COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 31 (1982) 1–18
NORTH-HOLLAND PUBLISHING COMPANY

NUMERICAL ANALYSIS OF THICK PLATES

Tarun KANT*
Department of Civil Engineering, Indian Institute of Technology, Bombay 400 076, India

Received 28 October 1980
Revised manuscript received 20 July 1981

Bending behaviour of a rectangular plate is analysed with the help of a refined higher order theory. The theory is based on a higher order displacement model and the three-dimensional Hooke's laws for plate material, giving rise to a more realistic quadratic variation of the transverse shearing strains and linear variation of the transverse normal strain through the plate thickness. It is shown that the segmentation method for the numerical analysis of plates simply supported on two opposite edges can be considered to be the most competitive method in terms of efficiency, economy, reliability and accuracy in such applications.

0. Introduction

This paper is specifically concerned with the numerical analysis of elastic plates by the segmentation method [1] based on a refined higher order theory. Motivation for the present study comes from a recent report [2]. Although Mindlin plate theory [3] provides a basis for the consideration of the effects of the transverse shear deformations on the behaviour of a plate, it has certain limitations: the transverse shearing strains (and thereby stresses) are assumed to be constant through the thickness of the plate and a fictitious warping coefficient is introduced; the classical contradiction whereby both the transverse normal stress (σ_z) and the transverse normal strain (ε_z) are neglected, is not resolved. A good review of the available plate theories is given by Lo, Christensen and Wu [4]. The present study is based on a theory incorporating a higher-order displacement model, quadratic variation of the transverse shearing strains (γ_{xz} and γ_{yz}) through the plate thickness, linear variation of the transverse normal strain (ε_z) through the plate thickness and finally consideration of the three-dimensional Hooke's laws. Thus the mathematical model which is conceived here for plate behaviour is free from any contradictions present in the well-known theories.

1. Theoretical formulation

The present study is specifically based on a refined higher-order theory for the bending behaviour of a plate. The only assumptions which enter into the development of the theory are those of small deflections, linear elasticity, homogeneity and isotropicity. The governing

*Presently a visitor at the Department of Civil Engineering, University of Wales, Swansea, SA2 8PP (U.K.) under the Janaharlal Nehru Memorial Trust (U.K.) Scholarship—1979 award.

equations of such a theory defining a properly posed boundary value problem are systematically derived and presented in Appendix A where the reduction process of the three-dimensional elasticity problem to the two-dimensional plate problem is clearly shown. Numerical integration of such a boundary value problem by the segmentation method [1], which was originally due to Goldberg et al. [5], involves first algebraic manipulation of the basic two-dimensional equations of the plate given in Appendix A comprising the equilibrium, the strain-displacement and constitutive relations, so as to obtain a system of first order differential equations—called the 'intrinsic equations' involving only some particular dependent variables—called the 'intrinsic variables', the number of which equals the order of the partial differential equation system of such a theory (twelfth order in the present case). Then out of the two independent coordinates which describe the problem, one is chosen to be the preferred one. In the present analysis, the x-coordinate is selected as the preferred one. Intrinsic equations are then derived consisting of a system of first order partial differential equations each of which contains necessarily the first derivative with respect to x of one of the so-called intrinsic dependent variables which also form the complete set of the boundary variables appearing naturally along the edge x = a constant. In the present analysis, $\boldsymbol{Y}$ defining the vector of intrinsic variables, consists of the dependent variables w, θ_x, θ_y, w^*, θ_x^*, θ_y^*, Q_x, M_x, M_{xy}, Q_x^*, M_x^* and M_{xy}^*. After the required manipulations which demand considerable ingenuity, the system of equations is obtained in the following form,

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= -\theta_x + \frac{2}{(1-2\nu)(K_1K_1^* - D_1^2)}(K_1^*Q_x - D_1Q_x^*)\,,\\
\frac{\partial \theta_x}{\partial x} &= -\frac{2\nu}{(1-\nu)}w^* - \frac{\nu}{(1-\nu)}\frac{\partial \theta_y}{\partial y} + \frac{1}{(1-\nu)(D_1D_1^* - K_1^{*2})}(D_1^*M_x - K_1^*M_x^*)\,,\\
\frac{\partial \theta_y}{\partial x} &= -\frac{\partial \theta_x}{\partial y} + \frac{2}{(1-2\nu)(D_1D_1^* - K_1^{*2})}(D_1^*M_{xy} - K_1^*M_{xy}^*)\,,\\
\frac{\partial w^*}{\partial x} &= -3\theta_x^* - \frac{2}{(1-2\nu)(K_1K_1^* - D_1^2)}(D_1Q_x - K_1Q_x^*)\,,\\
\frac{\partial \theta_x^*}{\partial x} &= -\frac{\nu}{(1-\nu)}\frac{\partial \theta_y^*}{\partial y} - \frac{1}{(1-\nu)(D_1D_1^* - K_1^{*2})}(K_1^*M_x - D_1M_x^*)\,,\\
\frac{\partial \theta_y^*}{\partial x} &= -\frac{\partial \theta_x^*}{\partial y} - \frac{2}{(1-2\nu)(D_1D_1^* - K_1^{*2})}(K_1^*M_{xy} - D_1M_{xy}^*)\,,\\
\frac{\partial Q_x}{\partial x} &= -\frac{K_1(1-2\nu)}{2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_y}{\partial y}\right) - \frac{D_1(1-2\nu)}{2}\left(\frac{\partial^2 w^*}{\partial y^2} + 3\frac{\partial \theta_y^*}{\partial y}\right) - (p_z^+ + p_z^- + \rho h)\,,\\
\frac{\partial M_x}{\partial x} &= Q_x - \frac{\partial M_{xy}}{\partial y}\,,\\
\frac{\partial M_{xy}}{\partial x} &= \frac{K_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial y} + \theta_y\right) + \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right)\\
&\quad - D_1\left[\nu\frac{\partial}{\partial x}\left(\frac{\partial \theta_x}{\partial y}\right) + (1-\nu)\frac{\partial^2 \theta_y}{\partial y^2} + 2\nu\frac{\partial w^*}{\partial y}\right] - K_1^*\left[\nu\frac{\partial}{\partial x}\left(\frac{\partial \theta_x^*}{\partial y}\right) + (1-\nu)\frac{\partial^2 \theta_y^*}{\partial y^2}\right],
\end{aligned}
\tag{1}
$$

((1) cont'd)

$$\frac{\partial Q_x^*}{\partial x} = -\frac{D_1(1-2\nu)}{2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_y}{\partial y}\right) - \frac{K_1^*(1-2\nu)}{2}\left(\frac{\partial^2 w^*}{\partial y^2} + 3\frac{\partial \theta_y^*}{\partial y}\right)$$
$$+ 2D_1\left[\nu\frac{\partial \theta_x}{\partial x} + \nu\frac{\partial \theta_y}{\partial y} + 2(1-\nu)w^*\right] + 2\nu K_1^*\left(\frac{\partial \theta_x^*}{\partial x} + \frac{\partial \theta_y^*}{\partial y}\right) - (p_z^+ + p_z^- + \tfrac{1}{3}\rho h)\frac{h^2}{4},$$

$$\frac{\partial M_x^*}{\partial x} = 3Q_x^* - \frac{\partial M_{xy}^*}{\partial y},$$

$$\frac{\partial M_{xy}^*}{\partial x} = \tfrac{3}{2}D_1(1-2\nu)\left(\frac{\partial w}{\partial y} + \theta_y\right) + \tfrac{3}{2}K_1^*(1-2\nu)\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right)$$
$$- K_1^*\left[\nu\frac{\partial}{\partial x}\left(\frac{\partial \theta_x}{\partial y}\right) + (1-\nu)\frac{\partial^2 \theta_y}{\partial y^2} + 2\nu\frac{\partial w^*}{\partial y}\right] - D_1^*\left[\nu\frac{\partial}{\partial x}\left(\frac{\partial \theta_x^*}{\partial y}\right) + (1-\nu)\frac{\partial^2 \theta_y^*}{\partial y^2}\right].$$

The other dependent variables are expressed as functions of intrinsic variables by simple algebraic relations, called 'auxiliary relations', in the following form,

$$Q_y = \frac{K_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial y} + \theta_y\right) + \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right),$$

$$M_y = D_1\left[\nu\frac{\partial \theta_x}{\partial x} + (1-\nu)\frac{\partial \theta_y}{\partial y} + 2\nu w^*\right] + K_1^*\left[\nu\frac{\partial \theta_x^*}{\partial x} + (1-\nu)\frac{\partial \theta_y^*}{\partial y}\right],$$

$$Q_y^* = \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial y} + \theta_y\right) + \frac{K_1^*(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right), \tag{2}$$

$$M_y^* = K_1^*\left[\nu\frac{\partial \theta_x}{\partial x} + (1-\nu)\frac{\partial \theta_y}{\partial y} + 2\nu w^*\right] + D_1^*\left[\nu\frac{\partial \theta_x^*}{\partial x} + (1-\nu)\frac{\partial \theta_y^*}{\partial y}\right],$$

$$M_z = D_1\left[\nu\frac{\partial \theta_x}{\partial x} + \nu\frac{\partial \theta_y}{\partial y} + 2(1-\nu)w^*\right] + \nu K_1^*\left(\frac{\partial \theta_x^*}{\partial x} + \frac{\partial \theta_y^*}{\partial y}\right).$$

The six displacement components and the corresponding six stress resultant components which form the vector $\boldsymbol{Y}$ of the intrinsic variables are functions of x and y, and for a rectangular plate with its two opposite edges, $y = 0$ and $y = b$ simply supported, these may be represented in the form of a Fourier series which automatically satisfies both the displacement and the force boundary conditions along these edges as follows:

$$\{w(x,y), \theta_x(x,y), w^*(x,y), \theta_x^*(x,y), Q_x(x,y), M_x(x,y), Q_x^*(x,y), M_x^*(x,y)\} =$$
$$= \sum_m \{w_m(x), \theta_{xm}(x), w_m^*(x), \theta_{xm}^*(x), Q_{xm}(x), M_{xm}(x), Q_{xm}^*(x), M_{xm}^*(x)\}\sin(2m-1)\frac{\pi y}{b}$$

and

$$\{\theta_y(x,y), \theta_y^*(x,y), M_{xy}(x,y), M_{xy}^*(x,y)\} =$$
$$= \sum_m \{\theta_{ym}(x), \theta_{ym}^*(x), M_{xym}(x), M_{xym}^*(x)\}\cos(2m-1)\frac{\pi y}{b}. \tag{3}$$

Substitution of (3) in the system of differential equations (1) and analytic integration of these equations with respect to the independent variable (coordinate) y coupled with the use of the orthogonality conditions of the basic beam functions used in the y-direction in the aforesaid expansions, reduces the system of partial differential equations (1) into the following set of simultaneous first order ordinary differential equations (say for the mth harmonic) involving only the twelve intrinsic variables.[1]

$$
\begin{aligned}
\frac{\mathrm{d}w_m}{\mathrm{d}x} &= -\theta_{xm} + \frac{2}{(1-2\nu)(K_1K_1^* - D_1^2)}(K_1^*Q_{xm} - D_1Q_{xm}^*)\,,\\
\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} &= -\frac{\nu}{(1-\nu)}\left[2w^* - (2m-1)\frac{\pi}{b}\theta_{ym}\right] + \frac{1}{(1-\nu)(D_1D_1^* - K_1^{*2})}(D_1^*M_{xm} - K_1^*M_{xm}^*)\,,\\
\frac{\mathrm{d}\theta_{ym}}{\mathrm{d}x} &= -(2m-1)\frac{\pi}{b}\theta_{xm} + \frac{2}{(1-2\nu)(D_1D_1^* - K_1^{*2})}(D_1^*M_{xym} - K_1^*M_{xym}^*)\,,\\
\frac{\mathrm{d}w_m^*}{\mathrm{d}x} &= -3\theta_{xm}^* - \frac{2}{(1-2\nu)(K_1K_1^* - D_1^2)}(D_1Q_{xm} - K_1Q_{xm}^*)\,,\\
\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} &= \frac{\nu}{(1-\nu)}(2m-1)\frac{\pi}{b}\theta_{ym}^* - \frac{1}{(1-\nu)(D_1D_1^* - K_1^{*2})}(K_1^*M_{xm} - D_1M_{xm}^*)\,,\\
\frac{\mathrm{d}\theta_{ym}^*}{\mathrm{d}x} &= -(2m-1)\frac{\pi}{b}\theta_{xm}^* - \frac{2}{(1-2\nu)(D_1D_1^* - K_1^{*2})}(K_1^*M_{xym} - D_1M_{xym}^*)\,,\\
\frac{\mathrm{d}Q_{xm}}{\mathrm{d}x} &= \frac{K_1(1-2\nu)}{2}(2m-1)\frac{\pi}{b}\left[(2m-1)\frac{\pi}{b}w_m + \theta_{ym}\right]\\
&\quad + \frac{D_1(1-2\nu)}{2}(2m-1)\frac{\pi}{b}\left[(2m-1)\frac{\pi}{b}w_m^* + 3\theta_{ym}^*\right] - \frac{4}{\pi(2m-1)}(p_z^+ + p_z^- + \rho h)\,,\\
\frac{\mathrm{d}M_{xm}}{\mathrm{d}x} &= Q_{xm} + (2m-1)\frac{\pi}{b}M_{xym}\,,\\
\frac{\mathrm{d}M_{xym}}{\mathrm{d}x} &= \frac{K_1(1-2\nu)}{2}\left[(2m-1)\frac{\pi}{b}w_m + \theta_{ym}\right] + \frac{D_1(1-2\nu)}{2}\left[(2m-1)\frac{\pi}{b}w_m^* + 3\theta_{ym}^*\right]\\
&\quad - D_1(2m-1)\frac{\pi}{b}\left[\nu\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym} + 2\nu w_m^*\right]\\
&\quad - K_1^*(2m-1)\frac{\pi}{b}\left[\nu\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym}^*\right],\\
\frac{\mathrm{d}Q_{xm}^*}{\mathrm{d}x} &= \frac{D_1(1-2\nu)}{2}(2m-1)\frac{\pi}{b}\left[(2m-1)\frac{\pi}{b}w_m + \theta_{ym}\right]\\
&\quad + \frac{K_1^*(1-2\nu)}{2}(2m-1)\frac{\pi}{b}\left[(2m-1)\frac{\pi}{b}w_m^* + 3\theta_{ym}^*\right]\\
&\quad + 2D_1\left[\nu\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} - \nu(2m-1)\frac{\pi}{b}\theta_{ym} + 2(1-\nu)w^*\right]
\end{aligned}
\qquad (4)
$$

[1]For the sake of brevity $w_m(x)$, $w_m^*(x)$, $Q_{xm}(x)$, $M_{xm}(x)$, etc. are hereafter written simply as w_m, w_m^*, Q_{xm}, M_{xm}, etc.

((4) cont'd)

$$+ 2\nu K_1^*\left[\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} - (2m-1)\frac{\pi}{b}\theta_{ym}^*\right] - \frac{4}{\pi(2m-1)}(p_z^+ + p_z^- + \tfrac{1}{3}\rho h)\frac{h^2}{4},$$

$$\frac{\mathrm{d}M_{xm}^*}{\mathrm{d}x} = 3Q_{xm}^* + (2m-1)\frac{\pi}{b}M_{xym}^*,$$

$$\frac{\mathrm{d}M_{xym}^*}{\mathrm{d}x} = \tfrac{3}{2}D_1(1-2\nu)\left[(2m-1)\frac{\pi}{b}w_m + \theta_{ym}\right] + \tfrac{3}{2}K_1^*(1-2\nu)\left[(2m-1)\frac{\pi}{b}w_m^* + 3\theta_{ym}^*\right]$$
$$- K_1^*(2m-1)\frac{\pi}{b}\left[\nu\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym} + 2\nu w_m^*\right]$$
$$- D_1^*(2m-1)\frac{\pi}{b}\left[\nu\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym}^*\right].$$

The auxiliary relations (2) take the following form,

$$Q_y = \sum_m\left\{\frac{K_1(1-2\nu)}{2}\left[(2m-1)\frac{\pi}{b}w_m + \theta_{ym}\right] + \frac{D_1(1-2\nu)}{2}\left[(2m-1)\frac{\pi}{b}w_m^* + 3\theta_{ym}^*\right]\right\}\cos(2m-1)\frac{\pi y}{b},$$

$$M_y = \sum_m\left\{D_1\left[\nu\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym} + 2\nu w_m^*\right] + K_1^*\left[\nu\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym}^*\right]\right\}\sin(2m-1)\frac{\pi y}{b},$$

$$Q_y^* = \sum_m\left\{\frac{D_1(1-2\nu)}{2}\left[(2m-1)\frac{\pi}{b}w_m + \theta_{ym}\right] + \frac{K_1^*(1-2\nu)}{2}\left[(2m-1)\frac{\pi}{b}w_m^* + 3\theta_{ym}^*\right]\right\}\cos(2m-1)\frac{\pi y}{b}, \tag{5}$$

$$M_y^* = \sum_m\left\{K_1^*\left[\nu\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym} + 2\nu w_m^*\right] + D_1^*\left[\nu\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} - (1-\nu)(2m-1)\frac{\pi}{b}\theta_{ym}^*\right]\right\}\sin(2m-1)\frac{\pi y}{b},$$

$$M_z = \sum_m\left\{D_1\left[\nu\frac{\mathrm{d}\theta_{xm}}{\mathrm{d}x} - \nu(2m-1)\frac{\pi}{b}\theta_{ym} + 2(1-\nu)w_m^*\right] + \nu K_1^*\left[\frac{\mathrm{d}\theta_{xm}^*}{\mathrm{d}x} - (2m-1)\frac{\pi}{b}\theta_{ym}^*\right]\right\}\sin(2m-1)\frac{\pi y}{b}.$$

It may be mentioned that unlike other analytical [6] and semi-analytical [7] approaches, the loads need not be expanded in Fourier series in the y direction here. Another important characteristic which needs to be mentioned is that the expansions (3) uncouple with respect to the harmonic m, so that a term-by-term analysis of the original problem can be conducted and then all the results are summed together. The system of first order ordinary differential equations (4) is numerically integrated by the segmentation method [1] for a particular value of the harmonic number m one at a time. The final results are obtained by carrying-out such integrations for a few harmonics to achieve a desired accuracy.

2. Numerical examples

A square plate of side 'a' thickness 'h' and loaded with a uniformly distributed transverse load 'p' is considered. The two opposite edges $y = 0$ and $y = a$ are always assumed to be simply supported (S) implying boundary conditions $w = \theta_x = M_y = w^* = \theta_x^* = M_y^* = 0$. Numerical integration is carried out for a set of four edge conditions along $x = 0$ and $x = a$. These are,

(1) Simply-supported (S): $w = \theta_y = M_x = w^* = \theta_y^* = M_x^* = 0$;
(2) Just-supported (S*): $w = M_x = M_{xy} = w^* = M_x^* = M_{xy}^* = 0$;
(3) Clamped (C): $w = \theta_x = \theta_y = w^* = \theta_x^* = \theta_y^* = 0$;
(4) Free (F): $Q_x = M_x = M_{xy} = Q_x^* = M_x^* = M_{xy}^* = 0$.

In these four examples symmetry conditions, $\theta_x = Q_x = M_{xy} = \theta_x^* = Q_x^* = M_{xy}^* = 0$, exist along $x = a/2$. Thus only half the plate is analysed. Fifty equal segments [1] and five sub-divisions within each segment for the Runge–Kutta–Gill algorithm [9] are found suitable here. Results are presented in Tables 1 to 7 for a set of four geometrical parameters ($a/h = 5$, 10, 50, 100) involving both thick and thin plates. The results of the present study are compared throughout with a similar Mindlin plate formulation [2] and available classical thin (Kirchhoff) plate solutions [6, 8].

EXAMPLE 1. The first example is a simply-supported square plate subjected to a uniform transverse load. The edge conditions are taken to be the vanishing of deflection, normal

Table 1
Central displacement[a] ($\times pa^4/D$) of a simply supported square plate under uniform load ($\nu = 0.3$): Example 1

$\sum_m$	Theory	$a/h = 5$	$a/h = 10$	$a/h = 50$	$a/h = 100$
1	Present	0.00488	0.00430	0.00411	0.00411
	Mindlin	0.00499	0.00433	0.00411	0.00411
5	Present	0.00480	0.00424	0.00407	0.00406
	Mindlin	0.00490	0.00427	0.00407	0.00406
10	Present	0.00480	0.00424	0.00407	0.00406
	Mindlin	0.00490	0.00427	0.00407	0.00406
15	Present	0.00480	0.00424	0.00407	0.00406
	Mindlin	0.00490	0.00427	0.00407	0.00406
20	Present	0.00480	0.00424	0.00407	0.00406
	Mindlin	0.00490	0.00427	0.00407	0.00406

[a]Classical thin plate solution [6] = 0.00406.

Table 2
Central bending moments[a] ($\times pa^2$) in a simply supported square plate under uniform load ($\nu = 0.3$): Example 1

$\sum_m$	Theory	$a/h = 5$		$a/h = 10$		$a/h = 50$		$a/h = 100$	
		M_x	M_y	M_x	M_y	M_x	M_y	M_x	M_y
1	Present	0.0501	0.0522	0.0494	0.0518	0.0492	0.0516	0.0492	0.0516
	Mindlin	0.0492	0.0516	0.0492	0.0516	0.0492	0.0516	0.0492	0.0516
5	Present	0.0485	0.0485	0.0480	0.0481	0.0479	0.0479	0.0479	0.0479
	Mindlin	0.0479	0.0479	0.0479	0.0479	0.0479	0.0479	0.0479	0.0479
10	Present	0.0484	0.0485	0.0480	0.0480	0.0478	0.0479	0.0478	0.0479
	Mindlin	0.0478	0.0479	0.0478	0.0479	0.0478	0.0479	0.0478	0.0479
15	Present	0.0484	0.0485	0.0480	0.0480	0.0478	0.0479	0.0478	0.0479
	Mindlin	0.0478	0.0479	0.0478	0.0479	0.0478	0.0479	0.0478	0.0479
20	Present	0.0484	0.0485	0.0480	0.0480	0.0478	0.0479	0.0478	0.0479
	Mindlin	0.0478	0.0479	0.0478	0.0479	0.0478	0.0479	0.0478	0.0479

[a]Classical thin plate solution [6] = 0.0479.

bending moment and tangential rotation quantities. Results are presented in Tables 1 to 4. The convergence of the transverse shears presented in Table 3 is not very satisfactory. The remaining quantities, as shown in Tables 1, 2 and 4, converge very rapidly. The increase in deflection in the present formulation over the classical thin plate solution is seen to be 15.4%, 4.2%, 0.3%, 0.0% for a/h ratios of 5, 10, 50 and 100, respectively. The values are a little less than those given by the Mindlin formulation. In the absence of an exact solution, these values appear to be reasonable. The only analytical solution available is that of the Reissner theory as

Table 3
Mid-edge shear forces[a] ($\times pa$) in a simply supported square plate under uniform load ($\nu = 0.3$): Example 1

$\sum_m$	Theory	$a/h = 5$–100	
		Q_x	Q_y
1	Present	0.371	0.244
	Mindlin	0.371	0.244
5	Present	0.339	0.317
	Mindlin	0.339	0.317
10	Present	0.337	0.327
	Mindlin	0.337	0.327
15	Present	0.338	0.331
	Mindlin	0.338	0.331
20	Present	0.337	0.332
	Mindlin	0.337	0.332

[a]Classical thin plate solution [6] = 0.338.

Table 4
Corner twisting moment[a] ($\times pa^2$) in a simply supported square plate under uniform load ($\nu = 0.3$): Example 1

$\sum_m$	Theory	$a/h = 5$	$a/h = 10$	$a/h = 50$	$a/h = 100$
1	Present	−0.0288	−0.0298	−0.0301	−0.0301
	Mindlin	−0.0301	−0.0301	−0.0301	−0.0301
5	Present	−0.0301	−0.0317	−0.0323	−0.0323
	Mindlin	−0.0323	−0.0323	−0.0323	−0.0323
10	Present	−0.0299	−0.0317	−0.0324	−0.0324
	Mindlin	−0.0324	−0.0324	−0.0324	−0.0324
15	Present	−0.0299	−0.0317	−0.0324	−0.0324
	Mindlin	−0.0324	−0.0324	−0.0324	−0.0324
20	Present	−0.0299	−0.0317	−0.0324	−0.0324
	Mindlin	−0.0324	−0.0324	−0.0324	−0.0324

[a]Classical thin plate solution [6] = −0.0325.

reported in [10], which gives an increase of 4.4% for an a/h ratio of 10. Thus the present higher order theory gives a comparatively less flexible solution than Mindlin and Reissner theories. A similar trend of results is also reported in two recent studies wherein this higher-order theory is used in conjunction with the finite element method [11, 12].

EXAMPLE 2. The second example is one for which results of the classical thin plate theory are not possible. The edge boundary conditions along $x = 0$, a for this example, are taken to be vanishing for deflection, normal bending moment and edge twisting moment quantities. The converged results of the present formulation are compared with that of the Mindlin formulation [2] in Table 5.

Table 5
Square plate under uniform load—simply supported on two opposite edges and just supported on the remaining two edges ($\nu = 0.3$): Example 2

a/h	Theory	Central displacement $\times pa^4/D$	Central bending moment $\times pa^2$ M_x	M_y	Mid edge shear forces $\times pa$ Q_x	Q_y
5	Present	0.00515	0.0512	0.0519	0.401	0.347
	Mindlin	0.00526	0.0507	0.0515	0.403	0.348
10	Present	0.00442	0.0493	0.0499	0.413	0.340
	Mindlin	0.00445	0.0491	0.0498	0.413	0.341
50	Present	0.00410	0.0481	0.0482	0.419	0.334
	Mindlin	0.00410	0.0481	0.0482	0.419	0.334
100	Present	0.00408	0.0480	0.0480	0.420	0.333
	Mindlin	0.00408	0.0480	0.0480	0.420	0.333

EXAMPLE 3. In the third example, the edge boundary conditions along $x = 0$, a are taken to be vanishing for deflection, normal rotation and tangential rotation quantities. Results of the present higher-order theory along with that of the Mindlin theory [2] and the Kirchhoff theory [6] are presented in Table 6.

EXAMPLE 4. In the last example, free edge conditions implying vanishing of transverse shear, normal bending moment and twisting moment along $x = 0$, a are assumed. Such a plate simulates an ideal bridge deck. Exact satisfaction of such an edge boundary condition in the present formulation is quite straightforward; which is not the case in classical mechanics and in the displacement based finite element methods. The relevant results are presented in Table 7.

In the preceding examples it is seen that the solutions of the present higher-order theory are in excellent agreement with both the classical thin plate solutions and the Mindlin solutions in the thin plate limit. This establishes the correctness of the present formulation and the resulting software. Further the general conclusions of the first example which are described in detail are valid for the remaining three examples. The convergence study which is limited here to a twenty harmonic analysis shows that most of the quantities converge quite rapidly. The values of the transverse shears, however, are seen to oscillate a little. We find it difficult to give a possible explanation for such a behaviour at the present moment.

In a nutshell the present numerical evaluation re-establishes the essential use of a shear-deformation theory (preferably the present higher-order one) for isotropic and homogeneous plates of over 2% thickness (it also eliminates the anomalies and inconsistencies present in the classical Kirchhoff theory [2, 6, 8]).

Table 6
Square plate under uniform load—simply supported on two opposite edges and clamped on the remaining two edges ($\nu = 0.3$): Example 3

a/h	Theory	Central displacement $\times pa^4/D$	Central bending moments $\times pa^2$ M_x	M_y	Mid-edge bending moment $\times pa^2$	Mid-edge shear force $\times pa$ Q_x	Q_y
5	Present	0.00293	0.0340	0.0301	−0.0616	0.474	0.253
	Mindlin	0.00302	0.0330	0.0292	−0.0626	0.475	0.251
10	Present	0.00218	0.0334	0.0260	−0.0679	0.501	0.243
	Mindlin	0.00221	0.0332	0.0258	−0.0679	0.500	0.243
50	Present	0.00192	0.0332	0.0244	−0.0698	0.513	0.239
	Mindlin	0.00193	0.0332	0.0244	−0.0697	0.513	0.239
100	Present	0.00192	0.0332	0.0244	−0.0698	0.515	0.239
	Mindlin	0.00192	0.0332	0.0244	−0.0698	0.515	0.239
	( )[a]	0.00192	0.0332	0.0244	−0.0697		

[a]Classical thin plate solution [6].

Table 7
Square plate uniform load—simply supported on two opposite edges and free on the remaining two edges ($\nu = 0.3$): Example 4

a/h	Theory	Central displacement $\times pa^4/D$	Central bending moments $\times pa^2$ M_x	M_y	Mid-edge shear force $\times pa$
5	Present	0.0143	0.0244	0.123	0.455
	Mindlin	0.0145	0.0237	0.123	0.456
10	Present	0.0134	0.0258	0.122	0.459
	Mindlin	0.0134	0.0256	0.122	0.460
50	Present	0.0131	0.0268	0.122	0.463
	Mindlin	0.0131	0.0268	0.122	0.463
100	Present	0.0131	0.0269	0.122	0.463
	Mindlin	0.0131	0.0269	0.122	0.463
	Ref. 8	0.0131	0.0271	0.122	0.463

3. Conclusions

The segmentation method is used in conjunction with a higher-order theory for linear elastic analysis of homogeneous and isotropic plates. It is well known that the effect of transverse shear deformation and normal stress in the thickness direction becomes important above a certain value of the thickness-length parameter. The present formulation incorporates a quadratic variation of transverse shearing strains and a linear variation of transverse normal strain through the thickness of the plate besides the use of the three-dimensional Hooke's laws. Results for both the displacements and the stress-resultants are in excellent agreement with those of the Mindlin theory [2] and the classical theory [6, 8] in the thin plate limit. One significant feature of the present formulation is its ability to satisfy exactly both the displacement and the force boundary conditions along an edge. This is in contrast with the popular displacement-based finite element formulations where only displacement boundary conditions can be satisfied exactly [13]. However, the formulation is not general. For plates with other than simply supported boundary conditions along the two opposite edges ($y = 0, b$), the present formulation can be modified by simply employing the appropriate beam functions [14, 15]. But in such a case the series terms will generally be coupled and the resulting system of equations will be more complex.

The advantage in the use of the higher-order theory presented here over the Mindlin theory hitherto used is not quite evident for the plates analysed. But such usage could be very effective in the analysis of nonhomogeneous, anisotropic, composite or sandwich systems, and relatively thicker plates as the mathematical model on which this theory is based is far superior to the earlier ones [3, 6]. It is, thus, seen that the formulation described here offers a convenient and concise method for the analysis of both thick and thin rectangular plates in bending. Further, it is believed that the results presented here will form a basis for future comparative studies.

Acknowledgment

The author, who is on leave from the Indian Institute of Technology, Bombay, gratefully acknowledges the 1979 Scholarship award of the Jawaharlel Nehru Memorial Trust (U.K.) and the hospitality of the Civil Engineering Department of the University College of Swansea.

The initial support given to this research by the Department of Science and Technology and the Department of Atomic Energy of the Government of India through their respective grants HCS/DST/198/76 and BRNS/ENGG/17/75 is acknowledged.

Appendix A. Brief description of a refined higher order plate theory

A.1. Displacement model

The components of the displacement vector $U(x, y, z)$, $V(x, y, z)$ and $W(x, y, z)$ are expressed as follows[2] (see Fig. A.1).

$$\begin{aligned} U(x, y, z) &= z\theta_x(x, y) + z^3\theta_x^*(x, y)\,, \\ V(x, y, z) &= z\theta_y(x, y) + z^3\theta_y^*(x, y)\,, \\ W(x, y, z) &= w(x, y) + z^2 w^*(x, y)\,. \end{aligned} \tag{A.1}$$

A.2. Strain-displacement relations

Stipulation of the displacement assumptions (A.1) in the six well-known strain-displacement relations of the three-dimensional elasticity gives the following relations

$$\varepsilon_x = z\,\frac{\partial\theta_x}{\partial x} + z^3\,\frac{\partial\theta_x^*}{\partial x}\,, \tag{A.2}$$

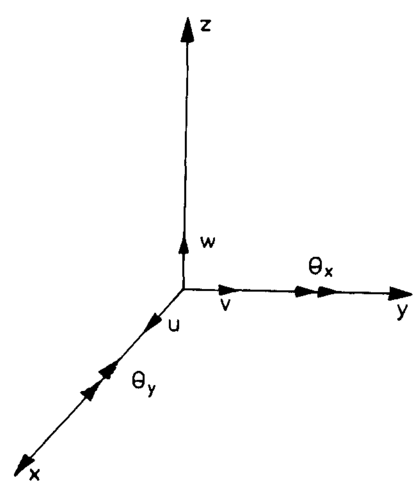

Fig. A.1. Positive set of displacement components.

[2] Hereafter $U(x, y, z)$, $\theta_x(x, y)$, $\theta_y^*(x, y)$, $w^*(x, y)$ etc. are written simply as U, θ_x, θ_y^*, w^* etc., respectively.

((A.2) cont'd)

$$\varepsilon_y = z\frac{\partial\theta_y}{\partial y} + z^3\frac{\partial\theta_y^*}{\partial y},$$

$$\varepsilon_z = 2zw^*,$$

$$\gamma_{xy} = z\Big(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\Big) + z^3\Big(\frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\Big),$$

$$\gamma_{xz} = \Big(\frac{\partial w}{\partial x} + \theta_x\Big) + z^2\Big(\frac{\partial w^*}{\partial x} + 3\theta_x^*\Big),$$

$$\gamma_{yz} = \Big(\frac{\partial w}{\partial y} + \theta_y\Big) + z^2\Big(\frac{\partial w^*}{\partial y} + 3\theta_y^*\Big).$$

The equations in (A.2) show that unlike the corresponding relations of the Mindlin plate theory, wherein the planar normal and shearing strain components vary linearly with the thickness coordinate z and the transverse normal and shearing strain components are zero and a constant respectively (see relations 1.7a–f of [2]), these variations are more realistic. This also eliminates the need for the introduction of a so-called warping coefficient as it is done in the Mindlin theory.

A.3. Equilibrium equations and boundary conditions

For equilibrium, the total potential energy Π for the plate must be stationary, i.e.

$$\delta\Pi = \delta(U - W_s - W_b - W_{ex} - W_{ey}) = 0 \qquad \text{(A.3)}$$

where U is the strain energy of the plate; W_s and W_b represent the work done by surface tractions and body forces respectively; W_{ex} and W_{ey} represent the work done by edge stresses on typical edges x = constant and y = constant, respectively. The individual terms of the relation (A.3) are evaluated as follows (see Figs. A.2, A.3, and A.4).

$$\delta U = \int_x\int_y\int_z (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})\,dx\,dy\,dz\,. \qquad \text{(A.4)}$$

Integration through the plate thickness with the use of (A.2) transforms the above relation (A.4) in the following form.

$$\begin{aligned}\delta U = &\oint_x (Q_y\delta w + M_{yx}\delta\theta_x + M_y\delta\theta_y + Q_y^*\delta w^* + M_{yx}^*\delta\theta_x^* + M_y^*\delta\theta_y^*)\,dx\\ &+ \oint_y (Q_x\delta w + M_x\delta\theta_x + M_{xy}\delta\theta_y + Q_x^*\delta w^* + M_x^*\delta\theta_x^* + M_{xy}^*\delta\theta_y^*)\,dy\\ &- \int_x\int_y \Big\{\Big(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\Big)\delta w + \Big(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x\Big)\delta\theta_x\\ &\quad + \Big(\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y\Big)\delta\theta_y + \Big(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z\Big)\delta w^*\\ &\quad + \Big(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^*\Big)\delta\theta_x^* + \Big(\frac{\partial M_{yx}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - 3Q_y^*\Big)\delta\theta_y^*\Big\}\,dx\,dy \qquad \text{(A.5)}\end{aligned}$$

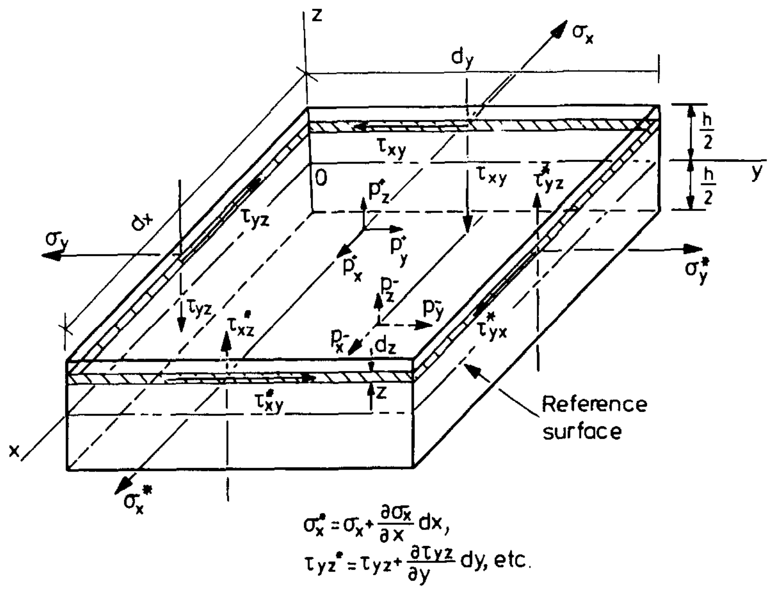

Fig. A.2. Positive set of stress components.

where

$$\{M_x, M_y, M_z, M_{xy}\} = \int_z \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}\} z \, dz \,, \qquad \{M_x^*, M_y^*, M_{xy}^*\} = \int_z \{\sigma_x, \sigma_y, \tau_{xy}\} z^3 \, dz \,,$$
$$\{Q_x, Q_y\} = \int_z \{\tau_{xz}, \tau_{yz}\} \, dz \,, \qquad \{Q_x^*, Q_y^*\} = \int_z \{\tau_{xz}, \tau_{yz}\} z^2 \, dz \,. \tag{A.6}$$

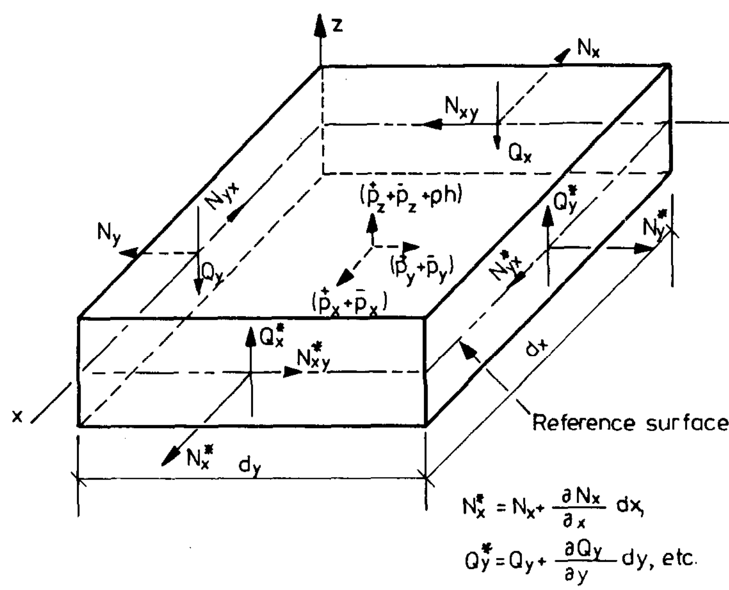

Fig. A.3. Positive set of stress resultants-forces.

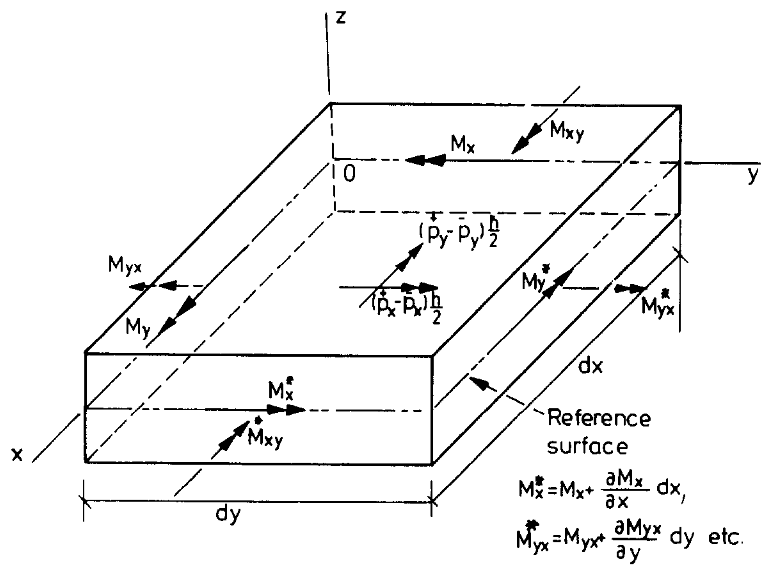

Fig. A.4. Positive set of stress resultants-couples.

The work done by surface tractions, W_s may be calculated as follows,

$$W_s = \int_x \int_y (p_z^+ W^+ + p_z^- W^-)\, dx\, dy\,. \tag{A.7}$$

But

$$W^+ = w + h^{+2} w^* \quad \text{and} \quad W^- = w + h^{-2} w^*\,. \tag{A.8}$$

Therefore

$$W_s = \int_x \int_y \{(p_z^+ + p_z^-) w + (p_z^+ h^{+2} + p_z^- h^{-2}) w^*\}\, dx\, dy \tag{A.9}$$

and the variation of W_s for $h^+ = h^- = h/2$,

$$\delta W_s = \int_x \int_y (p_z^+ + p_z^-)(\delta w + \tfrac{1}{4} h^2 \delta w^*)\, dx\, dy\,. \tag{A.10}$$

The work done by body forces, W_b is calculated as follows.

$$W_b = \int_x \int_y \int_z (\rho_z W)\, dx\, dy\, dz = \int_x \int_y \int_z \rho_z (w + z^2 w^*)\, dx\, dy\, dz\,. \tag{A.11}$$

If $\rho_z = \rho_z(x, y, z) = \rho(x, y) \equiv \rho$, that is, the unit weight of the plate material is assumed to be independent of z and $h^+ = h^- = h/2$, the variation of W_b can be written as

$$\delta W_b = \int_x \int_y (\rho h \delta w + \tfrac{1}{12} \rho h^3 \delta w^*)\, dx\, dy\,. \tag{A.12}$$

The work done by edge stresses is

$$W_{ex} = \int_y \int_z (\bar{\sigma}_x U + \bar{\tau}_{xy} V + \bar{\tau}_{xz} W)\, dy\, dz \tag{A.13a}$$

on an edge x = constant, and

$$W_{ey} = \int_x \int_z (\bar{\tau}_{xy} U + \bar{\sigma}_y V + \bar{\tau}_{yz} W)\, dx\, dz \tag{A.13b}$$

on an edge y = constant, where the bars on the quantities refer to edge values. On integration through the thickness the variation of these expressions take the form

$$\delta W_{ex} = \oint_y (\bar{Q}_x \delta w + \bar{M}_x \delta\theta_x + \bar{M}_{xy} \delta\theta_y + \bar{Q}_x^* \delta w^* + \bar{M}_x^* \delta\theta_x^* + \bar{M}_{xy}^* \delta\theta_y^*)\, dy$$

and (A.14)

$$\delta W_{ey} = \oint_x (\bar{Q}_y \delta w + \bar{M}_{yx} \delta\theta_x + \bar{M}_y \delta\theta_y + \bar{Q}_y^* \delta w^* + \bar{M}_{yx}^* \delta\bar{\theta}_y^* + M_y^* \delta\theta_y^*)\, dx\,.$$

The variational statement (A.3) takes the following form when the relevant foregoing expressions are substituted for its individual terms:

$$\begin{aligned}
&\int_x \int_y \Big[\Big(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ + p_z^- + \rho h\Big)\delta w \\
&\quad + \Big(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x\Big)\delta\theta_x + \Big(\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y\Big)\delta\theta_y \\
&\quad + \Big\{\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z + \tfrac{1}{4}(p_z^+ + p_z^-)h^2 + \tfrac{1}{12}\rho h^3\Big\}\delta w^* \\
&\quad + \Big(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^*\Big)\delta\theta_x^* + \Big(\frac{\partial M_{yx}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - 3Q_y^*\Big)\delta\theta_y^*\Big]\, dx\, dy \\
&+ \oint_x [(\bar{Q}_y - Q_y)\delta w + (\bar{M}_{yx} - M_{yx})\delta\theta_x + (\bar{M}_y - M_y)\delta\theta_y \\
&\quad + (\bar{Q}_y^* - Q_y^*)\delta w^* + (\bar{M}_{yx}^* - M_{yx}^*)\delta\theta_x^* + (\bar{M}_y^* - M_y^*)\delta\theta_y^*]\, dx \\
&+ \oint_y [(\bar{Q}_x - Q_x)\delta w + (\bar{M}_x - M_x)\delta\theta_x + (\bar{M}_{xy} - M_{xy})\delta\theta_y \\
&\quad + (\bar{Q}_x^* - Q_x^*)\delta w^* + (\bar{M}_x^* - M_x^*)\delta\theta_x^* + (\bar{M}_{xy}^* - M_{xy}^*)\delta\theta_y^*]\, dy = 0
\end{aligned} \tag{A.15}$$

The above relation (A.15) will be an identity only if each of the coefficients of the arbitrary variations vanishes. The vanishing of the surface integral defines six equilibrium equations, while that of the line integrals defines the consistent natural boundary conditions that are to be used with this theory along the two edges such that

$$
\begin{aligned}
&w = \bar{w} \quad \text{or } Q_x = \bar{Q}_x, && \theta_x = \bar{\theta}_x \quad \text{or } M_x = \bar{M}_x,\\
&\theta_y = \bar{\theta}_y \quad \text{or } M_{xy} = \bar{M}_{xy}, && w^* = \bar{w}^* \quad \text{or } Q_x^* = \bar{Q}_x^*,\\
&\theta_x^* = \bar{\theta}_x^* \quad \text{or } M_x^* = \bar{M}_x^*, && \theta_y^* = \bar{\theta}_y^* \quad \text{or } M_{xy}^* = \bar{M}_{xy}^*.
\end{aligned} \tag{A.16}
$$

On an edge x = constant and

$$
\begin{aligned}
&w = \bar{w} \quad \text{or } Q_y = \bar{Q}_y, && \theta_x = \bar{\theta}_x \quad \text{or } M_{yx} = \bar{M}_{yx},\\
&\theta_y = \bar{\theta}_y \quad \text{or } M_y = \bar{M}_y, && w = \bar{w}^* \quad \text{or } Q_y^* = \bar{Q}_y^*,\\
&\theta_x = \bar{\theta}_x^* \quad \text{or } M_{yx}^* = \bar{M}_{yx}^*, && \theta_y^* = \bar{\theta}_y^* \quad \text{or } M_y^* = \bar{M}_y^*,
\end{aligned} \tag{A.17}
$$

on the edge y = constant.

A.4. Constitutive relations

Hooke's laws for an isotropic and linear elastic material are,

$$
\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z],\\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu\varepsilon_z],\\
\sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + \nu\varepsilon_y + (1-\nu)\varepsilon_z],\\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}, \qquad \tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz}, \qquad \tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}.
\end{aligned} \tag{A.18}
$$

The following two-dimensional stress resultant-displacement relations are then obtained with the help of (A.2), (A.6) and (A.18).

$$
\begin{aligned}
M_x &= D_1\Big[(1-\nu)\frac{\partial\theta_x}{\partial x} + \nu\frac{\partial\theta_y}{\partial y} + 2\nu w^*\Big] + K_1^*\Big[(1-\nu)\frac{\partial\theta_x^*}{\partial x} + \nu\frac{\partial\theta_y^*}{\partial y}\Big],\\
M_y &= D_1\Big[\nu\frac{\partial\theta_x}{\partial x} + (1-\nu)\frac{\partial\theta_y}{\partial y} + 2\nu w^*\Big] + K_1^*\Big[\nu\frac{\partial\theta_x^*}{\partial x} + (1-\nu)\frac{\partial\theta_y^*}{\partial y}\Big],\\
M_z &= D_1\Big[\nu\frac{\partial\theta_x}{\partial x} + \nu\frac{\partial\theta_y}{\partial y} + 2(1-\nu)w^*\Big] + \nu K_1^*\Big(\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y}\Big),\\
M_{xy} &= M_{yx} = \frac{D_1(1-2\nu)}{2}\Big(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\Big) + \frac{K_1^*(1-2\nu)}{2}\Big(\frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\Big),\\
M_x^* &= K_1^*\Big[(1-\nu)\frac{\partial\theta_x}{\partial x} + \nu\frac{\partial\theta_y}{\partial y} + 2\nu w^*\Big] + D_1^*\Big[(1-\nu)\frac{\partial\theta_x^*}{\partial x} + \nu\frac{\partial\theta_y^*}{\partial y}\Big],
\end{aligned} \tag{A.19}
$$

((A.19) cont'd)

$$M_y^* = K_1^*\left[\nu\frac{\partial\theta_x}{\partial x} + (1-\nu)\frac{\partial\theta_y}{\partial y} + 2\nu w^*\right] + D_1^*\left[\nu\frac{\partial\theta_x^*}{\partial x} + (1-\nu)\frac{\partial\theta_y^*}{\partial y}\right],$$

$$M_{xy}^* = M_{yx}^* = \frac{K_1^*(1-2\nu)}{2}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) + \frac{D_1^*(1-2\nu)}{2}\left(\frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\right),$$

$$Q_x = \frac{K_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial x} + \theta_x\right) + \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial x} + 3\theta_x^*\right),$$

$$Q_y = \frac{K_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial y} + \theta_y\right) + \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right),$$

$$Q_x^* = \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial x} + \theta_x\right) + \frac{K_1^*(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial x} + 3\theta_x^*\right),$$

$$Q_y^* = \frac{D_1(1-2\nu)}{2}\left(\frac{\partial w}{\partial y} + \theta_y\right) + \frac{K_1^*(1-2\nu)}{2}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right)$$

where

$$K_1 = \frac{Eh}{(1+\nu)(1-2\nu)} \quad \left(K = \frac{Eh}{(1-\nu^2)}\right), \qquad D_1 = \frac{Eh^3}{12(1+\nu)(1-2\nu)} \quad \left(D = \frac{Eh^3}{12(1-\nu^2)}\right),$$

$$K_1^* = \frac{Eh^5}{80(1+\nu)(1-2\nu)} = \frac{3}{20}D_1h^2, \qquad D_1^* = \frac{Eh^7}{448(1+\nu)(1-2\nu)} = \frac{5}{28}K_1^*h^2. \qquad \text{(A.20)}$$

The system of equations comprising of six equilibrium equations in (A.15) and eleven stress resultant-displacement relations in (A.19), along with the consistent natural boundary conditions (A.16) and (A.17) constitute the complete set of governing equations and also constitute a completely defined and properly-posed boundary value problem. Thus there are seventeen equations containing seventeen unknowns, comprising of six displacement quantities and eleven force quantities. It can be shown through algebraic manipulation that these equations will ultimately give rise to a twelfth order partial differential equation system.

References

[1] T. Kant and C.K. Ramesh, Numerical integration of linear boundary value problems in solid mechanics by segmentation method, Internat. J. Numer. Meths. Engrg. 17 (1981) 1233–1256.

[2] T. Kant and E. Hinton, Numerical analysis of rectangular Mindlin plates by the segmentation Method, Civil Engineering Department, Rept. C/R/365/80, University of Wales, Swansea, 1980.

[3] R.D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic elastic plates, J. Appl. Mech., Trans. ASME 18 (1951) 31–38.

[4] K.H. Lo, R.M. Christensen and E.M. Wu, A high-order theory of plate deformation, J. Appl. Mech., Trans. ASME 99 (1977) 663–668.

[5] J.E. Goldberg, A.V. Setlur and D.W. Alspaugh, Computer analysis of non-circular cylindrical shells, Proc. IASS Symp. on Shell Structures in Engineering Practice, Budapest, Hungary, 1965.

[6] S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells (McGraw-Hill, New York, 2nd ed., NY, 1959).

[7] Y.K. Cheung, Finite Strip Method in Structural Analysis (Pergamon, Oxford, 1976).
[8] T. Kant, Numerical analysis of elastic plates with two opposite simply supported ends by segmentation method, Comput. and Structures 14 (1982) 195–203.
[9] A. Ralston and H.S. Wilf, eds., Mathematical Methods for Digital Computers (Wiley, New York, 1960).
[10] V.L. Salerno and M.A. Goldberg, Effect of shear deformation on the bending of rectangular plates, J. Appl. Mech. 27 (1960) 54–58.
[11] T. Kant, D.R.J. Owen and O.C. Zienkiewicz, A refined higher-order C^0 plate bending element, Comput. and Structures 15 (1982) 177–183.
[12] T. Kant, On evaluation of two plate theories—finite element solutions, Conf. on the Mathematics of Finite Elements and Applications (MAFELAP 1981), Brunel University, 1981.
[13] O.C. Zienkiewicz, The Finite Element Method (McGraw-Hill, New York, 1977).
[14] D. Young and R.P. Felgar, Jr., Tables of characteristic functions representing normal modes of vibration of a beam, The University of Texas Publication No. 4913, Austin, 1949.
[15] R.P. Felgar, Jr., Formulas for integrals containing characteristic functions of a vibrating beam, The University of Texas Publication, Austin, 1950.