \varsigma\ D\ u_e^2$$

$$= \frac{1}{2}\ 0.934\ \left(\frac{10.06}{9}\right)\ 1\ (0.4388)^2 = 0.0922\ {kN}/{m}$$

$$W = F_L + \frac{1}{\mu}\ (F_D + F_I)$$

$$= 0.0922 + \frac{1}{0.5}\ (0.1076) = 0.3074\ {kN}/{m}$$

$$= 31.34\ {kg}/{m}$$

Safety against vortex induced vibrations

Example:

Given, pipe diameter, ϕ = 0.324 m

wall thickness, t = 0.0127 m

pipe span, L = 30.488 m (s.s.)

flow velocity V = 0.61 m / sec

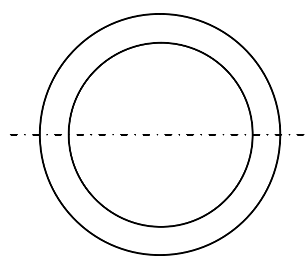
$\varsigma_{seawater} = 1030\ {kg}/{m^3}$ E = $2.09 \times 10^{11}\ {N}/{m^2}$; $\varsigma_{steel} = 7830\ \frac{kg}{m^3}$

Will there be vortex induced vibrations?

Sol^n : For cross vibrations, $f_e < 0.7\ f_n$

$$f_e = \frac{SV}{D}$$

$$= 0.2\ (0.61) / 0.324 = \underline{0.38}\ \text{Hz}$$



$$f_n = \frac{C}{L^2}\sqrt{\frac{EI}{M}}$$

C = 1.57

$$I = \frac{\pi}{64}\ \phi^4_{outer} - \frac{\pi}{64}\ \phi^4_{inner}$$

$$= \frac{\pi}{64}\ 0.324^4 - \frac{\pi}{64}\ 0.2986^4 = 0.00015048\ \text{m}^4$$

E I = $2.09 \times 11^{11} \times 0.00015048 = 3.14 \times 10^7$ Nm^2

M = actual mass in air +added mass

$$= \varsigma_{steel\frac{\pi}{4}}\ [\ \phi^2_{out} - \phi^2_{in}]\ 1 + 1\ (\text{displaced assume mass})$$

$$= (1830)\ \frac{\pi}{4}\ (0.324^2 - 0.2984^2) + 1030\ \frac{\pi}{4}\ (0.324)^2$$

$$= 97.985 + 84.921 = 182.91 \text{ kg/m}$$

$$\therefore f_n = \frac{1.57}{30.488^2}\ = \sqrt{\frac{3.14\ x\ 10^7}{182.91}}\ = 0.7$$

SINCE $f_e = 0.38$ Hz $< 0.7\ f_n$
$< 0.7\ (0.7)$
< 0.49 Hz

PIPE IS SAFE AGAINST VORTEX EXCITATIONS

CHAPTER 14

STATICS OF FLOATING BODIES

14.1 Introduction

One of the most common types of freely floating structures is a ship. For coastal and ocean construction a variety of ships and small barges are used as construction equipments and it then becomes necessary for an ocean engineer to understand basic design and analysis concepts of ships to know their capabilities and performance characteristics. Fig. 14.1 and 14.2 illustrate the various terms associated with the ship structure.

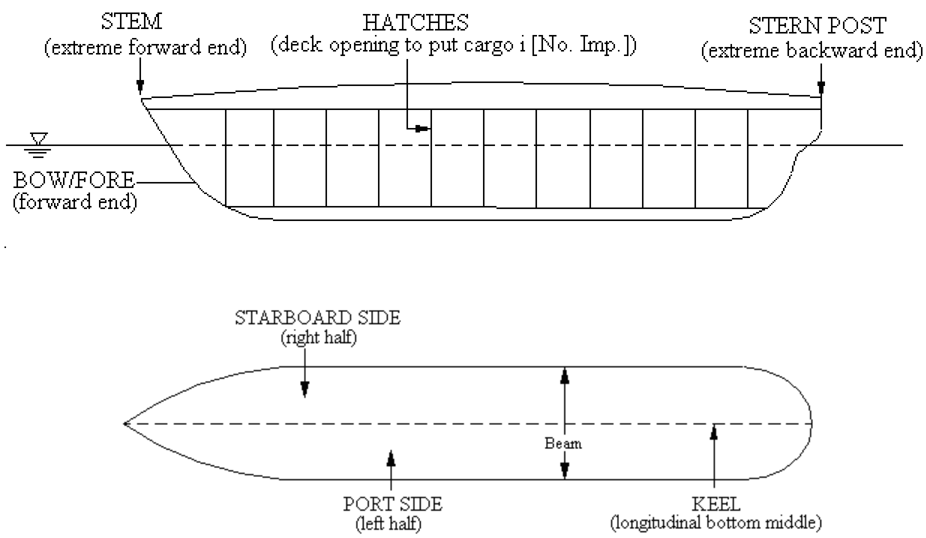


Fig. 14.1 Basic terminology of a ship structure (Elevation and plan)

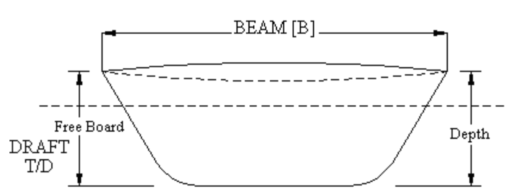


Fig. 14.2 Basic terminology of a ship structure (lateral section)

Displacement: It is the amount (in weight or volume) of water displaced by a ship.

Dead Weight: The weight that can be carried by a ship is called as dead weight.

Tonnage: **Tonnage is the capacity in volumes, its various terms are:**

1. GRT (Gross Registered Tonnage): **It's the total internal capacity or the total available space**

Unit: $100\text{ft}^3 \equiv 2.83\text{m}^3$

2. NRT (Net Registered Tonnage): **It's the commercial carrying capacity by volume**

NRT = GRT – (Volume of engine room, machinery, fuel, provisions…)

3. DWT (Dead Weight Tonnage): **It's known as the carrying capacity <u>by</u> <u>weight</u>**

Unit: 1 ton = 1016 kg.

DWT = (Displacement when loaded to full load line – Displacement when light without cargo, passengers, fuel…)

4. DT (Displacement Tonnage): **It's the actual ship weight. There are two divisions of DT.**

(a) Loaded DT: Ship weight with cargo, fuel etc…)

(b) Light DT: Ship weight without cargo, fuel etc…)

A typical empirical relationship between the draft and the DWT is shown in Fig. 14.3.

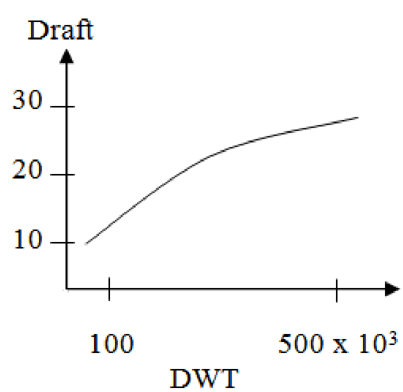


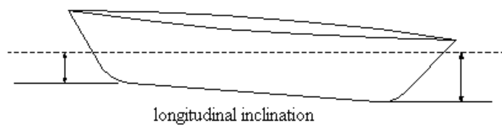
Fig. 14.3 Variation of draft with DWT

Typically a ship with a DWT of 17850, the vessel length = 150m, breadth = 21 m, depth = 10 m and its peed could be up to 14 knots.

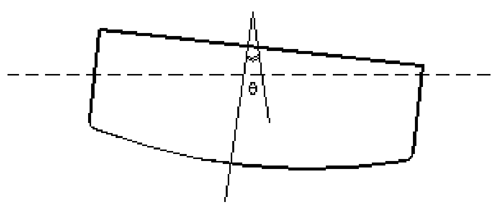
Rake : **Inclination, Knuckle**

Knuckle : **Sudden change of curvature of a ship in water under different wave heights.**

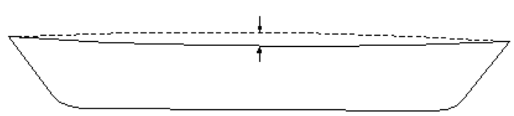
Trim : **Fore draft - Aft draft**



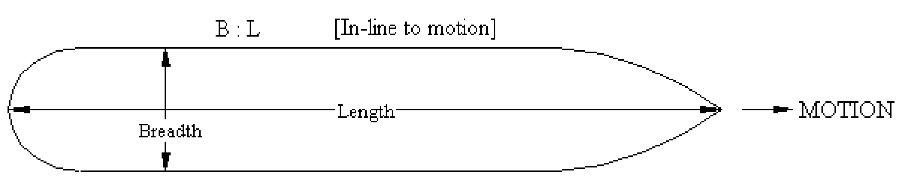
Heeling : Transverse inclination of a ship on the surface water level is heeling.



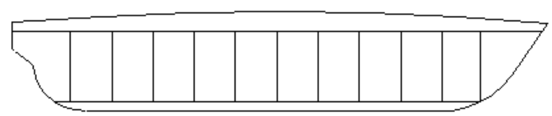
Sheer : Longitudinal (uppermost) curvature of the deck of a ship.



Aspect Ratio : Normal to In-line (to motion, in order) dimension.

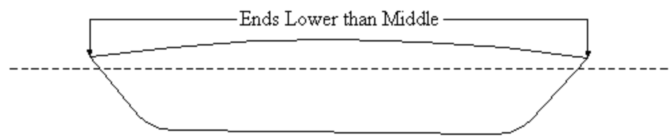


Bulkheads : Sub-division (compartments:-normal or watertight) of ship structures under various heads. Longitudinal, transverse-Uppermost & continuous Bulkhead Deck.



Margin Line : A level 3″ below Bulkhead deck.

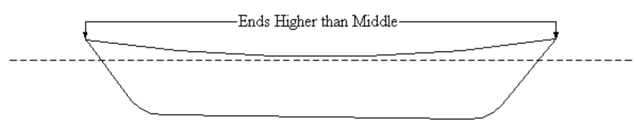
Hogging : When ship's both ends are lower than its middle portion in the sea water level, hogging takes place.



Sagging : When ship's both ends are higher than its middle portion in the sea water level, sagging takes place.

Boom : Pivoted spar (pole)

Water plane Coefficient : $C_W = \dfrac{\text{Water plane area}}{\text{length of ship} \times \text{breadth of ship}}$



Block coefficient : $C_B = \dfrac{\text{Volume displaced}}{\text{length} \times \text{breadth} \times \text{draft}}$

Prismatic coefficient : $C_P = \dfrac{\text{Displaced volume}}{\text{cross sectional} \times \text{lenght area amidship}}$

14.2 Stability of a Floating Body

It indicates the ability of a ship to return to its original position. It consists of 2 types.

1. Statical : When disturbance to a ship is steady in the water.

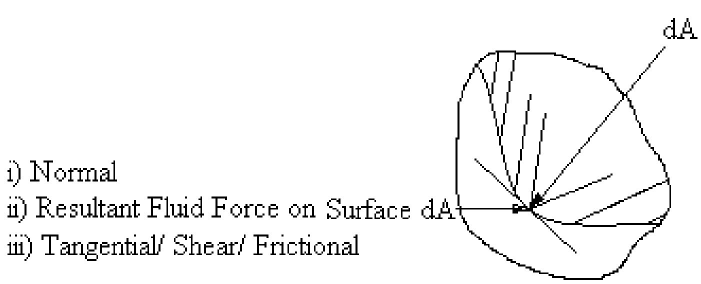
2. Dynamical : When disturbance is time varying or F(t).

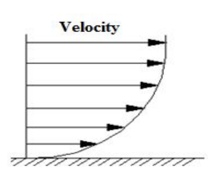
FLOATING EQUILIBRIUM

STABLE	UNSTABLE	NEUTRAL
Restoring forces are set up (after disturbance) bringing the body to equilibrium.	Overturning forces are set up.	No reactive forces. Only new position. (In horizontal displacement)

Consider fluid at rest:

The force acting on a surface can be locally divided into a normal and a tangential (shear or frictional) force as shown below:





The frictional force creates a velocity difference in between different layers of a moving fluid as shown in the above figure. The resulting frictional or shear stress is given by:

$$\text{Shear stress } = \tau = \mu \frac{\partial V}{\partial Y} \tag{14.1}$$

where, ∂ V = change in velocity and $\partial Y \approx 0$ (change in the vertical distance) as the fluid is at rest. Hence the only force acting is normal. Consider a freely floating or a submerged body (Fig. 14.1)

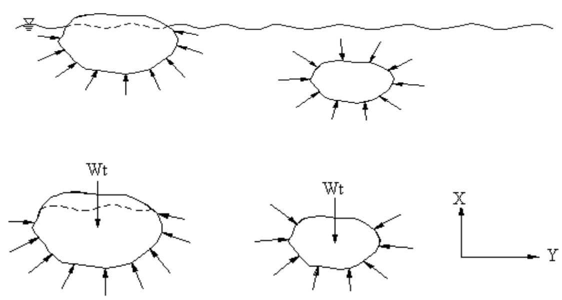


Fig. 14.4 Free body diagrams of a freely floating or a submerged body.

If the body is in equilibrium, $\Sigma F_x = 0 = \Sigma F_y$

$\Sigma F_x = 0$ indicates that there is no net horizontal fluid force and, $\sum F = 0$ indicates that the net vertical fluid force (Buoyant Force) = Body weight

It is possible to show that the body weight = weight of fluid displaced by body, or, net vertical fluid force = weight of water of submerged volume of body $\left(= \zeta_{\text{Fluid}} \cdot V\right)$

Principle of Floatation (or Buoyancy) : weight = displacement

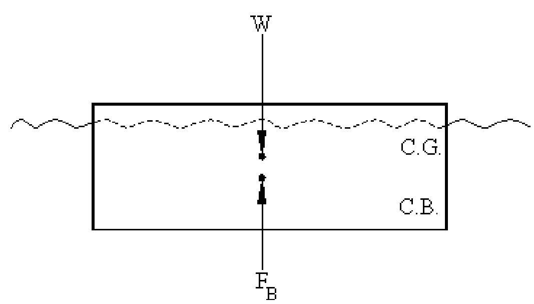


Fig. 14.5. Basic stability

Centre of gravity, CG is a function of the mass distribution while the centre of buoyancy, CB depends on the shape of the displaced fluid and it represents the centre of the mass of the displaced fluid. For basic stability the CG and the CB should lay on the same vertical.

For stability of the submerged objects the relative position of the CG and the CB matters. The CB should be above CB (Fig. 14.5) otherwise as shown in Fig. 14.6 overturning moments may be set up.

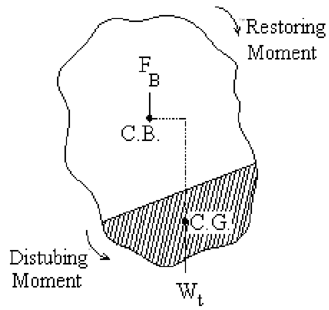


Fig. 14.5. Submerged instability

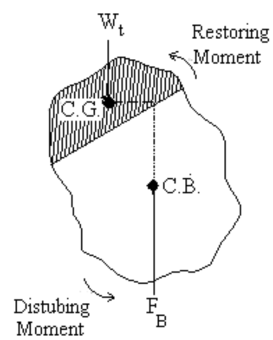


Fig. 14.6. Submerged stability

As against the above mentioned submerged stability, in case of the floating stability it is the relative position of the metacentre, M, that matters. As shown in Fig. 14.7 for initial transverse stability M should be above the CH or 'G' so that the distance GM would be positive. The location of 'M' changes with θ but for small values of it 'M' gets positioned and distance GM gets minimized. If 'M' goes below 'G', the overturning moments are set up which create further instability (Fig. 14.8).

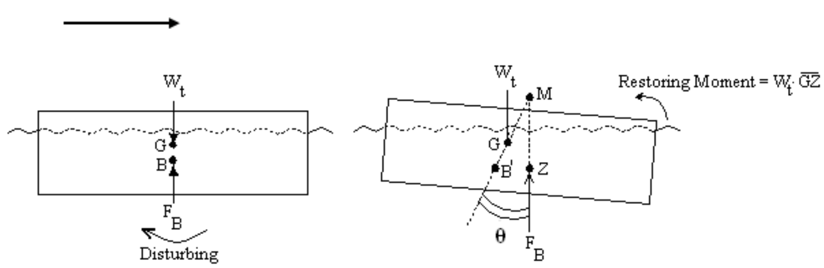


Fig 14.7 Initial Transverse Stability - I

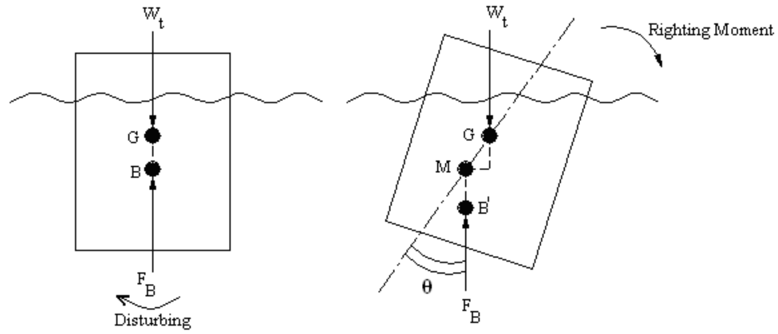


Fig. 14.8 Initial Transverse instability - II

To obtain GM : (at design stage)

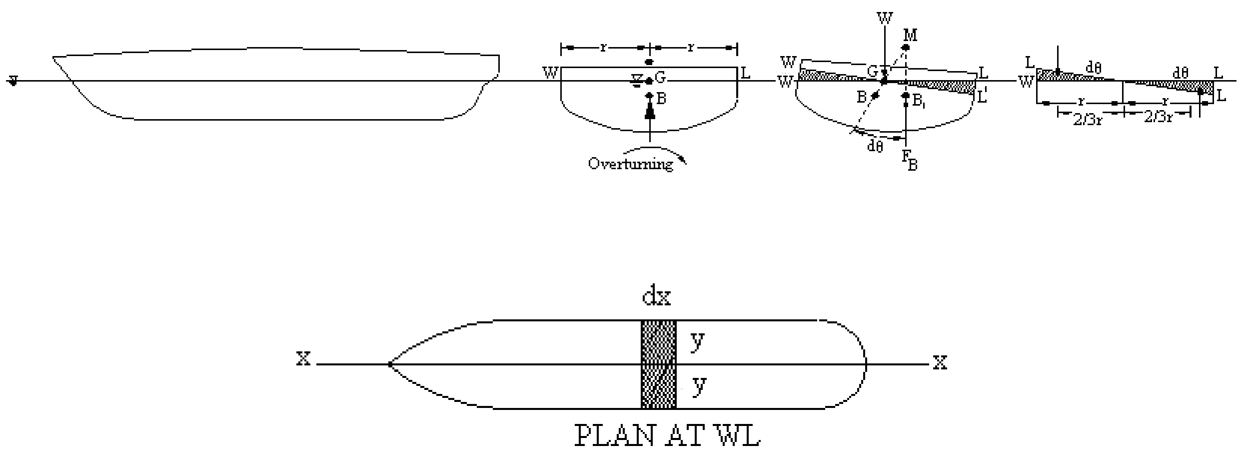


Fig. 14.9 Various sections of a ship structure

Let the ship heel (about xx) by small 'dθ'. When this happens the original water line WL becomes the new water line $W^{'}L^{'}$ and two wedges of the same volume dv get formed. Taking moments (for small θ),

$$F_B\left(\overline{BB_1}\right)=\int 2\left[\zeta(dV)\frac{2}{3}y\right]$$

where, ζ is the fluid density,

$$\therefore \zeta V\left(\overline{BM}d\theta\right)=\int 2\left[\zeta\left(\frac{1}{2}y\ yd\theta\ dx\right)\frac{2}{3}\right]$$

where, V is the total volume of the displaced fluid.

$$\therefore V\overline{BM} = \int \frac{2}{3} y^3 dx$$

$$= \int dI_{xx}$$ (moment of inertia of the elemental plan area about XX)

$$= I_{xx}$$ (m. i. of the total water plane area)

$$\therefore \overline{BM} = \frac{I_{xx}}{V} \tag{14.2}$$

$$\overline{GM} = \frac{I_{xx}}{V} - \overline{BG} \tag{14.3}$$

Note: If CB or 'B' is above G $\Rightarrow$ $\overline{GM} = \frac{I_{xx}}{V} + \overline{BG}$ (14.4)

In order to find B we can draw cross section of the ship structure on a mm x mm graph paper and count squares equal on bothe left and the right side. Typically a cargo vessel has GM = 3.1 m, a semi-sub has GM = 5.5 m.

Example: Obtain the increase in the draft requirement of a ship of 12000 t displacement and $1200 m^2$ waterplane area when it enters fresh water form sea water where the water density was $1020\,Kg/m^3$.

Solution: weight of ship = displacement

$$= \zeta_{\text{fresh water}} \cdot V_{\text{displaced in sea water}}$$

$$= \zeta_{\text{fresh water}} \cdot V_{\text{displaced in fresh water}}$$

since, ζ_{fresh} small , $V_{\text{fresh water}}$ is larger

$\therefore$ draft in fresh water more

$$\therefore (\text{Vol})_{\text{fresh}} = (\text{Vol})_{\text{sea}} \cdot \frac{\zeta\,\text{sea}}{\zeta\,\text{fresh}}$$

$$= (\text{Vol})_{\text{sea}} \cdot \frac{1.02}{1.00}$$

For 1000 Kg of water $(1t) \Rightarrow$ Vol. displaced in fresh water $= 1m^3$

$$\Rightarrow \text{Vol. in sea water} = 1 \cdot \frac{(1.00)}{(1.02)} = 0.98m^3$$

$\therefore$ change in volume $= 0.02m^3$ for 1t.

$\therefore$ total change in volume $= 0.02(12000) = 240m^3$

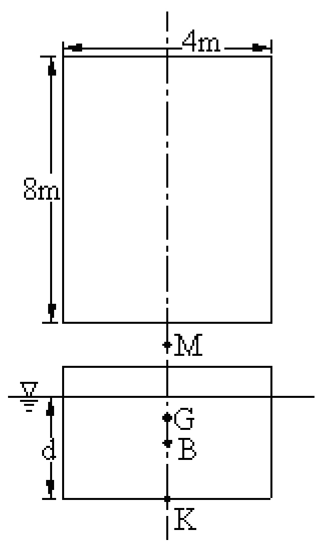
= A (change of draft)

= 1200 (ΔD)

$\therefore$ ΔD = 0.2*m* (Ans.)

Example: A barge of 40 t displacement has a rectangular waterplane area of 8*m* x 4*m*. Obtain the maximum permissible distance of its C.G. from its keel $\left(\zeta_{\text{water}} = 1020\, Kg/m^3\right)$.

Solution:



For stability 'M' should be above 'G' or should coincide with it.

i.e. $\overline{KM} = \overline{KG}$

$\therefore$ Obtain $\overline{KM}$:

KM = KB + BM   (KB is half of d)

<u>To obtain 'd':</u>   weight = displacement

40,000 = 1020 (V)

$$= 1020\ (8 \times 4 \times d)$$

$$\therefore \quad d = 1.225m$$

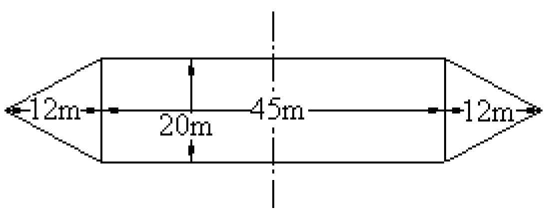
$$\therefore KB = \frac{d}{z} = 0.613m$$

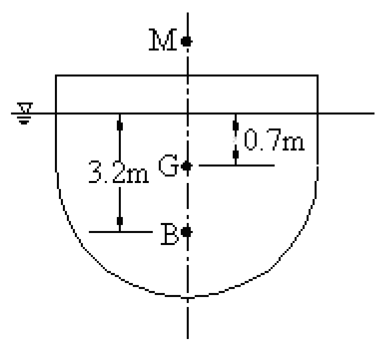
<u>To obtain ‘BM’:</u> $\quad BM = \frac{I}{V}$

$$= \frac{\frac{1}{12}(8)4^3}{8 \times 4 \times 1.225}$$

$$= 1.088m$$

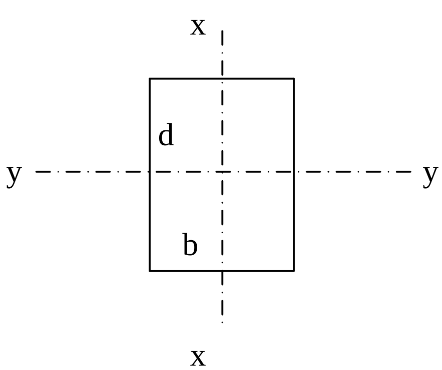
$\therefore$ KM = 0.613 + 1.088 = 1.701m



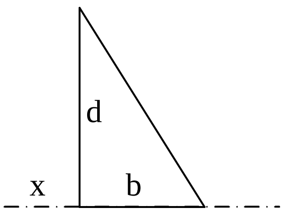


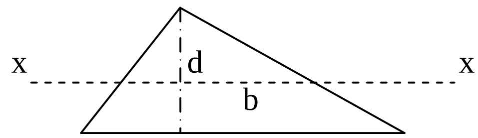
Example: Obtain $\overline{GM}$ for stable rolling & pitching for a ship of 3000 t displacement.

Solution:



$$I_{xx} = \frac{1}{12} db^3 \quad ; \quad I_{yy} = \frac{1}{12} bd^3$$





$I_{xx} = \frac{1}{12} db^3$; $I_{xx} = \frac{1}{36} bd^3$

GM = BM – BG

$$= \frac{I}{V} - 2.5$$

V: weight = displacement

3000 = 1.020V

$$\therefore V = 2941.2 m^3$$

Rolling: $I_{xx} = 4\left\{\frac{1}{12}(12)10^3\right\} + \frac{1}{12}(45)20^3$

$$= 34000 m^4$$

$$\therefore \frac{I_{xx}}{V} = \frac{34000}{2941.2}$$

= 11.56m = BM

$\therefore$ GM = 11.56 – 2.5

= 9.06m (Ans.)

Pitching: $I_{yy} = \frac{1}{12}(20)45^3 + 2\left\{\frac{1}{36}(20)(12)^3 + \frac{1}{2}(12)20\left[\frac{12}{3} + 22.5\right]^2\right\}$

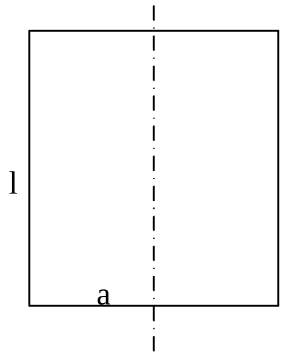
$$= 322335 m^4$$

$$\therefore \frac{I_{yy}}{V} = \frac{322335}{2941.2}$$

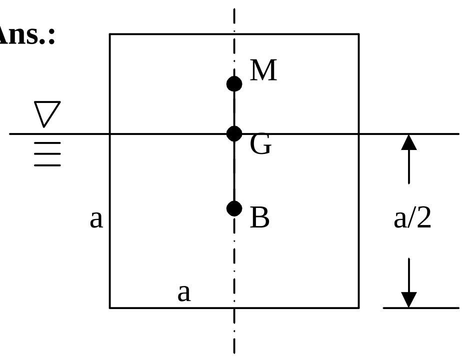
$$= 109.6m = \overline{BM}$$

$$\therefore \text{GM} = 109.6 - 2.5 = 107.1m \text{ (Ans.)}$$

Que.: How will a square c/s buoy with l > a float? (sp. gr. of buoy = 1/2)



Ans.:



For the above position: $\left(\text{weight} = \text{displacement} = \zeta_w \frac{1}{2} la^2 = \zeta_w V_d\right)$

$$BM = \frac{I}{V}$$

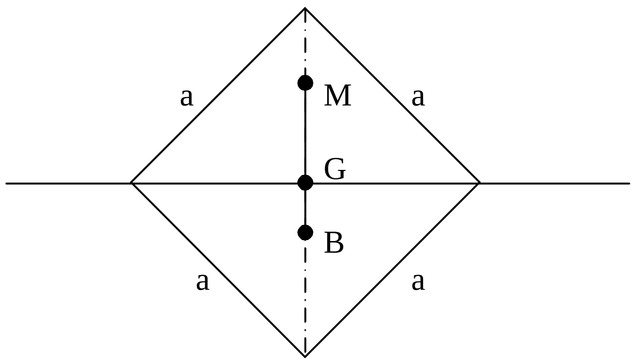
$$= \frac{\frac{1}{12} la^3}{\frac{1}{2} la^2}$$

$$= \frac{a}{6}$$

GM = BM – BG

$$= \frac{a}{6} - \frac{a}{4}$$

= Negative (unstable)



For the above position:

$$BM = \frac{I}{V}$$

$$= \frac{\frac{1}{12} l \left(a\sqrt{2}\right)^3}{\frac{1}{2} l a^2}$$

$$= \frac{\sqrt{2}}{3} a$$

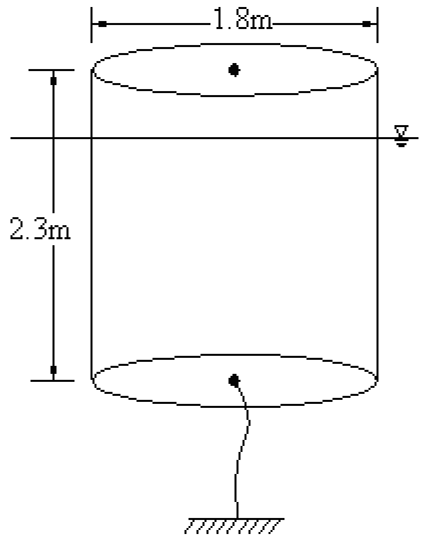
GM = BM – BG

$$= \frac{\sqrt{2}}{3} a - \frac{1}{3} \frac{a}{\sqrt{2}}$$

$$= \frac{a}{3} \left(\frac{1}{\sqrt{2}} \right)$$

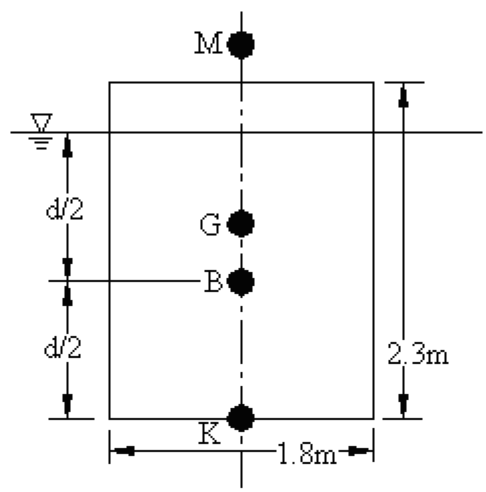
= Positive (stable)

Que.: State whether a floating cylindrical buoy of 1.8*m* diameter & 2.3*m* height and 1900 Kg displacement would be stable in sea?



If this buoy is to be held to the sea bed by a chain, what would be the minimum tension required to keep it stable?

Ans.:



GM = BM – BG

= BM – KG + KB where, $BM = \frac{I}{V}$; KG = 1.15m; KB (weight = displacement)

$$= \frac{\frac{\pi}{64}(1.8)^4}{\frac{1900}{1020}}$$

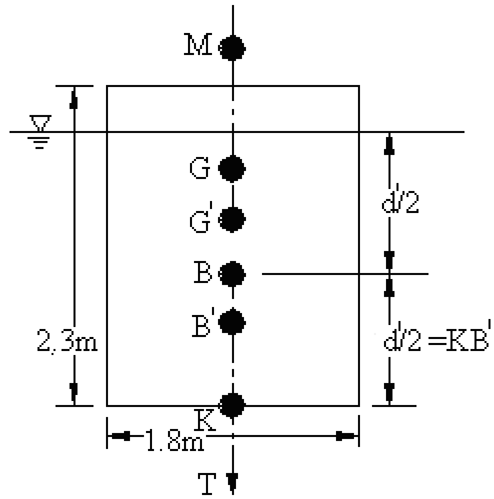
$$1900 = 1020\left(\frac{\pi}{4}18^2 \times d\right)$$

= 0.277m

$\therefore$ d = 0.732

$\therefore$GM = 0.277 – 1.15 + 0.366

= - 0.506 (unstable) (Ans.)



When ‘T’ is applied, d $\longrightarrow$ d′

B $\longrightarrow$ B′

G $\longrightarrow$ G′

For stability: B′M > B′G′

$$B'M = \frac{I}{V} = \frac{\frac{\pi}{64}(1.8)^4}{\frac{(1900+T)}{1020}}$$

$$= \frac{525.6}{(1900+T)}$$

Weight = Displacement

$$\therefore (1900+T) = 1020\left(\frac{\pi}{4}1.8^2 d'\right)$$

$$\therefore d' = \frac{(1900+T)}{2595.6}$$

B′G′ = KG′ - KB′ where, $KG' = \sum M_k = 0 \Rightarrow (1900+T)KG' = 1900(1.15)$

$$\therefore KG' = \frac{2185}{(1900+T)}$$

& where, $KB' = \dfrac{d}{2}$

$$\therefore B'G' = KG' - KB$$

$$= \frac{2185}{1900+T} - \frac{1900+T}{5195.6}$$

For stability: $B'M > B'G'$

$$= \frac{525.6}{1900+T} > \frac{2185}{1900+T} - \frac{1900+T}{5191.2}$$

$$\therefore \frac{1900+T}{5191.2} > \frac{2185}{1900+T} - \frac{525.6}{1900+T}$$

$$> \frac{1659.4}{1900+T}$$

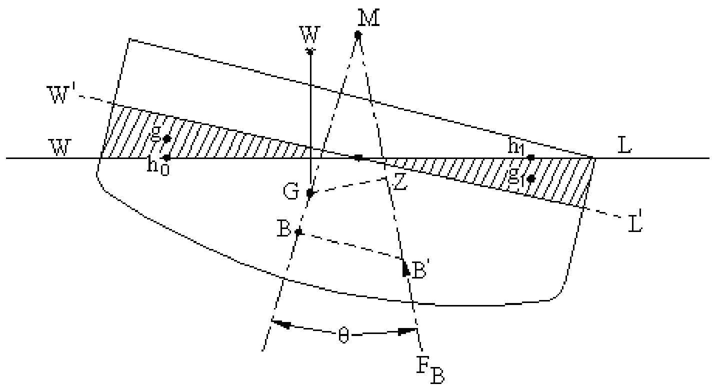
$$\therefore (1900+T)^2 > 5191.2(1659.4)$$

$$> 8614277.28$$

$\therefore$ 1900 + T > 2935

$\therefore$ T > 1035 Kg (Ans.)

14.3 Large angle displacements



For large ‘θ’ : ‘M’ not fixed, old & new water lines do not meet in the middle line.

Righting Lever, $\overline{GZ} \Rightarrow$ Atwood’s formula:

$$\overline{GZ} = \frac{v}{V}\overline{h_0 h_1} - \overline{BG}\sin\theta \tag{14.6}$$

v = volume of a wedge

V = total volume of displacement

$\overline{h_0 h_1}$ = horizontal projection of $\overline{g_0 g_1}$

g_0 = centroid of emerged wedge

g_1 = centroid of immersed wedge

Example: For a ship of 12000gN displacements it is noted that a 30° angle of heel produces the two wedges of displacement such that the volume of each wedge is 10% of the total displaced volume of the ship and that the line joining the centers of these wedges makes a projection of 15*m* on the horizontal.

Obtain the righting moment. (The C.G. is 0.5*m* vertically above the C.B.)

Solution: $\overline{GZ} = \frac{v}{V}\overline{h_0 h_1} - \overline{BG}\sin\theta$

$= 0.1\ (15) - 0.5 \sin 30°$

$= 1.5 - 0.25$

$= 1.25m$

moment = 12000g (1.25)

$$=147150N^{m}$$

14.4 Vertical shifting of weight:

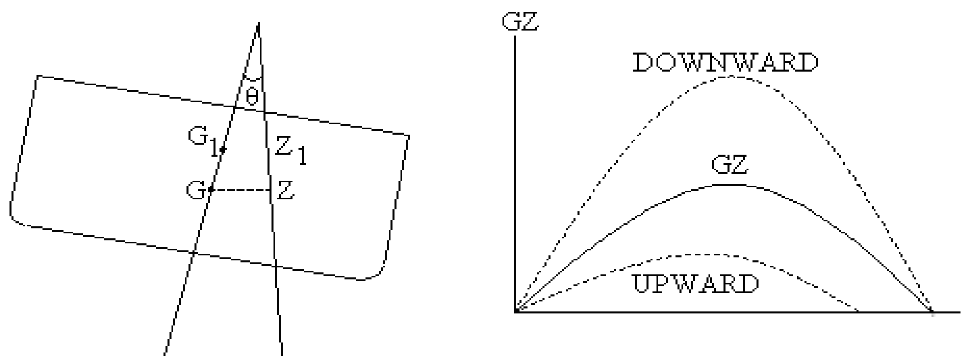


Fig. 14.8 Vertical shifting of weights

If weight ‘w’ on the ship
is moved up by ‘z’ CG goes up by $\frac{w}{W}z$

$$G_1Z_1 = GZ - GG_1 \sin\theta \tag{14.7}$$

(GG_1 +ve if ‘w’ upward)

14.5 Horizontal shifting of weight:

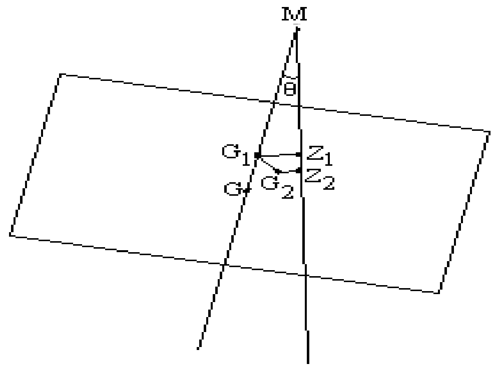


Fig. 14.9. Horizontal shifting of weight

Further to above, if ‘w’ moved horizontally, by ‘y’,

$$G_1G_2 = \frac{w}{W} y$$

$$G_2Z_2 = G_1Z_1 - G_1G_2 \cos\theta$$

$$= GZ - GG_1 \sin\theta - G_1G_2 \cos\theta \tag{14.9}$$

Static Stability Curve

One diagram for one ‘G’ & one ‘V’ value

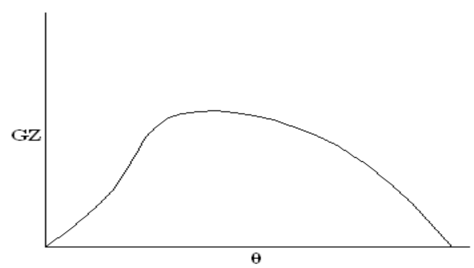


Fig. 14.10. Static stability curve

The shape of the above shown curve of GM versus displacement θ = ϕ (form of ship)

$$GZ = a + b\theta + c\theta^2 + d\theta^3 + ...$$

To obtain GM while afloat:

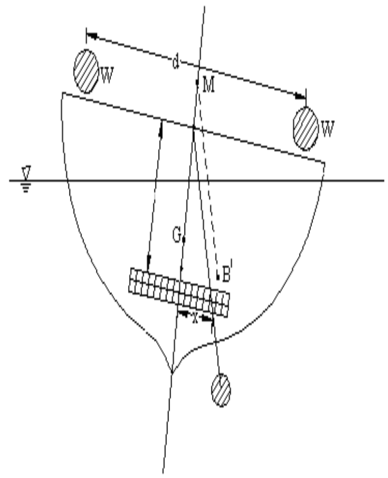


Fig. 14.11. Determination of GM

Roll known weight **'w'** across deck by known **'d'** ship will heel by θ

Measure it by plumb-bob hanging inside ship (Note: **l** ; **x**)

DISTURBING MOMENT = RESTORING MOMENT
(for rolling weight) (moment of weight at metacentre)

$$\therefore w \cdot d = W \cdot \overline{GM} \sin\theta$$

(where 'W' indicates ship weight)

$$= W \cdot \overline{GM} \tan\theta$$

$$= W \cdot \overline{GM} \cdot \frac{x}{l}$$

$$\therefore \overline{GM} = \frac{w}{W} \cdot \frac{d}{x} \cdot l \qquad (14.12)$$

14.6 Loss in GM due to partially flooded compartments

(Free Surface Effect)

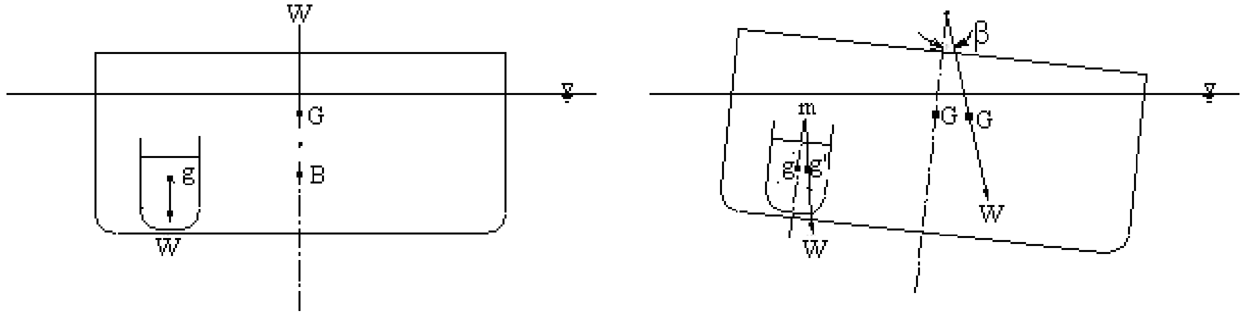


Fig. 14.12 Loss in GM due to partially flooded compartments

Many times ships contain ballast water or any other liquid stored in partially flooded compartments. If the tank is full it undergoes rigid body healing. Otherwise the CG ‘g’ of the liquid mass, the same ‘G’ of the ship as well as the centre of buoyancy B change to new positions of g' G' and B' respectively. The inside water piles up towards ‘G’ and the combined C.G. moves towards direction of C.B. shift. The lever arm between the CG and the CB reduces (Fig. 14.13) and this gives rise to the loss in the stability.

$$\Delta\overline{GM} = \frac{1}{\varsigma V}\sum_{i=1}^{n}\varsigma_i I_i \qquad (14.13)$$

where, n = no. of tanks

I = m.i. of inside water surface about an axis through centroid of free surface, parallel to axis of rotation of ship.

Loss in GM due to freely suspended weights

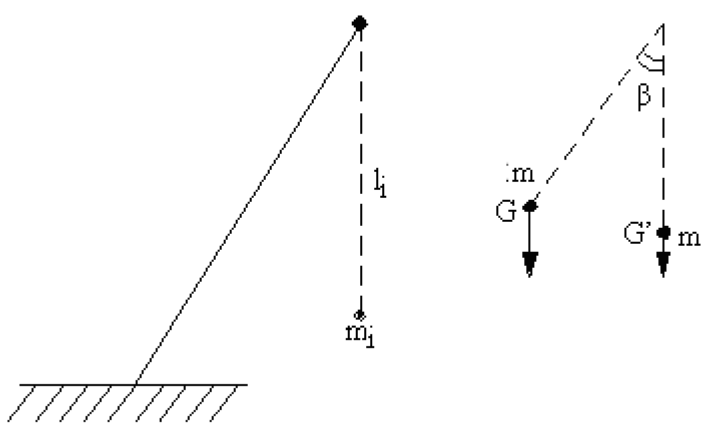


Fig. 14.13. Loss in Gm due to suspended weights

When suspended weights are present as the ship heels the CG shifts towards the direction of the shift in CB (Fig. 14.13), which gives rise to loss in GM and hence the stability. This loss is given by:

$$\Delta \overline{GM} = \frac{1}{\varsigma V} \sum_{i=1}^{n} m_i l_i \tag{14.15}$$

14.7 Flooded stability

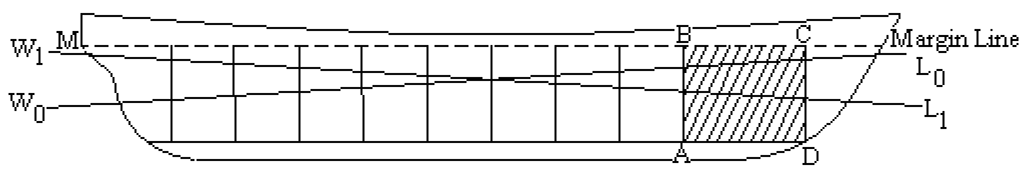
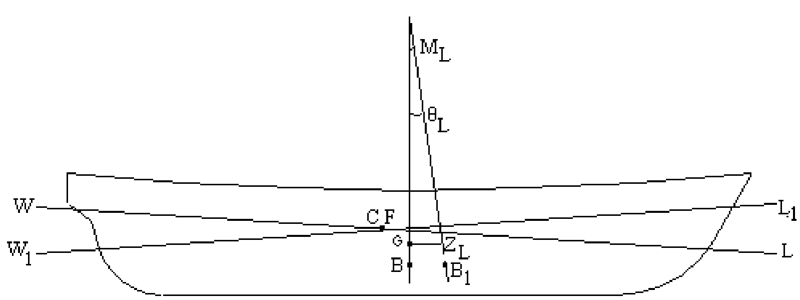


Fig. 14.16. Longitudinal stability

Consider Fig. 14.16 dealing with the longitudinal stability. The original waterline W_oL_0 changes to W_1L_1. The spacing in between the transverse bulkheads like ABCD is so decided that flooding in at least one compartment does not sink the ship. The maximum length of a compartment that can be flooded without submerging margin line in the water is called floodable length.



where, WL - Original water line
W_1L_1 - New water line
C.F. - Centre of flotation

Fig. 14.17 Longitudinal stability

The transverse and the longitudinal rotations assumed independent, which is true when ship is symmetrical about both axes.

$$\theta_L \sim \text{very small}$$

<u>Righting lever:</u> $\overline{GZ}_L = \overline{GM}_L \sin\theta$

Righting moment: $= W\,\overline{GM}_L \sin\theta$

Approximately $\overline{BM}_L \approx \overline{GM}_L$ where, $\overline{BM}_L = \dfrac{I_L}{V}$; I_L to be calculated at C.F.

Wind heel stability criteria

Wind attacking from the side of a ship creates overturning ot heeling moment at say the longitudinal centerline. The U.S. Coast Guard & American Bureau of Shipping Rule is as follows:

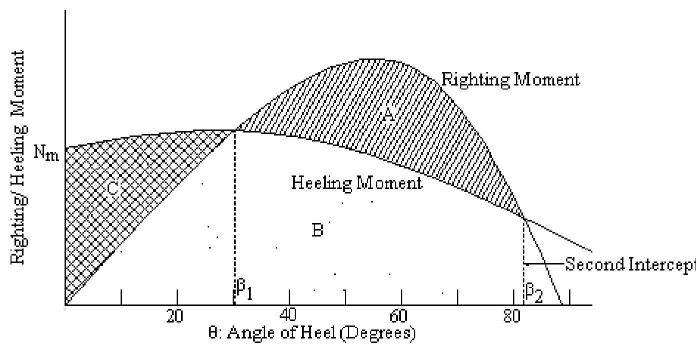


Fig. 14.17 Wind heel stability criteria

The righting moment above is the Static Stability Curve (One diagram for each 'G' & volume 'V')

$$= a + b\theta + c\theta^2 + \ldots$$

Area under the curve up to 'θ' = Work done by the moment up to that 'θ'
= Dynamical stability
(It controls heels due to sudden impulses)

For Stability, work done by righting moment > upsetting wind energy

$\therefore$ (A + B) > 1.4 (B + C) (Fig. 14.17)

Note:

(i) If any part of vessel develops allowable stress before 'θ' of second intercept, (or it catches water) areas A, B, C are to be calculated up to that θ (called Angle of downflooding).

(ii) The situation shown in Fig. 14.18 say for a leg raised jack up is unacceptable.

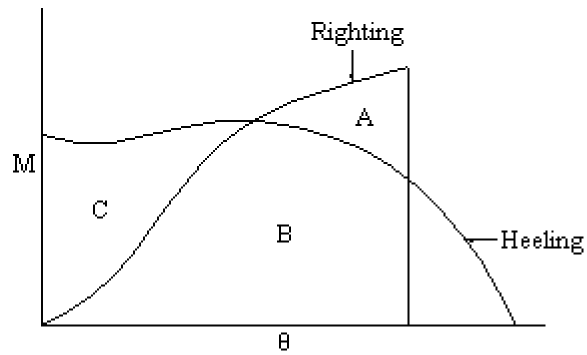


Fig.14.18 Areas A, B and C for a jack up with legs raised

(iii)

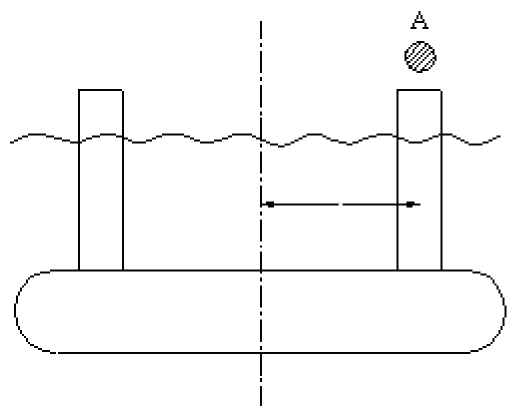


Fig.14.19 A schematic of a semi submarine

As per Fig. 14.19 for a semi sub, jack-up, the further located columns are the better it is since $M.I. \propto \sum Ar^2$ (A= cross sectional area and r = the radius of the leg)

(iv) As shown in Fig. 14.20 during the lowering of a weight module or a structure launched on the top of a launch barge when the deck goes below the water line there is a danger that the moment of inertia becomes very small and hence the stability may become very critical.

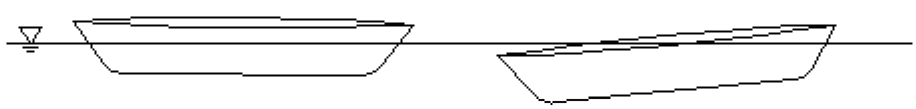


Fig. 14.20 Launch barge action.

(iii) For jacket-in-transit , the M I becomes very important (as the C.G. is high)

(iv) For gravity structures, semi-sub. an increase in draft indicates significant reduction in water plane.

14.8 The Pressure Integration Technique

Submerged V, A_w yields the stability information $\left(\text{like } \overline{GM},...\overline{GZ},...\right)$ for small θ, fixed form

If θ large form arbitrary the Pressure integration technique is used.

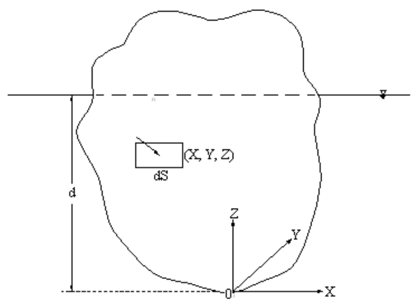


Fig. 14.21. Definition sketch

The distribution of 'p' around body surface obtained, integrated over surface, done by converting Surface Integrals to Volume Integrals through Divergence Theorem is the pressure integration technique.

[If P, Q, R $\Rightarrow$ functions of x, y, z in volume V bounded by surface S]
(single valued, continuous first derivatives)

$$\int_V \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV = \int_S PdS_x + QdS_y + RdS_z \qquad (14.18)$$

where, $PdS_x + QdS_y + RdS_z$ are outward projections of dS.

OR

$$\int_V \nabla\cdot\overline{f}\,dV = -\int_S \overline{f}\cdot\hat{n}\,dS \qquad ; \qquad \text{(- sign.) if } \hat{n} \text{ into the surface}$$

(14.19)

where, ∇ is the gradient operator & $\overline{f}$ is the vector field

At any point, $\overline{X}$ (x, y, z) p = $\varsigma g(d-z)$

Normal hydrostatic $\overline{dF} = \varsigma g(d-z)\hat{n}dS$

where, $\hat{n}$ = normal vector into the body $= f(x,y,z)$

Total hydrostatic $\overline{F} = \int_S \varsigma g(d-z)\hat{n}dS$ (14.20)

Moment $\overline{dM} = \overline{X} \times \overline{dF}$

Total moment $\overline{M} = \int \overline{X} \times [\varsigma g(d-z)\hat{n}\, dS]$

$$\left.\begin{array}{ll} \text{For equation:} & \overline{F} = W\hat{k} \\ \text{at C.G.} & \overline{M} = \overline{X}_G \times W\hat{k} \end{array}\right\} \quad [\text{W = body weight \& } \overline{X}_G = \text{C.G.}] \quad (14.21)$$

Using divergence theorem, equation (14.19): $\overline{F} = -\int \frac{\partial}{\partial z}[(\varsigma g(d-z))]dV\,\hat{k}$

$$= \varsigma g V\,\hat{k}$$

$$\text{Archimedes' Principal} \quad \left\{\begin{array}{l} = \text{displaced volume weight} \\ = \text{weight of body from (14.21)} \end{array}\right.$$

CHAPTER 15

VIBRATIONS

15.1 Introduction

Consider bodies undergoing time dependent motions, specifically the periodic or vibratory motions. The best method to study them is on the basis of a mass-spring system.

Mass-spring system

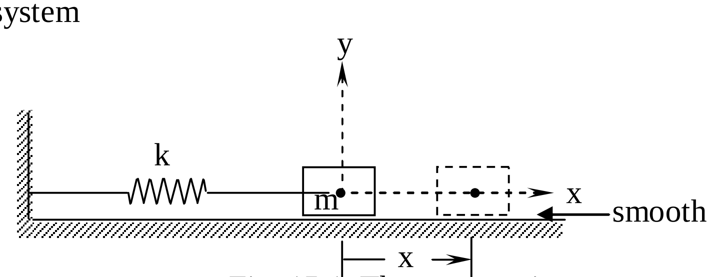


Fig. 15.1 The mass-spring system

Equation of motion of mass 'm': at any time 't'

Force = mass x acceleration

$$-kx = m\frac{d^2x}{dt^2}$$

$$x = x(t)$$

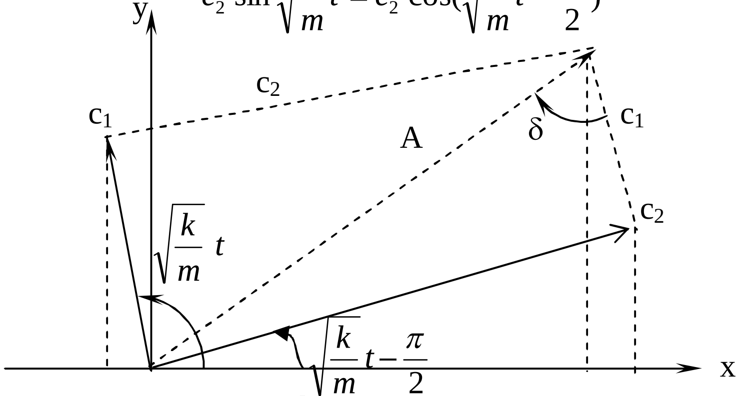
$$\therefore \quad m\frac{d^2x}{dt^2} + kx = 0$$

or $\boxed{m\ddot{x} + kx = 0}$ S.H.M. (15.1)

<u>has</u> <u>solution</u> $\boxed{x = c_1\cos\sqrt{\frac{k}{m}}t + c_2\sin\sqrt{\frac{k}{m}}t}$

at any time 't' (15.2)

$$c_2 \sin\sqrt{\frac{k}{m}}t = c_2 \cos\left(\sqrt{\frac{k}{m}}t - \frac{\pi}{2}\right)$$



$$\delta = \tan^{-1}\frac{c_2}{c_1}$$

phase angle

Fig. 15.2 Simple Harmonic Motion

Alternatively $\boxed{x = A\cos\left(\sqrt{\frac{k}{m}}t - \delta\right)}$

A = $\sqrt{c_1^2 + c_2^2}$ = $x_{\max}$ = amplitude of motion. (15.4)

If $\sqrt{\frac{k}{m}}\ t - \delta = \theta$

$\therefore$ x = A cos θ

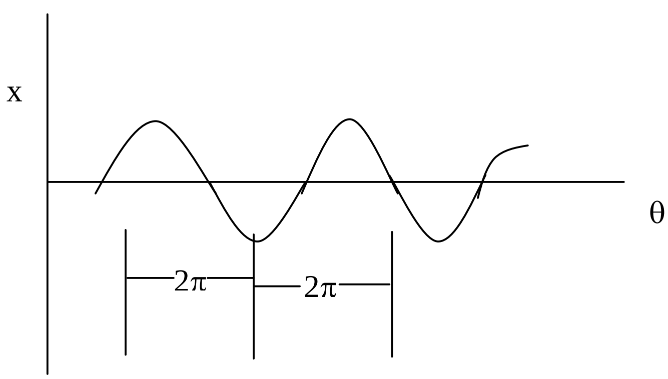


Fig 15.3 Displacement time history

$\theta = 0 \qquad \Rightarrow \sqrt{\frac{k}{m}}\ t - \delta = 0$

$\therefore \qquad t_1 = \delta \big/ \sqrt{k/m}$

$\theta = 2\pi \qquad \Rightarrow \sqrt{\frac{k}{m}}\, t - \delta = 2\pi$

$\therefore \qquad t_2 = (2\pi + \delta) \big/ \sqrt{k/m}$

$\therefore$ Time required to complete

one cycle $\quad = t_2 - t_1 = \boxed{T = \dfrac{2\pi}{\sqrt{k/m}}}$

Period of the motion

No. of cycles per unit time = $\dfrac{1}{T}$

= frequency (natural) of motion

$$\therefore \boxed{f = \frac{\sqrt{k/m}}{2\pi}}$$

No. of cycles per unit time in radians $\quad \boxed{\omega = \sqrt{\dfrac{k}{m}}}$

$\therefore$ Equation of motion : $\quad m\ddot{x} + kx = 0$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\boxed{\ddot{x} + \omega^2 x = 0}$$

To obtain 'ω' in $\qquad \omega = \sqrt{\dfrac{k_{equivalent}}{m}} \qquad (15.5)$

Practical problems

where $\qquad k_{eq}$ = Force / deflection

15.2 Free Vibrations with Damping

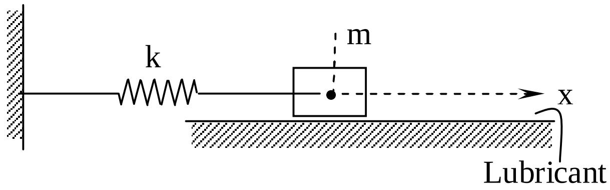


Fig. 15.4 Damped vibrations

Frictional force at support (not dry friction)

or damping force $\propto$ velocity of mass 'm'

$$\propto \frac{dx}{dt}$$

$$= -c\,\frac{dx}{dt} \qquad c = \text{coefficient of damping}$$

$\therefore$ Eq^{n} of motion:

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

$$m\ddot{x} = -kx - c\dot{x}$$

i.e. $$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

has standard solution $x \equiv x(t) = c_1 \, e^{\lambda t}$

substituting $m\, c_1\lambda^2 \, e^{\lambda t} + cc_1\lambda \, e^{\lambda t} + k\, c_1 e^{\lambda t} = 0$

$$\therefore \quad m\lambda^2 + c\lambda + k = 0$$

$$\therefore \quad \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\therefore \quad \lambda = \frac{\frac{-c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

$$= \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$= \lambda_1 \, , \, \lambda_2 \qquad \text{say}$$

In such a case $x = c_1 \, e^{\lambda_1 t} + c_2 \, e^{\lambda_2 t}$

$$\therefore \quad x = c_1 e^{\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right] t}$$

$$+ \; c_2 e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right] t}$$

Consider the case when $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0$

i.e. $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

i.e. c = $2m\sqrt{\frac{k}{m}}$

i.e. c = 2m ω_n $\qquad$ ω_n = <u>natural</u> <u>frequency</u> <u>of</u> <u>undamped</u> <u>free</u> <u>vibration</u>

$\downarrow$

critical damping coefficient = C_c

$\therefore$ <u>$C_c = 2m\ \omega_n$</u>

<u>(for given systems)</u>

ratio $\qquad \frac{C}{C_c}$ = d $\qquad$ <u>damping factor</u>

Now $\qquad \frac{c}{d} = 2m\ \omega_n$ $\qquad \therefore \frac{c}{2m} = d\ \omega_n$

$$\therefore \; x = c_1 e^{\left[-d\omega_n + \sqrt{d^2\omega_n^2 - \omega_n^2}\right] t}$$

$$+ \; c_2 e^{\left[-d\omega_n - \sqrt{d^2\omega_n^2 - \omega_n^2}\right] t}$$

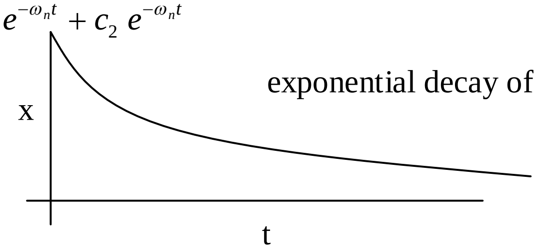
$$\therefore \; x = c_1 e^{\left[-d + \sqrt{d^2 - 1}\right] \omega_n t}$$

$$+ \; c_2 e^{\left[-d - \sqrt{d^2 - 1}\right] \omega_n t}$$

$$x = c_1 e^{\left[-d + \sqrt{d^2 - 1}\right] \omega_n t} \quad + \; c_2 e^{\left[-d - \sqrt{d^2 - 1}\right] \omega_n t}$$

$\boxed{\text{when d = 1}}$ $\qquad$ d = $\frac{c}{c_c} = 1$ $\quad \Rightarrow \quad$ CRITICAL DAMPING

$$\therefore \ x = c_1 \, e^{-\omega_n t} + c_2 \, e^{-\omega_n t}$$

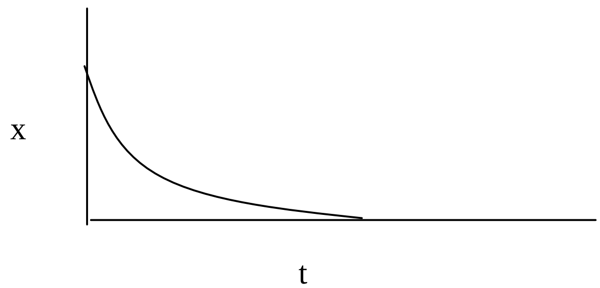


exponential decay of x

$\boxed{\text{when d > 1}}$ $\quad \sqrt{d^2 - 1} \quad \Rightarrow \quad$ positive

$\Rightarrow$ large exponential decay

$\Rightarrow$ <u>large</u> <u>damping</u>



$\boxed{\text{when d < 1}}$ $\quad d^2 - 1 \ \Rightarrow \quad$ negative

$$\therefore \ \sqrt{d^2 - 1} \ = \sqrt{(-1)(1 - d^2)} \ = i\sqrt{1 - d^2}$$

$$\therefore \ x \ = c_1 \, e^{\left[-d + i\sqrt{1-d^2}\right]\omega_n t} \ + \ c_2 \, e^{\left[-d - i\sqrt{1-d^2}\right]\omega_n t}$$

$$= e^{-d\omega_n t\left[c_1 e^{i\sqrt{1-d^2}\,\omega_n t} + c_2 \, e^{-i\sqrt{1-d^2}\,\omega_n t}\right]}$$

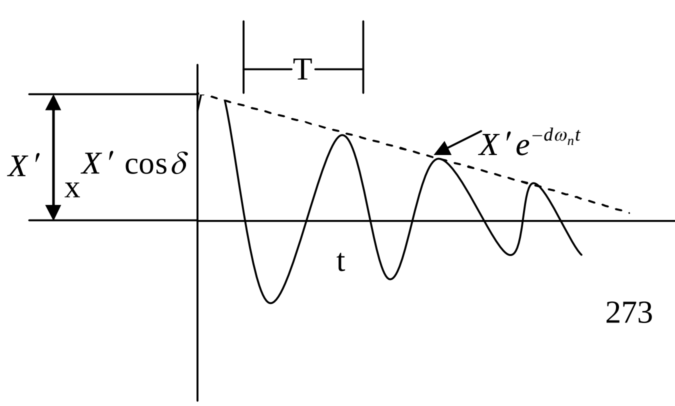
since $\quad e^{\pm i\theta} \ = \ \cos\theta \ \pm \ i\sin\theta$

$$x \ = e^{-d\omega_n t\left[(c_1 + c_2)\cos\left[\sqrt{1-d^2}\,\omega_n t\right] + i(c_1 - c_2)\sin\left[\sqrt{1-d^2}\,\omega_n t\right]\right]}$$

we have seen that if $x' = \text{A} \cos \ \omega t + B\sin\omega t$

$$= \sqrt{A^2 + B^2} \ \cos(\omega t - \delta)$$

$$\therefore x = e^{-d\omega_n t} \ \{ X' \cos\left\lfloor\sqrt{1 - d^2} \ \omega_n t - \delta\right\rfloor\} \qquad (15.8)$$



exponential decay of cosine function
$\Rightarrow$ <u>Light</u> <u>Damping</u>

NOTE: $\omega = \omega_n \sqrt{1-d^2}$

Fig. 15.5 Damped vibrations

15.3 Forced Vibrations

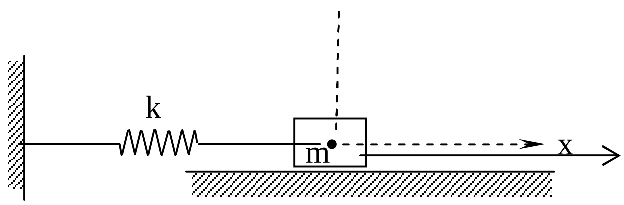


Fig. 15.6 Forced vibrations

Assume mass 'm' resting on smooth support acted by a hormonic force F cos ωt

<u>Eq^n of motion</u>: $m\ \ddot{x} = -kx + F\cos\omega t$

i.e. $\ddot{x} + \dfrac{k}{m}x = \dfrac{F}{m}\cos\omega t \qquad \rightarrow (1)$

(linear –non homogeneous –2order)

has sol^n $x = x_c + x_p$

$x_c =$ complementary or transient solution that makes $\ddot{x} + \dfrac{k}{m}x = 0$

and hence is $= c_1 \cos\omega_n t + c_2 \sin\omega_n t$

where ω_n = frequency of '<u>free</u>' vibrations = $\sqrt{k/m}$

$\Rightarrow$ vanishes with time 't' in practice.

$x_p =$ particular solution or steady state sol^n

that makes $\ddot{x} + \dfrac{k}{m}x = \dfrac{F}{m}\cos\omega t$

$= \mathrm{X}\cos\omega t$ X = amplitude of motion

putting in eq^n (1)

$$-\mathrm{X}\,\omega^2 \cos\omega t + \frac{k}{m} X \cos\omega t = \frac{F}{m}\cos\omega t$$

$$\therefore \mathrm{X}\left(-\omega^2 + \frac{k}{m}\right) = \frac{F}{m}$$

$$\therefore \mathrm{X} = \frac{F}{-m\omega^2 + k} = \frac{F/k}{1 - \frac{m}{k}\omega^2}$$

Now $\dfrac{F}{k}$ = static deflection of 'm'

= deflection of 'm' if force

$(F\cos\omega t = F\cos(o) = F)$

applied statically

$= \Delta$

and $\dfrac{k}{m} = \omega_n^2 \qquad \therefore \dfrac{m}{k} = \dfrac{1}{\omega_n^2}$

$$\therefore \mathrm{X} = \frac{\Delta}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$\therefore \mathrm{X}_\rho = \frac{\Delta}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos\omega t \qquad \text{steady state sol}^{\text{n}} \qquad (15.9)$$

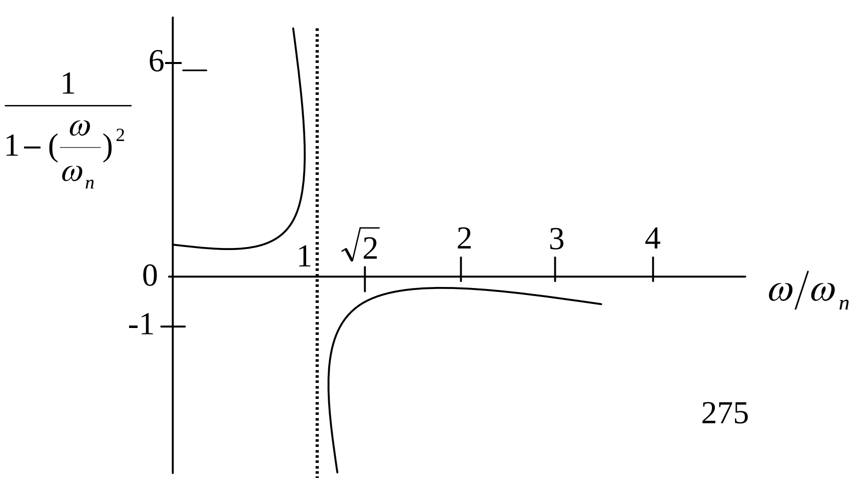
and complete sol$^{\text{n}}$ is

$$x = c_1 \sin\omega t + c_2 \cos\omega t + \frac{\Delta\cos\omega t}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Now $\dfrac{\text{max. 'steady state' deflection}}{\text{static deflection}}$ = magnification factor

$$= \frac{\Delta}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cdot \frac{1}{\Delta}$$

$$= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$



from graph:

-6-

(1) when $\frac{\omega}{\omega_n} \approx 1 \Rightarrow$ Very large magnification

$\Rightarrow$ Resonance

(2) If $\frac{\omega}{\omega_n} < 1 \Rightarrow$ m.f. +ve

$\frac{\omega}{\omega_n} > 1 \Rightarrow$ m.f. –ve (force & displacement opposite)

(3) when $\frac{\omega}{\omega_n} > \sqrt{2}$, m.f. <1 (fraction)

(4) when you start an electric motor from rest ($\omega = 0$) to any higher ω, resonance condition is met with and hence should be surpassed quickly.

15.4 Forced Damped Vibrations

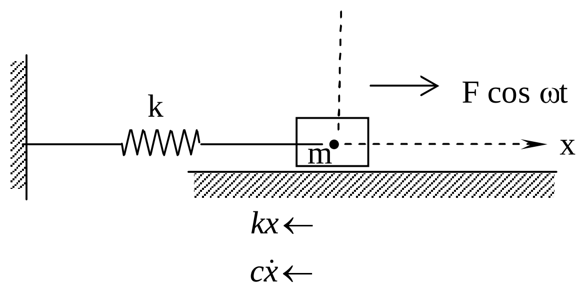


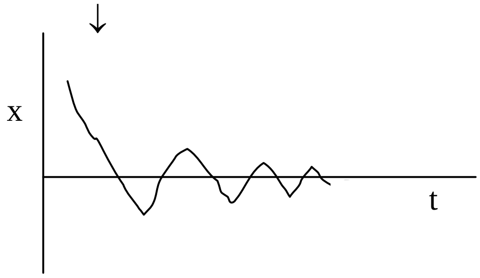
Fig. 15.7 Forced damped vibrations

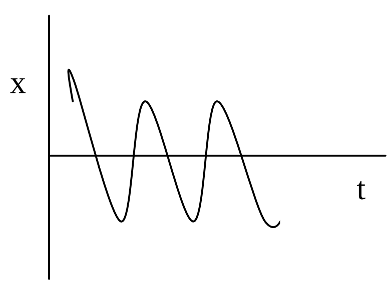
Equation of motion: $m\ddot{x} = F\cos\omega t - kx - c\dot{x}$

i.e. $m\ddot{x} + c\dot{x} + kx = F\cos\omega t$ ------(15.10)

has general solution

$x = x_{complementary} + x_{particular}$





represents transient motion lasting for very small time

steady state vibrations

$\underline{sol}^n \rightarrow x_p = X \cos(\omega t-\phi)$

$\quad\quad\quad\quad\downarrow \quad\quad\quad \downarrow$

amplitude phase angle

putting value of x in eq^n (1)

$$m(-\omega^2)X\cos(\omega t-\phi)+c(-\omega)X\sin(\omega t-\phi)+kX\cos(\omega t-\phi)$$

$$= F\cos\omega t \quad (15.12))$$

when $(\omega t-\phi)=0,\quad eq^n(2)\Rightarrow (k-m\omega^2)X = F\cos\phi$

when $(\omega t-\phi)=\dfrac{\pi}{2}\quad eq^n(2)\Rightarrow c\omega\ X = F\sin\phi$

squaring and adding, $X^2\left[(k-m\omega^2)^2+(c\omega)^2\right]=F^2$

$$\therefore\ X=\frac{F}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$$

& dividing $\tan\phi = \dfrac{c\omega}{(k-m\omega^2)}$

$$\therefore\quad X=\frac{F/k}{\sqrt{\left(1-\dfrac{m\omega^2}{k}\right)^2+\left(\dfrac{c\omega}{k}\right)^2}} \quad (15.13)$$

F/k = static deflection of spring by given force F =Δ

$\dfrac{m\omega^2}{k}=\dfrac{\omega^2}{\omega_n^2}\quad as\quad \sqrt{\dfrac{k}{m}}=\omega_n$

$\dfrac{c\omega}{k}=(\dfrac{c}{c_c})\,[c_c]\,\dfrac{\omega}{k}$

$= (d).\ [2m\ \omega_n]\ \dfrac{\omega}{k}$

$= 2d\ \dfrac{\omega}{\omega_n}$

$$\therefore X=\frac{\Delta}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]^2+\left(2d\,\dfrac{\omega}{\omega_n}\right)^2}}$$

and $\tan\phi = \dfrac{c\omega}{k - m\omega^2} = \dfrac{c\dfrac{\omega}{k}}{1-\dfrac{m\omega^2}{k}} = \dfrac{2d\left(\dfrac{\omega}{\omega_n}\right)}{1-\left(\dfrac{\omega^2}{\omega_n^2}\right)}$

$$\frac{X}{\Delta} = \frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2d\,\dfrac{\omega}{\omega_n}\right)^2}}$$ magnification factor

$$\frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(2d\,\dfrac{\omega}{\omega_n}\right)^2}}$$

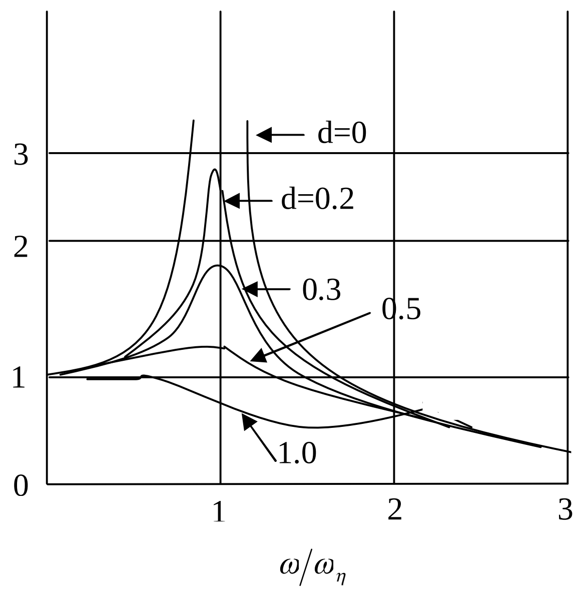


Fig. 15.9 Dynamic amplification of the motion

1) As damping reduces, magnification of motion increases
   ∴ To reduce vibrations →use large d or keep $\omega > \omega_n$

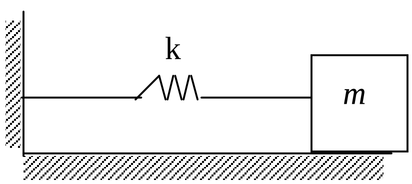
2) Peak magnification is at $\dfrac{\omega}{\omega_n} \approx 1$

   → exact value can be obtained by

$$\frac{d}{d\left(\omega/\omega_n\right)}\left[\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2d\,\frac{\omega}{\omega_n}\right)^2}}\right]=0$$

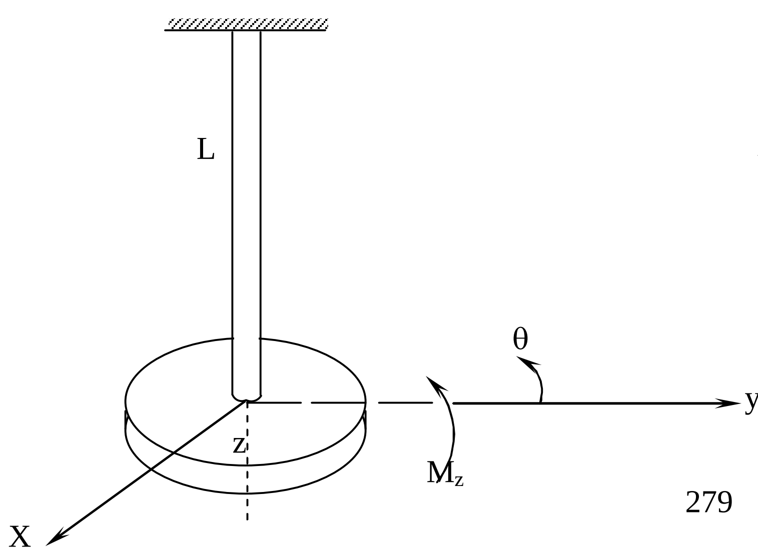
$$\Rightarrow\ \frac{\omega}{\omega_n}\sqrt{1-2d^2}$$

15.5 Torsional Vibrations



Vibrations → restoring force

Vibrations → restoring torque



Consider a circular disc attached to a tubular shaft (as shown) fixed at upper end.

Fig. 15.10 Torsional vibrations

Rotate disc (by applying moment M_z) by angle θ
Shaft will get twisted & supply a restoring
torque - M_z

causes torsional vibrations of disc.

Equation of Motion:

for translatory motion: Newton's eq^n
force = mass x accel^n
for rotational motion: Euler's eq^n
moment about axis of rotation = moment of inertia of disc x rotational accl^n

$$M_z = I_{zz} \ddot{\theta} \qquad \left(\ddot{\theta} = \frac{d^2\theta}{dt^2} \right) \tag{15.14}$$

We have,
restoring force = - (spring constant) displacement

Similarly
restoring torque = - (torsional spring constant) rotation

$$\therefore \quad M_z = -(K_t)\,\theta$$

$$= I_{zz} \ddot{\theta} \qquad \text{from eq}^n \text{ (15.14)}$$

$$\therefore I_{zz} \ddot{\theta} + k_t\theta = 0$$

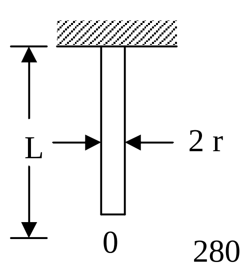
$$\therefore \ddot{\theta} + \frac{k_t}{I_{zz}}\theta = 0 \qquad \equiv \qquad \boxed{\ddot{\theta} + \omega^2\theta = 0}$$

$$\boxed{\therefore \ \omega \sqrt{\frac{k_t}{I_{zz}}}}$$

has sol^n: $\theta = c_1 \cos \omega t + c_2 \sin \omega t$

$= A \cos (\omega t - \delta)$

note: for circular shaft



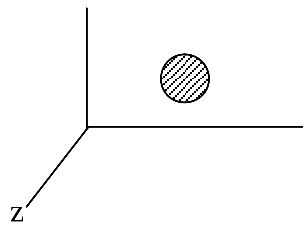
$$k_t = \frac{GJ}{L}$$

G = shear modulus of shaft material

J = polar moment of inertia of shaft c/s

= m. of i. about an axis normal to plane of c/s

$= \frac{1}{2} \pi \ r^4$

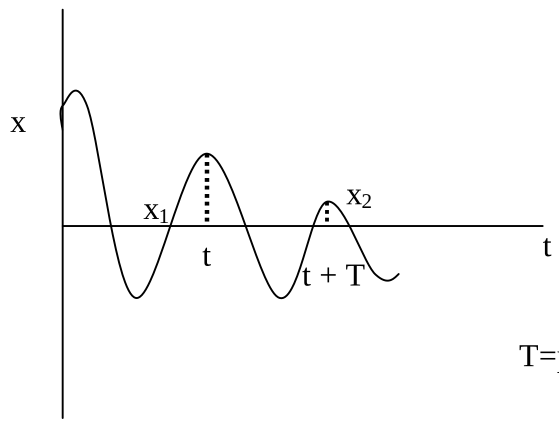


Rate of Damping

<u>Note:</u> In order to get vibratory motion

d<1.

Then $\quad x = x' e^{-dw_n t} \cos\left(\sqrt{1-d^2}\ \omega_n t - \delta\right)$



T=period

X_1 = amplitude at any 't'

X_2 = amplitude time (t+T)

(successive amplitude)

Rate of damping $\Rightarrow$ logarithmic decrement

$$\delta = ln \frac{x_1}{x_2}$$

$$= ln \ \frac{x' e^{-d\omega_n t} \cos\left(\sqrt{1-d^2}\omega_n t - \delta\right)}{x^1 e^{-d\omega_n (t+T)} \cos\left[\sqrt{1-d^2}\omega_n (t+T) - \delta\right]} \qquad (15.15)$$

since cosines of angles 1 cycle apart are same,

$$\delta = ln \ e^{d\omega_n T}$$

$$\therefore \delta = d\omega_n \frac{2\pi}{\omega} \qquad = 2\pi \ d \ \frac{\omega_n}{\omega}$$

CHAPTER 16

MOTIONS OF FREELY FLOATING BODIES

16.1 Encountering Wave Frequency

Encountering frequency is also called apparent or relative wave frequency, relative to the ship's speed and as encountered by the ship. Let μ is the angle made by the ship direction with the wave direction. Then μ = 0 means a 'following sea' ($T_{apparant} > T_{absolute)}$). Then μ = 180^0 means a 'Head sea' ($T_{apparant} < T_{absolute)}$). μ = 90^0 indicates 'Beam sea'.

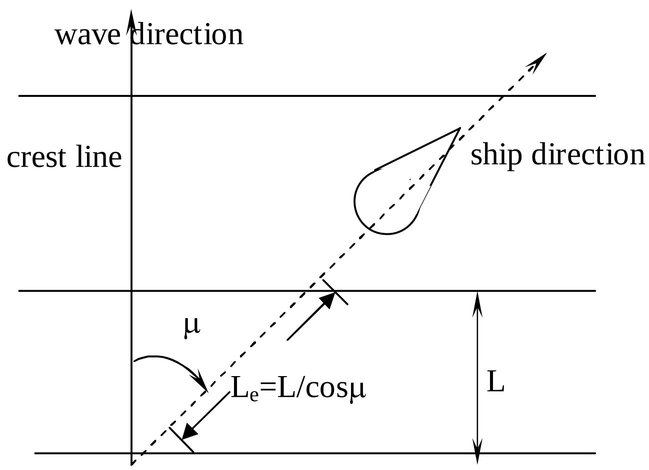


Fig. 16.1 Ship's movement

Let L = wave length
T = wave period (abs.)
C = wave speed
V= ship speed,

Relative speed of waves $= C - V\cos\mu$

Relative period of waves, T_e $= \dfrac{L}{C - V\cos\mu}$

(Bow. encounters waves after every T_e sec)

$$= \frac{C\ .\ T}{C - V\cos\mu}$$

$$= \frac{T}{1 - (V/C)\cos\mu}$$

$$\therefore\ \frac{2\pi}{\omega_e} = \frac{2\pi/\omega}{1 - (V/C)\cos\mu}$$

$$\therefore \quad \boxed{\omega_e = \omega\left[1-\left(\frac{V}{C}\right)\cos\mu\right]}$$

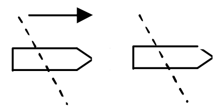
Since c = $\dfrac{g}{\omega}$

(deep water)

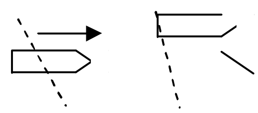
$$\boxed{\omega_e = \omega\left[1-\omega V\,\frac{\cos\mu}{g}\right]}$$

<u>Note</u>: (1) $\mu = 90^\circ \;\Rightarrow \omega_e = \omega$

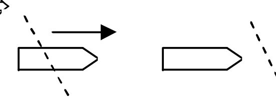
(2) $\omega_e = 0 \;\Rightarrow V\cos\mu = C \Rightarrow$ ship remains in same position w r.t. wave profile



$\omega_e < 0 \;\Rightarrow$ Following sea $\Rightarrow$



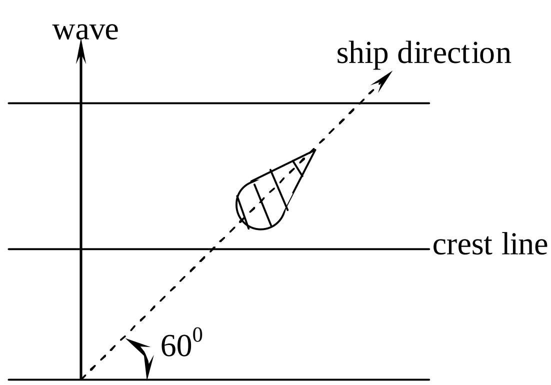
$\omega_e > 0 \;\Rightarrow$ Overtaking sea $\Rightarrow$



(3) For $(\omega_e)_{\max}$: $d/d_w\,[\omega_e]\,0 \Rightarrow$ V cos $\mu = \dfrac{c}{2}$

<u>Ex</u>. If the bow of the ship encounters wave crests after every 15 seconds and if the crests take 10 seconds to cross the ship from its bow to stern, obtain
(i) the wave length, (ii) wave speed, (iii) ship speed.
The length of the ship of 150 m makes an angle of 60° with crest lines.

<u>Sol^n</u> :



μ	= 30°
L_s	= 150 m
V	= ship's speed
L	= wave length
C	= wave speed

Along ship dir^n :

bow encounters waves after 15 sec.

$$\therefore \quad 15 = \frac{L}{C - V\cos\mu}$$

Along wave dir^n :

$$10 = \frac{\text{length along wave dir}^n}{\text{speed along wave dir}^n}$$

$$= \frac{150\cos 30}{C - V\cos\mu}$$

$$\therefore \; C - V\cos\mu = 15 \quad \cos 30 = \frac{L}{15}$$

$$\therefore \; \boxed{L = 194.86 \text{ m}}$$

In deep water, $C = \sqrt{\dfrac{9L}{2\pi}}$

$$= \sqrt{\frac{9.81(194.86)}{2\pi}}$$

$$= \boxed{17.44} \text{ m/s}$$

we have $\quad C - V\cos\mu = 15 \quad \cos 30 = 12.99$

$$= 17.44 - V\cos 30$$

$$\Rightarrow V = \boxed{5.14 \text{ m/s}}$$

$$= 10 \text{ knots}$$

16.2 Ship Motions

The ship motions could be classified into two types: translatory and rotational.

As shown in Fig. 16.2 there are 6 independent motions of a ship or 6 degrees of motions for it. In actual they are dependent on each other (coupled) but for mathematical simplicity we assume them to be independent or uncoupled. Out of these 6 types of motions the heave, pitch and roll are purely oscillatory and hence can be more dangerous.

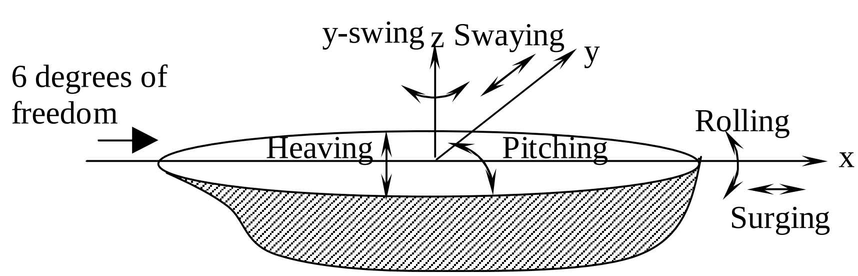


Fig. 16.2 The six degrees of freedom

UNCOUPLED MOTIONS

16.3 Heaving motions

Equation of free heaving motion : This takes place in absence of waves.

$$\underbrace{a\ \ddot{z}} + \underbrace{b\ \dot{z}} + \underbrace{c\ z} = 0$$

The first, second and the third term in the above equation represent the inertial, damping and the restoring force, respectively.
a = inertial coefficient, b = damping coefficient, c = restoring force coefficient, z, $\dot{z}$ and $\ddot{z}$ = vertical displacement, velocity and acceleration respectively of the ship.
The vertical motions are evaluated with respect to the CG.
The inertial coefficient is also called virtual mass and it has two components: Actual ship mass and the added mass.

k

It has sol^n (for oscillatory motion, d <1) :

$z = e^{-\vartheta t}\ A\sin\left(\omega_d t - \delta\right)$

damping factor $= \dfrac{b}{b_c} = \dfrac{b}{2a.\omega_z}$

damped frequency, $= \sqrt{\omega_z^2 - \vartheta^2} < \omega_z$

$\Downarrow$

$(T_d > T_z)$

ϑ = decaying constant $= \dfrac{b}{2a}$

free undamped frequency

$\omega_z = \sqrt{\dfrac{c}{a}}$

$e^{-ut}\left[c_1 \cos\left(\omega_d t\right) + c_2 \sin\left(\omega_d t\right)\right]$

c_1, c_2 obtained from initial conditions e.g.
t = 0 $\Rightarrow$ z = za ,
$\dot{z} = 0$

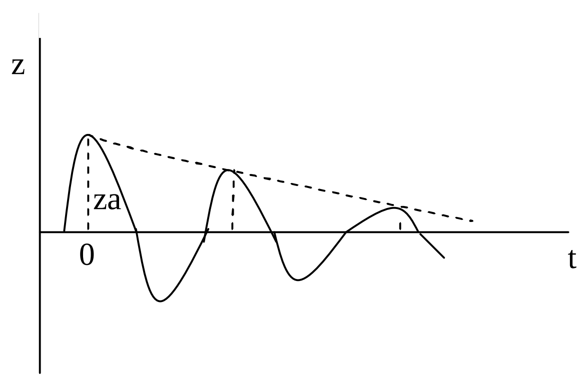


Fig. 16.3 Attenuation of displacements

INTERTIAL FORCE :

any mass, 'm', accelerating in fluid $(\ddot{z})$

$\Rightarrow$ experiences a force > m $\ddot{z}$

additional force = (added mass) $\ddot{z}$ → typically 80% of m

$\therefore$ inertial force = force accelerating body, 'm' + force accelerating liquid

$a_z\ \ddot{z} = m\ \ddot{z} + a_z'.\ \ddot{z}$

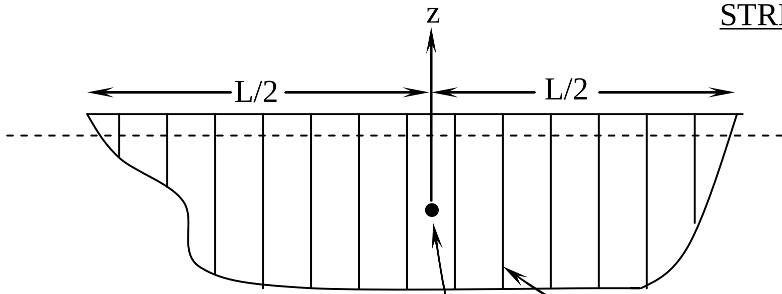
a_z → virtual mass [or inertial coefficient]

m → actual mass

a_z' → added mass (for heaving)

Lewis (1929) $\Rightarrow$ studied conventional hull c/s

$\Rightarrow$ a method to obtain a_z' by STRIP THEORY



For any n^{th} c/s :

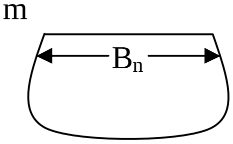
$$\boxed{a_n = C\ \underbrace{\frac{\rho\pi}{8}B_n^2}}$$

ρ – mass density of fluid

a_n – added mass coefficient for n^{th} c/s

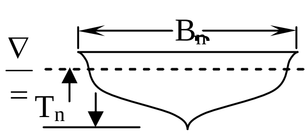
C – non-d section added mass coeff.

$\frac{\rho\pi}{8}B_n^2$ – mass of semi-circular c/s of diameter B_n and length 1 m $= \rho\frac{\pi\ B_n^2 1}{8}$



$= f(B_n, T_n, \omega_e, \beta_n)$

β_n – area coefficient



$= \frac{S_n}{B_n.T_n}$ (S_n – given c/s area)

$\therefore$ Total added mass for entire ship $\boxed{a_z^1 = \int_{-L/2}^{L/2} a_n.dx}$

DAMPING FORCE :

$\longrightarrow$ opposes motion

$\rightarrow$ due to radiated waves,
(generated by heaving) friction, eddies

Grim (1959) $\Rightarrow$ STRIP THEORY

for n^{th} c/s

Damping Coefficient, $\boxed{b_n = \dfrac{\rho g^2}{\omega_e^3}\left(\overline{A}\right)^2}$

$\overline{A}$ = amplitude ratio = $\dfrac{\text{amplitude of radiated waves}}{\text{amplitude of heaving motion}}$

$= f\ (\omega_e, B_n, T_n, S_n)$

for entire ship $\boxed{b = \int_{-L/2}^{L/} b_n\ dx}$

Note: 1) b $\propto$ (water plane area)^2, $\dfrac{B}{L}$, $\dfrac{1}{V_{ship}}$

2) $(\text{damping})_{\text{V type c/s}} > (\text{damping})_{\text{u type c/s}}$

RESTORING FORCE :

$\longrightarrow$ due to additional buoyancy caused by deeper submergence

$\longrightarrow$ = c. z = ρ g ($A_{\text{water plane}}$) Z

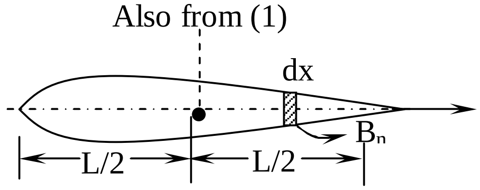
$\therefore$ c = ρ g A_{wp} $\longrightarrow$ (1)

If C_{wp} = coeff. of water plane area = $\dfrac{\text{actual water plane area, } A_{wp}}{L.B}$

$\boxed{c = \rho g\ L\ B\ C_{wp}}$

Note also: Block Coefficient, C_B = $\dfrac{\text{Actual volume displaced}}{\text{LBT}}$

Also from (1)



$\boxed{c = \rho g \int_{-L/2}^{L/2} B_n\ d_x}$

EQUATION OF FORCED HEAVING MOTION :

[in waves]

$$a\ddot{z} + b\dot{z} + cz = \underbrace{F_o \cos\omega_e t}_{\text{exciting force}}$$

has sol^n: $Z = \underset{\text{free damped}}{Z_{transient}} + Z_{steady\ state}$

$$= e^{-\vartheta t}\left[c_1 \cos\omega_d t + c_2 \sin\omega_d t\right] + z_a \cos(\omega_e t - \in)$$

$$= \underbrace{Ae^{-\vartheta t}\sin(\omega_d t - \beta)}_{\text{decays rapidly}} + z_a \cos(\omega_e t - \in)$$

we have $\underset{\substack{\text{amplitude of forced}\\ \text{heaving}}}{Z_a} = \underset{\substack{\text{amplitude of}\\ \text{static heaving}}}{Z_{st}} \times \underset{\substack{\text{magnification}\\ \text{factor}}}{\mu_z}$

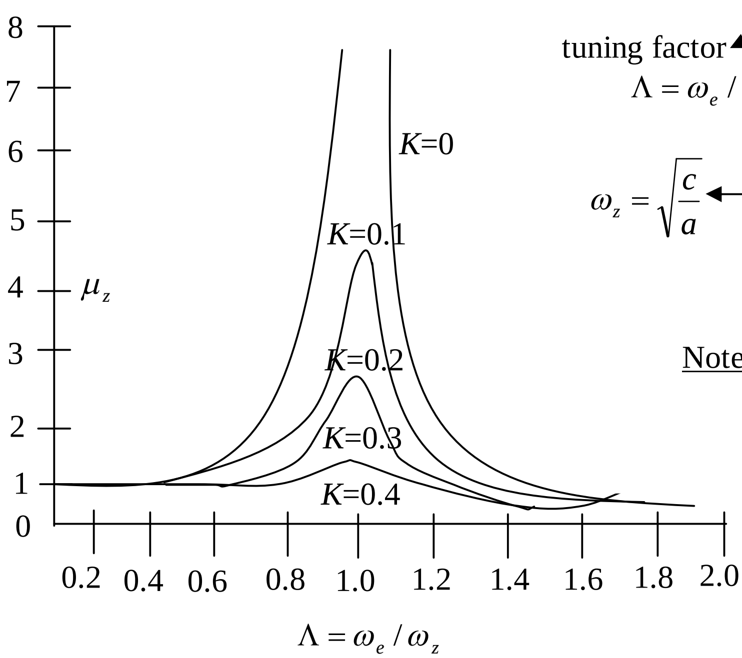
$$Z_{st} = \frac{F_\circ}{C} \qquad \mu_x = \frac{1}{\sqrt{(1-\Lambda^2)^2 + 4k^2\Lambda^2}}$$

tuning factor
$\Lambda = \omega_e / \omega_z$

$\omega_z = \sqrt{\dfrac{c}{a}}$

damping factor
$k = \vartheta / \omega_z$

decaying factor
$\vartheta = b/(2a)$



Note: (1) $\in = \tan^{-1} \dfrac{2k\Lambda}{1-\Lambda^2}$

(2) For max. response :
$\omega_e \approx \omega_z$
when damping large
$\omega_e / \omega_z \approx 1$

$$\frac{d}{d_\Lambda}\mu_z = 0 \Rightarrow \Lambda = \sqrt{1-2k^2}$$

(3) wave elevation w.r.t. ship, $\eta - z$ is imp. as deck wetness depends on it.

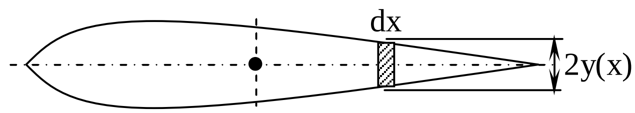
EXCITING FORCE :

due to waves causing excess buoyancy at 't'

assume → (i) at Load WL — ship wallsided

(ii) t = 0 ⇒ wave crest @ midship

(iii) ship remains still w.r.t. vertical motion
i.e. waves pass slowly by the ship.



$$\therefore F = \int_{-L/2}^{L/2} \rho\, g\,.2.\, \underbrace{y(x)}_{\text{half width at section @'x'}}\,.\eta.\, dx$$

$k_e = k\cos\mu$

$$= \int_{-L/2}^{L/2} \rho\, g\, 2\; y(x)\;\eta_a \cos(k_e x - \omega_e t)\,.dx$$

[assuming surface profile = effective profile, $\eta_a^{e^{-k_e T_m}} \cos(k_e x - \omega_e t)$]

$$= \left[2\rho\, g\eta_a \int_{-L/2}^{L/2} y(x).\cos(k_e x)\,.dx \right].\cos\omega_e t$$

$= [F_\circ].\cos\omega_e t$ [note: phase $\eta^\wedge F = 0$]

non-d force amplitude $\boxed{f_\circ = \dfrac{F_\circ}{\rho\, g\,\eta_a LB}}$

$$\boxed{f_\circ = \frac{2}{LB}\int_{-L/2}^{L/2} y(x).\cos(kx\cos\mu).dx}$$

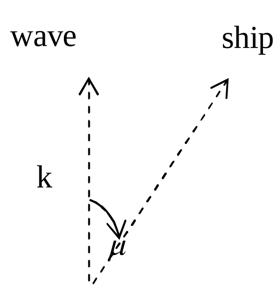
parametric studies⇒

effective wave length L_w':

If $L_w' < \dfrac{1}{2}$ L$_{ship}$ ⇒ small heaving force

< L$_{ship}$ ⇒ moderate force

$L_w' \to \infty$ ⇒ $f_\circ \to c_{wp}$



Ex. For a ship of length = 128 m,
beam = 17.07 m,
draft upto keel = 6.05 m,

obtain,
1) natural period of heaving in still water
2) heaving amplitude in still water
if t = 0 $\Rightarrow$ z = 0; $\dot{z} = 1.68$ m/s
3) maximum force exerted on the deck by a suspended 2 t anchor.

Assume added mass for heaving = 90% of ship mass $C_B = C_{wp}$,
damping negligible

1) Undamped free heaving frequency, $\omega_z = 2\pi / T_z$

$$T_z = 2\pi\sqrt{\frac{a}{c}}$$

$$= 2\pi\sqrt{\frac{.9M + M}{\rho g\ A_{wp}}}$$

$$= 2\pi\sqrt{\frac{1.9M}{\rho g\ C_{wp} BL}}$$

$$= 2\pi\sqrt{\frac{1.9\ \rho\ (LBT\,C_B)}{\rho g\ C_{wp}\ BL}}$$

$$= 2\pi\sqrt{\frac{1.9T}{g}}$$

$$= 2\pi\sqrt{\frac{1.9(6.05)}{9.81}}$$

$$= \boxed{6.8} \text{ sec.}$$

(2) For Free Undamped Heaving, z=A sin $\omega_z t + B\cos\omega_z t$

t=0 $\Rightarrow$ z=0 $\Rightarrow$ 0=B

$$\therefore\ \dot{z} = A\omega_z \cos\omega_z t$$

t=0 $\Rightarrow$ $\dot{z} = 1.68\ m/s \Rightarrow$ $1.68 = A\dfrac{2\pi}{6.8}$

$$\therefore \text{ A} = \boxed{1.818 \text{ m}}$$

(3) Maximum force by anchor = 2 g + 2. $\ddot{z}_{\max}$

z = 1.818 sin (0.924 t)

$$\therefore\ \ddot{z} = -1.818\,(0.924)^2\ \cos\,(0.924t)$$

$$\therefore\ \left|\ddot{z}\right|_{\max} = 1.552\quad m/s^2$$

$\therefore$ Max force = 2g + 2 (1.552) = $\boxed{22.7}$ kN

Ex.: A ship of length = 137.16 m,
beam = 21.336 m,
(mass) displacement = 12700 t
water plane area coefficient = 0.8
is attacked by 6.10 m high waves with 5.325 sec. encountering period.
Added mass for heaving = 80% of ship mass.

Non-dimensional damping coefficient, $\dfrac{b\sqrt{gL}}{mg} = 1.7$

Non-dimensional heaving force amplitude, $f_{\circ} = \dfrac{F_{\circ}}{\rho\, g \eta_a LB} = 0.17$

OBTAIN
1) Heaving amplitude
2) Phase angle between wave motion & heaving motion
3) Maximum acceleration in heaving

Take $\varsigma = 1030\, kg/m^3$
4) Relative motion of ship w.r.t. wave & its max. value.

Sol^n: (1) Amplitude of forced heaving, $Z_a = Z_{st} . \mu_z$

$$= \frac{F_{\circ}}{C} \cdot \frac{1}{\sqrt{\left(1-\Lambda^2\right)^2 + 4k^2\Lambda^2}}$$

$$\Lambda \rightarrow \frac{\omega_e}{\omega_z} \rightarrow \sqrt{\frac{c}{a}} \qquad k \rightarrow \vartheta/\omega_z \rightarrow \frac{b}{z_a}$$

c = ρ g A_{wp}

= ρ g L B C_{wp}

= 1030 (9.81) 137.16 (21.336) 0.8

= 23 6,5 5,7 49 N/m

$F_{\circ} = 0.17\, \varsigma\, g\, \eta_a\, L\, B$
= 0.17 (1030) 9.81 (3.05) 137.16 (21.336)
= 153, 31, 882 N

$\omega_e = 2\pi / 5.325$ = 1.18 r/s

a = (0.8 + 1) m
= 1.8 (12700) 10^3 kg
= 22860 x 10^3 kg

$\omega_z = \sqrt{c/a}$ = $\sqrt{236, 55, 749 / 228, 60, 000}$
= 1.017 r/s

$\Lambda = \omega_e / \omega_z$ = 1.18 / 1.017
= 1.16

$\dfrac{b\sqrt{gL}}{mg} = 1.7 \quad \Rightarrow b = [1.7\ (12700{,}000)\ 9.81] / \sqrt{(9.81)\, 137.16}$

= 57,73,955

$$k = \vartheta/\omega_z = [b/(2a)]/\omega_z = [5773955/2(22860000)]/1.017$$
$$= 0.126(1.017) = 0.124$$

$$\therefore Z_a = \frac{153\ 31\ 882}{236\ 55\ 749} \cdot \frac{1}{\sqrt{(1-1.16^2)^2 + 4(0.124)^2(1.16)^2}}$$
$$= \boxed{1.44\ \text{m}}$$

2. $$\phi_{\substack{\text{wave motion} \\ \text{heaving motion}}} = \underbrace{\phi_{\substack{\text{wave motion} \\ \text{exciting force}}}}_{0} + \phi_{\substack{\text{exciting force} \\ \text{heaving motion}}}$$

$$\in = \tan^{-1}\frac{2k\Lambda}{1-\Lambda^2}$$
$$= \tan^{-1}\frac{2(0.124)1.16}{1-1.16^2}$$
$$= \boxed{-39.77°}$$

3. Heaving motion : $z = z_a \cos(\omega_e t - \in)$

$$\ddot{z} = -z_a \omega_e^2 \cos(\omega_e t - \in)$$
$$\therefore \ddot{z}_{max} = 1.44(1.18)^2$$
$$= \boxed{2}\ \text{m/s}^2$$

4. Relative wave motion $= \eta - z$

at say $x = 0$ $= \eta_a \cos \omega_e t - z_a \cos(\omega_e t - \in)$

$$= \eta_a\left[\cos\omega_e t - \frac{z_a}{\eta_a}\cos(\omega_e t - \in)\right]$$

For max $(\eta - z)$, $\dfrac{d}{d(\omega_e t)}(\eta - z) = 0$

$$\therefore\ \eta_a\left[-\sin\omega_e t + \frac{za}{\eta_a}\sin(\omega_e t - \in)\right] = 0$$

$$\therefore -\sin\omega_e t + \frac{za}{\eta_a}\sin(\omega_e t - \in) = 0$$

$$\therefore -\sin\omega_e t + \frac{za}{\eta_a}[\sin\omega_e t \cdot \cos\in - \cos\omega_e t \cdot \sin\in] = 0$$

$$\therefore 1 = \frac{z_a}{\eta_a}\left[\cos\in - \frac{\sin\in}{\tan\omega_e t}\right]$$

$$1 = \frac{1.44}{3.05}\left[\cos -39.77^\circ - \frac{\sin(-39.77^\circ)}{\tan \omega_e t}\right]$$

$$\therefore \omega_e t = 25.36^\circ$$

$$Max(\eta - z) = 3.05\left[\cos\ 25.36^\circ - \frac{1.44}{3.05}\cos\left(25.36 + 39.77^\circ\right)\right]$$

$$= \boxed{2.15}\ \text{m}$$

16.4 Pitching motion

Free Pitching:

Equation of motion:

$$a\ddot{\theta} + b\dot{\theta} + c\theta = 0 \qquad 16.1$$

where, $a\ddot{\theta}$ is the Inertial Moment, in which a is virtual mass m.i. & $\ddot{\theta}$ is angular acceleration,

$b\dot{\theta}$ is the Damping Moment, in which b is damping moment coefficient & $\dot{\theta}$ is angular velocity.

$c\theta$ is the Restoring Moment, in which c is restoring moment coefficient & θ is angle of rotation.

To obtain 'a':

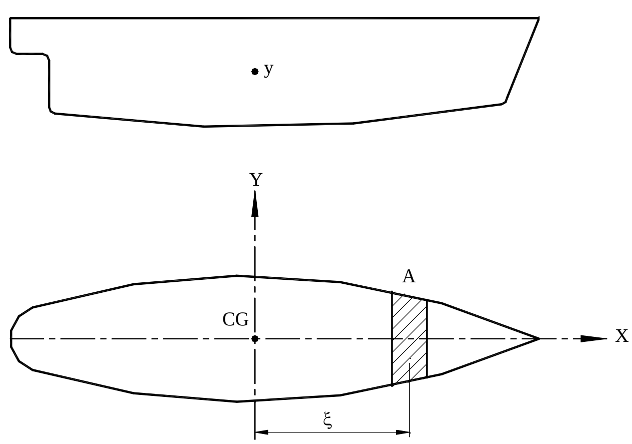


Fig. 16.1 Schematic vertical section and plan of a ship

$a = I_{yy} + \delta I_{yy}$ where, I_{yy} is m.i. of ship in pitching &

δI_{yy} is the added mass m.i. of ship in pitching.

$= I'_{yy}$ which also called the virtual mass moment of inertia

$I_{yy} = \int PA\xi^2 d\xi$ where, A is the sectional area & ξ^2 is distance along x direction from C.G. of the section.

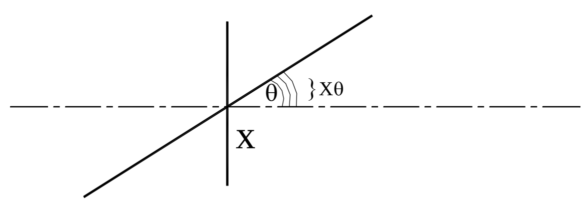
$\delta I_{yy} = \int a_n \xi^2 d\xi$, where, a_n is added mass for nth cross section

To obtain ‘b’:

$b = \int b_n \xi^2 d\xi$ where, b_n is the damping coefficient for nth section.

Note that damping is proportional directly to the beam width and the ‘V’ form of the ship and inversely to the draft, and natural frequency.

To obtain ‘c’:



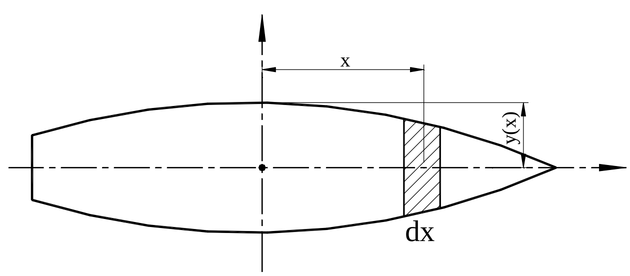


Fig.16.2 Schematic plan of the ship

$c = \int c_n \xi^2 d\xi$, where, c_n is the restoring coefficient for nth c/s.

Restoring Moment due to rotation ‘θ’

$= c\theta = \left\lfloor \int elemental\ moment \right\rfloor$

$= \left\lfloor \int \left(pg \cdot 2y(x)dx \cdot x\theta \right) x \right\rfloor$

$= pg\theta I_y$ where, I_y is m.i. of load water plane area.

$= pg\theta \overline{BM}_L \cdot V$ where, $\overline{BM}_L$ is the distance between C.B. and metacentre & V is displaced volume.

$\approx Mg\theta \overline{GM}_L \rightarrow$ for small 'θ' where, M is the ship mass for pitching & $\overline{GM}_L$ is metacentric height.

$\therefore C = Mg\overline{GM}_L$

Equation of free pitching:

$$a\ddot{\theta} + b\dot{\theta} + c\theta = 0 \qquad \textbf{(16.2)}$$

$$\therefore I'_{yy}\ddot{\theta} + b\dot{\theta} + Mg\overline{GM}_L\,\theta = 0$$

since decaying constant $v = \dfrac{b}{(za)}$

$$= \frac{b}{2I'_{yy}}$$

Undamped circular natural frequency for pitching: $w\theta = \sqrt{\dfrac{c}{a}}$

$$= \sqrt{\frac{Mg\overline{GM}_L}{I'_{yy}}}$$

Natural pitching period: $T_\theta = 2\pi / (\omega\,\theta)$

$$= 2\pi\sqrt{\frac{I'_{yy}}{Mg\overline{GM}_L}}$$

has the solution: $\theta = e^{-vt}\left[C_1 \cos w_d t + C_2 \sin w_d t\right]$ (16.3)

$= e^{-vt} A\sin(w_d t - \delta)$ where, $w_d t$ is damped pitching frequency $= \sqrt{w_\theta^2 - v^2}$

Damped pitching period: $T_d = \dfrac{2\pi}{w_d}$

$$= \frac{2\pi}{\sqrt{w_\theta^2 - v^2}}$$

$$= \frac{2\pi}{w_\theta} \frac{1}{\sqrt{1 - \left(\frac{v}{w_\theta}\right)^2}}$$

$$= \frac{T_\theta}{\sqrt{1 - \left(\frac{v}{w_\theta}\right)^2}}$$

NOTE: Generally $T_{pitching}$ 50% $T_{rolling}$ $\left\lfloor T_{pitching} \approx 7'' \; for \; 20000\, DWT \right\rfloor$

$T_{heaving}$

Forced Pitching motion

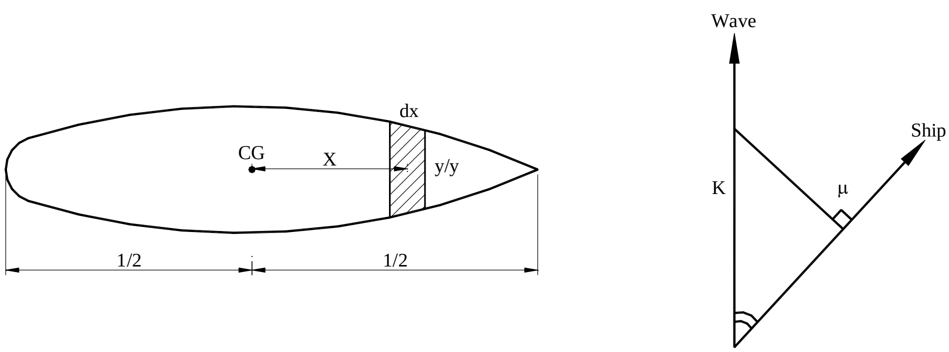


Fig. 16.3 The water plane area

The equation of motion is: $a\ddot{\theta} + b\dot{\theta} + c\theta = M_o \sin w_e t$ (16.4)

where, $M_o \sin w_e t$ is the exciting moment due to unbalanced moment about y-y

$$M_\theta = pg \int_{-L/2}^{L/2} 2 \cdot y(x) x dx [\eta]$$

Putting the value to above $\eta = \eta_a \cos(kx\cos\mu - w_e t)$ & assuming $\rightarrow$ ship is symmetrical about mid-ship section,

$$M_\theta = \left[2pg\eta_a \int_{-L/2}^{L/2} y(x) \cdot x \cdot \sin(kx\cos\mu)dx \right] \sin w_e t$$

$$= [M_\theta] \sin w_e t$$

where, $M_o = 2pg\eta_a \int y(x) \cdot x \cdot \sin(kx\cos\mu) \cdot dx$

Also, $M_\theta = M_o \cos(w_e t - \in_1)$

where, $\in_1 = 90^\circ \Rightarrow \textit{phase between}$ the exciting moment & wave motion

Non-d amplitude $\quad M_o = \dfrac{M_o}{\frac{1}{2} Rg\eta_a BL^2}$

Parametrically, $\quad L'_w < \dfrac{1}{2} L_{ship} \Rightarrow small\, M_o$

$$L_{ship} \Rightarrow high\, M_o$$

Equation (16.4): $\quad I'_{yy}\ddot{\theta} + b\dot{\theta} + mg\overline{GN}_L\,\theta = M_o \sin w_e t$

Has solution**:**

$$\theta = Ae^{-vt}\sin(w_d t + \gamma) + \theta_a \sin(w_e t - \in_2)$$

where, $Ae^{-vt}\sin(w_d t + \gamma)$ is transient

$\in_2$ is the phase-exciting moment & pitching motion,

The equation can be showed as:

$$\in_2 = \tan^{-1}\frac{2\kappa\wedge}{1-\wedge^2}$$

$\therefore$ steady state $\theta = \theta_a \sin(w_e t - \in_2)$

where, $\theta_a = \theta_s t \cdot M_\theta$

$$= \frac{M_o}{e} \cdot M_\theta$$

$$= \frac{M_o}{mg\,\overline{GM}_L} \cdot \frac{1}{\sqrt{\left(1-\wedge^2\right)+4\kappa^2\wedge^2}} \quad ; \text{ where, } \kappa = {}^{v}\!/_{w_\theta} \quad \& \quad \wedge = {}^{w_e}\!/_{w_\theta}$$

NOTE: Max amplitude when $\wedge = 1-2\kappa^2$

Normally $\wedge \Rightarrow 0.85 - \square\, 0.90 \Rightarrow$ RESONANCE

Example: Obtain natural pitching period as well as pitching angles for 4 cycles for a ship with displacement of 15241t & length of 152.4m sailing in calm water.

The radius of gyration along pitching axis is 25% of ship length. The metacentric height for pitching is 152.4m. The added mass n.i. is 90% of mass m.i. the coefficient of damping for pitching is $774192g\; kN \cdot m \cdot \sec$. The motion starts from rest when $\theta = 5°$.

Solution: Free Pitching: $T_\theta = {}^{2\pi}\!/_{w_\theta}$

where, $w_\theta = \sqrt{\dfrac{c}{a}}$

$= \sqrt{\dfrac{Mg\,\overline{GM}_L}{1.9\,Mk_{yy}^2}}$ after canceling M at numerator & denominator we get

$$= \sqrt{\frac{9.81(152.4)}{1.9[0.25(152.4)]^2}}$$

= 0.736 *rad/sec*

$$\therefore T_\theta = \frac{2\pi}{0.736} = 8.54\sec$$

Equation of Motion for Free Pitching: $\theta = e^{-vt} \cdot A \cdot \sin\left(w_d t - \delta\right)$

$w_d = \sqrt{w_\theta^2 - v^2}$; $v = \dfrac{b}{2I_{yy}} = \dfrac{774192g}{2(1.9)15241(0.25\times 152.4)^2}$

= 0.90

$= \sqrt{(0.736)^2 - (0.090)^2}$

= 0.73 *rad/sec*

$t = 0 \Rightarrow \dot{\theta} = 0 \qquad \therefore \dot{\theta} = e^{-vt} A w_d \cos(w_d t - \delta) - v e^{-vt} A \sin(w_d t - \delta)$

$$\therefore 0 = A w_d \cos\delta + vA \sin\delta$$

$$\therefore 0 = A w_d \cos\delta - 5v$$

$t = 0 \Rightarrow \theta = 5° \qquad \therefore 5 = -A \sin\delta$

$$\therefore \cos\delta = \frac{5v}{A w_d} \qquad ; \sin\delta = \frac{-5}{A} \qquad \text{where, A = 5.04}$$

$$\therefore \tan\delta = \frac{-w_d}{v} = \frac{-0.73}{0.090} = -82.97°$$

$$\therefore \theta = e^{-0.09t} \cdot 5.04 \cdot \sin(0.73t + 1.45)$$

$$\therefore t = 0 \Rightarrow \theta = 5°$$

$$= 8.54'' \Rightarrow \theta = 2.3°$$

$$= 17.08'' \Rightarrow \theta = 1.06°$$

$$= 25.62'' \Rightarrow \theta = 0.48°$$

Example: A ship with length = 137.16*m*, breadth = 21.336*m* & displacement = 12700t moves head sea against 6.096*m* high waves that have 5.32 Sec period [encountering].

It has pitching metacentric height = 137.16*m*, coefficient of water plane area = 80%. pitching radius of gyration, K_{yy} = 33.53*m*. added mass m.i. = 54% of mass m.i. Non-d. damping coefficient for pitching, $\frac{b\sqrt{9L}}{MgL^2} = 0.154$ Non-d. exciting moment amplitude, $\frac{M_o}{\frac{1}{2} pgn_a L^2 B} = 0.25$.

<u>OBTAIN</u>
- Amplitude for pitching motion,
- Phase difference between pitching & wave motions,
- Variation of wave motion, pitching motion & exciting motion with time.

Solution: *Steady state equation of pitching motion:*

$$\theta = \theta_a \sin\left(w_e t - \in_2\right)$$

where, $\theta_a = \theta_{st} \cdot M_\theta$

$$= \frac{M_o}{mg\,\overline{GM}_L} \cdot \frac{1}{\sqrt{\left(1-\wedge^2\right)^2 + 4\kappa^2 \wedge^2}}$$

$$M_o = 0.25\left(\frac{1}{2}\right)1030(9.81)3.048(137.16)^2 21.336$$

= 1545251.5 kNm

$$\therefore \theta_{st} = \frac{1545251.5}{12700\,g} = 0.09\,rad$$

$$\wedge = \frac{w_e}{w_\theta} \qquad ; \qquad w_\theta = \sqrt{\frac{c}{a}}$$

$$= \sqrt{\frac{mg\,\overline{GM}_L}{I'_{yy}}}$$

$$\frac{2\pi}{5.32} \qquad = \sqrt{\frac{m \cdot g(137.16)}{1.54\,m\,33.53^2}}$$

$$\therefore \wedge = \frac{1.18}{0.88} = 1.34 \qquad = 0.88\ rad/sec$$

$$\kappa = \frac{v}{w_\theta} \qquad \text{where, } v = \frac{b}{2I'_{yy}} \qquad \text{and where, } b = \frac{0.154\,mg\,L^2}{\sqrt{gL}}$$

$$= \frac{0.154\,mg\,(137.16)^2}{\sqrt{9.81(137.16)}}$$

= (78.98) mg kN-m sec

$$\therefore v = \frac{(78.98)\cancel{gm}}{2(\cancel{12700})1731.4}$$

$$=1.54\,K_{yy}^{2}$$

= 0.224

$$\therefore k = \frac{0.224}{0.882}$$

= 0.254

$$\therefore M_{\theta} = \frac{1}{\sqrt{(1-1.34)^2 + 4(0.254)^2 1.34^2}}$$

= 0.96

$$\theta_a = 0.96(0.09)$$

= 0.086 *rad* = 4.93

Phase

Pitching Motion

Wave Motion

= Phase - Pitching Motion   Exciting Moment + Phase-exciting Moment   Wave Motion

$$\in^2 = \tan^{-1}\frac{2\kappa\wedge}{1-\wedge^2}$$

= 90

$$= \tan^{-1}\frac{2(0.254)(1.34)}{1-1.34^2}$$

= - 41

= 49

EQUATION OF WAVE MOTION: $\eta = \eta_a \cos w_e t$

EQUATION OF PITCHING MOTION: $\theta = \theta_a \sin\left(w_e t + 41°\right)$

$$= \theta_a \cos\left(w_e t - 49°\right)$$

EQUATION OF EXCITING MOMENT: $M_\theta = M_o \cos\left(w_e t - 90°\right)$

16.5 Rolling Motions

Free Rolling:

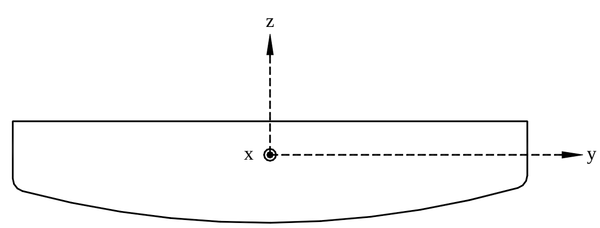
Equation of motion: $$a\ddot{\phi} + b\dot{\phi} + c\phi = 0 \tag{16.5}$$

where, $a\ddot{\phi}$ is the Inertial Moment, in which a is virtual mass m.i. for rolling & $\ddot{\phi}$ is angular acceleration.

$b\dot{\phi}$ is the Damping Moment, in which b is damping moment coefficient & $\dot{\phi}$ is angular speed.

$c\phi$ is the Restoring Moment, in which c is restoring moment coefficient & ϕ is angular displacement.

INERTIAL MOMENT:



a = m.i. of actual ship mass + m.i. of added mass

$$\left.\begin{aligned} &= mk_{xx}^2 + \delta m \cdot k_{xx}^2 \quad ; \\ &= mk_{xx}^2 \end{aligned}\right\}$$ where, δm < 20% of m □<u>Not Very Important</u>

DAMPING MOMENT:

The damping Moment is important as it may magnify rotation □□by□5 to 10 times. The damping moment is due to the generated waves (called the wave making resistance which is the largest in the magnitude), the friction between ship & water- (called the eddy making resistance), the bilge keel, fittings, the air resistance, the heat loss and the surface tension.

For wave making only: $$b_n = \frac{pg^2}{w_e^3}\left(\frac{Bn}{2}\right)^2 \overline{A}_\phi^2$$

Where for the Lewis forms:

$$\text{where, } \overline{A_\phi} = d_\phi \left(\frac{w_e^2 B_n}{2g}\right)^2$$

where, $d_\phi = f\left(\beta_n, \int_n'\right)$

where, $\beta_n = \frac{\int_n}{B_n T_n}$ in which 'n' is actual & where, $\int_n' = \frac{B_n}{2T_n}$ in which $\int_n'$ is not 'c/s'

<u>Note:</u> 1. Speeding ships have higher damping, 2. Damping moments $\neq f(b \cdot \phi)$ similarly, if large

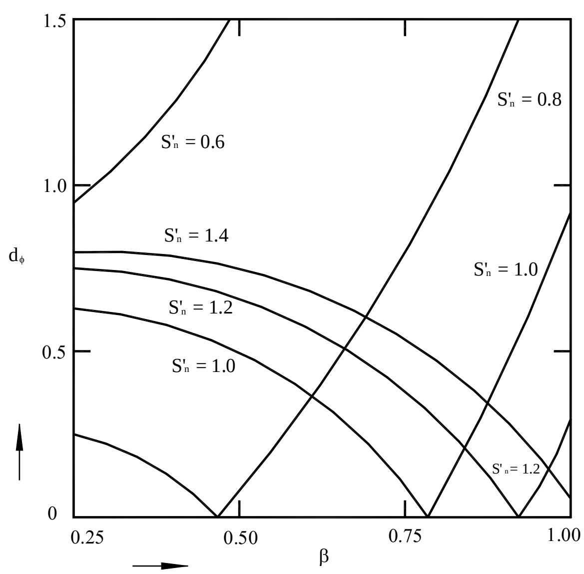
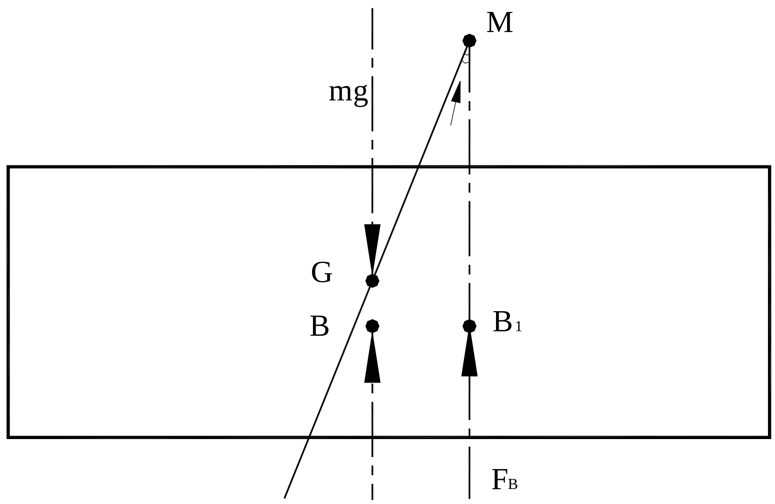


Fig. 16.4 Calculation of the damping moment

The restoring moment



$$C \cdot \phi = mg\,\overline{GM}\,\sin\phi$$

$$\approx mg\,\overline{GM}_T \cdot \phi \qquad \text{(if small)}$$

$$\therefore c = Mg\,\overline{GM}_T$$

Equation of motion: $$a\ddot{\phi} + b\dot{\phi} + c\phi = 0 \qquad (18.6)$$

$$\therefore \ddot{\phi} + \frac{b}{a}\phi + \frac{c}{a}\phi = 0$$

$$\therefore \ddot{\phi} + 2v\phi + w_\phi^2\phi = 0$$

where, $v = \dfrac{b}{2a} = \dfrac{b}{2I'_{xx}}$; $w_\phi = \sqrt{\dfrac{c}{a}} = \sqrt{\dfrac{Mg\,\overline{GM}_T}{I'_{xx}}} = \dfrac{2\pi}{T_\phi}$, T_ϕ is the undamped period

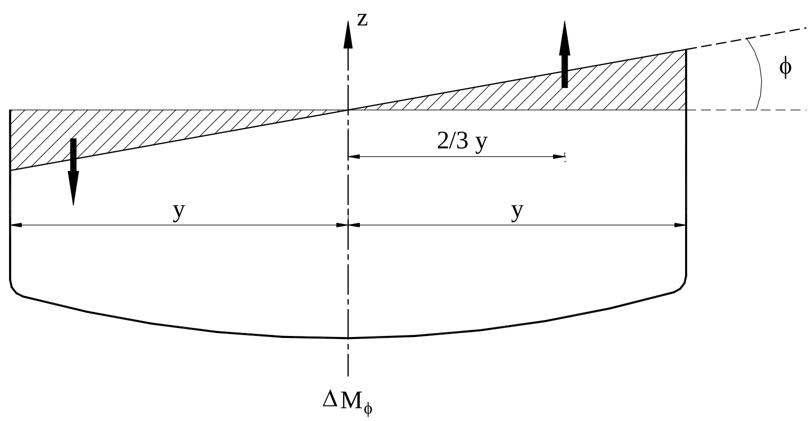
Solution: $$\phi = e^{-vt}\left[C_2\cos w_d t + C_2 \sin w_d t\right]$$

$$= e^{-vt} A\sin\left(w_d t - \delta\right)$$

where, $w_d = \sqrt{w_\phi^2 - v} = \dfrac{2\pi}{T_d}$; T_d is the damped period

The Forced Rolling motion:

Exciting Moment:



Equation of Motion: $a\ddot{\phi}+b\dot{\phi}+c\phi=M_{\phi}$ (16.6)

where, M_{ϕ} is the exciting moment due to change in buoyant force by waves.

For element of width d_x

$$\Delta M_{\phi}=2\left\{[\varsigma g\cdot volume]\frac{2}{3}y\right\}$$

Equation of Motion: $I'_{xx}\ddot{\phi}+b\dot{\phi}+mg\,\overline{GM}_T\,\phi=M_{\phi}\sin w_e t$

Solution: $\phi=e^{-vt}\left[C\cos w_d t+D\sin w_d t\right]+\phi_a\cdot\sin\left(w_e t-\in_2\right)$

where, $e^{-vt}\left[C\cos w_d t+D\sin w_d t\right]$ are free oscillations vanish

STEADY STATE $\phi=\phi_a\cdot\sin\left(w_e t-\in_2\right)$

where, $\phi_a=M_{\phi}\phi_{st}$ & where, $\in_2=\tan^{-1}\frac{2\kappa\wedge}{1-\wedge^2}$

in which $\phi_{st}=\frac{M_o}{c}$

$$=\frac{\propto'_M\cdot c}{c}$$

$$=\propto'_M$$

taking the above eqn.: $\phi_a = M_\phi \phi_{st}$; from which: $M_\phi = \dfrac{1}{\sqrt{\left(1-\wedge^2\right)+4\kappa^2\wedge^2}}$

where, $\wedge = \dfrac{w_e}{w_\phi}$ & $\kappa = \dfrac{v}{w_\phi}$, in which, $v = \dfrac{b}{2I'_{xx}}$

For max $M_\phi - \dfrac{\partial}{\partial \wedge}\left(M_\phi^2\right) = 0 \Rightarrow \wedge = 1-2\kappa^2$

$$\therefore M_\phi = \frac{1}{2\kappa\sqrt{1-\kappa^2}}$$

Example: A ship has length = 137.16*m*, mass, m = 12700t, radius of gyration, rolling, $k_{xx} = 9.39\text{m}$, transverse metacentric height, $\overline{GM}_T$ = 1.76*m*, coefficient of roll damping, b = 9910g kN-m-sec, its rolling added mass = 20% actual mass.

Obtain its natural & damped rolling period.

Solution: Undamped $T_\phi = \dfrac{2\pi}{w_\phi}$

$$w_\phi = \sqrt{\frac{c}{a}} = \sqrt{\frac{mg\,\overline{GM}_T}{I'_{xx}}}$$

$$= \sqrt{\frac{mg\,\overline{GM}_T}{1.2mk_{xx}^2}}$$

$$= \frac{1}{k_{xx}}\sqrt{\frac{g\,\overline{GM}_T}{1.2}}$$

= 0.404 rad/sec

$$\cdots T_\phi = \frac{2\pi}{0.404} = 15.55\,\text{sec}$$

$$T_d = \frac{2\pi}{w_d} = \frac{2\pi}{\sqrt{w_\phi^2 - v^2}}$$

$$v = \frac{b}{2I'_{xx}} = \frac{9910g}{2(1.2)mk_{xx}^2} = \frac{9910g}{2(1.2)12700(9.39)^2}$$

= 0.36/sec

$$\therefore T_d = \frac{2\pi}{\sqrt{0.404^2 - 0.036^2}}$$

= 15.61 sec

Example: A ship with 15240t displacement has transverse $\overline{GM}_T = 1.475m$ and rolling $k_{xx} = 90388m$. Its added mass m.i. = 20% of mass m.i. in rolling coefficient for roll damping=9910gkN-m-sec.

Plot the rolling motion time history for 3 periods if the ship is initially inclined at 7.

Solution: Equation of free rolling: $I'_{xx}\ddot{\phi} + b\dot{\phi} + mg\,\overline{GM}_T\,\phi = 0$

Has solution: $e^{-vt}\left[C\cos w_d t + D\sin w_d t\right]$

$$v = \frac{b}{2I'_{xx}} = \frac{9910g}{2(1.2)(15240)9.388^2}$$

= 0.0302

$$w_\phi = \sqrt{\frac{c}{a}} = \sqrt{\frac{mg\,\overline{GM}_T}{1.2\,m\,k_{xx}^2}}$$

$$= \sqrt{\frac{9.81(1.475)}{1.2(9.388)^2}}$$

= 0.37 *m*/*sec*

$$w_d = \sqrt{w_\phi^2 - v^2} = \sqrt{0.37^2 - 0.032^2}$$

= 0.37 *m*/*sec*

$t = 0 \Rightarrow \phi = 7^\circ \qquad \Rightarrow 7 = c$

$t = 0 \Rightarrow \dot{\phi} = 0 \qquad \Rightarrow \dot{\phi} = e^{-vt}\left[-Cw_d \sin w_d^t + Dw_d \cos w_d^t\right] + \left[C\cos w_d t + D\sin w_d^t\right](-v)e^{-vt}$

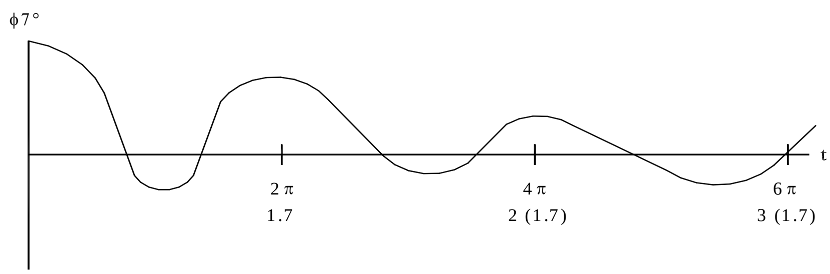
$$\therefore 0 = Dw_d + c(-v)$$

$$\therefore D = \frac{cv}{w_d} = \frac{7(0.0302)}{0.369}$$

$$= 0.573$$

$$\therefore \phi = e^{-0.0302t}\left[7\cos(0.369t) + 0.573\sin(0.369t)\right]$$

$$T_d = \frac{2\pi}{0.37} = 17''$$



Example: A ship of 137.16*m* length &12700t displacement has its transverse $\overline{GM}_T = 1.765m = 1$, radius of gyration, rolling = 9.388*m*, damping moment coefficient = 9910g kN-m-sec. Its added mass = 20% of mass m.i. in rolling.

If the ship is encountering 18.29*m* high waves at an angle of 150□□with wave direction with a speed of 20 knots. Plot $M_\phi \times \wedge$ for w_w= (0, 1.20) at 0.1 rad/sec intervals.

What is the amplitude of maximum roll?

Solution: $$M_\phi = \frac{1}{\sqrt{\left(1-\wedge^2\right)^2 + 4\kappa^2 \wedge^2}}$$

where, $\left(1-\wedge^2\right)^2 = \dfrac{w_e}{w_\phi}$ & $4\kappa^2 \wedge^2 = \dfrac{\left[b/2a\right]}{w_\phi}$

$$w_e = w_w\left(1 - \frac{w_w}{g} V\cos\mu\right)$$

$$= w_w\left[1 - \frac{w_w}{9.81}\,20(0.514)\cos 150^\circ\right]$$

$$= w_w\left[1 + w_w(0.9075)\right]$$

$$w_\phi = \sqrt{\frac{c}{a}}$$

$$= \sqrt{\frac{mg\,\overline{GM}_T}{1.2\,m\,k_{xx}^2}}$$

$$= \sqrt{\frac{9.81(1.756)}{1.2(9.388)^2}}$$

= 0.405 rad/sec

Damping factor $\kappa = \dfrac{v}{w_\phi} = \dfrac{\left\lfloor \frac{b}{2\,m'k_{xx}^2} \right\rfloor}{w_\phi}$

$$= \frac{\left\lfloor \frac{9910\,g}{2(1.2)12700\,(9.388)^2} \right\rfloor}{0.405}$$

$$= \frac{0.036}{0.405}$$

= 0.089

w_w $rad/\sec$	$w_e = w_w\left[1+0.9075w_w\right]$ $rad/\sec$	$\wedge = \frac{w_e}{w_\phi}$	$\mu = \frac{1}{\sqrt{\left(1-\wedge^2\right)^2 + 4\kappa^2\wedge^2}}$
0	0	0	1
0.1	0.109	0.269	1.078
0.2	0.236	0.583	1.497
0.3	0.382	0.942	4.923
0.4	0.545	1.346	1.179
0.5	0.727	1.795	0.445
0.6	0.927	2.288	0.235
0.7	1.145	2.827	0.143
0.8	1.381	3.410	0.094
0.9	1.635	4.038	0.065
1.0	1.908	4.711	0.047
1.1	2.198	5.428	0.035
1.2	2.507	6.191	0.027
1.3	2.834	6.998	0.021
1.4	3.179	7.850	0.016
1.5	3.543	8.747	0.013
1.6	3.924	9.689	0.011
1.7	4.324	10.676	0.009
1.8	5.177	11.707	0.007
1.9	5.177	12.783	0.006
2.0	5.681	13.904	0.005

Maximum Amplitude: $(\phi_a)_{\max.} = (M_\phi)_{\max.} : \phi_{st}$

$$\phi_{st} = \frac{M_o}{c} = \frac{mg\,\overline{GM}_T \cdot \propto'_m}{mg\,\overline{GM}_T} = \propto'_m = \propto_m \sin\mu = k \cdot n_a \cdot 0.5$$

(*) $k = \frac{2\pi}{L_w}$; (*) $L_w = \frac{gT_w^2}{2\pi} = \frac{9.81}{2\pi}\left[\frac{2\pi}{w_w\left(for\max.M_\phi\right)}\right]^2$

For max. M_ϕ ; $\wedge = \sqrt{1-2\kappa^2}$

$$= \sqrt{1-2(0.089)^2}$$

$= 0.992$

From table for $\wedge = 0.092 \Rightarrow w_w = 0.312 rad/\sec$; $\mu = 5.64$

(*)From the above equation: $L_w = 1.561\left(\frac{2\pi}{0.312}\right)^2$

$$= 633.07m$$

$$k = \frac{2\pi}{633.07}$$

$$= 0.0099$$

$\therefore (\phi_a)_{\max} = 5.64(0.5)0.0099(9.145)$ $\lfloor = M_\phi k \eta_a 0.5 \rfloor$

$= 0.255\ rad$ where, $\lfloor M_\phi = 5.64; k = 0.0099; \eta = 9.145 \rfloor$

$= 146$

Example: The equation of rolling motion of a ship is $\ddot{\phi} + 0.0724\dot{\phi} + 0.164\phi = \propto'_m\, w_\phi^2 \sin w_e t$ the maximum effective wave slope is $\frac{\pi}{20} rad$. Plot natural & forced rolling time histories for an encountering frequency of 0.2 *rad/sec* assuming that the ship is still and upright when hit by waves.

Solution: $\phi = e^{-vt}\left[C_1 \cos w_d t + C_2 \sin w_d t\right] + \phi_a \cdot \sin\left(w_e t - \in_2\right)$

where, $v = \frac{b}{2a}$;

$w_d = \sqrt{w_\phi^2 - v^2}$ in which, $w_\phi^2 = \sqrt{\frac{c}{a}}$;

$\phi_a = M_\phi \cdot \propto'_m$ in which, $M_\phi = \frac{1}{\sqrt{\left(1 - \wedge^2\right)^2 + 4\kappa^2 \wedge^2}}$;

$$\in_2 = \tan^{-1}\left(\frac{2\kappa \wedge}{1 - \wedge^2}\right)$$

$$\wedge = \frac{w_e}{w_\phi} = \frac{0.2}{\sqrt{\frac{0.164}{1}}} = \frac{0.2}{0.405}$$

$= 0.494$

$$v = \frac{b}{2a} = \frac{0.0724}{2.1} = 0.0362$$

$= 0.0362$

$$w_d = \left(0.485^2 - 0.0362^2\right)$$

$= 0.403$ *rad/sec*

$$\kappa = \frac{v}{w_\phi} = \frac{0.0362}{0.405}$$

$= 0.089$

$$M_\phi = \frac{1}{\sqrt{\left(1-\wedge^2\right)^2 + 4\kappa^2 \wedge^2}} = \frac{1}{\sqrt{\left(1-0.495^2\right)^2 + 4(0.089)^2 0.494^2}}$$

$= 1.316$

$$\phi_a = M_\phi \cdot \phi_{st} = M_\phi \cdot \propto'_m = 1.316\left(\frac{\pi}{20}\right)$$

$= 0.206$ *rad* $= 11.83$

$$\in_2 = \tan^{-1}\left(\frac{2\kappa \wedge}{1-\wedge^2}\right) = \tan^{-1}\left(\frac{2(0.089)0.494\kappa \wedge}{1-0.494^2}\right)$$

$= 0.116$ *rad* $= 6.659$

$$t = 0,\ \phi = 0 \Rightarrow 0 = C_1 - \phi_a \sin \in_2 \Rightarrow C_1 = \phi_a \sin \in_2 = 0.024$$

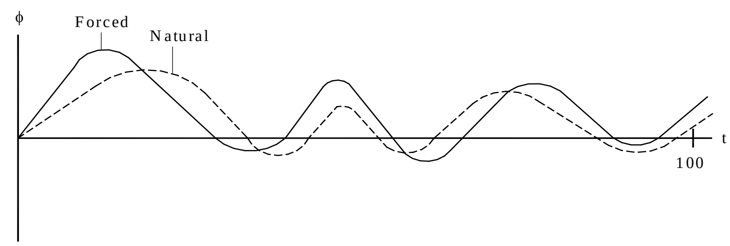
$$\dot{\phi} = e^{-vt}\left[-C_1 w_d \sin w_d t + C_2 w_d \cos w_d t\right] + \left(C_1 \cos w_d t + C_2 \sin w_d t\right)(-v)e^{-vt} + \phi_a w_e \cos\left(w_e t - \in_2\right)$$

$$t = 0,\ \dot{\phi} = 0 \Rightarrow 0 = C_2 w_d - C_1 v + \phi_a w_e \cos \in_2$$

$$\therefore C_2 = \frac{\left[C_1 v - \phi_a w_e \cos \in_2\right]}{w_d}$$

$= -0.099$

$$\therefore \phi = e^{-0.0362t}\left[0.024\cos(0.403t) - 0.099\sin(0.403t)\right] + 0.206\cdot\sin(0.2t - 0.116)$$



Example: At what heading angle against the waves, the largest rolling is expected, if a ship with a natural roll period of 15 seconds is sailing with a speed of 35 knots & length of waves is 274.32*m*.

Solution: For largest rolling, $\wedge = \dfrac{w_e}{w_\phi} \approx 1$

i.e. $w_e \approx w_\phi$

Now, $w_e = w_w\left[1 - \dfrac{w_w}{g}V\cos\mu\right]$

where, $w_e \approx w_\phi$; $w_w = \sqrt{\dfrac{2\pi g}{L_w}}$; $V = 0.514(35)$

$= \dfrac{2\pi}{15}$; $= \sqrt{\dfrac{2\pi g}{274.3^2}}$

= 0.419 *rad/sec* ; = 0.474 *rad/sec* ; = 17.99 *m/sec*

This gives $\mu = 82.3°$

[NOTE: To have $w_e \neq w_\phi$ change $\mu\, or\, V$]

CHAPTER 17

MOTION RESPONSE OF COMPLIANT STRUCTURES

17.1 Freely floating buoy:

Consider the sectional elevation of a freely floating cylindrical buoy before and after the wave attack as shown in Fig. 17.1 below:

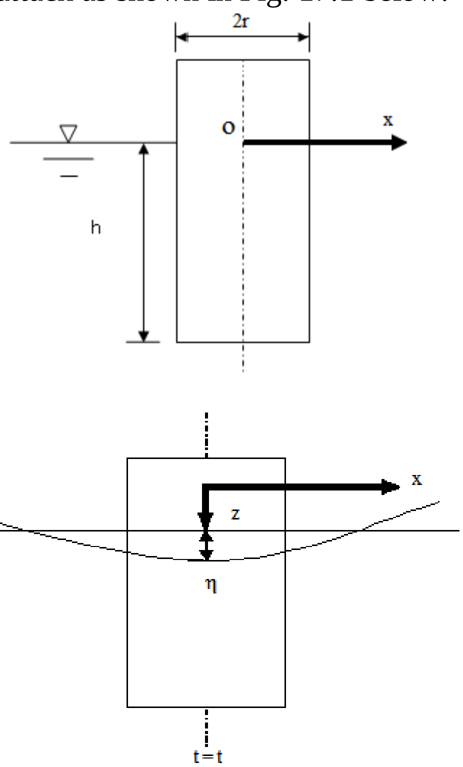


Fig. 17.1 Freely floating cylindrical buoy before and after the wave attack

Let z=heaving at time 't'; η = Sea surface elevation at 't'

Relative motion of the buoy with respect to sea water is due to forces induced by the added mass, damping , stiffness (hydrostatic).

Hence $m\ddot{z} = A\left(\ddot{\eta} - \ddot{z}\right) + B\left(\dot{\eta} - \dot{z}\right) + c(\eta - z)$ (17.1)

Where $m\ddot{z}$ is mass of buoy= $\rho\pi r^2 h$

$A\left(\ddot{\eta} - \ddot{z}\right)$ is added mass $\approx \frac{1}{3}\rho\pi r^2$ (Tables available for different shapes)

$B\left(\dot{\eta}-\dot{z}\right)$ is damping coefficient

$c(\eta - z)$ is stiffness coefficient = $\rho g \pi r^2$

(Note : cz= restoring force)

Hence $(m+A)\ddot{z}+B\dot{z}+cz = F(t) = A\ddot{\eta}+B\dot{\eta}+c\eta$ (17.2)

Let $\eta = a\cos\omega t$

$$\dot{\eta} = -a\omega\sin\omega t$$

$$\ddot{\eta} = -a\omega^2\cos\omega t$$

Hence $F(t) = A(-a\omega^2)\cos\omega t + B(-a\omega)\sin\omega t + ca\cos\omega t$

$$= a\left[(c - a\omega^2)\cos\omega t - B\omega\sin\omega t\right]$$

$= F_a\cos(\omega t+\varepsilon)$ ------ ε is between wave force and wave motion.

Where $\dfrac{F_a}{a} = \sqrt{(c-a\omega^2)^2 + (B\omega)^2}$

$$\varepsilon = \tan^{-1}\left[(-B\omega)/(c-a\omega^2)\right]$$

Hence equation of motion :

$$(m+A)\ddot{z}+B\dot{z}+cz = F_a\cos(\omega t+\varepsilon) \qquad (17.3)$$

has solution $z = z_a\cos(\omega t+\varepsilon+\phi)$ - ϕ is the angle between heaving motion and wave force.

$$\text{Where } \frac{z_a}{F_a} = \frac{1}{\sqrt{\left(c-(m+A)\omega^2\right)^2 + (B\omega)^2}} \qquad (17.4)$$

$$\phi = \tan^{-1}\left[B\omega/\left(c-(m+A)\omega^2\right)\right] \qquad (17.5)$$

Heaving of a semi-submersible

- It is not monohulls like ships
- For deeply submerged pontoons, wave pressure is low for low ‘z’.
- For Small A_{wp} and large submerged volume , Long T_n for heave , pitch roll is not of the order of T_{wave}

17.2 Semi-submersible: FREE UNDAMPED HEAVING

Circular cross-section pontoons

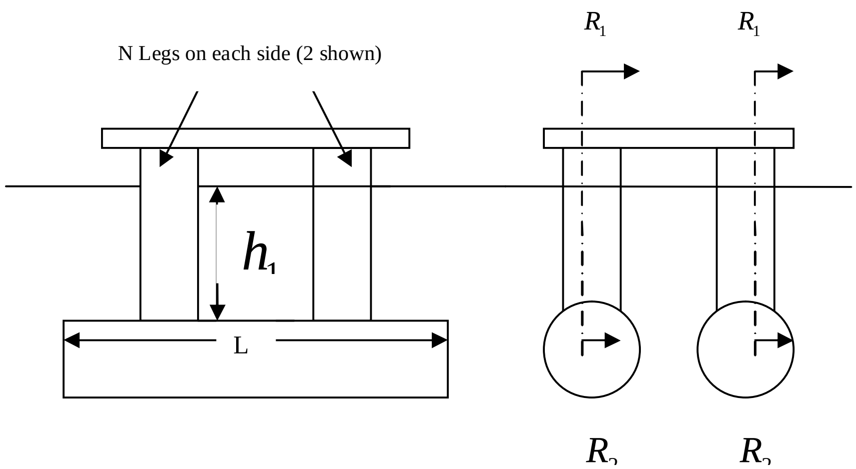


Fig. 17.2 Circular C/S Pontoons

The equation of motion is: $(m+A)\ddot{z}+cz=0$ (17.6)

Undamped natural period:

$$T_0 = 2\pi\sqrt{\frac{m+A}{c}}$$

$$= 2\pi\sqrt{\frac{2\rho\left[\pi R_2^2 L + n\pi R_1^2 h_1\right] + 2\pi R_2^2 L\rho}{2n\rho g\pi R_1^2}}$$

add/mass coefficient for vertical motion =1 for circle

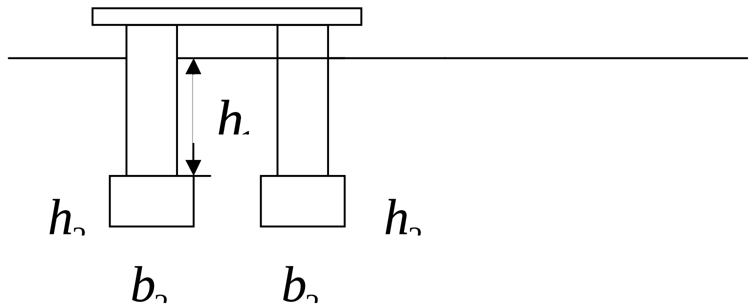
$$= 2\pi\sqrt{\frac{1}{g}\left(h_1 + \frac{2R_2^2 L}{nR_1^2}\right)}$$

Hence $T_0 = 2\pi\sqrt{\frac{h_1}{g}(1+2\alpha)}$ (17.8)

Where α = pontoon volume/ submerged column volume= $\frac{2\pi R_2^2 L}{2n\pi R_1^2 h_1}$

(17.9)

Rectangular cross-section pontoons



RECTANGULAR C/S PONTOONS

Fig. 17.3 Rectangular cross-section pontoons

$$T_0 = 2\pi\sqrt{\frac{2\rho b_2 h_2 L + 2\rho n\pi R_1^2 h_1 + 2C_M \rho b_2 h_2 L}{2n\rho g\pi R_1^2}}$$

$$= 2\pi\sqrt{\frac{1}{g}\left(h_1 + \frac{(1+C_m)b_2 h_2 L}{n\pi R_1^2}\right)}$$

Hence $T_0 = 2\pi\sqrt{\frac{h_1}{g}\left(1+(1+C_m)\alpha\right)}$ (17.10)

Where C_m is a function of $\left(\frac{b_2}{h_2}\right)$ (available in Tables) $\alpha = \frac{2b_2 h_2 L}{2n\pi R_1^2 h_1}$

Circular columns with circular base:

(Occur during the construction of concrete platforms. See Fig. 17.4)

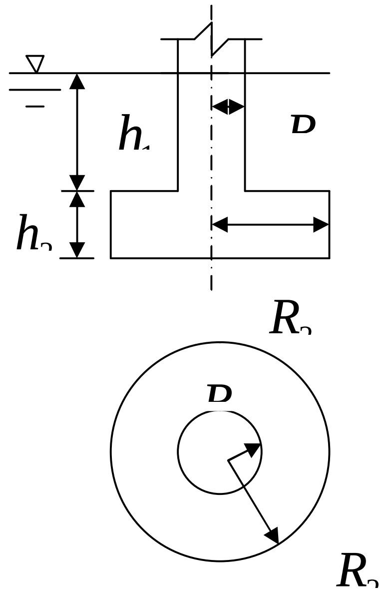


Fig. 17.4 Circular columns with circular base.

$$T_0 = 2\pi\sqrt{\frac{m+A}{c}}$$

$$= 2\pi\sqrt{\frac{\left(\rho\pi R_1^2 h_1 + \rho\pi R_2^2 h_2\right) + C_M\,\rho\pi R_2^3}{\rho g \pi R_1^2}} \quad \text{where } C_M = f\left(R_1 R_2 h_2\right)$$

$$= 2\pi\sqrt{\frac{h_1}{g}\left(1 + \frac{R_2^2 h_2}{R_1^2 h_1} + \frac{R_2^3 C_M}{R_1^2 h_1}\right)}$$

$$= 2\pi\sqrt{\frac{h_1}{g}\left(1 + \alpha + \frac{R_2^2 h_2 R_2 C_M}{R_1^2 h_1 h_2}\right)}$$

$$T_0 = 2\pi\sqrt{\frac{h_1}{g}\left(1 + \alpha\left(1 + \frac{R_2 C_M}{h_2}\right)\right)} \tag{17.11}$$

FORCED UNDAMPED HEAVING:

Consider a four column semi-submersible barge (Fig. 17.5)

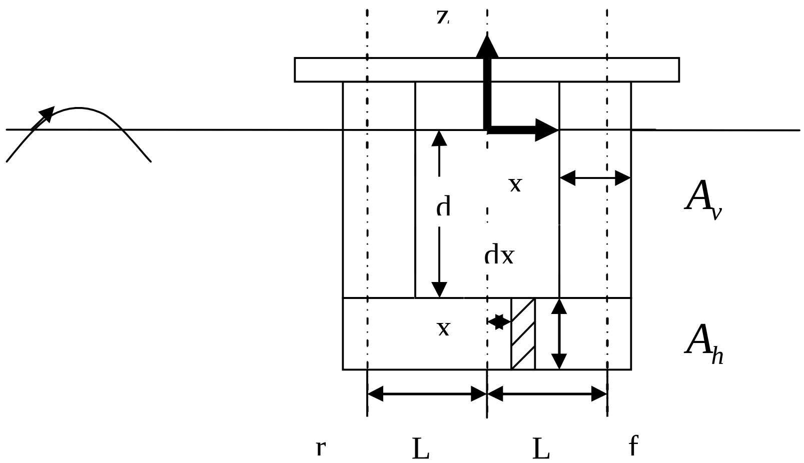


Fig. 17.5 A semi-sub barge

Equation of motion :

$$(m+A)\ddot{z}+cz=F(t)$$

$m=4\rho A_v d+4\rho A_h L$ --------- ($2\rho A_h 2L$ =2 pontoons)

$A=4\rho C_m A_h L$

$C=4\rho g A_v$

Vertical wave exciting force (neglecting drag) :

$F(t)=\int_{-L}^{L} 2\rho A_h(1+C_m)\dot{w}_x\,dx+2A_v(p_f+p_r)$ --------[(For $\dot{w}_x$ @ z=-d) , (p_f @ z=-d, x=L) , (p_r @ z=-d, x=-L)]

Assuming deep water conditions in linear wave theory,

$$\dot{w}=-a\omega^2 e^{-kd}\sin(kx-\omega t)$$

$$p=\frac{a\rho\omega^2}{k}e^{-kd}\sin(kx-\omega t)$$

Substituting

$$F(t)=\int_{-L}^{L}2\rho A_h(1+C_m)\left[-a\omega^2 e^{-kd}\sin(kx-\omega t)\right]dx+2A_v\left[\frac{a\rho\omega^2}{k}e^{-kd}\sin(kL-\omega t)\right]+2A_r\left[\frac{a\rho\omega^2}{k}e^{-kd}\sin(k(-L)-\omega t)\right]$$

$$=4a\rho g e^{-kd}A_h(1+C_m)\sin kL\sin\omega t+4a\rho g e^{-kd}(-A_v)(1+C_m)\cos kL\sin\omega t$$

$$=4a\rho g e^{-kd}\left[(-A_v)\cos kL+A_h(1+C_m)\sin kL\right]\sin\omega t \qquad (17.12)$$

Substituting in $F(t) = F_a \sin(\omega t + \varepsilon)$ and solving $(m + A)\ddot{z} + cz = F(t)$

We get $Z(t) = Z_a \sin(\omega t + \varepsilon + \phi)$

If damping due to drag is introduced, we get:

Forced damped heaving:

$$(m + A)\ddot{z} + B_v \left|\dot{z}\right| \dot{z} + cz = F(t)$$

To linearize this equation, the energy dissipated at resonant motion at frequency 'ω_0'by nonlinear and linear terms is equated , giving

$$B_v \left|\dot{z}\right| \dot{z} = \left(\frac{8}{3\pi} \dot{z}_{\max} B_v \right) \dot{z}$$

To solve subsequently :

Assuming B_v =10% of critical , then get $\dot{z}_{\max}$ accordingly.(i.e. maximum heaving)

$\dot{z}_{\max}$ is determined finally by iterations.

17.3 Articulated tower

It is a buoyant structure (its stability is due to its buoyancy).

It has a large tank at the surface, which brings it to the inertia ominant regime.

It is used in the early production system/SALM/SALS.

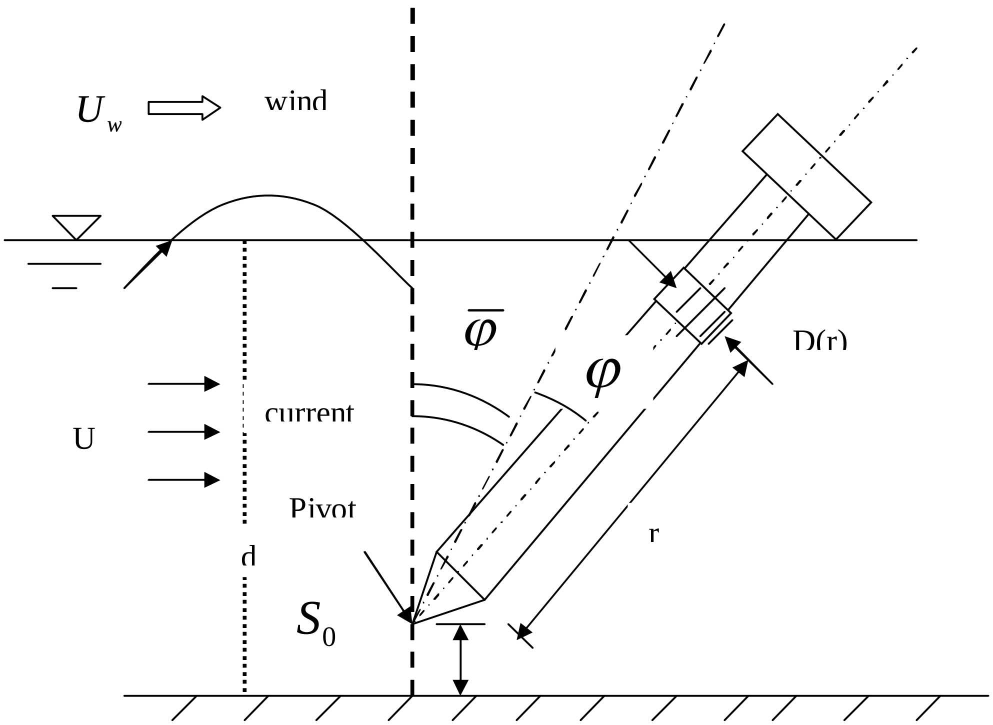


Fig. 17.6. Articulate tower

In its simplest form, the motion has one DOF along the wave direction.

Wind, current create disturbing moments which are restored by tank buoyancy. The static offset angle is $\overline{\varphi}$ where, φ is the angular displacement due to waves with respect to static equilibrium. The motion is damped by radiated waves and viscous drag while motion is restored by change in buoyancy at SWL and from equilibrium.

Case 1: Damping linear (no drag)

$$I\ddot{\varphi} + C\dot{\varphi} + R\varphi = M_0 e^{i(\delta - \omega t)} \tag{17.13}$$

Where $I\ddot{\varphi}$ is inertia force moment

$C\dot{\varphi}$ is damping force moment

$R\varphi$ is restoring force moment

M_0 is exciting moment amplitude

δ is phase angle of exciting force (obtained from exciting force given by the diffraction theory.)

The above equation has solution:

$$\varphi = \varphi_0 e^{i(\beta - \omega t)}$$

Where φ_0 is motion amplitude

β is motion phase angle

Hence $\dot{\varphi} = \varphi_0 (-i\omega) e^{i(\beta - \omega t)}$

$$\ddot{\varphi} = \varphi_0 (-i\omega)^2 e^{i(\beta - \omega t)} = -\varphi_0 \omega^2 e^{i(\beta - \omega t)}$$

Substituting, $-I\varphi_0 \omega^2 e^{i(\beta - \omega t)} + C\varphi_0 (-i\omega) e^{i(\beta - \omega t)} + R\varphi_0 e^{i(\beta - \omega t)} = M_0 e^{i(\delta - \omega t)}$

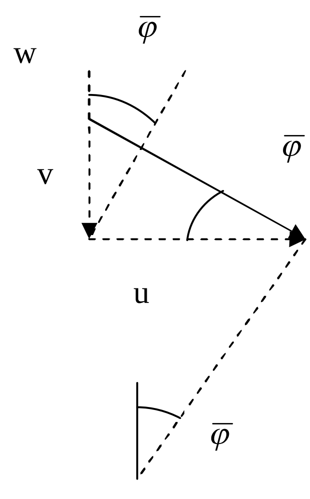
Hence $-I\varphi_0 \omega^2 e^{i\beta} + C\varphi_0 (-i\omega) e^{i\beta} + R\varphi_0 e^{i\beta} = M_0 e^{i\delta}$

$$\varphi_0 e^{i\beta} = \frac{M_0 e^{i\delta}}{(R - \omega^2 I) - i\omega C} \quad \text{-------} \qquad (17.14)$$

This equation will give φ_0 and β values.

Case 2: Damping Non-linear:

This is valid if drag (producing moment at pivot) is considered.



Normal velocity $w = u\cos\overline{\varphi} + v\sin\overline{\varphi}$

Hence $$B\left(\dot{\varphi}\right) = \frac{1}{2}\rho \int_0^{\frac{d-s_0}{\cos\overline{\varphi}}} C_D D(r)\left|r\dot{\varphi} - w\right|\left(r\dot{\varphi} - w\right) r dr \qquad (17.15)$$

Equation of motion:

$$I\ddot{\varphi} + B\left(\dot{\varphi}\right) + C\dot{\varphi} + R\varphi = M_0 e^{i(\delta - \omega t)} \qquad (17.16)$$

Equation (17.16)is solved numerically. The linearization of $B\left(\dot{\varphi}\right)$ can give closed form solution. This is good in inertia dominance and when $(T_n)_{Tower} \neq (T_n)_{waves}$

Thus, equation (17.16) has solution:

$$\varphi = \varphi_0 e^{i(\beta - \omega t)}$$

Let $w = w_0 e^{i(\alpha - \omega t)}$; where w_0 and α are known from $w = u\cos\overline{\varphi} + v\sin\overline{\varphi}$

Hence $$r\dot{\varphi} - w = r\varphi_0(-i\omega)e^{i(\beta - \omega t)} - w_0 e^{i(\alpha - \omega t)}$$
$$= \left(-i\omega r\varphi_0 e^{i\beta} - w_0 e^{i\alpha}\right)e^{-i\omega t}$$

Based upon Fourier Series Linearization:

$$u|u| \approx \frac{8}{3\pi} u_0 u$$

$$\left|\left(r\dot{\varphi} - w\right)\right|\left(r\dot{\varphi} - w\right) \approx \frac{8}{3\pi}\left|i\omega r\varphi_0 e^{i\beta} + w_0 e^{i\alpha}\right|\left(r\dot{\varphi} - w\right)$$

Substituting in (17.16):

$$I\ddot{\varphi} + \frac{1}{2}\rho \int_0^{\frac{d-s_0}{\cos\overline{\varphi}}} C_D D(r)\left[\frac{8}{3\pi}\left|i\omega r\varphi_0 e^{i\beta} + w_0 e^{i\alpha}\right|\left(r\dot{\varphi} - w\right)\right] r dr + C\dot{\varphi} + R\varphi = M_0 e^{i(\delta - \omega t)}$$

$$I\ddot{\varphi} + \left[\frac{4}{3\pi}\rho \int_0^{\frac{d-s_0}{\cos\overline{\varphi}}} C_D \left|i\omega r\varphi_0 e^{i\beta} + w_0 e^{i\alpha}\right| r^2 D(r) dr\right]\dot{\varphi}$$

$$-\left[\frac{4}{3\pi}\rho\int_0^{\frac{d-s_0}{\cos\overline{\varphi}}} C_D\left|i\omega r\varphi_0 e^{i\beta}+w_0 e^{i\alpha}\right| w_0 e^{i\alpha} rD(r)dr\right]e^{-i\omega t}+C\dot{\varphi}+R\varphi=M_0 e^{i(\delta-\omega t)}$$

Therefore $$I\ddot{\varphi}+B_1(\varphi_0,\beta)\dot{\varphi}+C\dot{\varphi}+R\varphi=M_0 e^{i(\delta-\omega t)}+B_2(\varphi_0,\beta)e^{-i\omega t} \qquad (17.17)$$

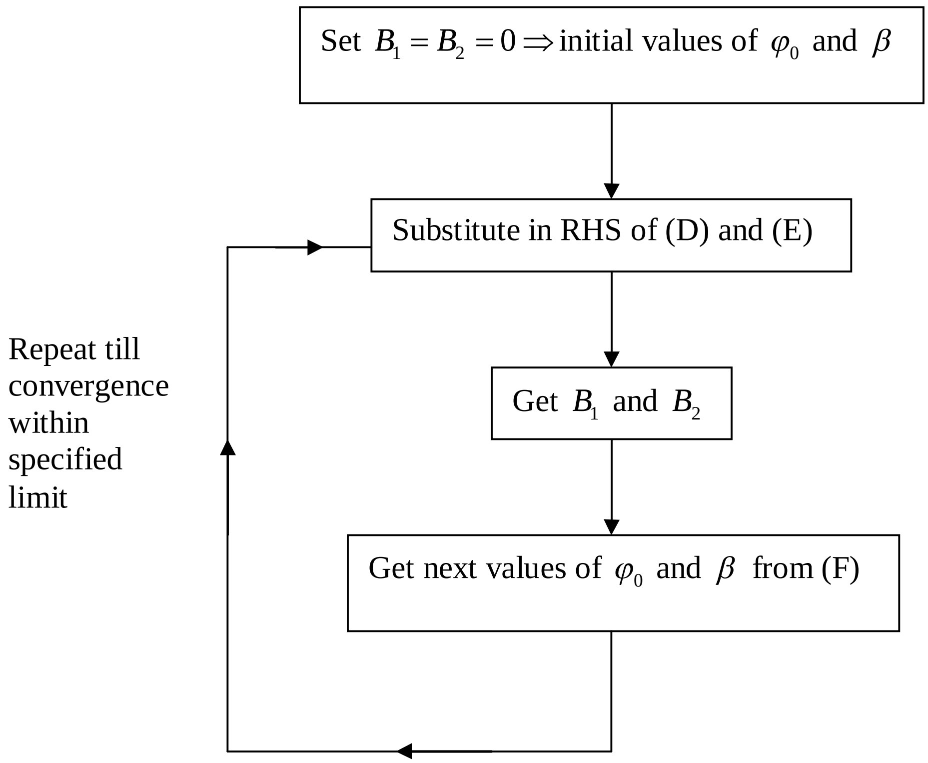
Where $$B_1(\varphi_0,\beta)=\frac{4}{3\pi}\rho\int_0^{\frac{d-s_0}{\cos\overline{\varphi}}} C_D\left|i\omega r\varphi_0 e^{i\beta}+w_0 e^{i\alpha}\right| r^2 D(r)dr \qquad (17.18)$$

$$B_2(\varphi_0,\beta)=\frac{4}{3\pi}\rho\int_0^{\frac{d-s_0}{\cos\overline{\varphi}}} C_D\left|i\omega r\varphi_0 e^{i\beta}+w_0 e^{i\alpha}\right| w_0 e^{i\alpha} rD(r)dr \qquad (17.19)$$

Substituting $\varphi=\varphi_0 e^{i(\beta-\omega t)}$ in (17.17),

$$\varphi_0 e^{i\beta}=\frac{M_0 e^{i\delta}+B_2(\varphi_0,\beta)}{(R-\omega^2 I)-i\omega[B_1(\varphi_0,\beta)+C]} \qquad (17.20)$$

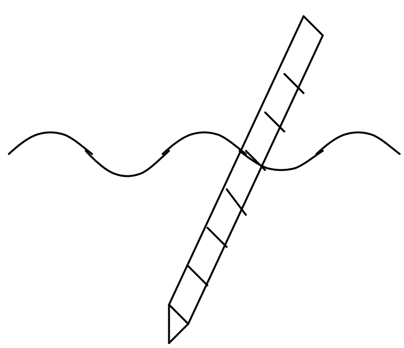
This has total iterative solution. To obtain φ_0 and β : follow the steps below:



To find the shear force and the bending moment at any point in the tower:

For each segment, (Point) from φ_0 and β,

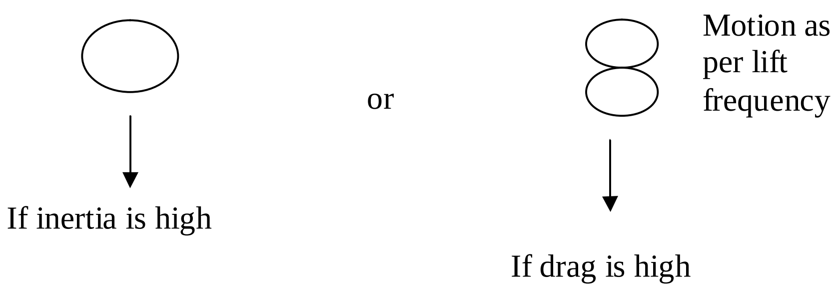
$I\ddot{\varphi}$, $B_1\dot{\varphi}$, $C\dot{\varphi}$, $R\varphi$ are determined normal to axis to any phase angle and added to steady wind and current load.



Case 3: IN-LINE AND TRANSVERSE FORCES

In actual, waves induce for the tower a transverse motion also:

Hence top of tower may move as follows:



This shows that there is 2-DOF system. Let φ =displacement in-line

φ_L = transverse displacement

Equation of motion (Transverse):

$$I\ddot{\varphi}_L + C\dot{\varphi}_L + R\varphi_L = \frac{1}{2}\int_0^{\frac{d-s_0}{\cos\bar{\varphi}}} C_L D(r)\left|u - r\dot{\varphi}\right|\left(u - r\dot{\varphi}\right) r\,dr \qquad (17.21)$$

Where $u = u_0 e^{i\omega t}$ (If phase $u\eta = 0$)

Similar to in-line motion:

$$\left|u - r\dot{\varphi}\right|\left(u - r\dot{\varphi}\right) \approx \frac{8}{3\pi}\left|i\omega r\varphi_0 e^{i\beta} + u_0\right|\left(u - r\dot{\varphi}\right)$$

Solution of equation of motion (Transverse) can be written as:

$$\varphi_{L_0} e^{i\beta_L} = \frac{B_2(\varphi_0, \beta) + i\omega B_1(\varphi_0, \beta)\varphi_0 e^{i\beta}}{\left[\left(R - \omega^2 I\right) - i\omega C\right]} \qquad (17.22)$$

Where
$$B_2(\varphi_0, \beta) = \frac{4}{3\pi}\rho\int_0^{\frac{d-s_0}{\cos\bar{\varphi}}} C_L\left|u_0 + i\omega r\varphi_0 e^{i\beta}\right| u_0 r D(r)\,dr$$

$$B_1(\varphi_0, \beta) = \frac{4}{3\pi}\rho\int_0^{\frac{d-s_0}{\cos\bar{\varphi}}} C_L\left|u_0 + i\omega r\varphi_0 e^{i\beta}\right| r^2 D(r)\,dr$$

(Note: φ_L depends up on in-line motion magnitude and phase

Equations can as well be derived using Taylor's series)

$$\left|u-\dot{x}\right|\left(u-\dot{x}\right)\approx|u|u-2|u|\,\dot{x}\approx\frac{8}{3\pi}u_0u-\frac{4}{\pi}u_0\,\dot{x} \qquad (17.23)$$

Where $\dot{x}$ is structure velocity

$|u|u$ is expanded by Fourier Series

$|u|=\frac{2}{\pi}u_0$ Temporal average

17.4 Tension Leg Platform

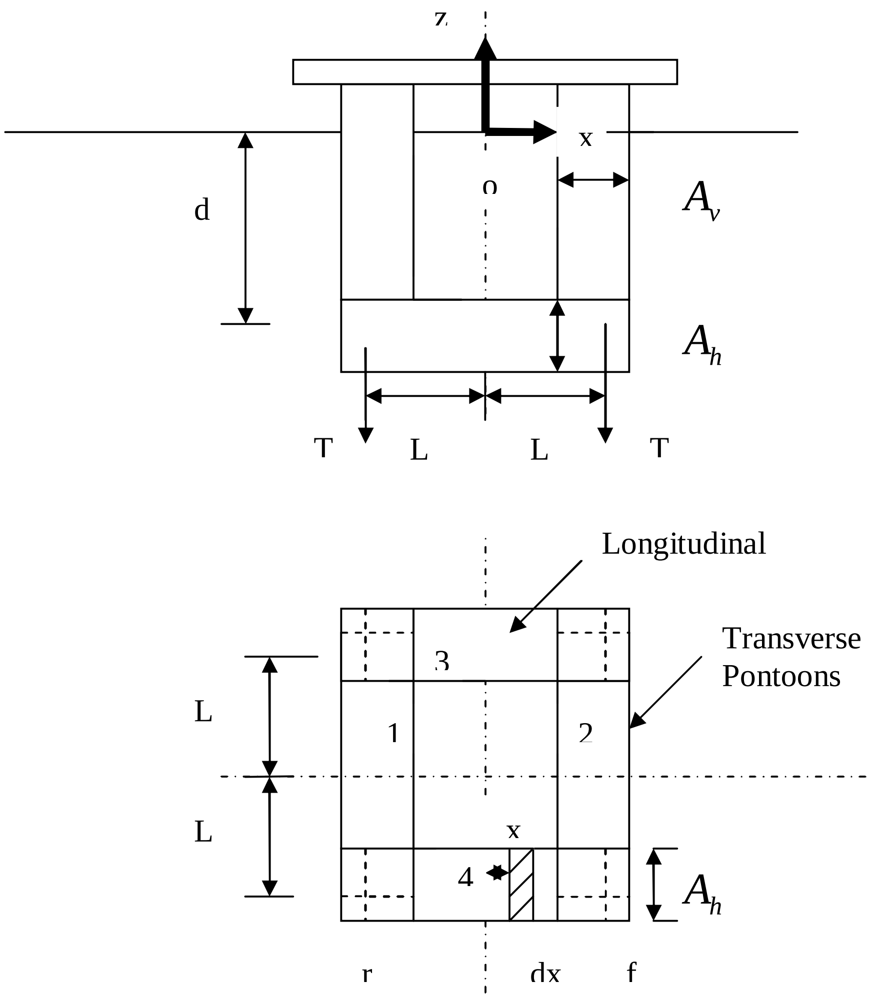
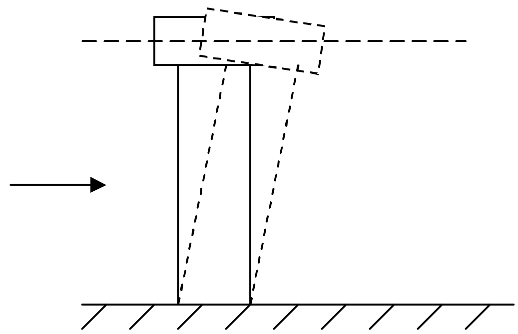


Fig. 17.7 Vertical section and plan (schematic) of a TLP

The TLP has advantage that its stiffness and tether-tension leads to less heave, pitch and roll caused by the waves. However the wind leads to surge and sway offsets. The current (steady) leads to lowering (set-down) as shown below:



The deeper locations of pontoons cause low wave pressure, which is advantages. Further for any buoyancy change at certain wave frequency, the upward wave force on vertical columns can equal the downward wave force on the horizontal pontoons and this can lead to cancellation of forces.

Analysis of a TLP is same as that of a semi-submersible, but involves additional tether stiffness matrix. Due to this, $T_{heave} \approx$ 3 to 5 sec. as against the semi-sub. $T_{heave} \approx 18$ to 25 sec. Even then for heave-sensitive operations, the heave study is important.

Equation of undamped forced heaving:

$$(m+A)\ddot{z}+(c+c_t)z = F_z(t) \tag{17.24}$$

Where $m = 4\rho A_v d + 4\rho A_h 2L - \dfrac{4T}{g}$

$A = 4\rho A_h 2LC_{mz}$ where, C_{mz} is added mass for pontoon cross-section in heave (to be considered for only members whose axis is normal to flow)

$C = 4\rho g A_v$ (members near SWL only to be considered). This is hydrostatic stiffness. (Note: c x z = additional buoyancy).

$C_t = 4\lambda^1$ (λ^1 is axial stiffness for each tether $= \dfrac{T}{Length}$)

Neglecting the wave drag force, the vertical wave force:

$$F_z(t) = \int_{-L}^{L} 2\rho A_h dx\, \dot{w}_x(1+C_m) + \rho A_h 2L\,\dot{w}_f(1+C_m) + \rho A_h 2L\,\dot{w}_r(1+C_m) + 2A_v p_f + 2A_v p_r$$

Using $\dot{w} = -a\omega^2 e^{-kd}\left[\sin(kx - \omega t)\right]$; $\quad p = a\rho \dfrac{\omega^2}{k} e^{-kd}\left[\sin(kx-\omega t)\right]$,

$$F_z(t) = 4a\rho g e^{-kd}\left[-A_v \cos kL + A_h(1+C_m)\sin kL + A_h kL(1+C_m)\cos kL\right]\sin\omega t \tag{17.24}$$

THE EQUATION OF UNDAMPED FORCED SURGING:

Consider the schematic of a surge as below:

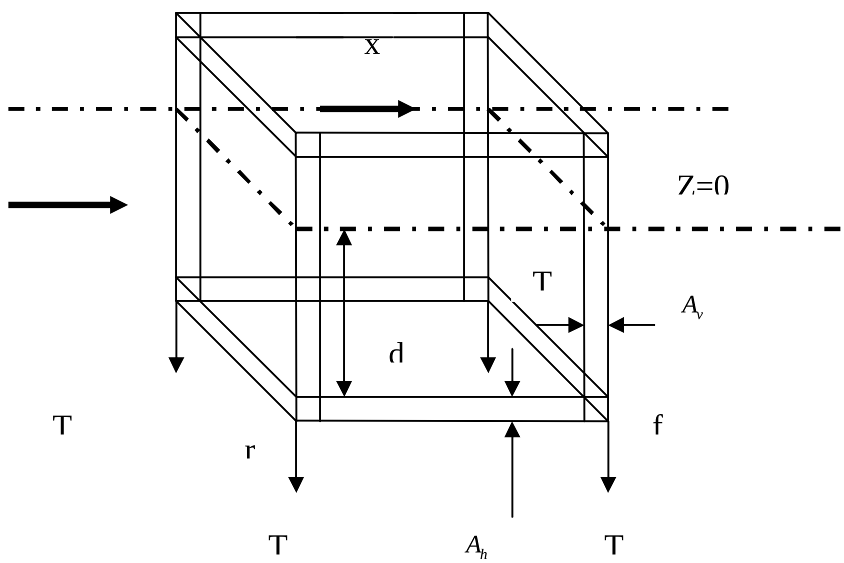


Fig. 17.8. A TLP under surge

We have the equation of motion as:

$$\left(m + A^1\right)\ddot{x} + \left(C^1 + C_t^1\right)x = F_x(t)$$

$$m = 4\rho A_v d + 4\rho A_h 2L - \frac{4T}{g}$$

$A^1 = 2C_{mx}\rho A_h 2L + 4\rho A_v d C_{mx}^1$ ------where C_{mx} is added mass coefficient for horizontal flow normal to pontoons.

$C_{mx}^1 = 1$

$C^1 \quad = 0$ (No buoyancy change); $C_t^1 =$ (4T/ Length of tether)

$$F_x(t) = \int_{-d}^{0} 2\rho A_v dz \dot{u}_f + 2\rho A_v dz \dot{u}_r (1+1) + \rho A_h qL(1+C_{mx})\dot{u}_f + \rho A_h qL(1+C_{mx})\dot{u}_r + 2A_h p$$

Where $\dot{u}_f$ @ x = L , $\dot{u}_r$ @ x =- L

C_{mx} is added mass coefficient for horizontal flow normal to pontoons.

Substituting $\dot{u} = -a\omega^2 e^{kz}\cos(kx - \omega t)$

$$p = a\rho \frac{\omega^2}{k} e^{-kd}\left[\sin(kx - \omega t)\right]$$

$$F_x(t) = -\left[\frac{8\rho a\omega^2 A_v}{k}\left(1 - e^{-kd}\right) + 2\rho a\omega^2 A_h L(1 + C_{mx})e^{-kd}\right]\cos kL \cos \omega t - 4\rho g a A_h e^{-kd} \sin kL \cos \omega t$$

17.5 Mooring Buoys:

They can be used to tie vessels and could be of catenary type Refer to Fig. 17.9 .

For static equilibrium $\sum F_x = 0$. Hence $T\cos\theta = (T + \Delta T)\cos(\theta + \Delta\theta)$

$$(T + \Delta T)\cos(\theta + \Delta\theta) - T\cos\theta = 0$$

$$\lim_{\Delta x \to o} \frac{(T + \Delta T)\cos(\theta + \Delta\theta) - T\cos\theta}{\Delta x} = 0$$

Hence $\frac{d}{dx}(T\cos\theta) = 0 \Rightarrow T\cos\theta = \text{constant} = T_0$

$$\sum F_y = 0 \Rightarrow (T + \Delta T)\sin(\theta + \Delta\theta) = W\Delta S + T\sin\theta$$

$$(T + \Delta T)\sin(\theta + \Delta\theta) - T\sin\theta = W\Delta S$$

$$\lim_{\Delta S \to o} \frac{(T + \Delta T)\sin(\theta + \Delta\theta) - T\sin\theta}{\Delta s} = W$$

$$\frac{d}{dS}(T\sin\theta) = W$$

$$T\sin\theta = WS$$

Dividing, $\tan\theta = \frac{dy}{dx} = \frac{WS}{T_0}$

Note: $T = \sqrt{T_0^{\ 2} + (WS)^2}$

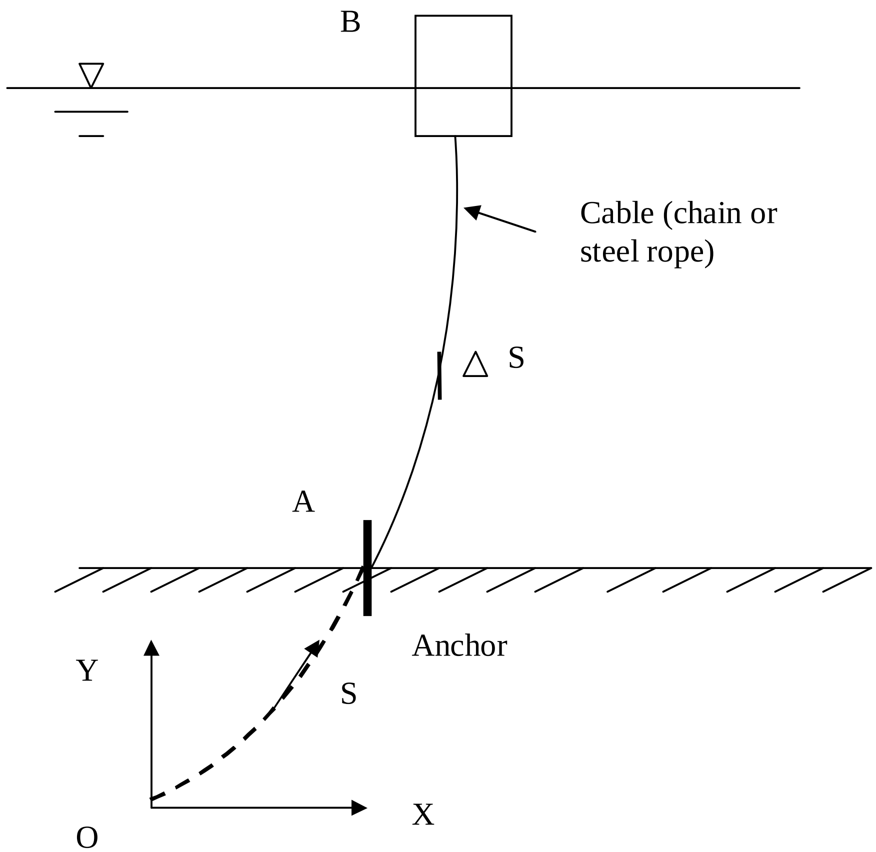
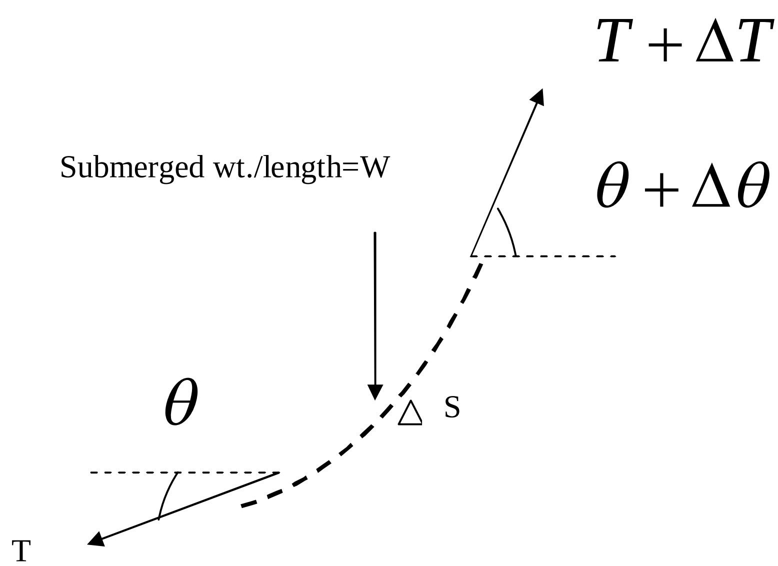


Fig. 17.9 a. A moored buoy



FBD OF ΔS

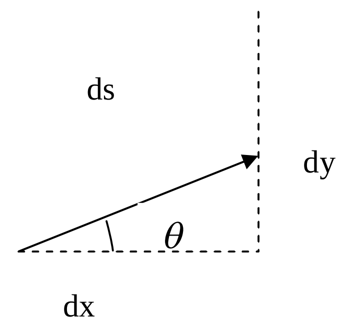


Fig. 17.9 b: The free body diagram of the cable element

$$X = \int_0^s dx \quad = \int_0^s ds\cos\theta \quad = \int \frac{ds}{\sec\theta} = \int \frac{ds}{\sqrt{1+\tan^2\theta}}$$

$$= \int \frac{ds}{\sqrt{1+\left(\frac{WS}{T_0}\right)^2}}$$; Assuming that the self weight as the only load,

Hence $X = \frac{T_0}{W}\sinh^{-1}\frac{WS}{T_0}$; (the constant of integration =0)

$$\sinh\frac{WX}{T_0}=\frac{WS}{T_0}$$

Hence $S=\frac{T_0}{W}\sinh\frac{WX}{T_0}$

We have $\frac{dy}{dx}=\frac{WS}{T_0}$

Hence $dy=\frac{WS}{T_0}dx=\sinh\frac{WX}{T_0}dx$

Hence $y=\int dy=\int\sinh\frac{WX}{T_0}dx$

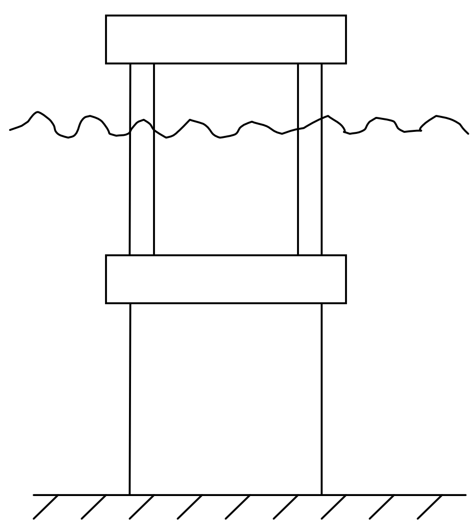
$$=\frac{T_0}{W}\cosh\frac{WX}{T_0}+c_1$$

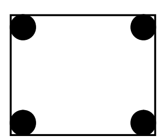
As x=0, y=0 and hence $c_1=-\frac{T_0}{W}$

Hence $y=\frac{T_0}{W}\left[\cosh\frac{WX}{T_0}-1\right]$

Note: For long lines, drag force should be added while considering equilibrium.

17.6 The T L P: Consider the following figure.





If the buoyancy is excessive, pretensioning of the legs shall be done. Because of this heave, pitch, roll is reduced. The response is mainly of three types: Usual motion at wave frequency, Low frequency drift (mainly in surge) and High frequency ‘SPRINGING’ of tethers

STATIC OFFSET :

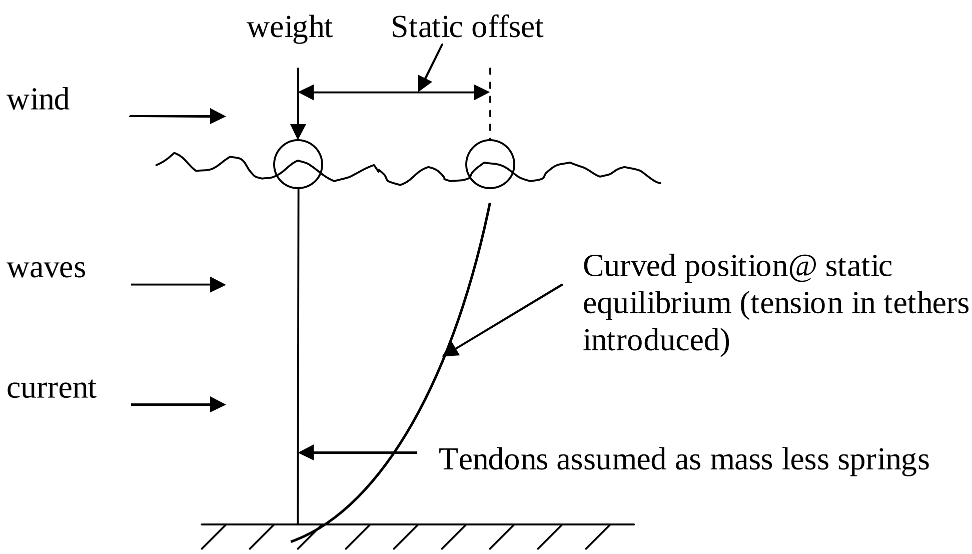
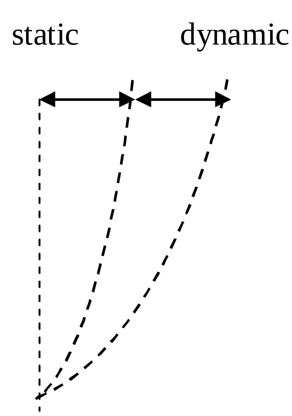


Fig. 17.10. TLP Offset

DYNAMIC OFFSET

This is also called as SET DOWN EFFECT :



This is due to wave drag and inertia. In this additional tether tension is introduced. Forces /motion in tethers should be calculated with respect to above equilibrium position.

TO OBTAIN 'B' FOR FREELY FLOATING STRUCTURES :

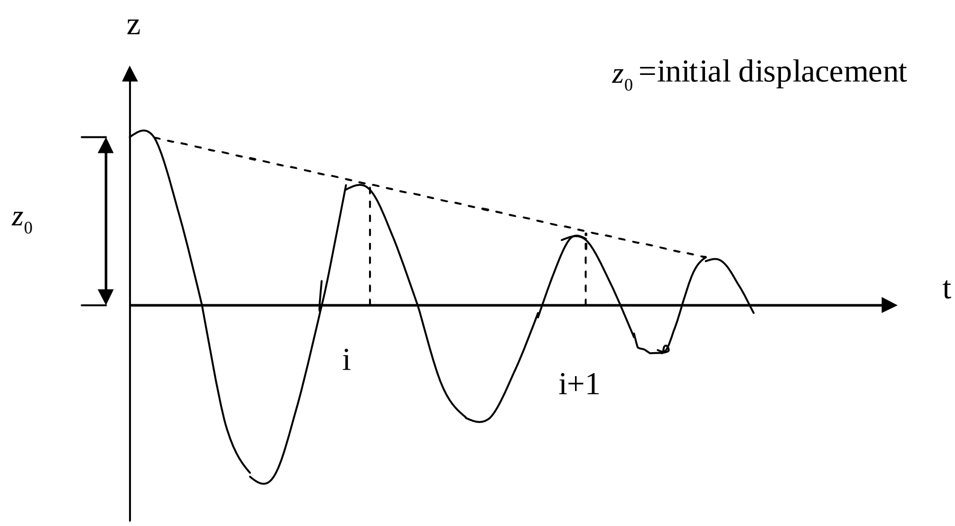
$k = \dfrac{v}{w_\phi}$ where $v = \dfrac{B}{2A}$ and $w_\phi = \sqrt{\dfrac{C}{A}}$

Obtain 'k' (5% to 25%for most offshore structures)

For $k_{\max}$, $w_d = \sqrt{1-0.25^2}\,w_\eta = 0.94 w_\eta$

Hence undamped $\omega \approx$ damped 'ω'

Experimentally, record time histories of 'z' of the scale model as below:



$$\delta = \ln\left(\frac{z_i}{z_{i+1}}\right)$$

We have seen , $\delta = 2\pi k \dfrac{\omega_\eta}{\omega_d}$

$$= 2\pi k \frac{\omega_\eta}{\omega_\eta \sqrt{1-k^2}}$$

$$= \frac{2\pi k}{\sqrt{1-k^2}}$$ ------ Since δ is known , k can be found out.

If k is assumed small:

Record t_i vs z_i

We have, $z_i = z_0 e^{-k\omega_\eta t}$

Hence $\ln z_i = \ln z_0 - k\omega_\eta t_i$

Since $\ln z_i$, $\ln z_0$, $\omega_\eta t_i$ are known , k can be found out by using least square fit to t_i , z_i .

REFERENCES

Baba, M. and Kurian, M.P (1988). "Ocean Waves And Beach Process", Centre of Earth Sciences Studies, Trivendrum.

Baba M. and Shahul Hameed, T.S. (1989). "Ocean Waves Studies And Applications", Centre of Earth Sciences Studies, Trivendrum.

Bendat, J.S. and Piersol, A.G. (1986). "Random Data Analysis And Measurement Procedures", John Wiley and Sons.

Brebbia, C.A. and Walker, S. (1979). "Dynamic Analysis of Offshore Structures", Sringer Verlag.

Bretschneider, C.L. (1963). "A One Dimensional Gravity Wave Spectrum", "Proceedings, Conference on Ocean Wave Spectra", Prentice Hall.

Bretschneider, C.L. (1977). "On the Determination of the Design Wave Spectrum", Look Lab, Hawaii 7(I), 1-23.

Bretschneider C L and Wibro P G (1976): "Tsunami inundation prediction", Proceedings, 15th Coastal Engineering Conference, ASCE, Honolulu

Chaudhary Anil (1985). "Wave Forces on Groups of Cylinders", M.Tech Thesis, Department of Civil Engineering, I.I.T., Bombay.

Copeland, G.J.M. (1985). "A Practical Alternative to the Mild Slope Equation", Coastal Engineering 9, pp 125-149.

CWPRS (1993). "Installation of Telemetered Wave Height Recorders…", Report on R and D scheme no. IV.MOST, Central Water and Power Research Station, Pune.

Dalrymple, R.A. and Kirby, J.T. (1988). "Models for Very Wide Angle Water Waves and Wave Diffraction", Journal of Fluid Mechanics, Vol. 192, pp 33-50.

Darbyshire, M. and Draper, L. (1963). “Forecasting Wind Generated Sea Waves”, Engineering (London), 1955 April 1963.

Dattatri, J. (1978). “Analysis of Regular and Irregular Waves”, Ph.D Thesis, Department of Civil Engineering, I.I.T., Madras.

Deo, M.C. and Burrows, R. (1986). “Extreme Wave Prediction Using Directional Data”, 20th International Conference on Coastal Engineering, Taipei, Taiwan, ASCE, Vol. 1, Ch. 11, pp 136-149.

Deo, M.C. and Venugopal, B. (1991). “Estimation of Design Wave Heights For Coastal Structures”, Second International Symposium on Coastal and Ocean Space Utilization, COSU-II, Long Beach, California.

Ebersole, B.A. (1985). “Refraction – Diffraction Model for Linear Water Waves”, ASCE Journal of WPCO Division, 111(6), pp 939-953.

Gaithwaite J (1981): “The marine environment and structural design”, Van Nostrand Reinhold, NY.

Gerwick, B.C. (1981) “Offshore Construction”.

Goswami, P. Deo, M.C. and Narasimhan, S. (1981) “Long Term Estimation of Waves Using Directional Data”, Fourth Indian Conference on Ocean Engineering, INCOE – 91, Goa, pp 33-40.

Gaythwaite, J. (1981). “ The Marine Environment and Structural Design”, Van Nostrand Reinhold Co, NY.

Hasselmann et al.(1973). “Measurement of Wind Wave Growth and Swell Decay During the Joint North Sea Wave Project”, Dent. Hydrogr. Z., Reihe, A-12.

Ippen, A.T. (1965). “Coastal hydrodynamics”, Mc Craw hill.

Kanetkar, C.N. (1996). "Combined Refraction – Diffraction of Short Wave Propagation", 1981 ISH Journal of Hydraulic Engineering, 2(1), pp 1-15.

Kankarej, S.A. (1992). "Development of Computer Software fro Stokes Fifth Order Theory", M.Tech Mini-project, Department of Civil Engineering, I.I.T., Bombay.

Kaplan K (1955): "Generalized laboratory study of tsunami run up", USACE, Coastal Engineering Research Centre, U S Army, Vicksberg, TM-60.

Kinsman, B. (1965). "Wind Waves", Prentice Hall.

Kiran Kumar,Deo, M.C. Pant, P.K. and Narasimhan, S. (1989). "Long Term Estimation of Waves", Third National Conference on Dock and Harbour Engineering, Surathkal, pp 57-46.

Kirby, J.T. (1986). "Higher Approximations in the Parabolic Equation Method for Water Waves", Journal of Geophysical Research, Vol. 91, pp 933-952.

Mader C (2004): "Numerical modeling of water waves", Second edition, CRC Press, London

Mandal, S. and Nayak, B.U. (1986). "Wave Prediction Techniques – A Review", Third Indian Conference on Ocean Engineering, I.I.T. Bombay, Vol. II, pp J21-J32.

Murty T S (1977): "Seismic sea waves – tsunamis", Bulletin 198, Department of fisheries and the Environment, Fisheries and Marine Service, Ottava, Canada

Narasimhan, S. and Deo, M.C. (1980) "Spectral Analysis of Ocean Waves", III IAHR Symposium on Stochastic Hydraulics, Tokyo, Japan, pp 747-758.

Narasimhan, S. and Deo, M.C. (1981) "Behaviour of Ocean Waves at Bombay High and their Statistical Predictions", First Indian Conference on Ocean Engineering, INCOE-81, IIT Madras, Vol. I, pp I.67 to I.74.

Narasimhan, S. and Deo, M.C. (1979). "Spectral Analysis of Ocean Waves – A Study", Civil Engineering in the Oceans IV Conference, San Fransisco, Vol. III, pp 877-892.

Newland, D.E. (1975) "Introduction to Random Vibration and Spectral Analysis", Longman, London.

NIO (2005): "Proceedings of the National workshop on Formulation of Science Plan for Coastal Hazard Preparedness", Feb. 18-19, 2005, National Institute of Oceanography, Goa.

Pagrut, P. and Deo, M.C. (1992). "Extreme Wave Height Estimation", Second International Conference on Offshore and Polar Engineering, San Fransisco, Session 54.

Pierson, W.J. and Muskowitz, L. (1964). "A Proposed Form for Fully Developed Wind Seas based on Similarity Theory of Kitaigorodskii", Journal of Geophysical Research, 69, pp 5181-5190.

Radder, A.C. (1979). "On the Parabolic Method for Water Wave Propagation", Journal of Fluid Mechanics, Vol 95, pp 159-176.

Sarpkaya, T. and Issacson, M. (1981). "Mechanics of Wave Induced Forces on Offshore Structures",Van Nostrand Reinhold.

Scott, J.R. (1965). "A Sea Spectrum for Model Tests and Long Term Ship Prediction", Journal of Ship Research, SNAME, 9(3), pp 145-152.

Seeling, W.N. and Ahrens, J.P. (1981). "Estimation of Wave Reflection and Energy Dissipation Coefficients for Beaches, Revetments and Breakwaters", TP 81-1, Coastal Engineering Research Centre, U.S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi.

Soni, H.R., Deo, M.C and Narasimhan, S. (1989). "Statistical Estimation of Extreme Wave Heights", Journal of the Institution of Engineers (India), CV, 70.

SPM (1984). "Shore Protection Manual", Coastal Engineering Research Centre, US Department of Army, Corps of Engineers, Waterways Experiment Station, Vicksburg, Mississippi.

Streeter, V.L. and Wylie, E.B. (1986). “Fluid Mechanics”, Prentice Hall.

Tucker, M.J. (1963). “Analysis of Records of Sea Waves”, Proceedings, Institution of Civil Engineers, London, 26, 305.

Wiegel, R.L. (1980). Lecture notes for short course on Small Harbour Engineering I.I.T Bombay.

Wiegel, R.L. (1965). “Oceanographical Engineering”, Engewood Cliffs.

WMO. (1988). “Guide to Wave Analysis and Forecasting”, Pub. NO. 702, World Meteorological Organization, Secretariate of WMO, Geneva.

Yeh (1996): “Long-wave run models”, In Proc. 2nd Int. Workshop on Long-wave Runup Models. World Scientific.

NOTATIONS

a	-	Wave Amplitude = H/2
a_j	-	Amplitude of j^{th} wave
A	-	Height of the Highest Crest in the Given Record Above SWL
b	-	Distance between Two Orthogonals after Refraction
b_1,b_2,a_1,a_2	-	Functioning of H,T,d
b^n,a^n	-	Unknown functions of H,T,d
b_0	-	Distance Between Two Orthogonals in Deep Water
B	-	Height of the Second Highest Crest Above SWL
B_{ij}	-	Unknown function of kd
c	-	Celerity
C	-	Depth of the Lowest Trough below SWL/Wave Speed in Intermediate Water depth
C_g	-	Group Velocity
C_i	-	Unknown Function of kd
C_0	-	Wave Celerity in Deep water
C_s	-	Wave Celerity in Shallow water
d	-	Water Depth
d_b	-	Depth of Water at the Breaking Point
D	-	d/dx / Depth of the Second Lowest Trough below SWL
E	-	Total Energy Per Unit Plan Area
f	-	Coriolis Parameter
F	-	Fetch of Wind
FFT	-	Fast Fourier * Transform
H	-	Wave Height / Individual Wave Heights
H_1	-	A+C
H_{10}	-	Average Wave Height of 10% of Largest Waves in a Random
H_2	-	B+D
H_i	-	Height of i^{th} Wave of Random Wave Series / Incident

Wave Height

H_m	-	Mean Wave Height
H_{ney}	-	Maximum Wave Height
H_0	-	Wave Height in Deep Water
H_{rms}	-	Root Means Square Wave Height
H_s	-	Significant Wave Height / Mean Significant Wave Height
His	-	Midpoint H_s value corresponding to i^{th} row of (H_s , T_z)
ΔH_s	-	Class Width of H_s in the (H_s , T_z) Scatter diagram
k_j	-	Wave number of j^{th} Wave
K	-	Wave number
$K^{'}$	-	Modified Wave number
K_r	-	Refraction Coefficient
K_s	-	Shoaling Coefficient
L	-	Wave Length
L_0	-	Deep Water Wave Length
m	-	Seabed slope
m_i	-	I^{th} Moment of Spectrum
M	-	Order of Expansion in Stream Function Theory / Number of Linear Waves added together
M^*	-	Corresponds to Limiting Value of H_s at the Site Say Due to
N	-	Total Number Of Observed η_J Value
N_z	-	Total number of Zero up-crosses in the Record
P	-	Wave Power
ΔP	-	Normal Pressure (= 760 mm of mercury) at Hurricane Center
P()	-	Cumulative Distribution Function of ()
$P(H)_{LT}$	-	Long Term Distribution of Individual Wave Height
R	-	Radius of Maximum Wind during Cyclone
$R_\eta(\tau)$	-	Auto Correction Function
R_L	-	Correction factor
R_p	-	Run up

R_s	-	Run up produced by a Regular Significant Wave of Height H_s
s	-	Source Function
S	-	d+z/d
ΔS	-	Total Set up at Shoe
$S_\eta(f)$	-	Spectral Density Function
$S_\eta(\omega_j)$	-	Spectral Density Fraction corresponding to the Frequency W^j
S_b	-	Set down at the Breaking Zone
S_{ds}	-	Wave Energy Dissipation Due to Bottom Friction and
S_{in}	-	Wind Energy Input
S_{ud}	-	Wave Energy Input transferred from One Wave Frequency
t	-	Time instant
t_{min}	-	Minimum Deviation of Wind blow for particular H_s x $T_{s\ for}$
T	-	Wave period
T_i	-	Period of i^{th} Wave of Random Wave Series
T_s	-	Significant Wave Period
T_z	-	Average Zero Crossing Period
T_{zj}	-	T_z value corresponding to the j^{th} Column of (H_s , T_z)
u	-	Wind Speed
u, v, w	-	The component of Fluid Velocity Along x,y,z Directions
U_{10}	-	Wind Speed at the height of 10 m above Ground Level
U_C	-	Corrected Wind Speed
U_F	-	Wind Speed along the forward direction of hurricane
U_L	-	Wind Speed over Land
U_{max}	-	Maximum Gradient Wind Speed at 10 m above MSL
U_R	-	Wind Speed at Radius Corresponding to Maximum Wind
U_w	-	Characteristic Wind Speed
w	-	Angular Speed of Earth Rotation
w_x, w_y, w_z	-	Rotation of Fluid element about x,y,z axis

W_i	-	$P(H_{is}+0.5\Delta H_s)-P(H_{is}-0.5\Delta H_s)$ Obtained from the
W_{ij}	-	Total Number of Occurrences of H_s values in the (i,j) Interval
ΔW	-	Energy in the Interval df
X	-	(x-ct)/L
X,Z,T	-	Unknown functions of x, z and t
α	-	Angle of Crelkine of refracted Wave
α, β	-	Philip's Conton
α_0	-	Angle of Crediline of Incident Wave
ε	-	Spectral Width Panel
ϕ	-	Velocity Potential
ϕ_i	-	Function of i^{th} Order of Stokes Waves Theory
γ	-	Specific Weight of Sea Water
η	-	Surface Elevations
$\eta(t)$	-	Sea Surface Elevation at Time t
$\eta(t+\tau)$	-	Sea Surface Elevation at Time $(t+\tau)$
$\eta(x,t)$	-	Sea Surface Elevation at x Distance from Reference Origin
η_j	-	j^{th} value of the Sea Surface Elevation
η_k	-	Complex Conjugate of η_k
η_r	-	Surface Elevation of Reflecting Wave
ρ	-	Fluid Mass Density
ψ	-	Stream Function
ω	-	Circular Wave Frequency
ω_j	-	Angular Wave Frequency of j^{th} Wave
ω_k	-	Circular Frequency of k^{th} wave
$\Delta\omega$	-	Frequency Step
ξ	-	Particle Displacement Along x direction
ζ	-	Particle Displacement Along y direction
θ	-	Phase Angle / Log N_z

θ	-	Angle of Reflection of Bed slope
θ_j	-	Phase of j^{th} wave
τ	-	Time Lag
ξ	-	Surf Similarity
φ	-	Complex Velocity Potential of Transmitted Wave