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Geometrically Non-Linear Analysis of Doubly Curved Laminated and Sandwich Fibre Reinforced Composite Shells with a Higher Order Theory and C° Finite Elements

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ABSTRACT: An isoparametric C° finite element formulation based on a higher order displacement model for linear and geometrically non-linear analysis (that accounts for large displacements in the sense of von Karman) of doubly curved laminated composite and sandwich shells under transverse loads is presented. The displacement model accounts for cubic and constant variation of tangential and transverse displacement components respectively through the shell thickness. The assumed displacement model eliminates the use of shear correction coefficients. The discrete element chosen is a nine-noded quadrilateral element with nine degrees of freedom per node. The accuracy of the formulation is then established by comparing the present results with the available analytical closed-form solutions, three-dimensional elasticity solutions and other finite element solutions.

1. INTRODUCTION

STRUCTURAL ELEMENTS MADE up of fibre reinforced composite materials are being extensively used in high and low technology areas in recent years. Their industrial applications are multiplying rapidly because of their superior mechanical properties. However, the engineering community is faced with many challenging problems associated with the use of these new materials. Of these, the geometric non-linear response of laminated composite shells is one of the major considerations in their design.

An accurate prediction of the behaviour of shell structures requires modeling of actual geometry and kinematic description of the components. The partial differential equations describing the large deflection behaviour of anisotropic composite shells of arbitrary geometry are not amenable to classical analytical methods. The finite element method has proved to be a very powerful tool for an-

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alyzing structural problems involving complex geometries, loadings, boundaries and non-linearities.

Many of the classical theories were developed originally for thin elastic shells and are based on Love-Kirchhoff assumptions and surveys of such classical shell theories can be seen in the works of Naghdi (1956) and Bert (1980).

The first analysis that incorporated the bending and stretching coupling is due to Ambartsumyan (1964). Ambartsumyan assumed that the individual orthotropic layers were oriented such that the principal axes of material symmetry coincided with that of the principal coordinates of the shell reference surface. The effects of transverse shear deformation, transverse normal stress and transverse normal strain on the behaviour of laminated shells can be incorporated on the basis of a mathematical model through the inclusion of higher order terms in the power series expansion of the assumed displacement field.

In the context of special orthotropic and homogeneous shells, Hildebrand, Reissner and Thomas (1949) were the first to make significant contributions by dispensing with all the Love's assumptions and assuming a three-term Taylor's series expansion for the displacement vector. Naghdi (1957) has employed Reissner's (1950) mixed variational principle to develop a complete shell formulation similar to that of Hildebrand et al. retaining two and three terms in the Taylor's series for tangential and transverse displacement components, respectively. Dong and Tso (1972) were perhaps the first to present a first order shear deformation theory, retaining one and two terms in the Taylor's series for transverse and tangential displacement components respectively which includes the effects of transverse shear deformation through the shell thickness, and then to construct a laminated orthotropic shell theory.

Further attempts at refining the theories for laminated anisotropic cylindrical shells have been made by Widera and Loagan (1980) and Whitney and Sun (1974) based on a displacement model similar to one used by Naghdi. Reddy (1984) extended Sanders (1959) theory for simply supported cross-ply laminated shells assuming five degrees of freedom per node. Kant (1976) developed complete governing equations for a thick laminated composite shell. The theory is based on a three-term Taylor's series expansion of the displacement vector and generalized Hook's law, and is applicable to orthotropic material laminas having planes of symmetry coincident with shell coordinates. Kant and Ramesh (1976) have presented a general orthotropic shell theory in orthogonal curvilinear coordinates based on a displacement model of Hildebrand et al. (1949). However, they concluded that the inclusion of the third terms in the tangential surface displacements would be of no practical significance for sufficiently thin shells. It was this result, perhaps had prompted Naghdi (1957) to truncate the Taylor's series expansion for tangential displacements after linear terms in the thickness coordinate and later many others followed him. This indeed was done without realizing that the flexural behaviour could only be improved if one retains terms up to cubic in the thickness coordinate in the expansion of the tangential displacements.

Kant (1981a,b) presented higher order theories for general orthotropic as well as laminated shells, which are derived from the three-dimensional elasticity equations by expanding the displacement vector in Taylor's series in the thickness coordinate. The theories account for the effects of transverse shear deformation, transverse normal stress and transverse normal strain with an implicit non-linear cubic distribution of the tangential displacement components through the shell thickness.

Reddy and Liu (1985) presented a higher order theory assuming constant and cubic variation of transverse and tangential displacement components respectively through the thickness of doubly curved shells and used Navier's approach for solution. Bhimaraddi (1986) and Murty and Reddy (1986), presented higher order displacement based shear deformation theories based on C^1 continuity. Kant and Pandya (1988) and Pandya (1987) presented different higher order shear deformation theories for static analysis of laminated composite plates using C° continuity. Kant and Menon (1989) presented various higher order theories for laminated composite cylindrical shells using C° finite elements. Kant (1988–1991) along with co-workers after doing extensive numerical investigations on laminated plates and shells, both static and dynamic analysis, using C° finite elements and different higher order theories, proved that the imposition of shear free boundary conditions at top and bottom bounding planes of the laminate gives stiffer solutions when compared to three-dimensional (3-D) elasticity solutions and also among various displacement models for flat laminates, the one having nine degrees of freedom per node produces results very close to 3-D elasticity solutions. As regards curved laminates the work is under progress and a definite conclusion would emerge after some more investigation. All of these studies are limited to small deformation theory.

Because of high modulus and high strength properties that composites have, structural composites undergo large deformations before they become inelastic. Therefore, an accurate prediction of displacements and stresses are possible only when one accounts for the geometric non-linearity. Horrigmoe and Bergan (1976) presented classical variational principles for non-linear problems by considering incremental deformations of a continuum. Wunderlich (1977) and Stricklen et al. (1973) have reviewed various principles of incremental analysis and solution procedures for geometrical non-linear problems respectively.

Noor and Hartley (1977) employed the shallow shell theory with transverse shear strains and geometric non-linearities to develop triangular and quadrilateral finite elements. Chang and Sawamiphakdi (1981) presented a formulation of the degenerate three-dimensional (3-D) shell element for geometrically non-linear analysis of laminated composite shells. Their presentation does not include any numerical results for laminated shells. Kim and Lee (1988) developed an 18-noded solid element to study the behaviour of laminated composite shells undergoing large displacements.

Chao and Reddy (1983), Reddy and Chandrasekhara (1985a,b) have presented a first order shear deformation theory based on kinematic and geometric assumptions of Sander's thin shell theory for geometrically non-linear analysis of doubly curved composite shells and Liu (1985) presented C^1 based higher order shear deformation theory for geometrically non-linear analysis of doubly curved anisotropic shells. Use of such elements in the non-linear analysis of composite shells inevitably leads to large storage requirements and computational costs and are not amenable to the popular and widely used isoparametric formulation in the present day finite element technology. Recently, Kant and Mallikarjuna (1991) presented a geometrically non-linear dynamic response of laminated plates with a higher order theory with seven degrees of freedom per node and C° finite elements.

In the present paper, a new higher order (third order) theory is presented that accounts for a parabolic distribution of the transverse shear strains and the von Karman strains. The tangential displacements have cubic variation through the thickness. Nine-node quadrilateral Lagrangian finite elements incorporating selective numerical integration have been used. Numerical results of this formulation have been compared with a parallel formulation of a first order shear deformation theory based on Sander's shell theory.

2. THEORY AND FORMULATION

Let (α_1, α_2, z) denote the orthogonal curvilinear coordinates (shell coordinates) such that α_1 and α_2 curves are lines of principal curvatures of middle surface $z = 0$ and α curves are perpendicular to the z-axis. For cylindrical and spherical shells the lines of principle curvatures coincide with the coordinates' lines. The values of principle radii are denoted by R_1 and R_2.

The position vector of a point on the middle surface is denoted by $\mathbf{r}$ and position of a point at a distance z from the middle surface is denoted by $\mathbf{R}$. The distance ds between two points $(\alpha_1, \alpha_2, 0)$ and $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, 0)$ is denoted by

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = (A_1 d\alpha_1)^2 + (A_2 d\alpha_2)^2 \tag{1}$$

where A_1 and A_2 are Lame's parameters.

The distance between two general points in shell space (α_1, α_2, z) and $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, z + dz)$ is given by

$$dS^2 = d\mathbf{R} \cdot d\mathbf{R} = h_1^2 d\alpha_1^2 + h_2^2 d\alpha_2^2 + h_3^2 dz^2 \tag{2}$$

where h_1, h_2 and h_3 are three-dimensional (3-D) Lame's coefficients, such that

$$h_1 = A_1\left(1 + \frac{z}{R_1}\right) \qquad h_2 = A_2\left(1 + \frac{z}{R_2}\right) \qquad h_3 = 1.0 \tag{3}$$

for a doubly curved shell.

A composite doubly curved shell consisting of thin homogeneous orthotropic layers having a total thickness of h is considered. The x, y are the curvilinear dimensional coordinates defining the doubly curved shell which coincides with the mid-surface of the shell and z-axis is oriented in the thickness direction as shown in Figure 1.

In the present theory displacement components of a generic point in the shell are assumed to be of the form given by Kant and Menon (1991) and Reddy and Liu (1985)

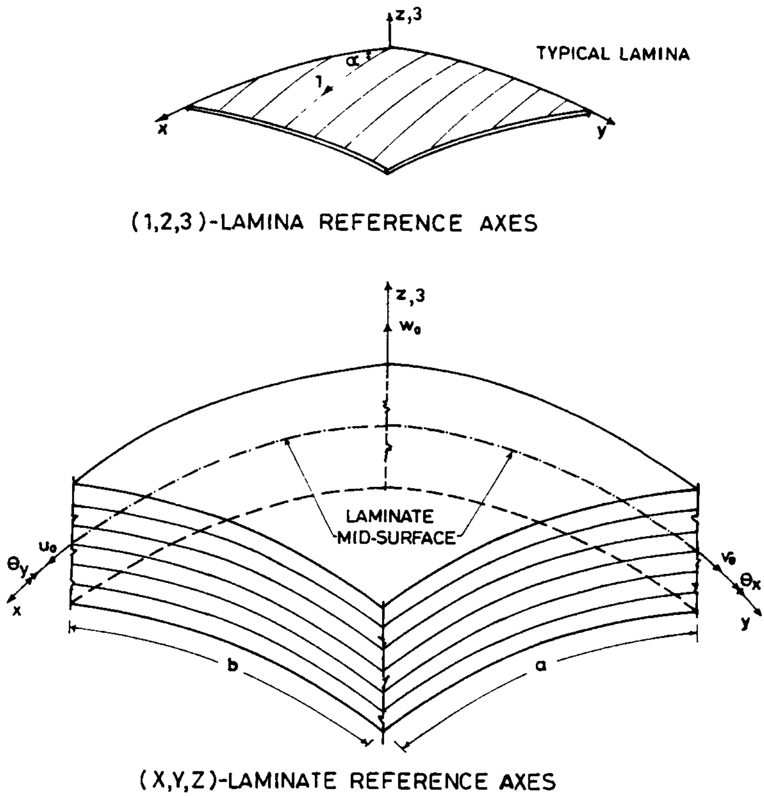

Figure 1. *Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.*

$$
\begin{aligned}
u(x,y,z) &= u_o(x,y) + z\theta_x(x,y) + z^2u_o^*(x,y) + z^3\theta_x^*(x,y) \\
v(x,y,z) &= v_o(x,y) + z\theta_y(x,y) + z^2v_o^*(x,y) + z^3\theta_y^*(x,y) \\
w(x,y,z) &= w_o(x,y)
\end{aligned}
\tag{4}
$$

where u_o, v_o and w_o are middle surface displacements of a generic point having displacements u, v, w in x, y and z directions respectively. θ_x and θ_y are rotations of the transverse normal in the xz and yz planes respectively, u_o^*, v_o^*, θ_x^* and θ_y^* are the corresponding higher order terms in the Taylor's series expansion.

A total Lagrangian approach is adopted and stress and strain descriptions used are those due to Piola-Kirchhoff and Green respectively. In the present theory large displacements in the sense of von Karman with small strains and rotations are considered. The following are the strain displacement relations.

$$\epsilon_x = \frac{\partial u_o}{\partial x} + \frac{w_o}{R_1} + z\frac{\partial \theta_x}{\partial x} + z^2\frac{\partial u_o^*}{\partial x} + z^3\frac{\partial \theta_x^*}{\partial x} + \frac{1}{2}\left(\frac{\partial w_o^2}{\partial x}\right)^2$$

$$\epsilon_y = \frac{\partial v_o}{\partial y} + \frac{w_o}{R_2} + z\frac{\partial \theta_y}{\partial y} + z^2\frac{\partial v_o^*}{\partial y} + z^3\frac{\partial \theta_y^*}{\partial y} + \frac{1}{2}\left(\frac{\partial w_o^2}{\partial y}\right)^2$$

$$\epsilon_z = 0$$

$$\gamma_{xy} = \left(\frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y}\right) + z\left(\frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y}\right) + z^2\left(\frac{\partial v_o^*}{\partial x} + \frac{\partial u_o^*}{\partial y}\right) \quad (5)$$

$$+ z^3\left(\frac{\partial \theta_y^*}{\partial x} + \frac{\partial \theta_x^*}{\partial y}\right) + \frac{\partial w_o}{\partial x}\frac{\partial w_o}{\partial y}$$

$$\gamma_{xz} = \left(\theta_x + \frac{\partial w_o}{\partial x} - \frac{u_o}{R_1}\right) + z\left(2u_o^* - \frac{\theta_x}{R_1}\right) + z^2\left(3\theta_x^* - \frac{u_o^*}{R_1}\right) + z^3\left(-\frac{\theta_x^*}{R_1}\right)$$

$$\gamma_{yz} = \left(\theta_y + \frac{\partial w_o}{\partial y} - \frac{v_o}{R_1}\right) + z\left(2v_o^* - \frac{\theta_y}{R_2}\right) + z^2\left(3\theta_y^* - \frac{v_o^*}{R_2}\right) + z^3\left(-\frac{\theta_y^*}{R_2}\right)$$

The total potential energy of the system Π with the middle surface area A enclosing the space of the volume V and loaded with an equivalent load vector $\mathbf{q}$ corresponding to nine degrees of freedom per node of a point on the middle surface can be written as

$$\Pi = \frac{1}{2}\int_v \underset{\sim}{\epsilon}^t \underset{\sim}{\sigma}\, dV - \int_A \mathbf{d}^t \mathbf{q}\, dA \quad (6)$$

$$\Pi = \frac{1}{2}\int_A \left(\int_z \underset{\sim}{\epsilon}^t \underset{\sim}{\sigma}\, dz\right) dA - \int_A \mathbf{d}^t \mathbf{q}\, dA \quad (7)$$

where $\mathbf{d}^t = (u_o, v_o, w_o, \theta_x, \theta_y, u_o^*, v_o^*, \theta_x^*, \theta_y^*)$.

By substituting the expression for the strain components in the above expression for total potential energy and minimizing the function while carrying out explicit integration through the shell thickness leads to the stress resultant vector $\underset{\sim}{\bar{\sigma}}$ [ref. Kant and Pandya (1988) and Kant and Menon (1989)]. $\underset{\sim}{\bar{\sigma}} = (N_x, N_y, N_{xy}, N_x^*, N_y^*, N_{xy}^*, M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*, Q_x, Q_y, Q_x^*, Q_y^*, S_x, S_y, S_x^*, S_y^*)$.

$$[\mathbf{N}, \mathbf{N}^*] = \begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} [1, z^2]\, dz$$

$$[\mathbf{M},\mathbf{M}^*] = \begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} [z, z^3] dz \tag{8}$$

$$[\mathbf{Q},\mathbf{Q}^*] = \begin{bmatrix} Q_x & Q_x^* & S_x & S_x^* \\ Q_y & Q_y^* & S_y & S_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} [1, z^2, z, z^3]$$

Then the laminate constitutive relations are obtained as

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{N}^* \\ \mathbf{M} \\ \mathbf{M}^* \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_m & \mathbf{D}_c & \mathbf{0} \\ \mathbf{D}_c^t & \mathbf{D}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_s \end{bmatrix} \begin{bmatrix} \underset{\sim}{\epsilon}_m \\ \underset{\sim}{\epsilon}_b \\ \underset{\sim}{\epsilon}_o \end{bmatrix}$$

$$\text{or } \bar{\underset{\sim}{\sigma}} = \mathbf{D}(\bar{\underset{\sim}{\epsilon}}_l + \bar{\underset{\sim}{\epsilon}}_{nl}) = \mathbf{D}\bar{\underset{\sim}{\epsilon}} \tag{9}$$

where the coefficients $D_{m_{ij}}$, $D_{c_{ij}}$, $D_{b_{ij}}$ and $D_{s_{ij}}$ are respectively in-plane, bending in-plane coupling, bending and transverse shear stiffness coefficients respectively and $\bar{\underset{\sim}{\epsilon}}_l$ and $\bar{\underset{\sim}{\epsilon}}_{nl}$ are generalized linear and non-linear strains respectively defined in the following manner.

$$\underset{\sim}{\epsilon}_m = \begin{bmatrix} \dfrac{\partial u_o}{\partial x} + \dfrac{w_o}{R_1} + \dfrac{1}{2}\left(\dfrac{\partial w_o^2}{\partial x}\right)^2 \\ \dfrac{\partial v_o}{\partial y} + \dfrac{w_o}{R_2} + \dfrac{1}{2}\left(\dfrac{\partial w_o^2}{\partial y}\right)^2 \\ \dfrac{\partial v_o}{\partial x} + \dfrac{\partial u_o}{\partial y} + \dfrac{\partial w_o}{\partial x}\dfrac{\partial w_o}{\partial y} \\ \dfrac{\partial u_o^*}{\partial x} \\ \dfrac{\partial v_o^*}{\partial y} \\ \dfrac{\partial v_o^*}{\partial x} + \dfrac{\partial u_o^*}{\partial y} \end{bmatrix} \quad \underset{\sim}{\epsilon}_b = \begin{bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_y}{\partial x} + \dfrac{\partial \theta_x}{\partial y} \\ \dfrac{\partial \theta_x^*}{\partial x} \\ \dfrac{\partial \theta_y^*}{\partial y} \\ \dfrac{\partial \theta_y^*}{\partial x} + \dfrac{\partial \theta_x^*}{\partial y} \end{bmatrix}$$

$$\underset{\sim}{\epsilon}_s = \begin{bmatrix} \theta_x + \dfrac{\partial w_o}{\partial x} - \dfrac{u_o}{R_1} \\ \theta_y + \dfrac{\partial w_o}{\partial y} - \dfrac{v_o}{R_2} \\ 3\theta_x^* - \dfrac{u_o^*}{R_1} \\ 3\theta_y^* - \dfrac{v_o^*}{R_2} \\ 2u_o^* - \dfrac{\theta_x}{R_1} \\ 2v_o^* - \dfrac{\theta_y}{R_2} \\ -\dfrac{\theta_x^*}{R_1} \\ -\dfrac{\theta_y^*}{R_2} \end{bmatrix} \tag{10}$$

and $\bar{\underset{\sim}{\epsilon}}^t = (\bar{\underset{\sim}{\epsilon}}_m^t, \bar{\underset{\sim}{\epsilon}}_b^t, \bar{\underset{\sim}{\epsilon}}_s^t)$.

As the basic equilibrium equation, the virtual work equation for laminated shell under the assumption of small strain and large displacement in total Lagrangian coordinate system is considered and can be written in compact form as

$$\int_A \delta \bar{\underset{\sim}{\epsilon}}^t \bar{\underset{\sim}{\sigma}} da + \mathbf{F} \delta \mathbf{d} = 0 \tag{11}$$

3. C° FINITE ELEMENT FORMULATION

The finite element used here is a nine-noded isoparametric quadrilateral (Lagrangian family) element. The laminate displacement field in the element can be expressed in terms of the nodal variables as

$$\mathbf{d}(\xi,\eta) = \sum_{i=1}^{NN} \mathbf{N}_i(\xi,\eta)\mathbf{d}_i \tag{12}$$

where NN is number of nodes per element, $N_i(\xi,\eta)$ contains interpolation functions associated with node i in terms of the local coordinates ξ and η, $\mathbf{d}_i$ is nodal displacement vector. The generalized strain vector $\bar{\epsilon}$ can be expressed in terms of mid surface nodal displacements $\mathbf{d}$, displacement gradient θ_{nl}, Cartesian derivatives of shape function matrix $\mathbf{N}$ and its variation $\delta\bar{\epsilon}$ can be written in the form of

$$\bar{\epsilon} = \left(\mathbf{B}_o + \frac{1}{2}\mathbf{B}_{nl}\right)\mathbf{d}$$
$$\delta\bar{\epsilon} = (\mathbf{B}_o + \mathbf{B}_{nl})\delta\mathbf{d} \qquad (13)$$

where $\mathbf{B}_o$ is the strain matrix giving linear strains. $\mathbf{B}_{nl}$ is linearly dependent upon the nodal displacements $\mathbf{d}$ such that $\mathbf{d}^t = (\mathbf{d}_1^t, \mathbf{d}_2^t, \mathbf{d}_3^t, \ldots, \mathbf{d}_n^t)$ and substituting Equations (12) and (13) in Equation (11), the following discrete equation is obtained.

$$\mathbf{K}_o\mathbf{d} + H(\mathbf{d})\mathbf{d} = \mathbf{F} \qquad (14)$$

where $\mathbf{K}_o$ is linear elastic stiffness matrix, $\mathbf{F}$ is force vector, $\mathbf{H}(d)$ is generalized non-linear stiffness matrix which is given by

$$H(d) = \int_A \mathbf{B}_o^t\bar{\sigma}_{nl}dA + \int_A \mathbf{B}_{nl}^t\bar{\sigma}dA \qquad (15)$$

where $\bar{\sigma}_{nl}$ is non-linear stress vector, the stresses are induced by the non-linear part of the strain and $\mathbf{F}$ is force vector due to external loads.

4. NUMERICAL RESULTS

In order to demonstrate the versatility of the refined theory and C° finite elements developed, several examples drawn from the literature are evaluated and discussed. Computer programs have been developed for first order shear deformation theory (FOST) with five degrees of freedom (u_o, v_o, w_o, θ_x, θ_y) and a new higher order shear deformation theory (HOST) with nine degrees of freedom (u_o, v_o, w_o, θ_x, θ_y, u_o^*, v_o^*, θ_x^*, θ_y^*) per node for both linear and geometrically non-linear analysis of doubly curved shells. All the computations were carried out in single precision with a 16-significant digit word length on CDC CYBER 180/840 computer at Indian Institute of Technology, Bombay, India.

The results to be discussed are grouped into two categories, viz., 1) linear analysis and 2) non-linear analysis. Due to biaxial symmetry only one quadrant of shell was modeled with 2 × 2 uniform mesh to model cross-ply shells whereas to model the angle-ply shells a 4 × 4 uniform mesh in full shell is adopted. In the present study the nine-noded Lagrangian quadrilateral isoparametric element was employed. Selective integration scheme, based on Gauss quadrature rules viz. 3 × 3 for integration of membrane, flexure and coupling between membrane and flexure terms, and 2 × 2 for shear terms in the energy expression, was employed in the evaluation of element stiffness property. All the stress values are reported at the Gauss points nearest to their maximum value locations. The shear correction coefficient used in first order shear deformation theory (FOST) is assumed as 5/6.

The material properties, unless otherwise specified are assumed as

$$E_1 = 25E_2 \qquad G_{12} = G_{13} = 0.5E_2 \qquad G_{23} = 0.2E_2 \qquad \nu = 0.25 \tag{16}$$

These properties do not satisfy the symmetry condition, i.e., $\nu_{12}/E_1 \neq \nu_{21}/E_2$ is selected to facilitate the comparison of present results with available results [see Pagano (1970)].

The boundary conditions used in the present investigation are classified and explained in Table 1.

4.1 Linear Analysis

4.1.1 COMPARISON OF PRESENT RESULTS WITH 3-D ELASTICITY RESULTS

Example 1: A square simply supported (S1) cross-ply (0°/90°) laminate under sinusoidal transverse load is considered. The present results are compared with Pagano (1970) as well as corresponding higher order results presented by Pandya and Kant (1988) and these are presented in Table 2. The following non-dimensional quantities are used

$$\overline{w}_o = \left(\frac{h^3 E_2}{q_o a^4} * 10\right) w_o \qquad \overline{\sigma}_1 \text{ or } \overline{\tau}_{12} = \left(\frac{h^2}{q_o a^2}\right)(\sigma_1 \text{ or } \tau_{12}) \tag{17}$$

Example 2: An infinite long orthotropic cylindrical shell of radius and arc lengths respectively 10 in, 10.472 in subjected to sinusoidal transverse load of $q = q_o \sin(\pi x/l)$ is considered. The boundaries are free along circumferential direction and simply supported along longitudinal direction. A 1 × 10 mesh discretization for one quadrant of shell is taken. The results are compared with exact solution given by Ren (1987), finite element higher order theory and classical lamination theory given by Dennis and Plazotto (1991) and all these are presented in Table 3. The following non-dimensional quantities are used

$$\overline{w}_o = \left(\frac{h^3 E_2}{q_o R^4} * 10\right) w_o \qquad \overline{\sigma}_\theta = \left(\frac{h^2}{q_o R^2}\right)(\sigma_\theta) \tag{18}$$

Example 3: A simply supported (S1) cross-ply (0°/90°) cylindrical shell of length equal to 4 times radius of shell, subjected to an internal sinusoidal pressure of intensity $q = q_o \sin(\pi x/l) \cos[4(y/R)]$ is considered. A 2 × 2 mesh discretization in 1/8 along circumference and 1/2 along longitudinal is adopted. The following non-dimensional quantities are used for presenting the results

Table 1.

Type	$x = 0/x = a$		$x = a/2$		$y = 0/y = b$		$y = b/2$	
S1	$v_o = 0$	$v_o^* = 0$	$u_o = 0$	$u_o^* = 0$	$u_o = 0$	$u_o^* = 0$	$v_o = 0$	$v_o^* = 0$
	$\theta_y = 0$	$\theta_y^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_y = 0$	$\theta_y^* = 0$
	$w_o = 0$				$w_o = 0$			
S2	$u_o = 0$	$u_o^* = 0$			$u_o = 0$	$u_o^* = 0$		
	$v_o = 0$	$v_o^* = 0$	$u_o = 0$	$u_o^* = 0$	$v_o = 0$	$v_o^* = 0$	$v_o = 0$	$v_o^* = 0$
	$\theta_y = 0$	$\theta_y^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_y = 0$	$\theta_y^* = 0$
	$w_o = 0$				$w_o = 0$			
S3	$u_o = 0$	$u_o^* = 0$	$u_o = 0$	$u_o^* = 0$	$v_o = 0$	$v_o^* = 0$	$v_o = 0$	$v_o^* = 0$
	$\theta_y = 0$	$\theta_y^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_y = 0$	$\theta_y^* = 0$
	$w_o = 0$				$w_o = 0$			
C	$u_o = 0$	$u_o^* = 0$			$u_o = 0$	$u_o^* = 0$		
	$v_o = 0$	$v_o^* = 0$	$u_o = 0$	$u_o^* = 0$	$v_o = 0$	$v_o^* = 0$	$v_o = 0$	$v_o^* = 0$
	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_x = 0$	$\theta_x^* = 0$	$\theta_y = 0$	$\theta_y^* = 0$
	$\theta_y = 0$	$\theta_y^* = 0$			$\theta_y = 0$	$\theta_y^* = 0$		
	$w_o = 0$				$w_o = 0$			
C1	$\theta_x = 0$	$\theta_x^* = 0$	$u_o = 0$	$u_o^* = 0$	$\theta_y = 0$	$\theta_y^* = 0$	$v_o = 0$	$v_o^* = 0$
	$w_o = 0$		$\theta_x = 0$	$\theta_x^* = 0$	$w_o = 0$		$\theta_y = 0$	$\theta_y^* = 0$

Table 2. Maximum displacement and in-plane stresses for a simply supported (S1) square cross-ply (0°/90°) laminate under sinusoidal transverse load.

$\frac{a}{h}$	Variable		Exact Pagano (1970)	Pandya and Kant (1988)	Present FOST	Present HOST
4	$\bar{\sigma}_x$	top	0.7807	0.8056	0.7244	0.8056
	$\bar{\sigma}_x$	bot	0.1098	0.0969	0.0852	0.0969
	$\bar{\tau}_{xy}$	top	0.0591	0.0597	0.0527	0.0597
	$\bar{\tau}_{xy}$	bot	0.0588	0.0597	0.0527	0.0597
	$\bar{w}_o$		—	0.2055	0.2149	0.2055
10	$\bar{\sigma}_x$	top	0.7300	0.7390	0.7240	0.7390
	$\bar{\sigma}_x$	bot	0.0890	0.0871	0.0852	0.0871
	$\bar{\tau}_{xy}$	top	0.0538	0.0540	0.0528	0.0540
	$\bar{\tau}_{xy}$	bot	0.0536	0.0540	0.0528	0.0540
	$\bar{w}_o$		—	0.1224	0.1238	0.1224

Table 3. Maximum displacement and circumferential stress for an orthotropic infinite long cylindrical shell under sinusoidal transverse load.

$\frac{R}{h}$	Variable	Exact Ren (1987)	HSDT Dennis and Plazotto (1991)	CLT Dennis and Plazotto (1991)	Present FOST	Present HOST
4	$\overline{w}_o$	0.312	0.2780 (10.89)	0.0752 (75.90)	0.3093 (0.86)	0.3102 (0.57)
	$\overline{\sigma}_{\theta\,bot}$	1.331	1.219 (8.415)	0.7810 (41.32)	0.7840 (41.09)	1.1610 (12.77)
	$\overline{\sigma}_{\theta\,top}$	1.079	0.950 (11.96)	0.719 (33.36)	0.717 (33.55)	1.0280 (4.73)
10	$\overline{w}_o$	0.115	0.108 (6.09)	0.0749 (34.87)	0.113 (1.74)	0.1130 (1.74)
	$\overline{\sigma}_{\theta\,bot}$	0.890	0.839 (5.73)	0.7630 (14.27)	0.765 (14.04)	0.8490 (4.60)
	$\overline{\sigma}_{\theta\,top}$	0.807	0.773 (4.21)	0.7380 (8.55)	0.736 (8.80)	0.8020 (0.62)
50	$\overline{w}_o$	0.077	0.0762 (1.04)	0.0748 (2.86)	0.076 (1.30)	0.076 (1.30)
	$\overline{\sigma}_{\theta\,bot}$	0.767	0.761 (0.78)	0.753 (1.83)	0.760 (0.91)	0.761 (0.78)
	$\overline{\sigma}_{\theta\,top}$	0.752	0.744 (1.06)	0.748 (0.53)	0.740 (1.60)	0.745 (0.93)
100	$\overline{w}_o$	0.0755	0.0751 (0.53)	0.0748 (0.93)	0.075 (0.66)	0.0751 (0.53)
	$\overline{\sigma}_{\theta\,bot}$	0.758	0.759 (0.13)	0.7510 (0.09)	0.766 (1.05)	0.7650 (0.09)
	$\overline{\sigma}_{\theta\,top}$	0.751	0.742 (1.19)	0.749 (0.27)	0.733 (2.39)	0.734 (2.20)

Values in parentheses give percentage difference with respect to the elasticity solution [Ren (1987)].

$$\bar{w}_o = \left(\frac{h^3E_2}{q_oR^4} * 10\right) \frac{w_o}{\sin\left(\frac{\pi x}{1}\right)\cos\left(4\frac{y}{R}\right)}$$

$$\bar{\sigma}_x \text{ or } \bar{\sigma}_y = \left(\frac{10h^2}{q_oR^2}\right)(\sigma_x \text{ or } \sigma_y)/\sin\left(\frac{\pi x}{1}\right)\cos\left(4\frac{y}{R}\right) \qquad (19)$$

The present results for displacement and stresses are compared with Varadan and Bhaskara (1991) and are presented in Table 4.

4.1.2 COMPARISON OF PRESENT RESULT WITH 2-D CLOSED FORM RESULTS

To show further the validity of present higher order theory the following spherical shells with different boundary conditions are considered.

Example 4: A square simply supported (S1) cross-ply (0°/90°) spherical shell subjected to uniform/sinusoidal transverse load is considered. The results are compared with Reddy and Liu (1985) and are presented in Table 5. The non-dimensional quantity for representing displacement is as follows.

$$\bar{w}_o = \left(\frac{h^3E_2}{q_oa^4} * 10^3\right) w_o \qquad (20)$$

Example 5: A simply supported (S2) cross-ply (0°/90°) and a clamped (C) angle-ply spherical shell with $R/a = 10$ subjected to uniform transverse load are considered. The non-dimensional quantities used are as follows

Table 4. *Maximum displacement and extreme fibre stresses of an unsymmetric cross-ply (0°/90°) cylindrical shell of finite length subjected to sinusoidal transverse load.*

$\frac{R}{h}$	Theory	$\bar{w}_o$	$\bar{\sigma}_{x\,top}$	$\bar{\sigma}_{x\,bot}$	$\bar{\sigma}_{y\,top}$	$\bar{\sigma}_{y\,bot}$
2	Exact	14.0340	2.6600	0.2510	3.036	9.775
	FOST	17.9590	1.5897	0.2904	1.4890	10.2573
	HOST	16.3658	1.8720	0.2848	2.4849	13.5439
4	Exact	6.1000	0.9610	0.2120	1.7800	10.3100
	FOST	6.6361	0.7828	0.2164	1.3781	10.4345
	HOST	6.1923	0.8516	0.2146	1.6458	11.4735
10	Exact	3.3300	0.1689	0.1930	1.3430	10.5900
	FOST	3.1532	0.1598	0.1660	1.2090	9.0266
	HOST	3.2447	0.1651	0.1769	1.2404	9.2835

Exact result given by Varadan and Bhaskar (1991).

Table 5. Maximum displacement for a simply supported (S1) unsymmetric cross-ply (0°/90°) spherical shell.

		Sinusoidal Transverse Load				Uniform Load			
		$a/h = 100$		$a/h = 10$		$a/h = 100$		$a/h = 10$	
$\frac{R}{a}$	Theory	Pres.	Reddy and Liu (1985)	Pres.	Reddy and Liu (1985)	Pres.	Reddy and Liu (1985)	Pres.	Reddy and Liu (1985)
5	FOST	1.1948	1.1909	11.429	11.4303	1.7535	1.7505	19.944	17.9678
	HOST	1.1937	1.1909	11.166	11.3146	1.7519	1.7506	17.566	17.8003
10	FOST	3.5760	3.5668	12.123	12.1269	5.5428	5.5363	19.065	19.0938
	HOST	3.5733	3.5667	11.896	11.9925	5.5388	5.5363	18.744	18.8963
20	FOST	7.1270	7.1193	12.309	12.3146	11.2730	11.2754	19.365	19.3972
	HOST	7.1236	7.1187	12.094	12.1749	11.2680	11.2746	19.064	19.1912
50	FOST	9.8717	9.8729	12.362	12.3682	15.7140	15.7358	19.452	19.4838
	HOST	9.8692	9.8718	12.150	12.2269	15.7110	15.7341	19.155	19.2753
100	FOST	10.4460	10.4505	12.370	12.3759	16.6450	16.6717	19.464	19.4963
	HOST	10.4440	10.4491	12.158	12.2344	16.6420	16.6697	19.168	19.2874
Plate	FOST	10.6530	10.6583	12.373	12.3784	16.9800	17.0085	19.469	19.5004
	HOST	10.6510	10.6569	12.161	12.2368	16.9770	17.0065	19.172	19.2914

$$\overline{w}_o = \left(\frac{h^3 E_2}{q_o a^4} * 10^3 \right) w_o \qquad \overline{M} = \left(\frac{10^3}{q_o a^2} \right) M \tag{21}$$

The results for maximum displacement and stress resultants are compared with Kabir (1990) and are plotted in Figures 2a–2c.

From the above results it is clear that all the theories predict the same results in the case of geometrically thin shells, but as the thickness increases the classical thin shell theory (CLT) and first order shear deformation theory (FOST) underpredict displacements and stresses, whereas present HOST predicts the displacement and stresses very close to the 3-D elasticity as well as closed form results. This indicates that as the shear deformation increases, the present simple C° higher order finite element theory is the best alternate to the classical thin shell theory and first order shear deformation theory.

4.2 Non-Linear Analysis

4.2.1 COMPARISON OF PRESENT RESULTS WITH FEM RESULTS

To the authors' knowledge there is no closed form solution available for geometrically non-linear analysis of shells in open literature. Hence to check the validity of present theories in non-linear analysis the following problems which are having finite element two-dimensional (2-D) as well as three-dimensional (3-D) solutions are considered.

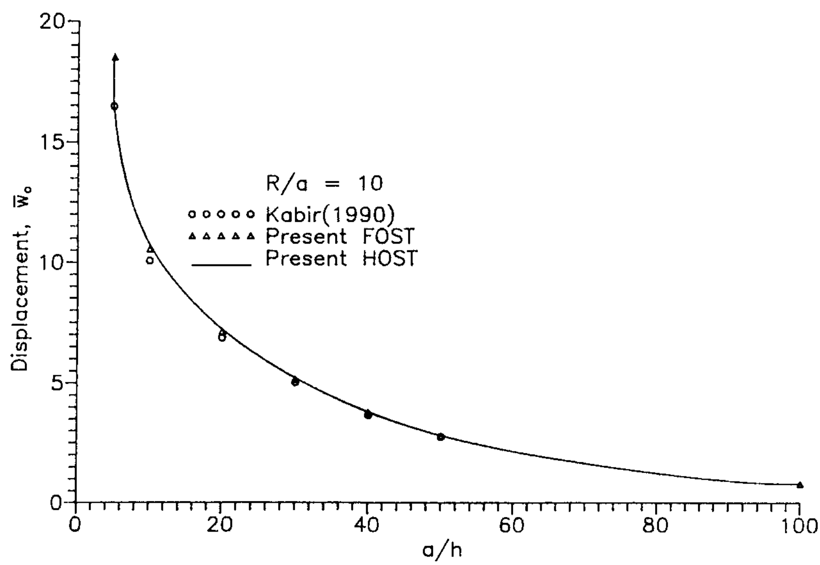

***Figure 2a**. Displacement vs. a/h ratio for a simply supported (S2) cross-ply (0°/90°) spherical shell under uniform load.*

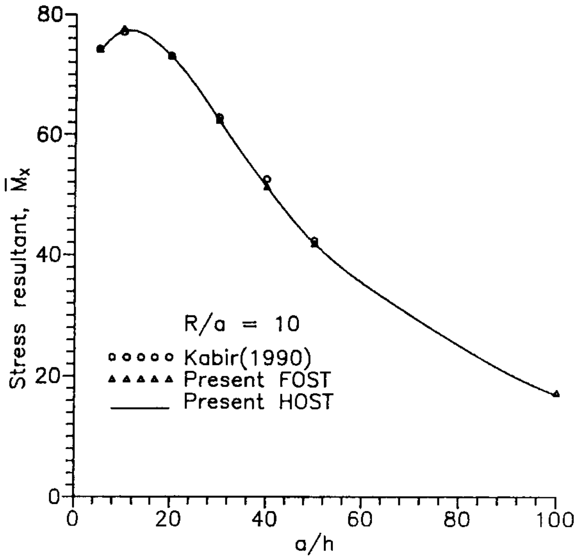

Figure 2b. *Stress resultant vs. a/h ratio for a simply supported (S2) cross-ply (0°/90°) spherical shell under uniform load.*

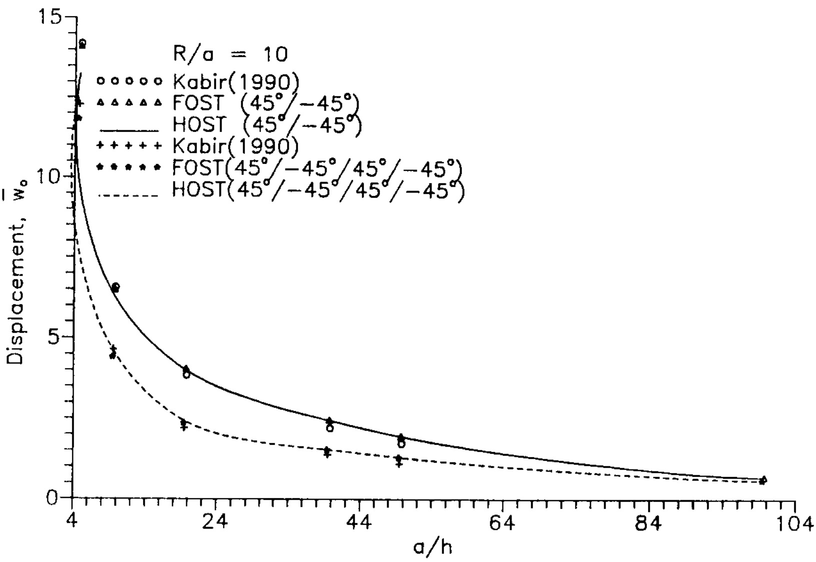

Figure 2c. *Displacement vs. a/h ratio for a clamped (C) angle-ply spherical shell under uniform load.*

Firstly, an unsymmetric cross-ply (0°/90°) spherical shell of $R/a = 10$, $a/h = 100$ with different simply supported boundaries subjected to uniform load is considered. This problem is considered to show the validity of present theories by comparing present results with tabulated results of Chandrasekhara (1985) and also to study the effect of boundary conditions. The results are presented in Table 6. It is clear from the results that the transverse deflection is sensitive to the boundary conditions on the in-plane displacements of simply supported shells. Boundary conditions S2 and S3 give almost same deflection. Boundary conditions Sl and Cl give deflection of an order of magnitude higher than that given by S2 and S3. Thus, boundary condition S2 and S3 make the shell quite stiff.

A clamped (C) unsymmetric cross-ply (0°/90°) cylindrical shell of $R = 2540$ in, $a = b = 508$ in, $h = 2.54$ in, and subjected to uniform load is considered. The present results are compared with finite element first order shear deformation theory given by Reddy and Chandrasekhara (1985a) and three-dimensional (3-D) finite element result given by Kim and Lee (1988) and these are plotted in Figure 3a. Even though the present results are very close to the Reddy and Chandrasekhara (1985a) results, there is marginal difference when compared with 3-D finite element results. This may be due to 2-D and 3-D models of the problem.

A clamped (C) 8-layer quasi-isotropic cylindrical shell with two different types of laminations of $(0°/45°/90°/-45°)_s$ and $(0°/\pm45°/90°)_s$, geometry $a = b = 508$ in, $h = 2.54$ in, $R = 2540$ in and the material properties are $E_1 = 25 * 10^6$ psi, $E_2 = 2 * 10^6$ psi, $G_{12} = G_{13} = 10^6$ psi, $G_{23} = 0.4 * 10^6$ psi, $\nu = 0.25$ is considered. The one quarter of the shell is descretized using the 2×2 uniform mesh and the results are compared with Reddy and Chandrasekhara (1985b) and are plotted in Figure 3b.

From the above results it can be concluded that the present simple C° displacement models are valid in geometrically non-linear analysis.

4.2.2 UNSYMMETRIC CROSS-PLY (0°/90°) SPHERICAL SHELL UNDER UNIFORM LOAD

This problem is selected to carry out the convergence study by taking 2×2, 3×3 and 4×4 uniform meshes in a quadrant of the shell. A simply supported (Sl) unsymmetric cross-ply (0°/90°) spherical shell of $R/a = 10$, $a/h = 5$ subjected to uniform load is considered. The plots for displacement and stresses versus load, respectively, are shown in Figures 4a and 4b. Since the variation in results seems to be very marginal, it is concluded that a 2×2 mesh gives reasonably converged results. The following non-dimensional quantities are used

$$\bar{w}_o = \frac{w_o}{h} \qquad \bar{p} = \frac{q_o}{E_2}\left(\frac{a}{h}\right)^4 \qquad \bar{\sigma} = \frac{\sigma}{E_2}\left(\frac{a}{h}\right)^2 \tag{22}$$

4.2.3 A SIMPLY SUPPORTED SPHERICAL SHELL

A simply supported (Sl) spherical shell of $R/a = 10$, $a/h = 10$ with different laminations (0°/90°) and (45°/−45°) subjected to sinusoidal transverse load is considered. The non-dimensional quantities are as per Equation (22). The results are plotted in Figures 5a and 5b. The results show that the angle-ply shell gives

Table 6. Maximum displacement for a square unsymmetric cross-ply (0°/90°) spherical shell under uniform transverse load.

Load (psi)	Theory	Displacement in inches							
		S1		S2		S3		C1	
		FSDT*	Pres.	FSDT*	Pres.	FSDT*	Pres.	FSDT*	Pres.
0.25	FOST	—	0.1507	—	0.02057	—	0.02061	—	0.2021
	HOST		0.1507		0.02062		0.02066		0.2021
0.50	FOST	0.3344	0.3353	0.04246	0.04249	0.04257	0.04260	0.4692	0.4616
	HOST		0.3353		0.04260		0.04270		0.4616
0.75	FOST	0.5757	0.5799	0.06599	0.06604	0.06617	0.06622	0.8255	0.8316
	HOST		0.5799		0.06622		0.06639		0.8315
1.00	FOST	0.9485	0.9614	0.09144	0.09156	0.09171	0.09184	1.3845	1.3977
	HOST		0.9612		0.09181		0.09207		1.3975
1.25	FOST	1.6529	1.6898	0.11926	0.11954	0.11966	0.11993	1.9589	1.9630
	HOST		1.6891		0.11985		0.12024		1.9626

(continued)

Table 6. (continued).

Load (psi)	Theory	S1 FSDT*	S1 Pres.	S2 FSDT*	S2 Pres.	S3 FSDT*	S3 Pres.	C1 FSDT*	C1 Pres.
1.50	FOST	2.2826	2.2914	0.15063	0.15064	0.15063	0.15121	2.3597	2.3450
	HOST		2.2907		0.15104		0.15160		2.3446
1.75	FOST	2.6421	2.6349	0.18478	0.18590	0.18556	0.18673	2.5951	2.6220
	HOST		2.6343		0.18641		0.18722		2.6210
2.00	FOST	2.8499	2.8797	0.22473	0.22700	0.22584	0.22822	2.8074	2.8419
	HOST		2.8791		0.22765		0.22880		2.8415
2.25	FOST	3.0764	3.0743	0.27425	0.27707	0.27593	0.27887	3.0284	3.0267
	HOST		3.0737		0.27785		0.27968		3.0262
2.50	FOST	3.2432	3.2383	0.33534	0.34271	0.33795	0.34576	3.1948	3.1876
	HOST		3.2377		0.34374		0.34685		3.1870
2.75	FOST	3.4214	3.3872	0.42970	0.44580	0.43487	0.45292	3.3719	3.3310
	HOST		3.3806		0.44746		0.45479		3.3304

Displacement in inches.

*Discrete values given by Chandrasekhara (1985).

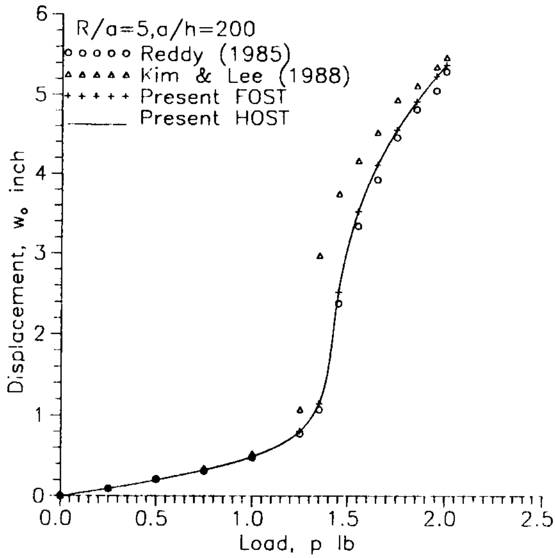

Figure 3a. *Displacement vs. load curves for an unsymmetric cross-ply (0°/90°) clamped (C) cylindrical shell under uniform load.*

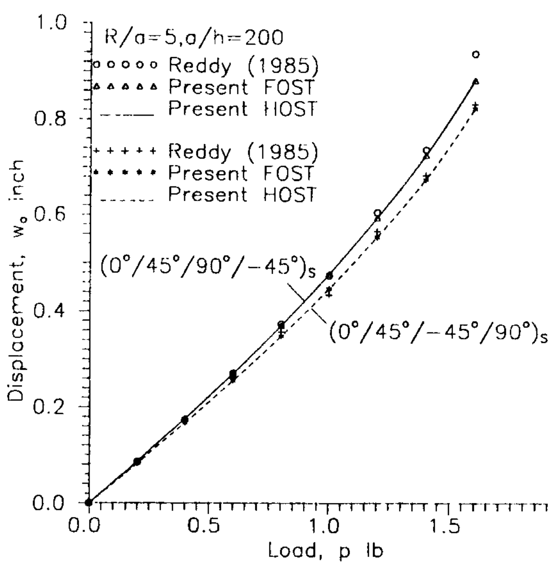

Figure 3b. *Displacement vs. load curves for quasi-isotropic clamped (C) cylindrical shells under uniform load.*

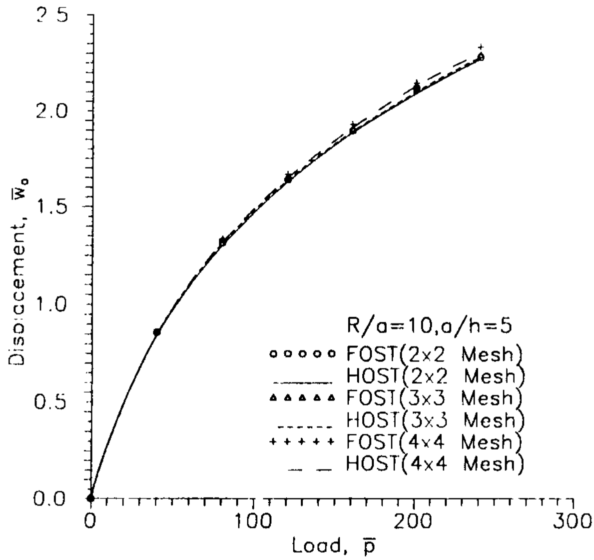

Figure 4a. *Displacement vs. load curves for a simply supported (S1) cross-ply (0°/90°) spherical shell under uniform load.*

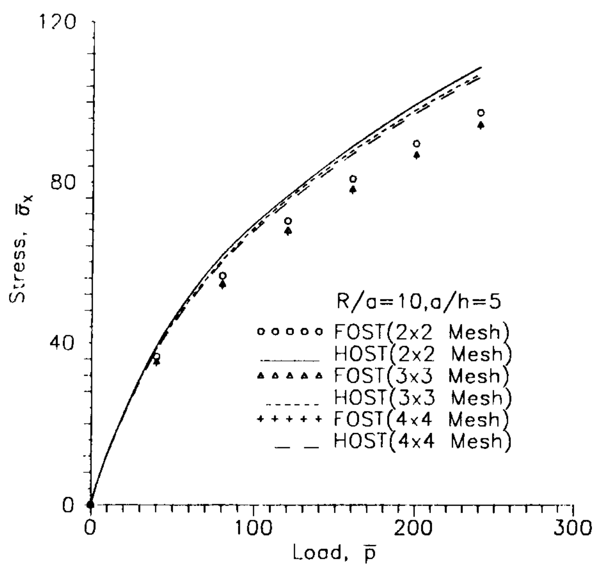

Figure 4b. *Stress vs. load curves for a simply supported (S1) cross-ply (0°/90°) spherical shell under uniform load.*

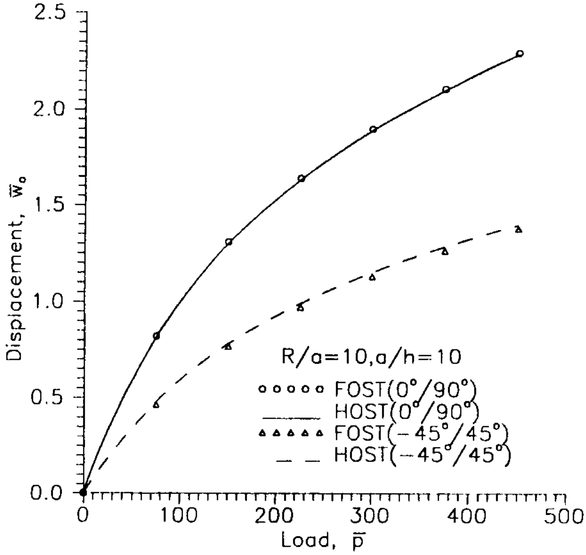

***Figure 5a.** Displacement vs. load curves for a simply supported (S1) spherical shell under sinusoidal transverse load.*

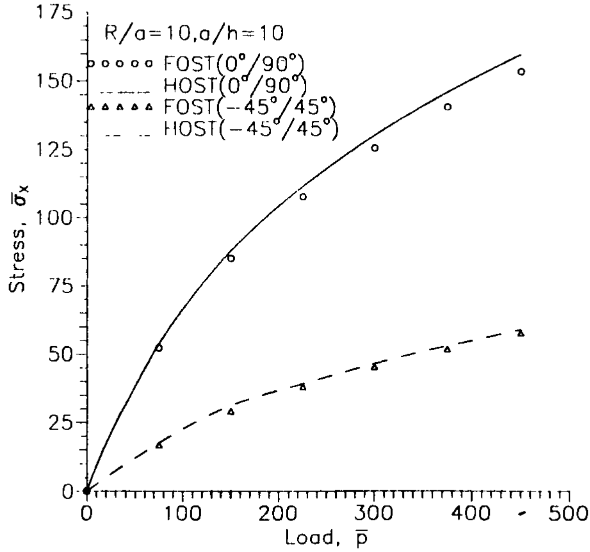

***Figure 5b.** Stress vs. load curves for a simply supported (S1) spherical shell under sinusoidal transverse load.*

stiffer solution when compared to the cross-ply shell. The maximum stress predicted in the case of angle-ply shell is far lower than the cross-ply shell for the same load, geometry and boundary conditions.

4.2.4 ANGLE-PLY SPHERICAL SHELL

A simply supported (S1) angle-ply spherical shell of $R/a = 10$, $a/h = 5$ with different laminations subjected to uniform load is considered. The non-dimensional quantities used are as per Equation (22). The results are presented in Figures 6a–6c. It is observed from the results that as the lamination angle increases the shell is observed to be stiffer until $\theta = 45°$; a further increase in the lamination angle makes the shell flexible. This difference is observed to be more as the magnitude of load increases.

4.2.5 SANDWICH SPHERICAL SHELL

A clamped (C) angle-ply sandwich (0°/45°/90°/CORE/90°/45°/30°/0°) spherical shell of $R/a = 5$ subjected to uniform transverse load is considered. The geometry and material properties taken from Allen (1969) are as follows

$$a = b = 100 \text{ cm}$$

For face sheets, the assumed ply data is based on Hercules ASI/3501-6/graphite/epoxy prepreg system [Mallikarjuna and Kant (1989)]

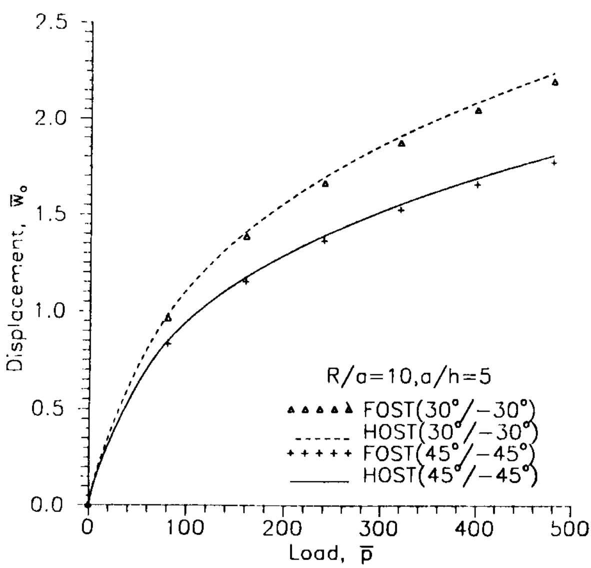

***Figure 6a.** Displacement vs. load curves for a simply supported (S1) angle-ply spherical shell under uniform load.*

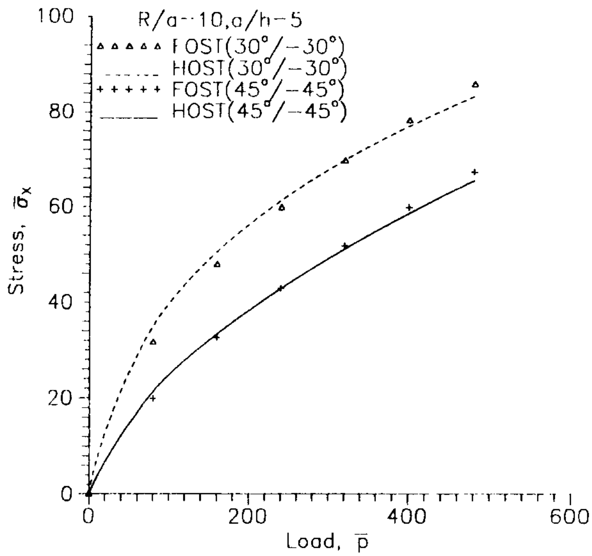

Figure 6b. *Stress vs. load curves for a simply supported (S1) angle-ply spherical shell under uniform load.*

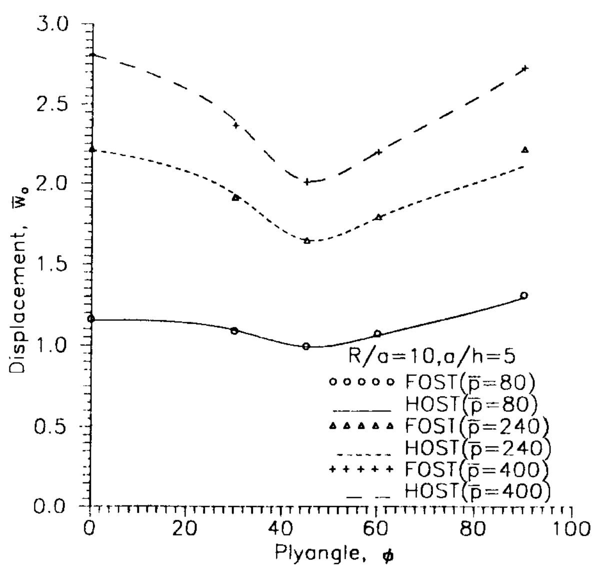

Figure 6c. *Displacement vs. ply-angle for a simply supported (S1) spherical shell under uniform load (0°/φ°).*

$$E_1 = 13.08 * 10^6 \text{ N/cm}^2 \qquad E_2 = E_3 = 1.06 * 10^6 \text{ N/cm}^2$$

$$G_{12} = G_{13} = 0.6 * 10^6 \text{ N/cm}^2 \qquad G_{23} = 0.39 * 10^6 \text{ N/cm}^2$$

$$\nu_{12} = \nu_{13} = 0.28 \qquad \nu_{23} = 0.34$$

Thickness of each top stiff layer = 0.025 h

Thickness of each bottom stiff layer = 0.08125 h

Core material is of U.S. commercial aluminum honeycomb (1/4 inch cell size, 0.003 inch foil) [Allen (1969)]

$$G_{23} = 1.772 * 10^4 \text{ N/cm}^2 \qquad G_{13} = 5.206 * 10^4 \text{ N/cm}^2$$

$$E_3 = 3.013 * 10^5 \text{ N/cm}^2$$

Thickness of core = 0.6 h

The non-dimensional quantities are defined as per Equation (22). The results for displacement and extreme fibre stresses are presented in Figures 7a and 7b for different side to thickness ratios. It is observed that for $a/h = 100$, the results predicted by first order shear deformation (FOST) and higher order shear deformation theory (HOST) are almost the same, whereas for $a/h = 10$, the results

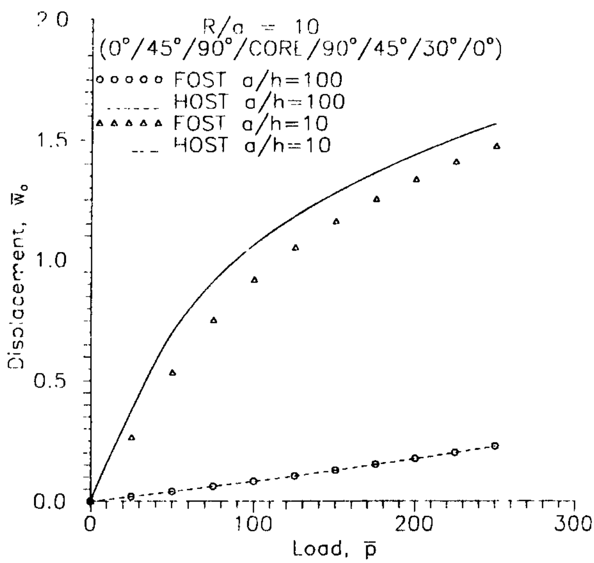

***Figure 7a.** Displacement vs. load curves for a clamped (C) sandwich spherical shell under uniform load.*

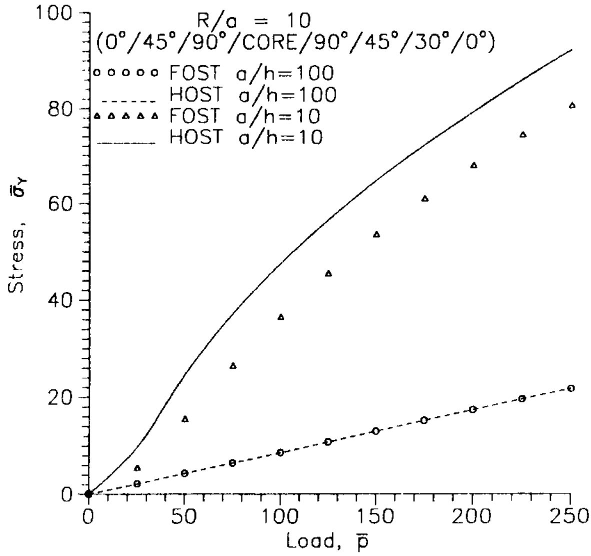

Figure 7b. *Stress vs. load curves for a clamped (C) sandwich spherical shell under uniform load.*

predicted by HOST and FOST differ considerably. This is due to predominant shear deformation leading to warping of transverse cross sections. In earlier examples it was proven that as shear deformation effect increases, the results predicted by HOST are more reliable.

5. CONCLUSIONS

A refined shear flexible C° finite element including the effect of geometric nonlinearity is employed in the static analysis of laminated doubly curved shells. The theory accounts for parabolic variation of transverse shear strains through the thickness and large displacements in the sense of von Karman and therefore no shear correction factors are needed in the present theory.

Numerical results are presented for both linear and geometrically non-linear analysis. Deflections, stresses and stress resultants of square/rectangular doubly curved shells subjected to various loadings, edge conditions, laminations, etc., are presented. The present results are compared with the closed-form two-dimensional laminate solutions, 3-D elasticity solutions and other available finite element solutions in open literature. The results presented here have proved the simplicity and accuracy of this element in geometrically non-linear analysis.

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