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New theories for symmetric/unsymmetric composite and sandwich beams with C^0 finite elements

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In this paper, a new set of higher order theories for the analysis of composite and sandwich beams by using C^0 finite elements is presented. These theories incorporate the more realistic non-linear variation of displacements through the beam thickness, thus eliminating the use of shear correction coefficients. Discrete Lagrangian four-noded cubic element models having five, six and seven degrees of freedom per node are used. The computer program developed incorporates the realistic prediction of interlaminar stresses from equilibrium equations. By comparing the results obtained with the elasticity solution and the classical plate theory, it is shown that the present higher order theories give a much better approximation to the behaviour of thick to thin laminated composite beams. Numerical results for unsymmetric sandwich beams are also presented for future reference.

1 INTRODUCTION

Fibre-reinforced composite materials are continuing to replace the conventional metals in primary and secondary aerospace structural elements owing to their high strength-to-weight and stiffness-to-weight ratios. Their structural response is, however, rendered complex by the many planes of weakness, where local failure can initiate and grow until structural failure occurs. The interface between adjacent layers in a laminate presents one such plane of weakness where an interlaminar delamination can initiate and grow. The delamination occurring near the free edges is mostly due to the interlaminar stresses present at the interface of the beam. In this paper, a set of theories is developed which can evaluate accurately the interlaminar stresses for composite and sandwich beams.

Euler-Bernoulli (see Ref. 1) and Timoshenko[2] theories have been used in the past for the analysis of beams. The former assumes that the transverse normals to the reference middle plane remain so and undergo no changes in length during deformation. Thus, transverse shear strain becomes zero, transverse normal rotation becomes a first derivative of transverse displacement w and transverse displacement field becomes C^1 continuous. Compatible and incompatible complicated higher order C^1 elements have been derived in the past.[1] But this theory leads to serious discrepancies for thick beams, as shear effects cannot be neglected. Further, it is established beyond doubt that this theory is computationally inefficient from the point of view of simple C^0 finite element formulations.[3]

Timoshenko[2] has improved the Euler-Bernoulli theory by incorporating the effect of transverse shear strain into the governing equations. The resulting transverse shear strain was constant through the thickness and thus a shear correction coefficient which was somewhat arbitrary, was used to correct the strain energy of shear deformation. Many investigators have improved the Timoshenko theory[4-6] by giving new expressions for shear correction coefficient for different cross-sections of the beam. But even after refining the values of shear correction coefficients, the discrepancies between the results of this theory and the elasticity theory is seen to be large for built-up beams.

Cowper[7] has defined the displacement w and the rotation θ which appear in Timoshenko's beam equation in two different ways. The tradi-

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tional definition takes w as the displacement of the centroid of the cross-section and θ as the rotation of the normal to the cross-section through the centroid. Alternatively, w may be defined as the mean value of the transverse displacement of all points of the cross-section and θ as the first moment of the normal displacement of points of the cross-section divided by the moment of inertia. He has demonstrated by examples that far superior accuracy may be achieved by adopting this latter definition in preference to the former.

Stephen and Levinson[8] have given a second-order beam theory which is similar to the Timoshenko theory, but this theory contains two coefficients: one of which depends on cross-sectional warping and the other on the transverse direct stresses. Levinson[9-11] has given a new fourth-order beam theory which requires two boundary conditions at each end of the beam. Here, transverse shear deformation and cross-sectional warping are taken into account and thus no shear correction coefficients are used. But this theory is too poor to adequately describe the two-dimensional displacement pattern.

Rychter[12] has used Levinson's theory, and improved it by embedding in it the two-dimensional linear theory of elasticity. He has proved that the corresponding relative mean square error is proportional to the square of the beam depth and the shear contribution to the error is proportional to the cube of beam depth. Bickford[13] used Hamilton's principle to derive a consistent higher order theory of the elastodynamics of a beam based upon the kinematic and stress assumptions previously used by Levinson.[9-11]

Pagano[14] has given an elasticity solution for composite laminates in cylindrical bending. The elasticity solution given by Schile[15] is more general compared to Gerstner.[16] In the former, the solution is valid for a beam with variable Young's modulus (E) and Poisson's ratio (v) in the transverse direction and in addition, is not restrictive in terms of assumed interfacial boundary conditions. Using the theory of Gerstner,[16] Sierakowski and Ebcioglu[17] have obtained a solution for the case of a multilayered cantilever sandwich beam under end load with symmetric distribution of material properties. They have used piecewise constant transverse properties. Goran[18] has given an analytical solution for simple fiber reinforced curved laminated beams.

A review of the literature indicates that no work on finite element analysis of composite beams is available.[19,20] Thus, recognising the need for a more refined theoretical model, in the present paper, a set of simple but efficient and accurate higher order theories is developed which includes all the secondary effects such as warping of the transverse cross-section, the transverse shear stress, shear strain and their variation across the beam thickness. Our models use four-noded cubic C^0 finite elements in the numerical study.

2 THEORY AND FORMULATIONS

The development of the present higher order shear deformation theories begins with the assumption of the displacement fields in the following form (Fig. 1),

HOT1

$$\begin{aligned} u(x,z) &= u_0(x) + z\theta_x(x) \\ w(x,z) &= w_0(x) + z\theta_z(x) + z^2 w_0^*(x) \end{aligned} \tag{1}$$

HOT2

$$\begin{aligned} u(x,z) &= u_0(x) + z\theta_x(x) + z^2 u_0^*(x) \\ w(x,z) &= w_0(x) + z\theta_z(x) + z^2 w_0^*(x) \end{aligned} \tag{2}$$

HOT3

$$\begin{aligned} u(x,z) &= u_0(x) + z\theta_x(x) + z^3 \theta_x^*(x) \\ w(x,z) &= w_0(x) + z\theta_z(x) + z^2 w_0^*(x) \end{aligned} \tag{3}$$

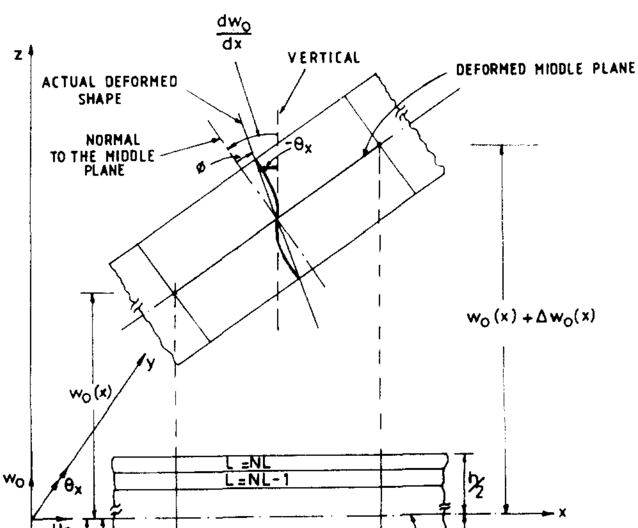

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes and displacement components.

HOT4

$$u(x,z)=u_0(x)+z\theta_x(x)+z^2u_0^*(x)+z^3\theta_x^*(x)$$
$$w(x,z)=w_0(x)+z\theta_z(x)+z^2w_0^*(x) \quad (4)$$

Here u and w are the displacements at any point (x,z) in the beam domain in the x and z directions, respectively. The parameters u_0, w_0, θ_x, θ_z, u_0^*, w_0^* and θ_x^* are the appropriate one-dimensional terms in the Taylor series and are defined along the x axis at $z=0$. In the case of pure bending of a beam with no axial deformation of the reference x axis, the parameters u_0 and u_0^* will vanish. In these expressions u_0, w_0 and θ_x are the axial x-displacement of a point on the reference x axis, the transverse z-displacement of a point on the reference x axis and the rotation of the transverse normal to the reference x axis in the x–z plane, respectively. While these parameters are physical quantities, the parameters θ_z, u_0^*, w_0^* and θ_x^* are the higher order terms in the Taylor series expansion and their physical interpretation is difficult indeed, except that they represent higher order transverse cross-sectional deformation modes.

Here, derivations for the model HOT4 given in eqn (4) only are presented. Other theoretical models become special cases of HOT4. The variations in the case of HOT1, 2 and 3 are given concisely in the Appendix. Under these assumptions, the strain displacement relations of three-dimensional elasticity[21] for a point at a distance z from the middle surface of the laminate are given by

$$\varepsilon_x=\varepsilon_{x0}+z\kappa_x+z^2\varepsilon_{x0}^*+z^3\kappa_x^*$$
$$\varepsilon_z=\varepsilon_{z0}+z\kappa_z^*$$
$$\gamma_{xz}=\theta+z\kappa_{xz}+z^2\theta^* \quad (5)$$

where

$$(\varepsilon_{x0},\kappa_x,\varepsilon_{x0}^*,\kappa_x^*,\varepsilon_{z0},\kappa_z^*)=\left(\frac{\partial u_0}{\partial x},\frac{\partial \theta_x}{\partial x},\frac{\partial u_0^*}{\partial x},\frac{\partial \theta_x^*}{\partial x},\theta_z,2w_0^*\right)$$

$$(\phi,k_{xz},\phi^*)=\left(\theta_x+\frac{\partial w_0}{\partial x},2u_0^*+\frac{\partial \theta_z}{\partial x},3\theta_x^*+\frac{\partial w_0^*}{\partial x}\right) \quad (6)$$

Each lamina in the laminate is orthotropic and is assumed to be in a two-dimensional state of plane stress. The stress–strain relation for a typical lamina L can be written as follows:

$$\begin{bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_z\\ \gamma_{xz}\end{bmatrix} \quad (7)$$

Here $\sigma_x, \sigma_z, \tau_{xz}$ are the stress and $\varepsilon_x, \varepsilon_z, \gamma_{xz}$ are the strain components referred to the lamina/laminate coordinates $(x$–$z)$. The coefficients, C_{ij} are the reduced elastic constants. The following relations hold between these and the physical engineering elastic constants:

$$C_{11}=\frac{E_1}{1-v_{12}v_{21}},\ C_{12}=\frac{v_{21}E_1}{1-v_{12}v_{21}},$$
$$C_{22}=\frac{E_2}{1-v_{12}v_{21}}, \quad (8)$$

The total potential energy, $\prod$, of the beam is given by,

$$\prod=\frac{1}{2}\int_z\int_l \boldsymbol{\varepsilon}^t\boldsymbol{\sigma}\,\mathrm{d}x\,\mathrm{d}z-\int_z\int_l(\boldsymbol{u})^t\boldsymbol{p}\,\mathrm{d}x\,\mathrm{d}z \quad (9)$$

where

$$\boldsymbol{\varepsilon}=(\varepsilon_x,\varepsilon_z,\gamma_{xz})^t \qquad \boldsymbol{\sigma}=(\sigma_x,\sigma_z,\tau_{xz})^t$$
$$\boldsymbol{u}=(u,w)^t \qquad \boldsymbol{p}=(p_x,p_z)^t \quad (10)$$

The superscript t denotes the transpose of a vector/matrix. The expressions for strain components given by eqn (5) are substituted in eqn (9). The following relation is obtained when an explicit integration is carried out through the beam thickness.

$$\prod=\frac{1}{2}\int_l(\varepsilon_{x0}N_x+\varepsilon_{x0}^*N_x^*+\varepsilon_{z0}N_z+\kappa_xM_x+\kappa_x^*M_x^*$$
$$+\kappa_z^*M_z)\,\mathrm{d}x+\frac{1}{2}\int_l(\phi Q+\phi^*Q^*+\kappa_{xz}S)\,\mathrm{d}x$$
$$-\int_l(\boldsymbol{d})^t\boldsymbol{p}_0\,\mathrm{d}x \quad (11)$$

Or, this can be written in a compact form as

$$\prod=\frac{1}{2}\int_l\bar{\boldsymbol{\varepsilon}}^t\bar{\boldsymbol{\sigma}}\,\mathrm{d}x-\int_l(\boldsymbol{d})^t\boldsymbol{p}_0\,\mathrm{d}x \quad (12)$$

in which

$$\bar{\sigma}=(N_x, N_x^*, N_z, M_x, M_x^*, M_z, Q, Q^*, S)^t$$
$$\bar{\varepsilon}=(\varepsilon_{x0}, \varepsilon_{x0}^*, \varepsilon_{z0}, \kappa_x, \kappa_x^*, \kappa_z^*, \phi, \phi^*, \kappa_{xz})^t$$
$$d=(u_0, w_0, \theta_x, \theta_z, u_0^*, w_0^*, \theta^*)^t$$
$$p_0=(p_{x0}, p_{z0}, m_{x0}, m_{z0}, p_{x0}^*, p_{z0}^*, m_{x0}^*)^t \tag{13}$$

The stress resultants in eqn (11) are defined as follows:

$$\begin{bmatrix} N_x & M_x & N_x^* & M_x^* \\ N_z & M_z & 0 & 0 \\ Q_x & S & Q_x^* & 0 \end{bmatrix} = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix} \times (1, z, z^2, z^3)\, \mathrm{d}z \tag{14}$$

After integration, these relations are written in a matrix form which define the stress resultant–strain relations of the laminate and is given by

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} \mathbf{D}_M & \mathbf{D}_c & 0 \\ \mathbf{D}_c^t & \mathbf{D}_B & 0 \\ 0 & 0 & \mathbf{D}_s \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \varkappa \\ \phi \end{bmatrix}$$

or

$$\bar{\sigma}=\mathbf{D}\bar{\varepsilon} \tag{16}$$

where

$$N=(N_x, N_x^* N_z)^t \qquad M=(M_x, M_x^*, M_z)^t$$
$$Q=(Q_x, Q_x^*, S)^t \qquad \varepsilon_0=(\varepsilon_{x0}, \varepsilon_{x0}^*, \varepsilon_{z0})^t$$
$$\kappa=(\kappa_x, \kappa_x^* \kappa_z^*)^t \qquad \phi=(\phi_x, \phi_x^*, \kappa_{xz})^t \tag{17}$$

$$\mathbf{D}_M = \sum_{L=1}^{NL} \begin{bmatrix} C_{11}h_1 & C_{11}h_3 & C_{12}h_1 \\ & C_{11}h_5 & C_{12}h_3 \\ \text{symm} & & C_{22}h_1 \end{bmatrix} \tag{18}$$

$$\mathbf{D}_s = \sum_{L=1}^{NL} \begin{bmatrix} Gh_1 & Gh_3 & Gh_2 \\ & Gh_5 & Gh_4 \\ \text{symm} & & Gh_3 \end{bmatrix} \tag{19}$$

$$h_i=\tfrac{1}{i}(h_{L+1}^i - h_L^i) \qquad i=1,2,\ldots,5 \tag{20}$$

The coefficients of the $\mathbf{D}_c$ matrix can be obtained by replacing h_1, h_3, h_5 by h_2, h_4, h_6, respectively, in the $\mathbf{D}_M$ matrix. Similarly, the coefficients of $\mathbf{D}_B$ matrix can be obtained by replacing h_1, h_3, h_5 by h_3, h_5, h_7, respectively, in the $\mathbf{D}_M$ matrix.

The transverse shear stress τ_{xz} and transverse normal stress σ_z cannot be accurately given by eqn (7) as the continuity condition at the interface of any two layers is not satisfied for laminates. The three- or two-dimensional analysis becomes very complex due to thickness variation of constitutive laws and continuity requirements across the interface. For this reason, the interlaminar shear and normal stress at any layer L at z is obtained by integrating the two equilibrium equations of elasticity of each layer over the lamina thickness and summing over layers 1 to L as shown below.

The equations of equilibrium representing the pointwise equilibrium can be written as

$$\tau_{i,j}=0, \qquad i,j=x,z \tag{21}$$

The interlaminar shear stress can be found by substituting the lamina stresses in eqn (21) and integrating through the thickness of the laminate. The following equation results after integration:

$$\tau_{xz}^L\big|_{z=h_{L+1}} = -\sum_{i=1}^{L} \int_{h_i}^{h_{i+1}} \frac{\partial \sigma_x}{\partial x}\, \mathrm{d}z + C_1 \tag{22}$$

The variation of σ_z is determined in terms of inplane stress after eliminating the interlaminar shear stress from the second equilibrium equation of elasticity given by eqn (21). A second-order differential equation results which can be written as follows:

$$\frac{\partial^2 \sigma_z}{\partial z^2} = \frac{\partial^2 \sigma_x}{\partial x^2} \tag{23}$$

After integrating eqn (23) twice with respect to the thickness of the beam, the following equation for σ_z is obtained:

$$\sigma_z^L\big|_{z=h_{L+1}} = \sum_{i=1}^{L} \int_{h_i}^{h_{i+1}} \left(\int_z \frac{\partial^2 \sigma_x}{\partial x^2}\, \mathrm{d}z \right) \mathrm{d}z + zC_2 + C_3 \tag{24}$$

Thus, interlaminar stresses can be obtained by using stress equilibrium equations. Inplane stresses and strains can be obtained by using constitutive relations. In this manner stresses in the laminate can be evaluated. But in the case of interlaminar shear stress, eqn (22), it is seen that the values obtained through the beam thickness may not, in general, satisfy beam boundary conditions at $z = \pm h/2$, as only one constant of integration is present.

In the case of interlaminar normal stress this problem does not arise, as here two constants of integration obtained by integrating twice, can be determined by substituting two boundary conditions at $z = \pm h/2$. Equation (22) is substituted in the second equilibrium equation to get the continuity of σ_z across the thickness. Equation (24) is solved as a boundary value problem instead of an initial value problem as in eqn (22). This requires the use of a cubic element, as third derivatives of displacements can be calculated. Thus, four-noded cubic elements are used in this numerical study.

3 FINITE ELEMENT FORMULATION

In the standard finite element technique, the total solution domain is subdivided into non-overlapping NE subdomains (elements), such that

$$\prod(\boldsymbol{d}) = \sum_{e=1}^{NE} \prod{}^{e}(\boldsymbol{d}) \tag{25}$$

where $\prod$ and $\prod^e$ are the total potential energies of the system and the element, respectively. The potential energy for an element e can be expressed in terms of internal strain energy U^e and the external work done W^e, such that

$$\prod{}^{e}(\boldsymbol{d}) = U^e - W^e \tag{26}$$

in which $\boldsymbol{d}$ is the vector of unknown displacement variable at a point on the mid-surface in the problem and it is defined in eqn (13). In C^0 finite element theory, the continuum displacement vector within the element is discretised such that

$$\boldsymbol{d} = \sum_{i=1}^{NN} N_i \boldsymbol{d}_i \tag{27}$$

where NN is the number of nodes in an element, N_i is the interpolating function associated with node i and $\boldsymbol{d}_i$ is the generalised displacement vector corresponding to the ith node of an element.

Knowing the generalised displacement vector, $\boldsymbol{d}$, at a general point x within the element, the generalised strain at any point, given by eqn (6), can be expressed in the matrix form as follows:[1]

$$\bar{\boldsymbol{\varepsilon}} = \sum_{i=1}^{NN} \mathbf{B}_i \boldsymbol{d}_i \tag{28}$$

where

$$\bar{\boldsymbol{\varepsilon}} = (\varepsilon_{x0}, \varepsilon^*_{x0}, \varepsilon_{z0}, \kappa_x, \kappa^*_x, \kappa^*_z, \phi, \phi^*, \kappa_{xz})^t \tag{29}$$

The $\mathbf{B}_i$ matrix has a dimension of 9×7 in which the non-zero elements are

$$\mathrm{B}_{1,1} = \mathrm{B}_{2,5} = \mathrm{B}_{4,3} = \mathrm{B}_{5,7} = \mathrm{B}_{7,2} = \mathrm{B}_{8,6} = \mathrm{B}_{9,4} = \frac{\partial N_i}{\partial x}$$

$$\mathrm{B}_{3,4} = \mathrm{B}_{7,3} = N_i, \mathrm{B}_{6,6} = \mathrm{B}_{9,5} = 2N_i, \mathrm{B}_{8,7} = 3N_i \tag{30}$$

Upon evaluating the $\mathbf{D}$ and $\mathbf{B}_i$ matrices as given by eqns (15) and (30), respectively, the element stiffness matrix is computed by using the following standard relation:[1]

$$\mathbf{K}^e_{ij} = \int_{-1}^{+1} \mathbf{B}^t_i \mathbf{D} \mathbf{B}_j \,|\mathbf{J}|\, \mathrm{d}\xi \tag{31}$$

The computation of element stiffness matrix $\mathbf{K}^e$ is economised by explicit multiplication of $\mathbf{B}_i$, $\mathbf{D}$ and $\mathbf{B}_j$ matrices instead of carrying out the full matrix multiplication of the triple product. In addition, due to symmetry of the stiffness matrix, only the blocks $\mathbf{K}_{ij}$ lying on one side of the main diagonal are formed. The integral is evaluated by a selective integration technique with four- and three-point Gauss quadrature rules for membrane flexure and shear parts, respectively, as follows:

$$\mathbf{K}^e_{ij} = \sum_{\alpha=1}^{g} W_\alpha \, \mathbf{B}^t_i \mathbf{D} \mathbf{B}_j \,|\mathbf{J}| \tag{32}$$

where W_α is the weighting coefficient, g is the number of numerical quadrature points and $|\mathbf{J}|$ is the Jacobian conversion.

The consistent load vector p_i due to uniformly distributed transverse load q can be written as

$$\boldsymbol{p}_i = \int_{-1}^{+1} N^t_i \, \boldsymbol{p}_0 \,|\mathbf{J}|\, \mathrm{d}\xi \tag{33}$$

The integral of eqn (33) is evaluated numerically using the four-point Gauss quadrature rule. The result is

$$\boldsymbol{p}_i = \sum_{\alpha=1}^{g} W_\alpha N^t_i (0, q, 0, qh/2, 0, qh^2/4, 0)\,|\mathbf{J}| \tag{34}$$

In the consistent load vector, sinusoidal transverse load can be obtained by using the following substitution in eqn (34).

$$q = q_0 \sin\left(\frac{m\pi x}{l}\right) \quad (35)$$

where l is the beam dimension, x is the Gauss point coordinate and m is the usual harmonic number.

4 NUMERICAL RESULTS AND DISCUSSION

A set of computer programs incorporating the present higher order theories is developed for the analysis of symmetric and unsymmetric laminated composite and sandwich beams. A four-noded Lagrangian cubic element with varying degrees of freedom per node is employed. All the computations are carried out on a CYBER 180/840 computer in single precision with a word length of 16 significant digits. Validity of these theories is established by comparing the results with available elasticity and classical plate theory (CPT) solutions.[14] The following properties which simulate a high modulus graphite/epoxy composite have been used.

$$E_L = 25\times10^6 \qquad E_T = 1\times10^6$$

$$G_{LT} = 0{\cdot}50\times10^6 \qquad v_{LT} = 0{\cdot}25$$

where L signifies the direction parallel to the fibres, T is the transverse direction and v_{LT} is the Poisson's ratio measuring strain in the transverse direction under uniaxial normal stress in the L direction. In all the problems that follow only sinusoidal loading is considered. The displacement, inplane and interlaminar stresses are presented here in the non-dimensional form by using the following multipliers:

$$\bar{\sigma}_x = \frac{\sigma_x(l/2, z)}{q_0}, \; \bar{\sigma}_z = \frac{\sigma_z(l/2, z)}{q_0},$$

$$\bar{\tau}_{xz} = \frac{\tau_{xz}(0, z)}{q_0}$$

$$\bar{u} = \frac{E_T u(0, z)}{h q_0}, \; \bar{w} = \frac{100 E_T h^3 w(l/2, 0)}{q_0 l^4} \quad (36)$$

Example 1 — A simply supported orthotropic beam with fibres orientated in the x-direction is considered for the comparison of displacement and stresses. The variation of displacement, inplane stresses and interlaminar shear stress for different values of l/h ratios are shown in Figs 2–5. These plots show that the values obtained by models HOT3 and HOT4 are close to elasticity solution compared to HOT1 and HOT2. The CPT underestimates the values and gives very poor estimates for relatively low values of l/h.

Example 2 — A simply supported bidirectional laminate with the T and L directions aligned paralled to the x-axis in the top and bottom layers, respectively, is considered. The layers are of equal thickness. The variation of displacement with l/h ratios and the variation of σ_x, τ_{xz} and σ_z through laminate thickness for $l/h=4$ are shown in Figs 6–9. These plots show that the values obtained by

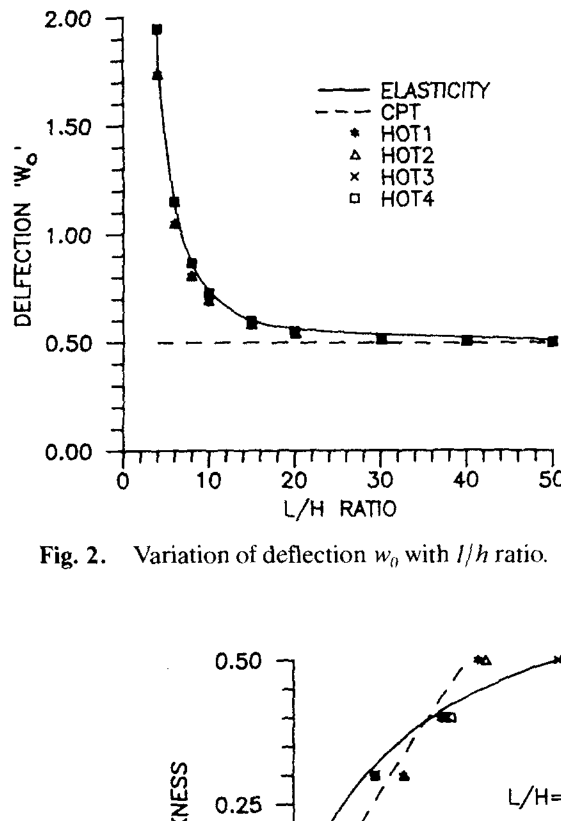

Fig. 2. Variation of deflection w_0 with l/h ratio.

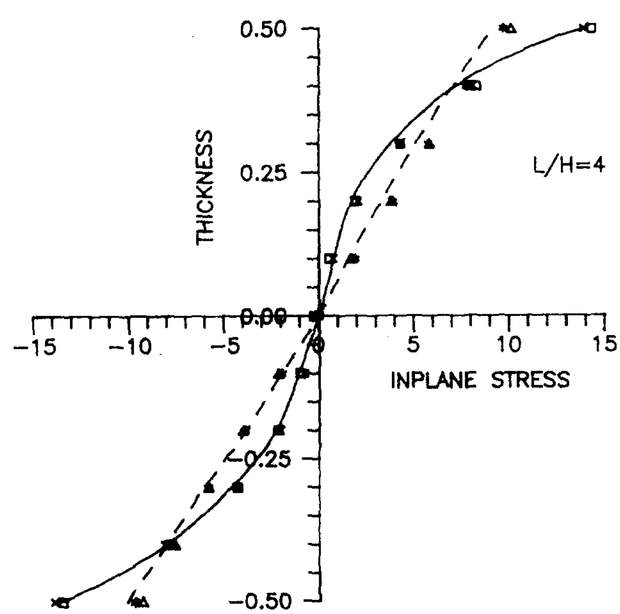

Fig. 3. Variation of inplane stress through the thickness.

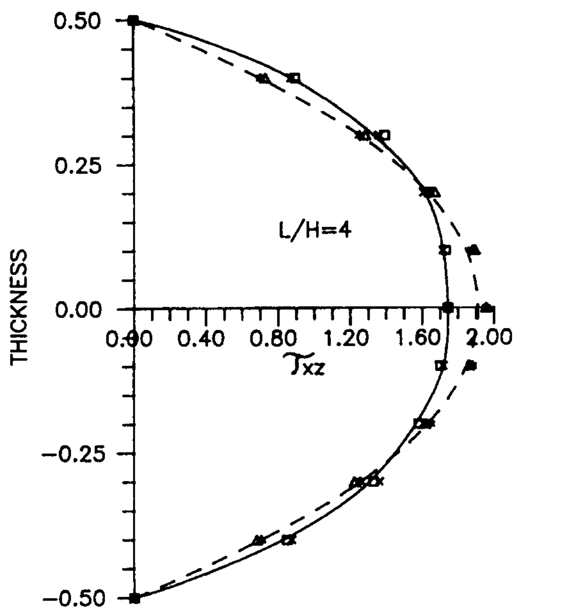

Fig. 4. Variation of interlaminar shear stress through the thickness.

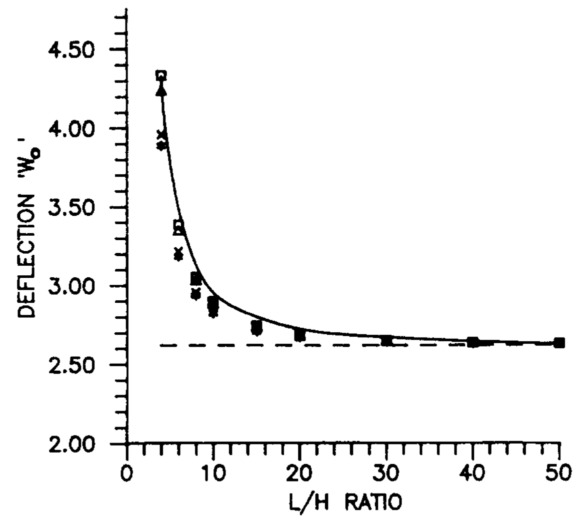

Fig. 6. Variation of deflection w_0 with l/h ratio.

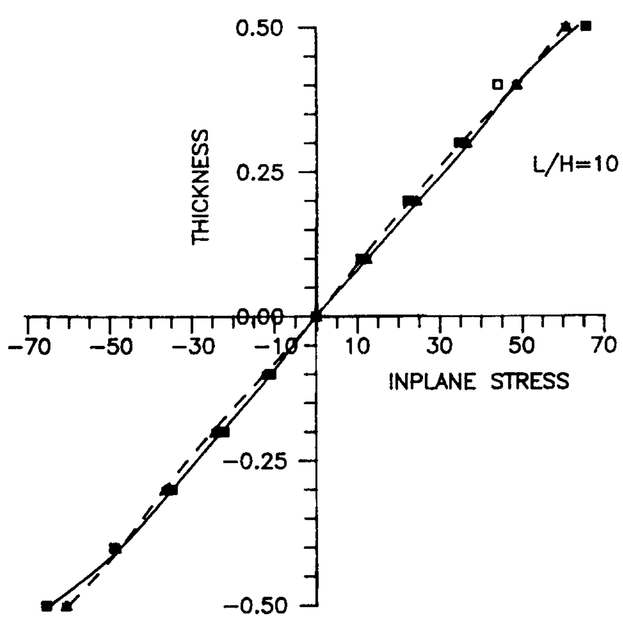

Fig. 5. Variation of inplane stress through the thickness.

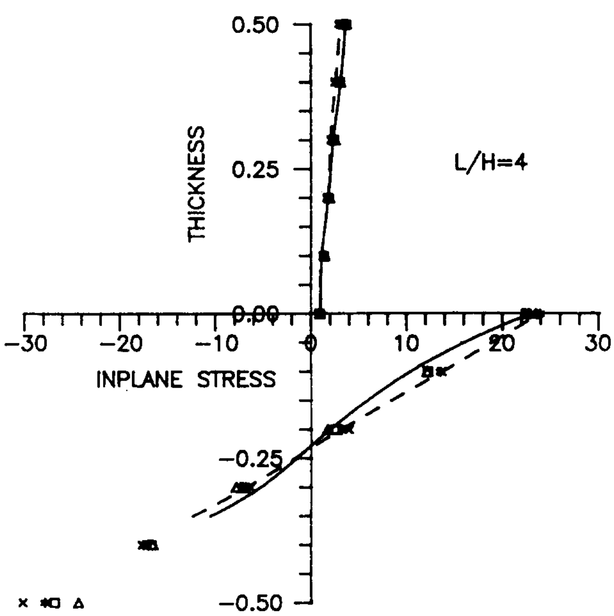

Fig. 7. Variation of inplane stress through the thickness.

model HOT4 are close to the elasticity solution compared to other models.

Example 3 — A simply supported symmetric three-ply orthotropic beam with the L direction coinciding with the x in the outer layers while T is parallel to x in the central layer is considered with layers having equal thickness. The results of displacement, σ_x, τ_{xz} and σ_z are shown in Figs 10–18. These plots show that the values obtained by models HOT3 and HOT4 are close to the elasticity solution compared to other models and the CPT. But the difference in result is more significant for σ_x, τ_{xz} and inplane displacement for thick beams ($l/h=4$). In Figs 13 and 14, the distribution of inplane stress and interlaminar shear stress through the thickness, respectively, for $l/h=10$ and in Figs 17 and 18 the variation of inplane displacement for $l/h=6$ and 10, respectively, are shown. The same trend of results as obtained for thick beam ($l/h=4$) is obtained here too. But the discrepancy with the elasticity results is not so significant.

Example 4 — A simply supported unsymmetric sandwich beam is considered next for comparison of displacement, inplane and interlaminar

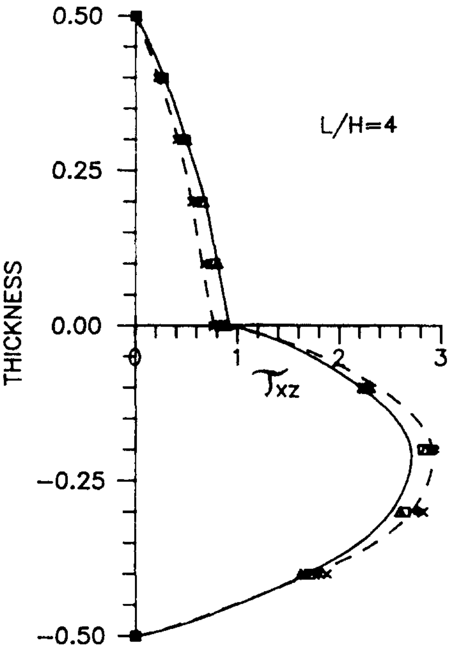

Fig. 8. Variation of interlaminar shear stress through the thickness.

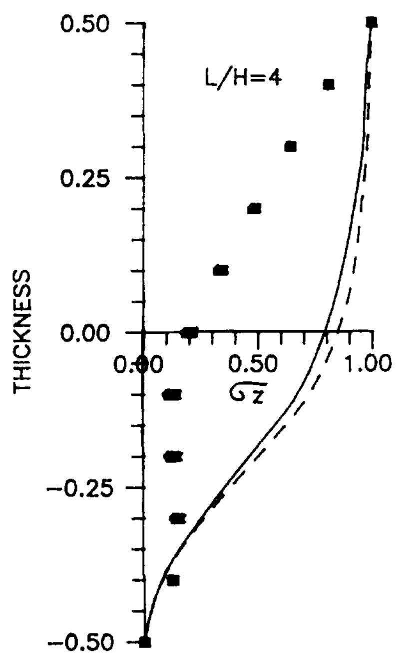

Fig. 9. Variation of interlaminar normal stress through the thickness.

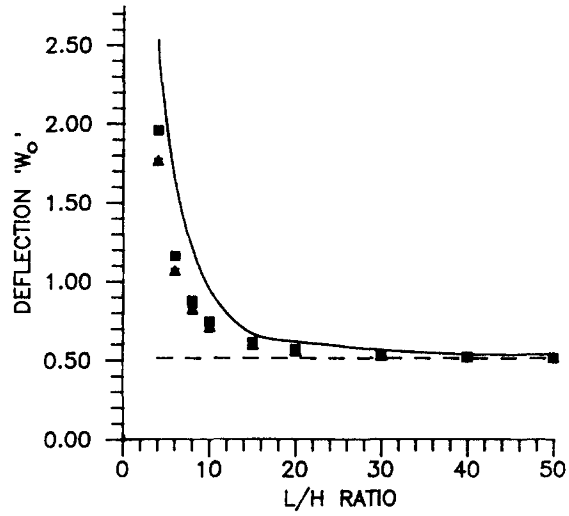

Fig. 10. Variation of deflection w_0 with l/h ratio.

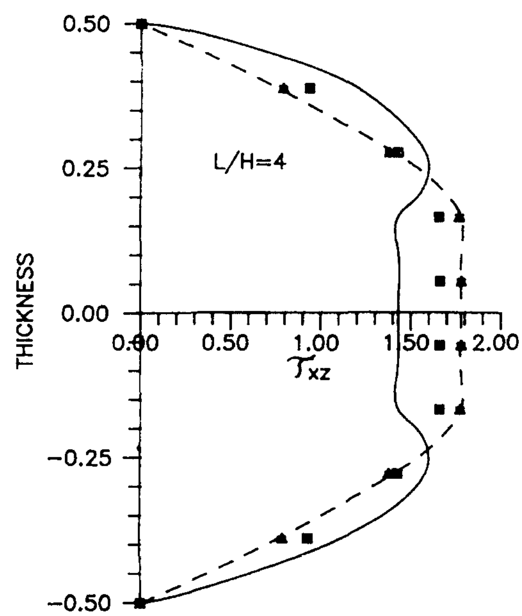

Fig. 11. Variation of interlaminar shear stress through the thickness.

stresses. The following material properties[22] are used:

Stiff layers:

$$E_1 = 0{\cdot}1308 \times 10^8, \quad E_2 = 0{\cdot}106 \times 10^7,$$

$$G = 0{\cdot}6 \times 10^6, \quad \nu = 0{\cdot}28$$

$$\frac{h_{\text{core}}}{h_{\text{stiff}\,b}} = 7{\cdot}0, \frac{h_{\text{core}}}{h_{\text{stiff}\,t}} = 3{\cdot}5, (h_{\text{stiff}\,b}/h_{\text{core}}/h_{\text{stiff}\,t})$$

Core layers:

$$E_1 = E_2 = 0, \qquad G = 0{\cdot}1772 \times 10^5, \qquad \nu = 0,$$

The results of transverse displacement, inplane stress and interlamimar shear stress for different l/h ratios are presented in Table 1. The variations of inplane displacement, τ_{xz} and σ_z through the beam thickness are shown in Figs 19–21 for $l/h = 10$. These results show large differences in the values of displacement and stresses for HOT4 and HOT3 compared with other models and Timoshenko's theory for thick laminates ($l/h \leqslant$

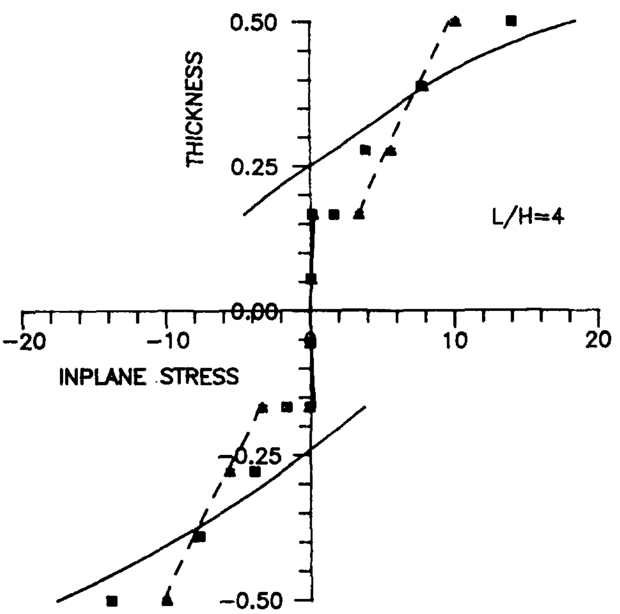

Fig. 12. Variation of inplane stress through the thickness.

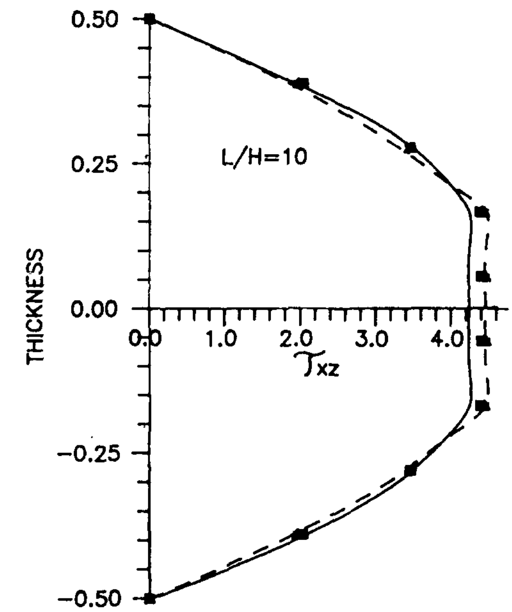

Fig. 14. Variation of interlaminar shear stress through the thickness.

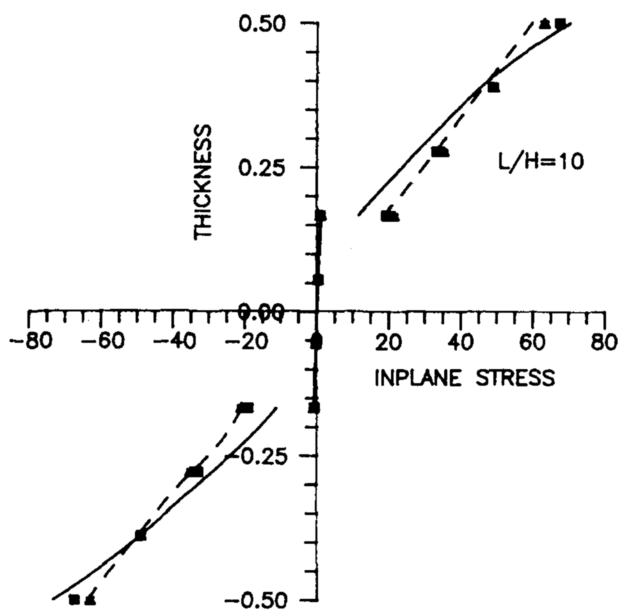

Fig. 13. Variation of inplane stress through the thickness.

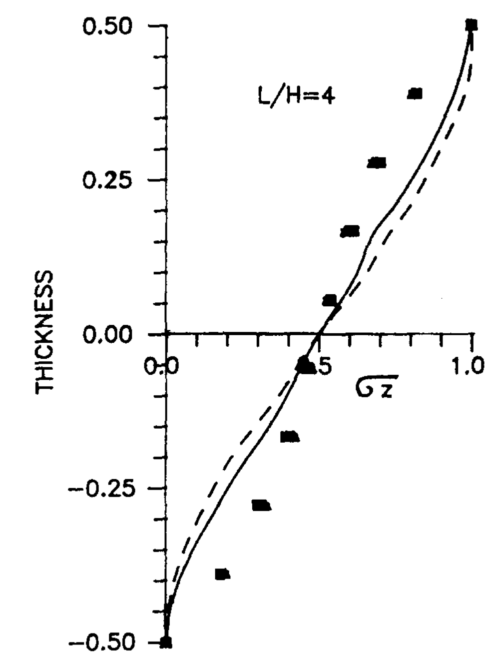

Fig. 15. Variation of interlaminar normal stress through the thickness.

10). For relatively thin laminates ($l/h \geqslant 50$) almost the same results are obtained for all the models.

5 CONCLUSIONS

A new set of higher order theories for computer analyses of composite and sandwich beams by using C^0 finite elements is presented. These theories do not require the usual shear correction coefficients which are generally associated with the Timoshenko theory. By comparing results obtained with the elasticity solution and the CPT, it is shown that the present higher order theories give a much better approximation to the behaviour of composite laminates. This is especially true in the case of relatively thick laminates where the effects of transverse components of stresses and strains cannot be neglected. While only simply supported sinusoidally loaded beams are

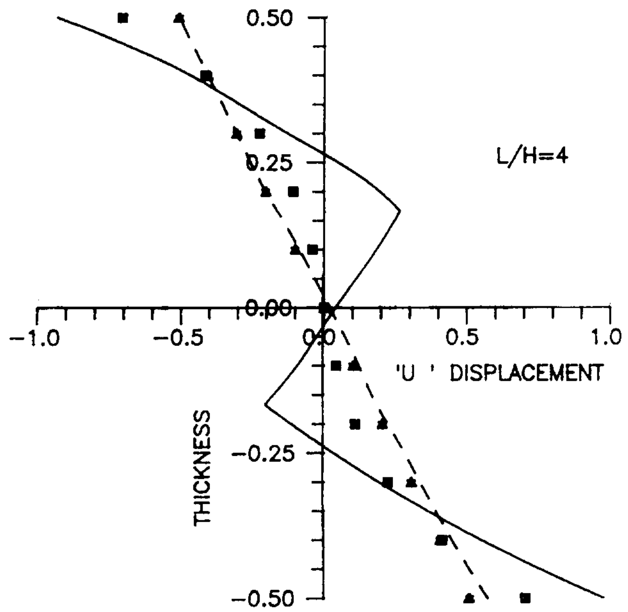

Fig. 16. Variation of inplane displacement through the thickness.

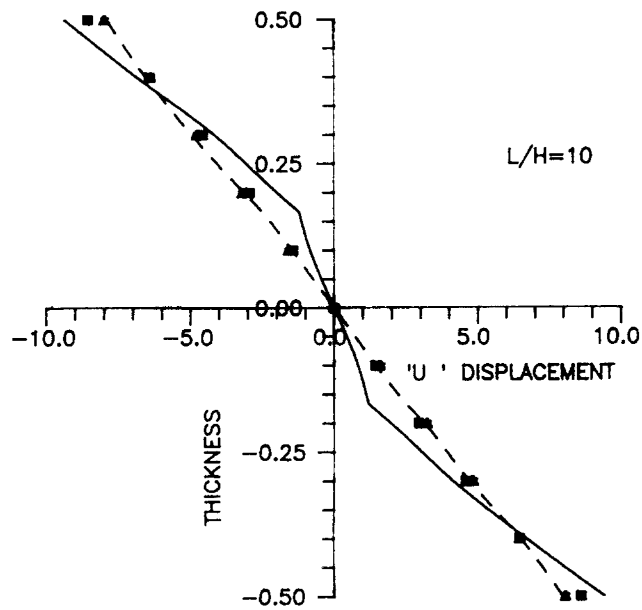

Fig. 18. Variation of inplane displacement through the thickness.

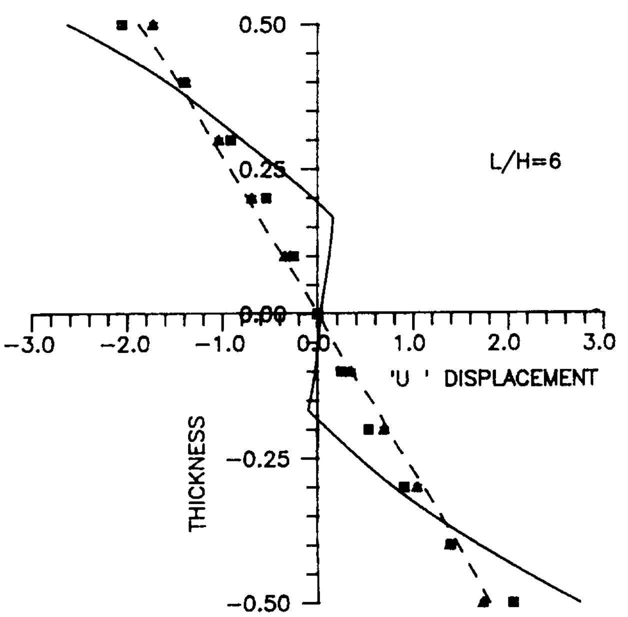

Fig. 17. Variation of inplane displacement through the thickness.

Table 1. Variation of Displacement and Stresses for Unsymmetric Sandwich Beams for Different Values of l/h ratios

Models	l/h	$\bar{w}_0$	$\bar{\sigma}_x$	$\bar{\tau}^c_{xz}$
HOT1	10	2·26121	127·70	3·712
HOT2	10	2·31953	126·30	3·706
HOT3	10	4·17540	143·10	3·647
HOT4	10	4·45053	141·10	3·631
TIMO	10	2·23956	95·55	4·836
HOT1	25	1·79060	798·00	9·279
HOT2	25	1·79971	796·70	9·277
HOT3	25	2·10171	813·70	9·253
HOT4	25	2·14799	811·80	9·246
TIMO	25	1·67689	597·20	12·090
HOT1	50	1·72322	3192·00	18·56
HOT2	50	1·72549	3191·00	18·56
HOT3	50	1·80117	3208·00	18·55
HOT4	50	1·81282	3206·00	18·54
TIMO	50	1·59651	2389·00	24·18
HOT1	100	1·70637	12770·00	37·12
HOT2	100	1·70693	12770·00	37·12
HOT3	100	1·72587	12780·00	37·11
HOT4	100	1·72878	12780·00	37·11
TIMO	100	1·57642	9555·00	48·36
HOT1	200	1·70216	51080·00	74·23
HOT2	200	1·70230	51070·00	74·23
HOT3	200	1·70703	51090·00	74·23
HOT4	200	1·70776	51090·00	74·23
TIMO	200	1·57139	38220·00	96·72

considered here for comparison with elasticity solution, these theories are general and thus can be used to tackle any type of loading and boundary conditions. The aim here was to establish the credibility of the formulations to compute, especially, the interlaminar shear and normal stresses. For this reason, this paper is limited to problems for which elasticity solutions were available. The numerical estimates of the interlaminar normal stress which is of paramount importance in the delamination studies of laminates are not available to-date. The use of cubic elements seems to have given fairly accurate estimates of these stresses as third derivatives of displacements can be calculated by using this element.

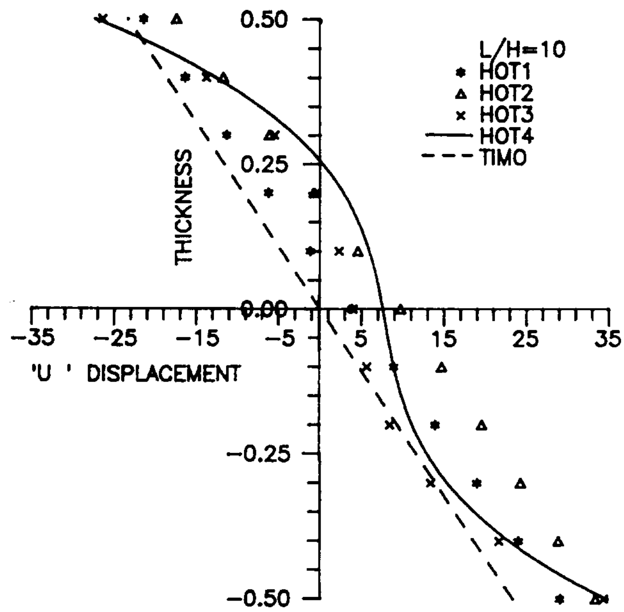

Fig. 19. Variation of inplane displacement through the thickness.

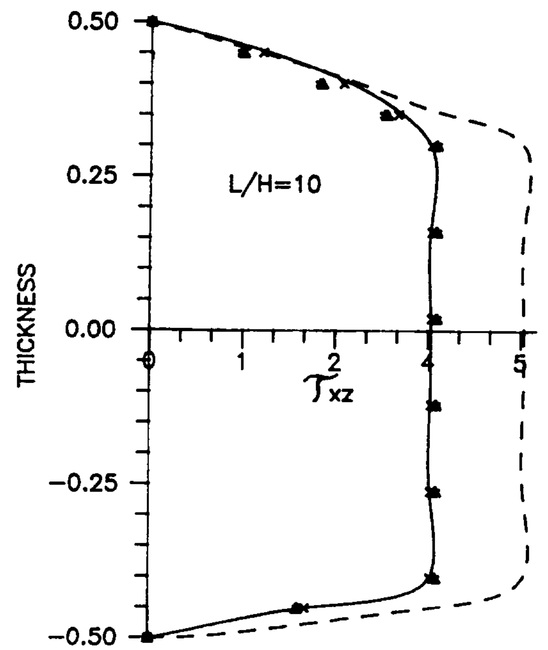

Fig. 20. Variation of interlaminar shear stress through the thickness.

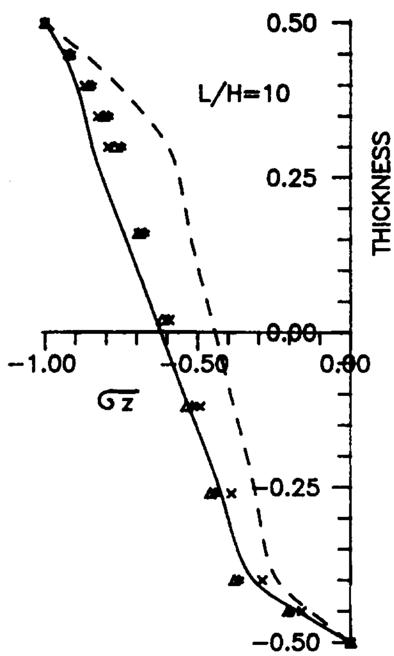

Fig. 21. Variation of interlaminar normal stress through the thickness.

In the symmetric case, the results of HOT1 and HOT3 follow closely to the results of HOT2 and HOT4, respectively, with the model HOT3 solution being close to elasticity solutions. This is due to the inplane displacements of the middle surface becoming negligibly small because of the symmetry of material properties and loading. Thus, model HOT3 is recommended for symmetric composite and sandwich beams.

In the unsymmetric case, the results obtained by HOT4 match well with the elasticity solution. But for the sandwich plate, the results show a large variation for thick plate $(a/h \leqslant 10)$ compared to Timoshenko's theory. This is due to the simplifying assumptions made in the latter theory. However, the values are almost equal to the latter theory for thin plate $(a/h \geqslant 50)$. Thus, this model is recommended for unsymmetric composite and sandwich beams.

The displacement model HOT4 is the most general one. It contains two degrees of freedom for the axial deformation and another two degrees of freedom for the flexural deformation modes of the transverse normal beam cross-section in the definition of the axial displacement component $u(x,z)$. Out of these four degrees of freedom, HOT1 has only two degrees of freedom, one each for axial and flexural modes of deformation, HOT2 has three degrees of freedom, two for axial and one for flexural modes of deformation and HOT3 contains three degrees of freedom, one for axial and two for flexural modes of deformation. Thus, it is seen that HOT1 to HOT3 are subsets of HOT4.

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APPENDIX

HOT 1

$$\mathbf{d} = (u_0, w_0, \theta_x, \theta_z, w_0^*)^t \tag{A1}$$

The membrane flexure and coupling matrix can be written as follows:

$$\sum_{L=1}^{NL} \begin{bmatrix} C_{11}h_1 & C_{12}h_1 & C_{11}h_2 & C_{12}h_2 \\ & C_{22}h_1 & C_{12}h_2 & C_{22}h_2 \\ \text{symm.} & & C_{11}h_3 & C_{12}h_3 \\ & & & C_{22}h_3 \end{bmatrix} \tag{A2}$$

The $\mathbf{D}_s$ matrix is same as that for model HOT4.

The non-zero elements of the $\mathbf{B}_i$ (7×5) matrix can be written as follows:

$$B_{1,1} = B_{3,3} = B_{5,2} = B_{6,5} = B_{7,4} = \frac{\partial N_i}{\partial x}$$

$$B_{2,4} = B_{5,3} = N_i$$

$$B_{4,5} = 2N_i \tag{A3}$$

HOT 2

$$\mathbf{d} = (u_0, w_0, \theta_x, \theta_z, u_0^*, w_0^*)^t \tag{A4}$$

The membrane flexure and coupling matrix can be written as follows:

$$\sum_{L=1}^{NL} \begin{bmatrix} C_{11}h_1 & C_{12}h_1 & C_{11}h_3 & C_{11}h_2 & C_{12}h_2 \\ & C_{22}h_1 & C_{12}h_3 & C_{12}h_2 & C_{22}h_2 \\ \text{symm.} & & C_{11}h_5 & C_{11}h_4 & C_{12}h_4 \\ & & & C_{11}h_3 & C_{12}h_3 \\ & & & & C_{22}h_3 \end{bmatrix} \tag{A5}$$

The $\mathbf{D}_s$ matrix is the same as that for the model HOT4.

The $\mathbf{B}_i$ matrix has a dimension of 8×6 in which the non-zero elements are

$$B_{1,1} = B_{3,5} = B_{4,3} = B_{6,2} = B_{7,6} = B_{8,4} = \frac{\partial N_i}{\partial x}$$

$$B_{2,4} = B_{6,3} = N_i$$

$$B_{5,6} = B_{8,5} = 2N_i \tag{A6}$$

HOT 3

$$\mathbf{d} = (u_0, w_0, \theta_x, \theta_z, w_0^*, \theta_x^*)^t \tag{A7}$$

The membrane flexure and coupling matrix can be written as follows:

$$\sum_{L=1}^{NL} \begin{bmatrix} C_{11}h_1 & C_{12}h_1 & C_{11}h_2 & C_{12}h_2 & C_{11}h_4 \\ & C_{22}h_1 & C_{12}h_2 & C_{22}h_2 & C_{12}h_4 \\ \text{symm.} & & C_{11}h_3 & C_{12}h_3 & C_{11}h_5 \\ & & & C_{22}h_3 & C_{12}h_5 \\ & & & & C_{11}h_7 \end{bmatrix} \tag{A8}$$

The $\mathbf{D}_s$ matrix is the same as that for model HOT4.

The non-zero elements of $\mathbf{B}_i$ (8×6) can be written as follows:

$$\begin{aligned} &\mathrm{B}_{1,1}=\mathrm{B}_{3,3}=\mathrm{B}_{5,6}=\mathrm{B}_{6,2}=\mathrm{B}_{7,5}=\mathrm{B}_{8,4}=\frac{\partial N_i}{\partial x} \\ &\mathrm{B}_{2,4}=\mathrm{B}_{6,3}=N_i \\ &\mathrm{B}_{4,5}=2\,N_i \\ &\mathrm{B}_{7,6}=3\,N_i \end{aligned} \tag{A9}$$