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Different Numerical Techniques for the Estimation of Multiaxial Stresses in Symmetric/Unsymmetric Composite and Sandwich Beams with Refined Theories

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ABSTRACT: A set of simple but efficient and accurate higher-order displacement models are used to evaluate the multiaxial stress behavior in symmetric and unsymmetric composite and sandwich laminates. These theories incorporate a more realistic non-linear variation of displacements through the beam thickness, thus eliminating the use of shear correction coefficient(s). Constitutive relations are used to evaluate inplane stresses. The computer program developed incorporates the realistic prediction of transverse stresses from equilibrium equations. The integration of the equilibrium equations is attempted through direct integration method, forward and central direct finite difference technique and a new approach called an exact curve fitting method. The versatility of the present higher-order theories is demonstrated by comparing the results with the available elasticity and other closed-form solutions for cross-ply and sandwich laminates. The results show that the exact curve fitting method gives good estimates of multiaxial stresses compared to finite difference and direct integration methods.

1. INTRODUCTION

DELAMINATION, ALSO KNOWN as interlaminar cracking is widely recognized as a critical failure mechanism for laminated composite materials or multilayered structures during their service. Delamination is the failure of laminate in the form of separated laminae. This delamination can occur in a laminate subjected to any or combination of mechanical or thermal loads or environmental effects. Most favorable sites for delamination cracks are the geometric boundaries, such as cutouts, free edges, notches, and holes. One of the causes of delamination is the existence of transverse/interlaminar stresses which develop near the free edges or holes in composite laminates (Jones, 1975; Calcote, 1969). A theory

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which can predict all these stresses accurately becomes necessary for understanding the failure mechanism of fibre reinforced composite structures.

In the past, structural behavior of beams has been approximated by using the elementary Euler-Bernoulli and Timoshenko theories. The main assumption in the Euler-Bernoulli theory is that the transverse normal to the reference middle plane remains so during bending, implying that the transverse shear strain becomes zero. Thus the bending rotation becomes a first derivative of transverse displacement w and hence the theory requires the transverse displacement field to be C^1 continuous (Zienkiewicz and Taylor, 1989). But this theory leads to serious discrepancies in the case of composites where shear effects are significant and further the resulting finite element formulation turns out to be computationally inefficient from the point of view of simple finite element procedures.

Timoshenko (1921) has improved the Euler-Bernoulli theory by incorporating the effect of transverse shear strain into the governing equations. But the resulting transverse shear stress remains constant through the beam thickness. Thus a fictitious shear correction coefficient is used to correct the strain energy of deformation. Many investigators (Cowper, 1966, 1968; Krishna Murty, 1970, 1970a; Tessler and Dong, 1981; Heppler and Hansen, 1988; Suman and Dipak, 1988) have given some new expressions for this coefficient for different cross-sections of the beam. But for composite beams, the discrepancies between the Timoshenko theory and elasticity solution is large even after modifying the values of shear correction coefficient. As an improvement over the Timoshenko theory, Stephen and Levinson (1979) have given a second-order beam theory. But they have used two coefficients. One depends on cross-sectional warping while the other includes terms dependent on the transverse direct stresses. As a further improvement, Levinson (1981, 1981a, 1985) has given a fourth-order beam theory which takes into account transverse shear deformation. Here shear correction coefficient(s) is/are not used. Levinson's theory however, fails to adequately describe the two-dimensional displacement pattern.

Rychter (1987) has improved the consistency and accuracy of Levinson's theory by embedding in it the two-dimensional linear theory of elasticity. He has proved that the corresponding relative mean square error is, in general, proportional to the square of the beam depth. The shear contribution to the error, which is comprised of terms multiplied by the shear modulus G turns out to be proportional to the cube of the beam depth. Bickford (1982) used Hamilton's principle to derive a consistent higher-order theory of the elastodynamics of the beam based upon the kinematic and stress assumptions previously used by Levinson (1981, 1981a, 1985). Yuan and Miller (1988, 1989) have presented a new finite element model which includes separate degrees of freedom for each lamina. The displacements are expressed in a polynomial form, thus allowing the cross-section to deform into a shape that can be described by a function which includes quadratic and cubic terms as well as the linear one. Shear deformation is thus included but not interfacial slip or delamination.

The discrepancies in the above theories are rectified by introducing the higher-order functions in the Timoshenko theory leading to the development of higher-order theories (Reissner, 1975; Lo et al., 1977, 1978; Kant, 1982). Kant (1982) and

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Kant et al. (1982) were the first to present a C° finite element formulation of this theory. Kant and Gupta (1988) and Kant and Manjunatha (1989, 1989a, 1990) have extended this theory for symmetric and unsymmetric one-dimensional laminates. Further, in stress evaluation of composite laminates, lamina stresses can be accurately evaluated by using constitutive relations. But the same relations can not be used for the evaluation of transverse stresses, as it violates the continuity of transverse stresses at the interfaces. Thus, two-dimensional equilibrium equations are used in the present work to evaluate the transverse shear and normal stresses. The integration of equilibrium equations is attempted through direct integration, finite difference methods and a new approach called "exact curve fitting" method. The numerical results obtained by these methods are compared with available elasticity (Pagano, 1969) and other finite element solutions (Engblom and Ochoa, 1985; Spilker, 1982; Toledano and Murakami, 1987; Wen-Jinn and Sun, 1987).

2. THEORY AND FORMULATIONS

The Taylor's series expansion method is used to deduce a one-dimensional formulation of a two-dimensional elasticity problem (Hilderbrand et al., 1949) and the following set of equations are obtained by expanding the displacement components $u(x,z)$ and $w(x,z)$ of any point in the laminate space in terms of the thickness coordinate and these are designated as HOSTB3 to HOSTB8 (see Figure 1).

Symmetric Laminates

HOSTB3

$$u(x,z) = z\theta_x(x) + z^3\theta_x^*(x)$$

$$w(x,z) = w_o(x) \tag{1a}$$

HOSTB4

$$u(x,z) = z\theta_x(x) + z^3\theta_x^*(x)$$

$$w(x,z) = w_o(x) + z^2w_o^*(x) \tag{1b}$$

Unsymmetric Laminates

HOSTB5

$$u(x,z) = u_o(x) + z\theta_x(x) + z^2u_o^*(x) + z^3\theta_x^*(x)$$

$$w(x,z) = w_o(x) \tag{2a}$$

HOSTB6

$$u(x,z) = u_o(x) + z\theta_x(x) + z^2u_o^*(x) + z^3\theta_x^*(x)$$

$$w(x,z) = w_o(x) + z\theta_z(x) \tag{2b}$$

HOSTB7

$$u(x,z) = u_o(x) + z\theta_x(x) + z^2u_o^*(x) + z^3\theta_x^*(x)$$

$$w(x,z) = w_o(x) + z\theta_z(x) + z^2w_o^*(x) \tag{2c}$$

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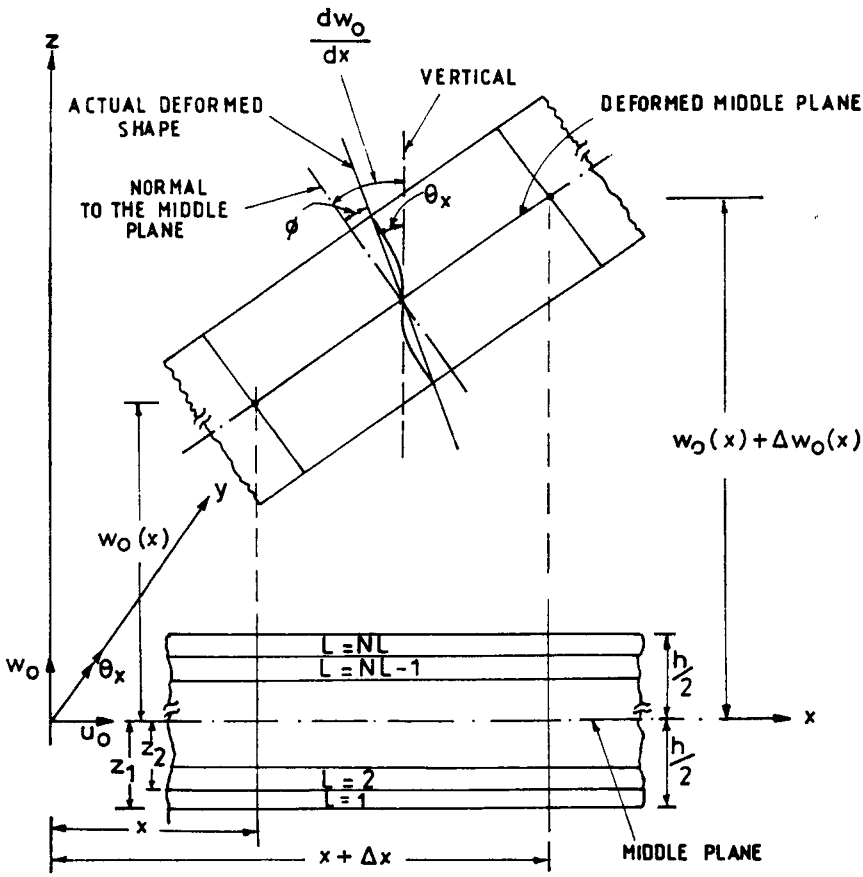

Figure 1. *Laminate geometry with positive set of lamina/laminate reference axes and displacement components.*

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$$\text{HOSTB8} \qquad u(x,z) = u_o(x) + z\theta_x(x) + z^2u_o^*(x) + z^3\theta_x^*(x)$$

$$w(x,z) = w_o(x) + z\theta_z(x) + z^2w_o^*(x) + z^3\theta_z^*(x) \tag{2d}$$

Where the parameters u and w define the displacement components of a general point (x,z) in the coordinate directions x and z respectively at any point in the beam domain. Here only the formulation for unsymmetric laminate (HOSTB8) is presented and other theoretical models become special cases of HOSTB8. The variations in the cases of other models (HOSTB3 to HOSTB7) are given elsewhere (Kant and Manjunatha, 1989, 1989a, 1990). The following relations are obtained by substituting Equation (2d) into the linear strain displacement relations of two-dimensional elasticity (Timoshenko and Goodier, 1985).

$$\begin{aligned} \epsilon_x &= \epsilon_{xo} + z\kappa_x + z^2\epsilon_{xo}^* + z^3\kappa_x^* \\ \epsilon_z &= \epsilon_{zo} + z\kappa_z^* + z^2\epsilon_{zo}^* \\ \gamma_{xz} &= \phi_x + z\kappa_{xz} + z^2\phi_x^* + z^3\kappa_{xz}^* \end{aligned} \tag{3}$$

where,

$$\begin{aligned} (\epsilon_{xo}, \kappa_x, \epsilon_{xo}^*, \kappa_x^*) &= \left(\frac{\partial u_o}{\partial x}, \frac{\partial \theta_x}{\partial x}, \frac{\partial u_o^*}{\partial x}, \frac{\partial \theta_x^*}{\partial x}\right) \\ (\phi_x, \kappa_{xz}, \phi_x^*, \kappa_{xz}^*) &= \left(\theta_x + \frac{\partial w_o}{\partial x}, 2u_o^* + \frac{\partial \theta_z}{\partial x}, 3\theta_x^* + \frac{\partial w_o^*}{\partial x}, \frac{\partial \theta_z^*}{\partial x}\right) \\ (\epsilon_{zo}, \kappa_z^*, \epsilon_{zo}^*) &= (\theta_z, 2w_o^*, 3\theta_z^*) \end{aligned} \tag{4}$$

Each lamina in the laminate is in a two-dimensional stress state. The constitutive relation for a typical lamina L is thus written simply as,

$$\begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & G \end{bmatrix}^L \begin{bmatrix} \epsilon_x \\ \epsilon_z \\ \gamma_{xz} \end{bmatrix}^L \tag{5}$$

where,

$$\begin{aligned} C_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}} \\ C_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \\ C_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}} \end{aligned} \tag{6}$$

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The relations given by Equation (5) are used to develop the theories based on the displacement models HOSTB4 [Equation (lb)] and HOSTB6 to HOSTB8 [Equations (2b) to (2d)].

The stress-strain relation for the theory based on the displacement models HOSTB3 [Equation (la)] and HOSTB5 [Equation (2a)] can be written as,

$$\begin{bmatrix} \sigma_x \\ \tau_{xz} \end{bmatrix}^L = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}^L \begin{bmatrix} \epsilon_x \\ \gamma_{xz} \end{bmatrix}^L \tag{7}$$

where (σ_x, σ_y, τ_{xz}) are the stresses and (ϵ_x, ϵ_y, γ_{xz}) are the strain components referred to the laminate coordinates (x,z).

The total potential energy Π of the beam can be written as,

$$\Pi = \frac{b}{2}\int_z\int_l \underset{\sim}{\epsilon}^t \underset{\sim}{\sigma} dx dz - b\int_z\int_l (\underset{\sim}{u})^t \underset{\sim}{P} dx dz \tag{8}$$

where,

$$\underset{\sim}{\epsilon} = (\epsilon_x, \epsilon_z, \gamma_{xz})^t \qquad \underset{\sim}{\sigma} = (\sigma_x, \sigma_z, \tau_{xz})^t$$
$$\underset{\sim}{u} = (u, w)^t \qquad \underset{\sim}{P} = (p_x, p_z)^t \tag{9}$$

The expressions for the strain components given by Equation (3) are substituted in Equation (8). The following relations result when an explicit integration is carried out through the beam thickness.

$$\Pi = \frac{b}{2}\int_l \bar{\underset{\sim}{\epsilon}}^t \bar{\underset{\sim}{\sigma}} dx - b\int_l (\bar{\underset{\sim}{u}})^t \bar{\underset{\sim}{P}} dx \tag{10}$$

where,

$$\bar{\underset{\sim}{\sigma}} = (N_x, N_x^*, N_z, N_z^*, M_x, M_x^*, M_z^*, Q_x, Q_x^*, S_x, S_x^*)^t$$
$$\bar{\underset{\sim}{\epsilon}} = (\epsilon_{xo}, \epsilon_{xo}^*, \epsilon_{zo}, \epsilon_{zo}^*, \varkappa_x, \varkappa_x^*, \varkappa_z^*, \phi_x, \phi_x^*, \varkappa_{xz}, \varkappa_{xz}^*)^t$$
$$\bar{\underset{\sim}{u}} = (u_o, w_o, \theta_x, \theta_z, u_o^*, w_o^*, \theta_x^*, \theta_z^*)^t \tag{11}$$
$$\bar{\underset{\sim}{P}} = (p_{xo}, p_{zo}, m_{xo}, m_{zo}, p_{xo}^*, p_{zo}^*, m_{xo}^*, m_{zo}^*)^t$$

The stress-resultants in Equation (10) are defined as follows:

$$\begin{bmatrix} N_x & M_x & N_x^* & M_x^* \\ N_z & M_z & N_z^* & 0 \\ Q_x & S_x & Q_x^* & S_x^* \end{bmatrix} = \sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix} [1, z, z^2, z^3]\, dz \tag{12}$$

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Upon integration, these are written in matrix form as follows:

$$\begin{bmatrix} \underset{\sim}{N} \\ \hline \underset{\sim}{M} \\ \hline \underset{\sim}{Q} \end{bmatrix} = \left[\begin{array}{c|c|c} \underline{D}_M & \underline{D}_{MF} & 0 \\ \hline \underline{D}^t_{MF} & \underline{D}_F & 0 \\ \hline 0 & 0 & \underline{D}_S \end{array}\right] \begin{bmatrix} \underset{\sim}{\epsilon} \\ \hline \underset{\sim}{\chi} \\ \hline \underset{\sim}{\phi} \end{bmatrix} \tag{13}$$

or,

$$\bar{\underset{\sim}{\sigma}} = \underline{D}\,\bar{\underset{\sim}{\epsilon}} \tag{14}$$

where,

$$\begin{aligned} &\underset{\sim}{N} = (N_x, N_x^*, N_z, N_z^*)^t && \underset{\sim}{M} = (M_x, M_x^*, M_z^*)^t \\ &\underset{\sim}{Q} = (Q_x, Q_x^*, S_x, S_x^*)^t && \underset{\sim}{\epsilon} = (\epsilon_{xo}, \epsilon_{xo}^*, \epsilon_{zo}, \epsilon_{zo}^*)^t \\ &\underset{\sim}{\varkappa} = (\varkappa_x, \varkappa_x^*, \varkappa_z^*)^t && \underset{\sim}{\phi} = (\phi_x, \phi_x^*, \varkappa_{xz}, \varkappa_{xz}^*)^t \end{aligned} \tag{15}$$

The inplane stress can be accurately evaluated by using constitutive relations. But, the transverse stresses (τ_{xz}, σ_z) cannot be accurately estimated by the same relations. This is mainly due to the fact that the constitutive laws are discontinuous whereas transverse stresses have to maintain continuity across the interface of layers. For these reasons, the transverse stresses between the layers L and $L + 1$ at z are obtained by integrating the two equilibrium equations of two-dimensional elasticity for each layer over the lamina thickness and summing over layer 1 to L as follows.

The equations of equilibrium representing the pointwise equilibrium can be written as,

$$\tau_{ij,j} = 0 \qquad i,j = x,z \tag{16}$$

The integration of the equilibrium equations is attempted here through different novel approaches: direct integration method, forward and central direct finite difference techniques and a new approach called an exact curve fitting method. These techniques are explained below.

DIRECT INTEGRATION METHOD

The two differential equations of equilibrium given by Equation (16) gives two relations, viz.,

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} \tag{17}$$

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and

$$\frac{\partial^2 \sigma_z}{\partial z^2} = \frac{\partial^2 \sigma_x}{\partial x^2} \tag{18}$$

from which the transverse stresses τ_{xz} and σ_z can be evaluated through integration with respect to the laminate thickness z. The right-hand sides of Equations (17) and (18) contain lamina inplane stress σ_x which is easily computed through constitutive relations. Equation (17) is then essentially an initial value problem in τ_{xz} requiring only a prescribed value of τ_{xz} at either of the two boundary surfaces of the laminate. However, this problem is vexing because τ_{xz} is normally known at both top and bottom boundary surfaces of the laminate. Thus one obtains only a non-unique solution for τ_{xz} as two prescribed conditions for τ_{xz} cannot be simultaneously enforced in the solution. Equation (18), on the other hand, represents a second-order boundary value problem in σ_z requiring two prescribed values of σ_z on the bounding planes of the laminate, which are available. Thus a unique solution for σ_z is obtained.

The computational algorithms for τ_{xz} and σ_z take the form,

$$\tau_{xz}^L \big|_{z_{L+1}} = -\sum_{l=1}^{L} \int_{z_l}^{z_{l+1}} \frac{\partial \sigma_x}{\partial x}\, dz + C_1 \tag{19}$$

$$\sigma_z^L \big|_{z_{L+1}} = \sum_{l=1}^{L} \int_{z_l}^{z_{l+1}} \left(\int_z \frac{\partial^2 \sigma_x}{\partial x^2}\, dz \right) dz + zC_2 + C_3 \tag{20}$$

The constant C_1 is obtained from the known values of τ_{xz} at either of the two boundaries at $z = \pm h/2$ while constants C_2 and C_3 are obtained from the known values of σ_z at $z = \pm h/2$. However, Equation (20) requires the use of third derivatives of displacements. For this reason cubic four-noded quadrilateral Lagrangian elements are used here. The presence of second and third derivatives of displacements in the stress evaluation dictates the use of high degree polynomial elements and thereby, increasing the numerical error in the estimation of transverse shear and normal stresses.

To overcome these problems, forward and central direct finite difference (FDM) methods and a new approach called exact curve fitting method (ESFM) are proposed.

FINITE DIFFERENCE METHOD

The inplane stress σ_x is first evaluated by using constitutive relations at different Gauss points in an element (four points). Then a forward difference operator (stresses are maximum at the edge of the laminate) is used to evaluate the derivatives of inplane stress at a particular Gauss point inside the element and either a forward difference or a backward difference operator is used for the evaluation

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of the same at the edges of the laminate depending on whether the edge is a positive or negative one. The following relations corresponding to Equation (17) is obtained:

$$\frac{\partial \tau_{xz}(GP)}{\partial z} = -AA \tag{21}$$

with

$$AA = \left[\frac{\sigma_x(GP + 1) - \sigma_x(GP)}{\Delta X}\right] \tag{22}$$

where GP is the Gauss point number at which stresses are evaluated and $GP + 1$ is the next Gauss point number in the x-direction.

The following equation is obtained by using a forward difference operator along the thickness direction in Equation (19) for a particular layer and this we designate as the "forward difference-direct method."

$$\tau_{xz}^{L}\big|_{z_{L+1}} = \tau_{xz}^{L}\big|_{z_L} - [AA]^{L} * (z_{L+1} - z_L) \tag{23}$$

An alternate equation is obtained by substituting a central difference operator in Equation (21) and we designate this as the "central difference-direct method" (Salvadori and Baron, 1961).

$$\left[\frac{\tau_{xz(z_{L+1})} - (1 - \alpha^2)\tau_{xz(z_L)} - \alpha^2\tau_{xz(z_{L-1})}}{\alpha(\alpha + 1)(z_{L+1} - z_L)}\right] = -[AA] \tag{24}$$

where,

$$\alpha = \frac{(z_{L+1} - z_L)}{(z_L - z_{L-1})} \tag{25}$$

The use of central difference operator gives a two-step, non-self starting method, forward difference method is used to evaluate the transverse stress at the first step and for subsequent steps central difference method is used. These methods are very effective for isotopic laminates. However, in the case of laminates having different isotropic, orthotropic or anisotropic laminae, the inplane stress is discontinuous and two values are obtained at an interface of two layers. As the transverse stresses are continuous through the interface of two layers, the derivatives of inplane stress must also be continuous through the interface. Thus, an average of the two values at the interface is used in the above techniques.

EXACT CURVE FITTING METHOD

An exact curve fitting method is proposed here. The inplane stress is evaluated through constitutive relations at different Gauss points in an element. Having ob-

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tained the inplane stress acting on lower and upper surfaces of a particular layer, the variation of this stress over a particular surface of an element can be expressed as a polynomial in x as follows:

$$\sigma_x(z) = C_1^z + C_2^z x + C_3^z x^2 + C_4^z x^3 \tag{26}$$

Here four Gauss points are assumed. The parameter z in Equation (26) refers to a particular surface in the laminate at a distance "z" from the middle surface and this may be the top or bottom surface of a particular lamina or a subset of particular lamina having same ply orientations. On substituting the inplane stress values at different Gauss points in an element, the following equation results,

$$\underset{(4 \times 4)}{[\underline{A}_{ij}]}\ \underset{(4 \times 1)}{[\underline{C}_j^z]} = \underset{(4 \times 1)}{[\underline{\sigma}_{xi}^z]} \tag{27}$$

The above equation is solved and the four polynomial constants are obtained. Equation (26) is differentiated with respect to x and thus derivatives of inplane stress are obtained. These can be written as,

$$\frac{\partial \sigma_x}{\partial x} = C_2^z + 2C_3^z x + 3C_4^z x^2 \tag{28}$$

$$\frac{\partial^2 \sigma_x}{\partial x^2} = 2C_3^z + 6C_4^z x \tag{29}$$

These derivatives are then used in Equations (19) and (20) and the same procedure as used for direct finite difference method is used following Equations (21)–(25).

The transverse normal stress is evaluated by using a central difference operator in Equation (20) and the following difference equation is obtained:

$$2\left[\frac{\sigma_{z(z_{L+1})} - (1 + \alpha)\sigma_{z(z_L)} + \alpha\sigma_{z(z_{L-1})}}{\alpha(\alpha + 1)(z_{L+1} - z_L)^2}\right] = \frac{\partial^2 \sigma_x}{\partial x^2} \tag{30}$$

The double derivative of inplane stress is substituted in Equation (30) and this equation is solved as a boundary value problem by substituting the two boundary conditions for σ_z at top and bottom surfaces of the laminate.

3. FINITE ELEMENT FORMULATION

The standard finite element technique is followed. The total solution domain Ω is subdivided into "NE" subdomains (elements) $\Omega_1, \Omega_2, \ldots, \Omega_{NE}$, such that,

$$\Pi(\underline{u}) = \sum_{e=1}^{NE} \Pi^e(\underline{u}) \tag{31}$$

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in which, Π and Π^e are the total potential energies of the system and the element, respectively. The potential energy for an element e can be written as,

$$\Pi^e(\bar{\underset{\sim}{u}}) = U^e - W^e \tag{32}$$

where U^e, W^e and $\bar{\underset{\sim}{u}}$ are the internal strain energy, external work done and generalized displacement vector, respectively, for an element e. In C° finite element theory, the continuum displacement vector within an element is discretized such that (Zienkiewicz and Taylor, 1989),

$$\bar{\underset{\sim}{u}} = \sum_{i=1}^{NN} N_i \bar{\underset{\sim}{u}}_i \tag{33}$$

where N_i is the interpolating function associated with node i, NN is the number of nodes in an element and $\bar{\underset{\sim}{u}}_i$ is the generalized displacement vector corresponding to the i^{th} node of an element. Here four-noded cubic elements are considered in numerical study.

Knowing the generalized displacement vector $\underset{\sim}{u}$ at all points within the element, the generalized strain at any point given by Equation (3) can be expressed in matrix form as follows (Zienkiewicz and Taylor, 1989):

$$\bar{\underset{\sim}{\epsilon}} = \sum_{i=1}^{NN} \underline{B}_i \, \underset{\sim}{u}_i \tag{34}$$

where,

$$\bar{\underset{\sim}{\epsilon}} = (\epsilon_{xo}, \epsilon_{xo}^*, \epsilon_{zo}, \epsilon_{zo}^*, \chi_x, \chi_x^*, \chi_z^*, \phi_x, \phi_x^*, \chi_{xz}, \chi_{xz}^*)^t \tag{35}$$

The matrix $\underline{B}_i$ has a dimension of (11 × 8) in which the non-zero elements are,

$$\begin{aligned} B_{1,1} &= B_{2,5} = B_{6,3} = B_{7,7} = B_{8,2} \\ &= B_{9,6} = B_{10,4} = B_{11,8} = \frac{\partial N_i}{\partial x} \\ B_{3,4} &= B_{8,3} = N_i \\ B_{4,6} &= B_{10,5} = 2N_i \\ B_{7,8} &= B_{9,7} = 3N_i \end{aligned} \tag{36}$$

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Upon evaluating the $\underline{D}$ and $\underline{B}_i$ matrices as given by Equations (13) and (34) respectively, the element stiffness matrix is computed by using the standard relation:

$$\underline{K}^e = \int_A (\underline{B}^t \underline{D} \underline{B}) dA \tag{37}$$

In isoparametric representation, the geometry and the displacement field are interpolated using the same shape functions. The isoparametric concept allows any arbitrary geometry to be closely approximated thereby reducing any error associated with modelling the geometry and without resorting to use of fine mesh along the boundaries. In this method the rigid body displacement as well as constant strain criteria are satisfied and numerical integration can be carried out conveniently as a standard procedure for evaluating the integrals.

The final form of the Equation (37) after using the standard isoparametric relation to change the coordinate system from x to ξ coordinate system can be written as,

$$\underline{K}^e_{ij} = \int_{-1}^{+1} \underline{B}^t_i \underline{D} \underline{B}_j |J| d\xi \tag{38}$$

The computation of element stiffness matrix $\underline{K}^e$ is economised by explicit multiplication of the $\underline{B}^t_i$, $\underline{D}$ and $\underline{B}_j$ matrices instead of carrying out the full matrix multiplication of the triple product (Kant et al., 1982). In addition, due to symmetry of the stiffness matrix only the blocks $\underline{K}_{ij}$ lying on one side of the main diagonal are formed. The integral is evaluated by using different Gauss quadrature rules for membrane-flexure and shear parts as follows:

$$\underline{K}^e_{ij} = \sum_{g=1}^{NG} W_g \underline{B}^t_i \underline{D} \underline{B}_j |\underline{J}| \qquad i,j = 1,NN \tag{39}$$

where W_g is the weighting coefficient, NG is the number of numerical quadrature points in the ξ-direction and $|\underline{J}|$ is the determinant of the standard Jacobian matrix.

The consistent load vector $\underset{\sim}{p}$ due to uniformly distributed transverse load P can be written as,

$$\underset{\sim}{p} = \int_{-1}^{+1} N^t P |\underline{J}| d\xi \tag{40}$$

The integral of Equation (40) is evaluated numerically using the four Gauss quadrature rule. The result is

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$$\underset{\sim}{p} = \sum_{a=1}^{NG} \sum_{a=1}^{NG} W_a W_b N_i^t (0,P,0,Ph/2,0,Ph^2/4,0,Ph^3/8)^t |\underline{J}| \quad (41)$$

The consistent load vector for sinusoidal transverse load can be obtained by using the following substitution in expression Equation (41).

$$P = P_o \sin \frac{m\pi x}{a} \quad (42)$$

where a is the beam dimension, x is the Gauss point coordinate and m is the usual harmonic number.

4. NUMERICAL RESULTS AND DISCUSSION

A set of computer programmes incorporating the present higher-order theories are developed for the numerical computation of various types of examples in symmetric and unsymmetric composite and sandwich laminates. All the computations are carried out on a CYBER 180/840 computer in single precision with sixteen significant digits word-length. A computer programme based on the Timoshenko theory has been developed to support the numerical evaluations of the present formulations. A shear correction coefficient 5/6 is used for all the materials in the Timoshenko theory to correct the transverse shear energy terms. The numerical results obtained by the present higher-order theories are compared with the Timoshenko theory for problems where the elasticity, closed-form and other numerical solutions are not available. Lagrangian one-dimensional four-noded cubic elements are used. Selective numerical integration techniques, based on Gauss-Legendre product rules, namely 4 for flexure/membrane and 3 for shear terms, have been used in the analysis.

In the present numerical examples, the values of inplane and transverse stresses are evaluated at the Gauss points, whereas the displacements are computed at the nodal points. The displacements, as well as inplane and transverse stresses are presented here in the non-dimensional form using the following multipliers.

$$\bar{\sigma}_x = \frac{\sigma_x(a/2, z)}{q_o} \quad \bar{\sigma}_z = \frac{\sigma_z(a/2,z)}{q_o} \quad \bar{\tau}_{xz} = \frac{\tau_{xz}(0,z)}{q_o}$$
$$u = \frac{E_T u(0,z)}{hq_o} \quad \bar{w}_o = \frac{100 E_T h^3 w_o(a/2,0)}{q_o a^4} \quad (43)$$

The percentage difference in results are calculated as follows:

$$\text{Percentage difference (PD)} = \left[\frac{\text{Approximate value} - \text{True value}}{\text{True value}}\right] \times 100 \quad (44)$$

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Boundary Conditions

These are clearly specified for simply supported beam as follows:

Model	At $x = 1$	At $x = 0$
HOSTB8	$u_o = u_o^* = w_o = w_o^* = 0$	$w_o = w_o^* = 0$
	$\theta_z = \theta_z^* = 0$	$\theta_z = \theta_z^* = 0$

SYMMETRIC LAMINATES

A simply supported symmetric 3-ply orthotropic beam with L-direction coinciding with x in the outer layers while T is parallel to x in the central layer with layers having equal thickness is considered. A sinusoidal load is applied and the material properties as given below are used (Pagano, 1969).

$$E_L = 0.25 \times 10^8 \text{ psi} \qquad E_T = 0.1 \times 10^7 \text{ psi}$$
$$G_{LT} = 0.5 \times 10^6 \text{ psi} \qquad \nu_{LT} = 0.25 \tag{45}$$

where L signifies the direction parallel to the fibres, T is the transverse direction and ν_{LT} is the Poisson's ratio measuring strain in the transverse direction under uniaxial normal stress in the L-direction.

The maximum transverse displacement $\bar{w}_o$, inplane and transverse shear and normal stresses are presented in Tables 1 and 2 for the two a/h ratios ($a/h = 4$, 10). The convergence of transverse displacement $\bar{w}_o$ with a/h ratio is shown in Figure 2. The variation of transverse shear stress ($\bar{\tau}_{xz}$) through the laminate thickness is shown in Figures 3 and 4 for the two a/h ratios ($a/h = 4$, 10). The variation of inplane stress through the laminate thickness for $a/h = 4$ is shown in Figure 5.

A comparison of present results with the elasticity (Pagano, 1969) and other solutions (Spilker, 1982) shows good convergence for transverse displacement $\bar{w}_o$ and in particular model HOSTB3 results (-22.2504 PD) are better compared to other models. The CPT (classical plate theory) underestimates the values and gives a very poor estimate for relatively low values of a/h (-79.5919 PD) (Figure 1 and Table 1).

The transverse shear stress variation shows that the values obtained by the present theory are better compared to the CPT and other solutions (Engblom and Ochoa, 1985; Spilker, 1982) and in particular the results of central difference direct method [3.3004 PD for $a/h = 4$ (HOSTB4) and 3.0966 PD for $a/h = 10$ (HOSTB3)] are in good agreement with elasticity solution results compared to other methods and CPT (11.43108 PD) (Figures 3 and 4).

Table 2 shows that the transverse normal stress obtained by the present theory are in good agreement with the elasticity and other solutions and in particular the results obtained by model HOSTB4 are better compared to other model. The inplane stress variation (Figure 5 and Table 1) show that model HOSTB4 results are

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Table 1. ***Comparison of maximum transverse displacement, inplane stress and transverse shear stress ($\bar{\tau}_{xz}$) for simply supported laminate under sinusoidal loading (a/h = 4, 10) (0/90/0).***

Source	a/h Ratio	$\bar{w}_o$	$\bar{\sigma}_x$	Constitutive	Direct Integration	Direct Method		Curve Fitting Method	
						F.D.	C.D.	F.D.	C.D.
HOSTB3	4	1.970547	13.8900 −13.8900	1.78200	1.66300	1.71271	1.65071	1.72974	1.66712
HOSTB4	4	1.960297	13.9400 −13.9600	1.76000	1.66200	1.71231	1.65012	1.72928	1.66648
Spilker (1982)	4	2.841000			1.56364				
Toledano and Murakami (1987)	4	2.881000	—		—				
Lo et al. (1978)	4	—			1.55556				
Engblom and Ochoa (1985)	4	—	10.0090 −10.1081		1.77340				
Pagano (1969)	4	2.534480	18.4102 −17.6923		1.59740				
CPT (Pagano, 1969)	4	0.517240	9.6410 −10.2564		1.78000				
HOSTB3	10	0.749174	67.4000 −67.4000	4.64800	4.39500	4.48452	4.36582	4.52912	4.40923
HOSTB4	10	0.747944	67.4100 −67.4200	4.64800	4.39500	4.48420	4.36590	4.52880	4.40932
Spilker (1982)	10	0.931200	—		4.52922				
Engblom and Ochoa (1985)	10	—	63.7344 −63.4025		4.45902				
Pagano (1969)	10	0.956896	70.2564 −73.4359		4.23469				
CPT (Pagano, 1969)	10	0.517240	59.6923 −63.7948		4.45195				

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Table 2. ***Transverse normal stress*** **($\bar{\sigma}_z$)** ***for simply supported laminate under sinusoidal loading*** **(a/h = 4) (0/90/0).**

Thickness	Direct Integration Method		Exact Curve Fitting Method		Pagano (1969)		
	HOSTB3	HOSTB4	HOSTB3	HOSTB4	CPT	Elasticity	
−0.5000	0	0	0	0	0	0	(−.5)
−0.3889	0.043137	0.043164	0.037804	0.037929	0.026316	0.042105	(−.4)
−0.2778	0.145215	0.145323	0.138309	0.138647	0.105263	0.144737	(−.3)
−0.1667	0.277127	0.277166	0.270914	0.271326	0.215789	0.268421	(−.2)
−0.0556	0.425662	0.425489	0.414171	0.414559	0.368420	0.394737	(−.1)
0.0556	0.574338	0.573901	0.557658	0.557980	0.657895	0.626316	(.1)
0.1667	0.722873	0.722022	0.701033	0.701150	0.789474	0.736842	(.2)
0.2778	0.854785	0.854046	0.837536	0.837428	0.900000	0.863158	(.3)
0.3889	0.956863	0.956475	0.946431	0.946342	0.986840	0.952630	(.4)
0.5000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	(.5)

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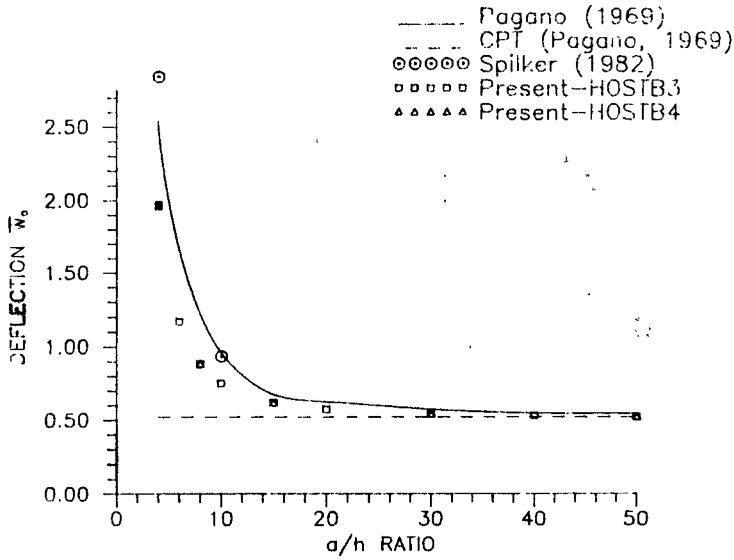

Figure 2. *Convergence of transverse displacement* $\overline{w}_o$ *with* a/h *ratio for simply supported laminate under sinusoidal loading (0/90/0).*

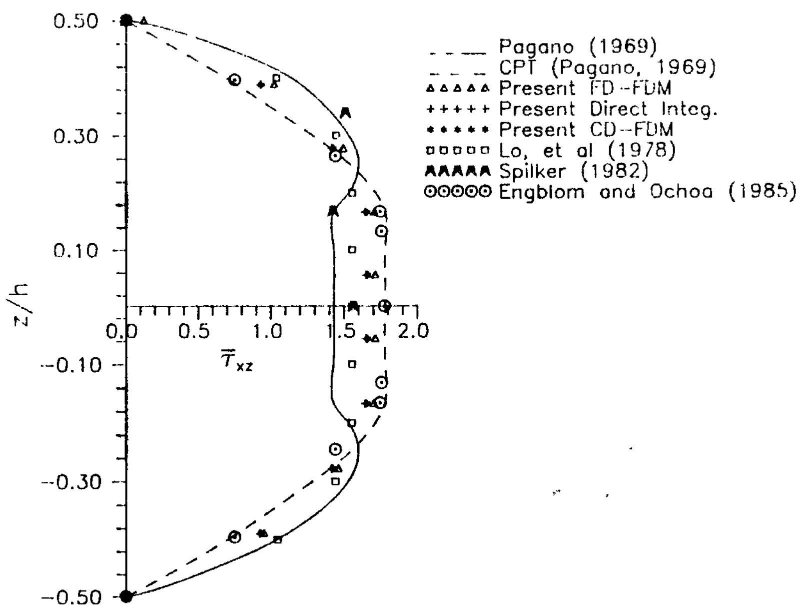

Figure 3. *Variation of transverse shear stress* ($\overline{\tau}_{xz}$) *through the thickness of a simply supported laminate under sinusoidal loading* (a/h = 4) *(0/90/0) (HOSTB4).*

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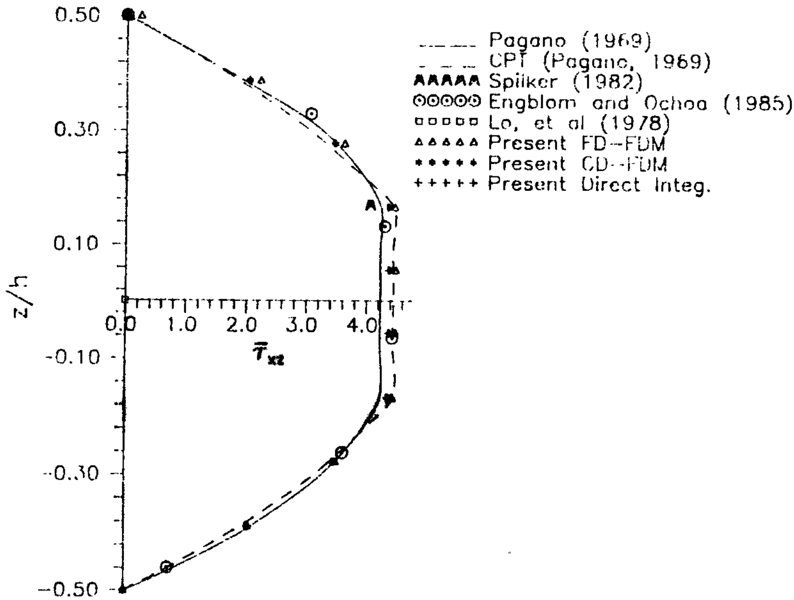

Figure 4. *Variation of transverse shear stress ($\bar{\tau}_{xz}$) through the thickness of a simply supported laminate under sinusoidal loading (a/h = 10) (0/90/0) (HOSTB3).*

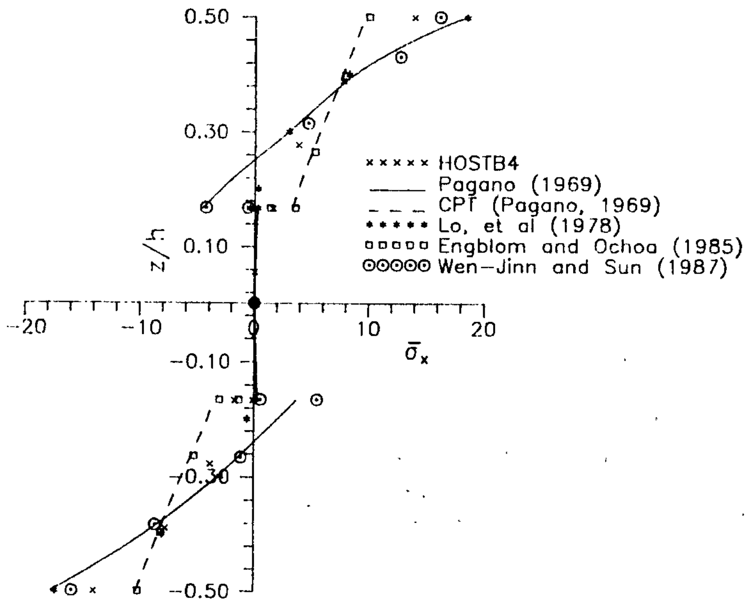

Figure 5. *Variation of inplane stress ($\bar{\sigma}_x$) through the thickness of a simply supported laminate under sinusoidal loading (a/h = 4) (0/90/0).*

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in good agreement with elasticity and other solutions (−24.2811 PD) compared to model HOSTB3 and the CPT (−47.6323 PD).

A simply supported symmetric sandwich laminate under transverse loading is considered. The following material properties are used (Khatua and Cheung, 1972).

$$\left.\begin{array}{l} \text{Stiff layers} \\ \qquad E_L = E_T = 0.1 \times 10^8 \text{ psi} \qquad G = E_L/2(1 + \nu) \qquad \nu = 0.30 \\ \text{Core layers} \\ \qquad G = 0.5 \times 10^4 \text{ psi} \qquad \nu = 0 \qquad E = 2G \qquad h_c/h_{tf} = 18 \end{array}\right\} \quad (46)$$

The maximum transverse displacement $\bar{w}_o$, inplane and transverse stresses for different a/h ratios (a/h = 4, 10 and 25) are presented in Tables 3 to 5. The variation of inplane displacement and inplane stress through the laminate thickness is shown in Figures 6 and 7 for a/h = 4. The variations of transverse shear and normal stresses through the laminate thickness for a/h = 4 are shown in Figures 8 and 9, respectively.

Since the elasticity and other closed-form solutions are not available for the present problem, the results obtained by the present theory are compared with those based on the Timoshenko theory. These results show large differences in the values of displacement (− 11.9131 PD), inplane (−42.05276 PD) and transverse shear (−44.4361 PD) and normal stresses for present theory compared to the Timoshenko theory for thick laminates (a/h = 4). This is due to the simplifying assumption made in the Timoshenko theory. For relatively thin laminates ($a/h \geq 25$), the discrepancies between the results of present theory and the Timoshenko theory are seen to decrease and for very thin laminates almost the same results are obtained for all the models. Figure 6 shows the actual warping of the cross-section of the laminate. But the Timoshenko theory gives an unrealistic straight line variation through the thickness of the laminate. The transverse normal stress variation shows that the central difference exact curve fitting method gives a good estimate of the stress compared to the direct integration method in which the stress changes its sign (Figure 9).

UNSYMMETRIC LAMINATES

A simply supported bidirectional orthotropic laminate with the T and L direction aligned parallel to x-axis in the top and bottom layers respectively is considered. The layers are of equal thickness. The beam is subjected to sinusoidal loading and the material properties as defined in Equation (45) are used (Pagano, 1969).

The maximum transverse displacement $\bar{w}_o$, inplane and transverse stresses are presented in Tables 6 to 8 for a/h = 4 and 10. The variation of maximum transverse displacement $\bar{w}_o$ with a/h ratio is shown in Figure 10. The variations of

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Table 3. Comparison of maximum transverse displacement, inplane stress and transverse shear stress ($\bar{\tau}_{xz}$) for simply supported symmetric sandwich laminate under transverse loading (a/h = 4, 10, 25).

Source	a/h Ratio	$\bar{w}_o/10^3$	$\bar{\sigma}_x$	Constitutive	Direct Integration	Direct Method		Curve Fitting Method	
						F.D.	C.D.	F.D.	C.D.
HOSTB3	4	9.26316	73.350 −73.350	1.310	2.011	2.41851	1.96366	2.57222	2.02599
HOSTB4		9.24752	76.190 −76.160	1.330	2.015	2.45126	1.96272	2.61852	2.03321
TIMO		8.15963	44.150 −44.150	1.921	2.047	2.10624	2.00133	2.17846	2.06995
HOSTB3	10	1.98144	305.10 −305.10	3.411	5.083	5.33749	4.96923	5.54502	5.13611
HOSTB4		1.98021	308.00 −307.90	3.426	5.063	5.33728	4.95859	5.53050	5.10903
TIMO		0.61357	275.90 −275.90	4.803	5.118	5.26561	5.00333	5.44614	5.17486
HOSTB3	25	0.80047	1754.0 −1754.0	10.21	12.78	13.1780	12.4924	13.6344	12.9154
HOSTB4		0.80019	1757.0 −1757.0	10.21	12.75	13.1682	12.4796	13.6021	12.8798
TIMO		0.58190	1725.0 −1725.0	12.01	12.79	13.1640	12.5083	13.6153	12.9372

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Table 4. Transverse normal stress ($\bar{\sigma}_z$) for simply supported symmetric sandwich laminate under transverse loading (a/h = 4).

Thickness	Direct Integration Method			Exact Curve Fitting Method		
	HOSTB3	HOSTB4	TIMO	HOSTB3	HOSTB4	TIMO
−0.5000	0	0	0	0	0	0
−0.4500	0.195916	−0.172392	−0.174329	0.018079	0.018038	0.018063
−0.3500	0.306381	−0.023337	−0.024684	0.113403	0.113361	0.113389
−0.2500	0.417000	0.125871	0.125115	0.214290	0.214245	0.214277
−0.1500	0.527729	0.275189	0.275025	0.315289	0.315243	0.315281
−0.0500	0.638522	0.424572	0.425001	0.416358	0.416311	0.416355
0.0500	0.749338	0.573978	0.574999	0.517454	0.517406	0.517455
0.1500	0.860131	0.723362	0.724975	0.618532	0.618483	0.618538
0.2500	0.970860	0.872682	0.874885	0.719549	0.719501	0.719559
0.3500	1.081480	1.021890	1.024680	0.820461	0.820415	0.820474
0.4500	1.191940	1.170950	1.174330	0.921224	0.921181	0.921239
0.5000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

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Table 5. ***Transverse normal stress ($\bar{\sigma}_z$) for simply supported symmetric sandwich laminate under transverse loading (a/h = 10).***

	Direct Integration Method			Exact Curve Fitting Method		
Thickness	HOSTB3	HOSTB4	TIMO	HOSTB3	HOSTB4	TIMO
−0.5000	0	0	0	0	0	0
−0.4500	0.519912	−0.174427	−0.174329	0.018061	0.018075	0.018063
−0.3500	0.596516	−0.024761	−0.024684	0.113386	0.113399	0.113389
−0.2500	0.673275	0.125060	0.125115	0.214275	0.214285	0.214277
−0.1500	0.750145	0.274992	0.275025	0.315279	0.315287	0.315281
−0.0500	0.827081	0.424990	0.425001	0.416354	0.416359	0.416355
0.0500	0.904039	0.575010	0.574999	0.517456	0.517457	0.517455
0.1500	0.980975	0.725008	0.724975	0.618539	0.618538	0.618538
0.2500	1.057840	0.874939	0.874885	0.719561	0.719558	0.719559
0.3500	1.134600	1.024760	1.024680	0.820477	0.820471	0.820474
0.4500	1.211210	1.174430	1.174330	0.921242	0.921233	0.921239
0.5000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

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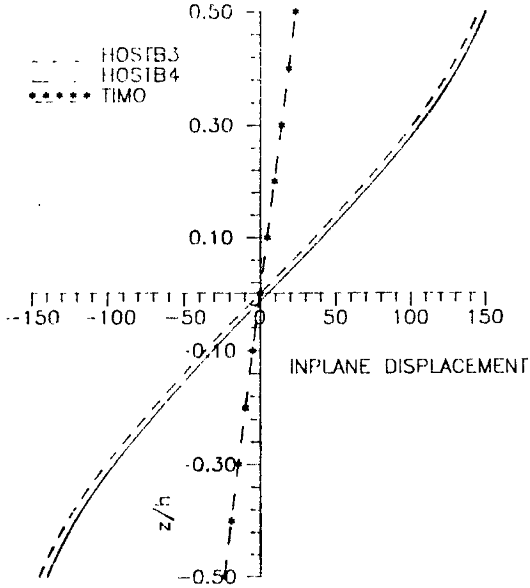

Figure 6. *Variation of inplane displacement through the thickness of a simply supported symmetric sandwich laminate under uniformly distributed loading (a/h = 4).*

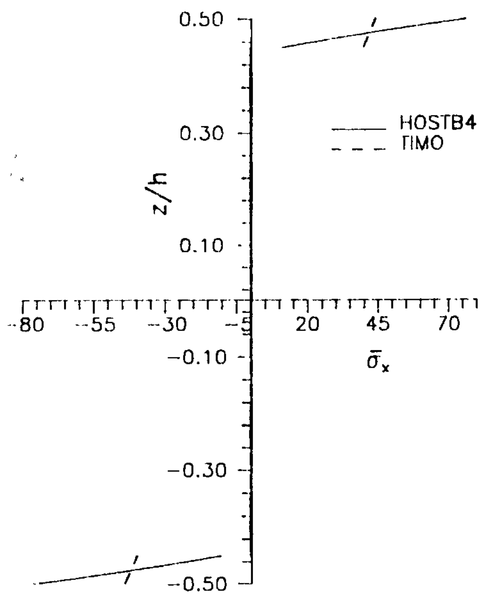

Figure 7. *Variation of inplane stress ($\bar{\sigma}_x$) through the thickness of a simply supported symmetric sandwich laminate under uniformly distributed loading (a/h = 4).*

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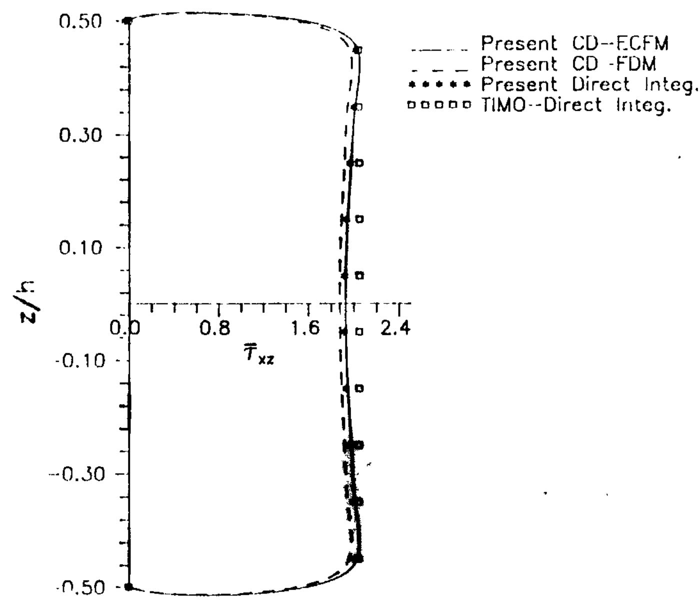

Figure 8. *Variation of transverse shear stress ($\bar{\tau}_{xz}$) through the thickness of a simply supported symmetric sandwich laminate under uniformly distributed loading (*a/h *= 4) (HOSTB4).*

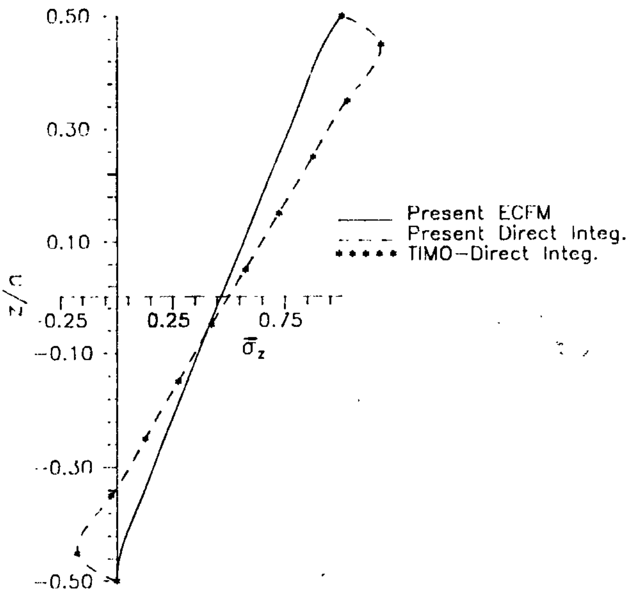

Figure 9. *Variation of transverse normal stress ($\bar{\sigma}_z$) through the thickness of a simply supported symmetric laminate under uniformly distributed loading (*a/h *= 4) (HOSTB4).*

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Table 6. Comparison of maximum transverse displacement, inplane stress and transverse shear stress ($\bar{\tau}_{xz}$) for simply supported laminate under sinusoidal loading (a/h = 4, 10) (0/90).

Source	a/h Ratio	$\bar{w}_o$	$\bar{\sigma}_x$	Constitutive	Direct Integration	Direct Method F.D.	Direct Method C.D.	Curve Fitting Method F.D.	Curve Fitting Method C.D.
HOSTB5		4.282781	3.7490 −27.0500	1.927	2.824	2.92130	2.80523	2.95033	2.83311
HOSTB6		4.275031	3.7570 −27.0200	1.926	2.822	2.91905	2.80317	2.94807	2.83105
HOSTB7	4	4.283969	3.7500 −26.9700	1 926	2.823	2.92725	2.81168	2.95636	2.83964
HOSTB8		4.290359	3.7680 −26.9200	1.927	2.822	2.91835	2.80300	2.94748	2.83098
Engblom and Ochoa (1985)		—	2.9821 −27.7932						
Wen-Jinn and Sun (1987)		4.595000	—						
Pagano (1969)		4.327586	3.6207 −10.5517		2.700				
CPT (Pagano, 1969)		2.620690	3.1034		2.910				
HOSTB5		2.898570	19.7100 −173.0000	4.913	7.282	7.53603	7.23401	7.61097	7.30595
HOSTB6		2.892930	19.7100 −173.1000	4.913	7.283	7.53739	7.23525	7.61233	7.30719
HOSTB7	10	2.894700	19.6700 −173.1000	4.914	7.285	7.54476	7.24268	7.61979	7.31471
HOSTB8		2.896500	19.7300 −173.0000	4.915	7.284	7.53763	7.23565	7.61269	7.30771
Wen-Jinn and Sun (1987)		2.952000	—		—				
Pagano (1969)		2.956897	—		—				

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Table 7. Transverse normal stress ($\bar{\sigma}_z$) for simply supported laminate under sinusoidal loading (direct integration method) (a/h = 4) (0/90).

Thickness	HOSTB5	HOSTB6	HOSTB7	HOSTB8	Pagano (1969)	
					CPT	Elasticity
−0.5	0	0	0	0	0	0
−0.4	0.071610	0.071535	0.071495	0.071409	0.078900	0.078940
−0.3	0.243201	0.242947	0.242844	0.242583	0.263150	0.250000
−0.2	0.457761	0.457326	0.457239	0.456858	0.500000	0.460526
−0.1	0.658058	0.657531	0.657547	0.657142	0.723580	0.671050
0.0	0.784767	0.784317	0.784446	0.784081	0.847368	0.789474
0.1	0.850834	0.850548	0.850714	0.850403	0.921050	0.868421
0.2	0.908662	0.908504	0.908647	0.908413	0.952636	0.921053
0.3	0.955179	0.955112	0.955200	0.955065	0.978947	0.960526
0.4	0.986928	0.986916	0.986946	0.986905	0.989470	0.973684
0.5	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table 8. Transverse normal stress ($\bar{\sigma}_z$) for simply supported laminate under sinusoidal loading (exact curve fitting method) (a/h = 4) (0/90).

Thickness	HOSTB5	HOSTB6	HOSTB7	HOSTB8	Pagano (1969)	
					CPT	Elasticity
−0.5	0	0	0	0	0	0
−0.4	0.058569	0.058510	0.057701	0.058416	0.078900	0.078940
−0.3	0.222878	0.222645	0.221367	0.222309	0.263150	0.250000
−0.2	0.435837	0.435418	0.433886	0.434958	0.500000	0.460526
−0.1	0.640321	0.639798	0.638279	0.639400	0.723580	0.671050
0.0	0.777300	0.776833	0.775498	0.776589	0.847368	0.789474
0.1	0.844730	0.844429	0.843322	0.844280	0.921050	0.868421
0.2	0.903555	0.903385	0.902528	0.903292	0.952636	0.921053
0.3	0.951392	0.951318	0.950729	0.951268	0.978947	0.960526
0.4	0.984828	0.984811	0.984509	0.984798	0.989470	0.973684
0.5	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

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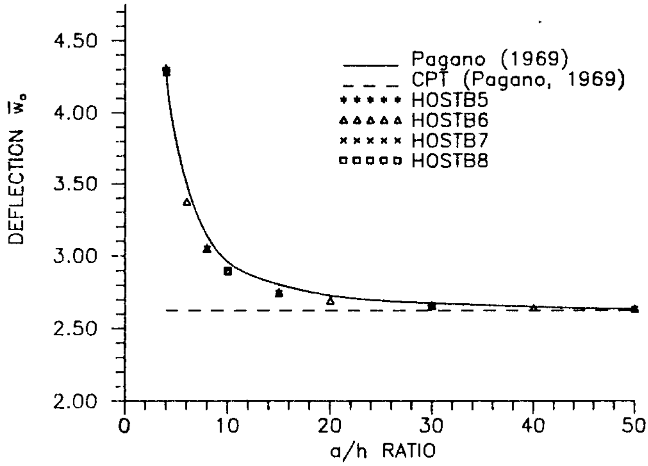

Figure 10. *Convergence of transverse displacement* $\overline{w}_o$ *with* a/h *ratio for simply supported laminate under sinusoidal loading (0/90).*

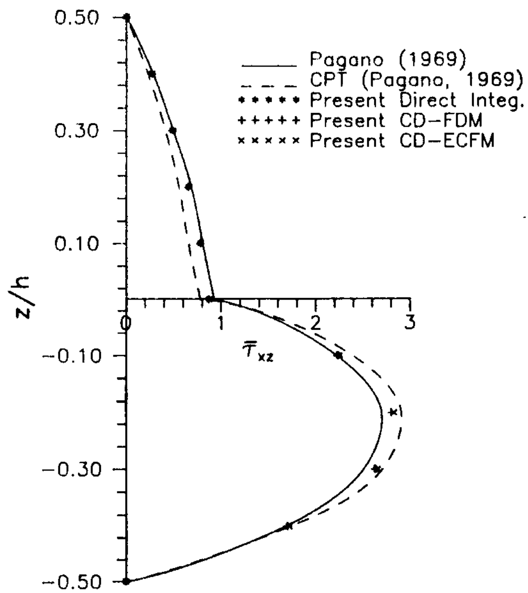

Figure 11. *Variation of transverse shear stress* ($\overline{\tau}_{xz}$) *through the thickness of a simply supported laminate under sinusoidal loading* (a/h = 4) (0/90) (HOSTB8).

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transverse shear and inplane stresses through the laminate thickness are shown in Figures 11 and 12, respectively for $a/h = 4$.

Figure 10 shows that the maximum transverse displacement $\bar{w}_o$ obtained by the present theory follows closely the elasticity solution (Pagano, 1969) and, in particular, the model HOSTB8 gives a better estimate (−0.860226 PD) compared to other models and the CPT. As seen in previous problems, the CPT underestimates the values here also and gives a very poor estimate for relatively low value of a/h (−39.4422 PD). The transverse shear stress results show that (Table 6) model HOSTB8 gives good estimates of this stress compared to other models. Thus, in Figure 11, the variation of transverse shear stress obtained by different methods for model HOSTB8 is compared with elasticity and CPT solutions. This shows that the results obtained by central difference direct finite difference method are better (3.81481 PD) compared to other methods and the CPT (7.7778 PD). The inplane stress variation shows that (Table 6 and Figure 12) model HOSTB5 (3.5435 PD) gives good estimates of this stress compared to other models and the CPT (− 14.2873 PD) in comparison with the elasticity solution. The transverse normal stress results (Tables 7 and 8) show that the present models give a very good estimate of this stress, when compared with the elasticity solution and all the methods give almost the similar variation through the thickness of the laminate.

A simply supported unsymmetric sandwich laminate under transverse loading is considered. The following material properties are used (Allen, 1969; Cairns and Lagace, 1987).

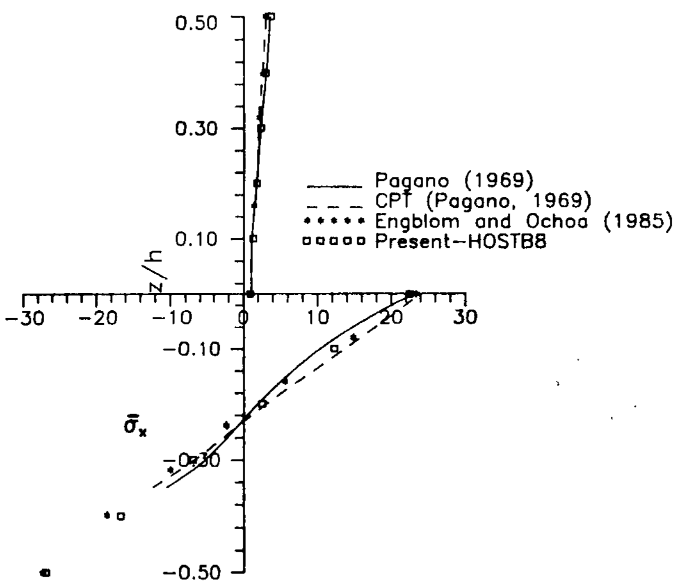

Figure 12. *Variation of inplane stress ($\bar{\sigma}_x$) through the thickness of a simply supported laminate under sinusoidal loading (a/h = 4) (0/90).*

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$$\left.\begin{aligned}
&\text{Stiff layers}\\
&\qquad E_L = 0.1308 \times 10^8 \text{ psi} \qquad E_T = 0.106 \times 10^7 \text{ psi}\\
&\qquad G_{LT} = 0.6 \times 10^6 \text{ psi} \qquad \nu_{LT} = 0.28\\
&\text{Core layers below middle plane}\\
&\qquad E = 2G \quad G_{LT} = 0.3 \times 10^5 \text{ psi} \quad \nu = 0\\
&\text{Core layers above middle plane}\\
&\qquad E = 2G \quad G_{LT} = 0.5 \times 10^4 \text{ psi} \quad \nu = 0 \quad h_c/h_{tf} = 8
\end{aligned}\right\} \tag{47}$$

The maximum transverse displacement $\overline{w}_o$, inplane and transverse stresses for different a/h ratios are presented in Tables 9 to 11 ($a/h = 4$, 10 and 25). The variations of inplane displacement and inplane stress through the thickness of the laminate are shown in Figures 13 and 14, respectively for $a/h = 4$. The variations of transverse shear and normal stresses through the laminate thickness for $a/h = 4$ are shown in Figures 15 and 16, respectively.

In this problem, the displacement and stresses obtained by various models are compared with the displacement model based on Timoshenko theory, as the results of elasticity and other closed-form solutions were not available. It can be observed from Tables 9 to 11 and Figures 13 to 16 that large variation in displacement (−73.005 PD), inplane (−56.193 PD) and transverse shear stresses (10.6047 PD) are seen between the present theory and the Timoshenko theory for thick laminates ($a/h = 4$). This is due to the simplifying assumption made in the latter theory. But it can be seen that, as the laminate thickness is reduced ($a/h \geq 25$ and above), all the theories almost give the same results (−20.808 PD), thus showing the validity of the present higher-order theory.

The variation of inplane displacement (Figure 13) shows the actual warping of the cross-section and it can be seen from the same figure that the Timoshenko theory gives an unrealistic straight line variation through the laminate thickness. The transverse normal stress results show that the exact curve fitting method gives a good estimate of this stress and the direct integration method gives a very high value and follows a different path. This error may be due to the use of third derivative of displacements in the evaluation of the transverse normal stress (Figure 16).

5. CONCLUSIONS

A set of simple but efficient and accurate higher-order theories with C° finite elements is presented with a view to provide accurate evaluation of inplane and transverse shear and normal stresses in composite and sandwich beams. These

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Table 9. Comparison of maximum transverse displacement, inplane stress and transverse shear stress ($\bar{\tau}_{xz}$) for simply supported unsymmetric sandwich laminate under transverse loading (a/h = 4, 10, 25).

Source	a/h Ratio	$\bar{w}_o$	$\bar{\sigma}_x$	Constitutive	Direct Integration	Direct Method		Curve Fitting Method	
						F.D.	C.D.	F.D.	C.D.
HOSTB5		3.699781	54.38	15.92	2.013	2.74441	2.01658	2.60653	1.96284
HOSTB6		3.698704	54.58	18.83	1.924	2.40613	1.89039	2.46402	1.92284
HOSTB7	4	3.694663	54.57	18.79	1.950	2.52304	1.99895	2.60967	2.05500
HOSTB8		3.705727	55.95	19.81	1.972	2.24452	1.92323	2.31299	1.97668
TIMO		1.000360	24.51	8.703	2.162	2.23031	2.11383	2.30677	2.18630
HOSTB5		0.830877	182.8	41.70	5.264	5.64897	5.14739	5.86514	5.30844
HOSTB6		0.829258	183.0	41.49	5.207	5.61807	5.12070	5.77577	5.23256
HOSTB7	10	0.830340	183.0	41.47	5.229	5.71408	5.21549	5.92640	5.38010
HOSTB8		0.830454	184.0	49.86	5.246	5.59802	5.12892	5.79514	5.29113
TIMO		0.377407	153.2	21.76	5.406	5.57578	5.28456	5.76694	5.46570
HOSTB5		0.350854	988.3	23.20	13.39	13.9123	13.1009	14.3949	13.5312
HOSTB6		0.349215	988.2	23.20	13.35	13.8962	13.0800	14.3409	13.4736
HOSTB7	25	0.350731	988.2	23.20	13.36	13.9909	13.1749	14.4764	13.6085
HOSTB8		0.350713	987.5	23.33	13.40	13.9284	13.1023	14.4183	13.5374
TIMO		0.277736	957.6	54.39	13.51	13.9394	13.2114	14.4173	13.6644

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Table 10. ***Transverse normal stress ($\bar{\sigma}_z$) for simply supported unsymmetric sandwich laminate under transverse loading (direct integration method) (a/h = 4).***

Thickness	HOSTB5	HOSTB6	HOSTB7	HOSTB8	TIMO
−0.5	0	0	0	0	0
−0.4	−0.022873	−0.051704	−0.054665	0.820535	−0.099300
−0.3	0.110706	0.080573	0.078362	0.962696	0.050006
−0.2	0.244495	0.213076	0.211624	1.104360	0.199650
−0.1	0.378380	0.345706	0.345021	1.245070	0.349519
0.0	0.512294	0.478413	0.478493	1.384550	0.499500
0.1	0.646202	0.611141	0.611990	1.523230	0.649497
0.2	0.780100	0.876582	0.878965	1.661680	0.799475
0.3	0.913976	1.009260	1.012410	1.799920	0.949416
0.4	1.047810	1.009260	1.012410	1.937990	1.099300
0.5	1.000000	1.000000	1.000000	1.000000	1.000000

Table 11. ***Transverse normal stress ($\bar{\sigma}_z$) for simply supported unsymmetric sandwich laminate under transverse loading (exact curve fitting method) (a/h = 4).***

Thickness	HOSTB5	HOSTB6	HOSTB7	HOSTB8	TIMO
−0.5	0	0	0	0	0
−0.4	0.048552	0.048424	0.048087	0.059356	0.045847
−0.3	0.158088	0.157819	0.157537	0.172059	0.155618
−0.2	0.268259	0.267873	0.267656	0.284460	0.266226
−0.1	0.378537	0.378062	0.377917	0.395947	0.377070
0.0	0.488848	0.488328	0.488259	0.506216	0.488038
0.1	0.599155	0.598623	0.598630	0.615587	0.599029
0.2	0.709452	0.708918	0.709000	0.724722	0.710002
0.3	0.819729	0.819201	0.819358	0.833645	0.820940
0.4	0.929970	0.929456	0.929684	0.942392	0.931824
0.5	1.000000	1.000000	1.000000	1.000000	1.000000

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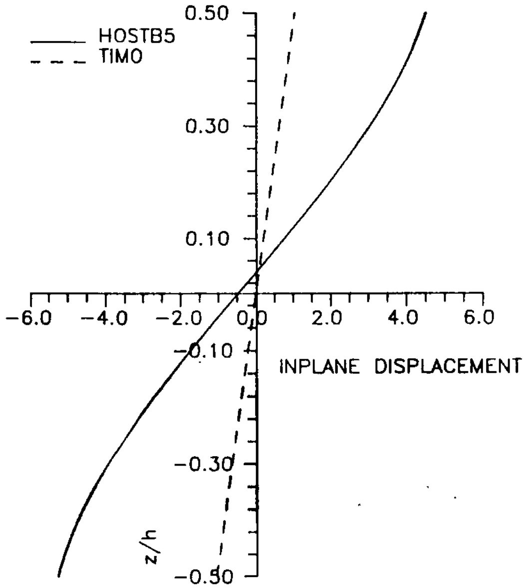

Figure 13. *Variation of inplane displacement through the thickness of a simply supported unsymmetric sandwich laminate under uniformly distributed loading (a/h = 4).*

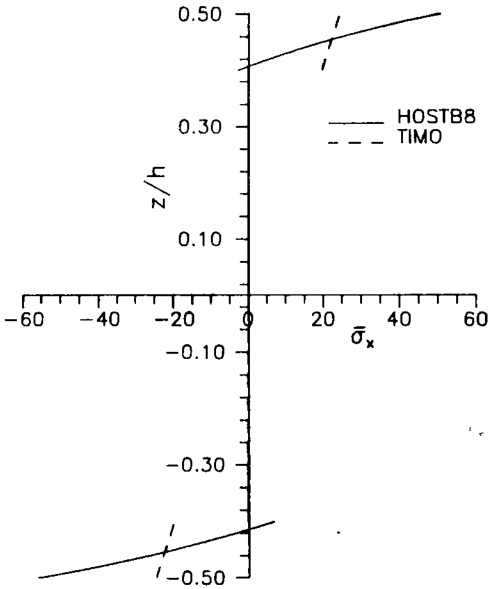

Figure 14. *Variation of inplane stress ($\bar{\sigma}_x$) through the thickness of a simply supported unsymmetric sandwich laminate under uniformly distributed loading (a/h = 4).*

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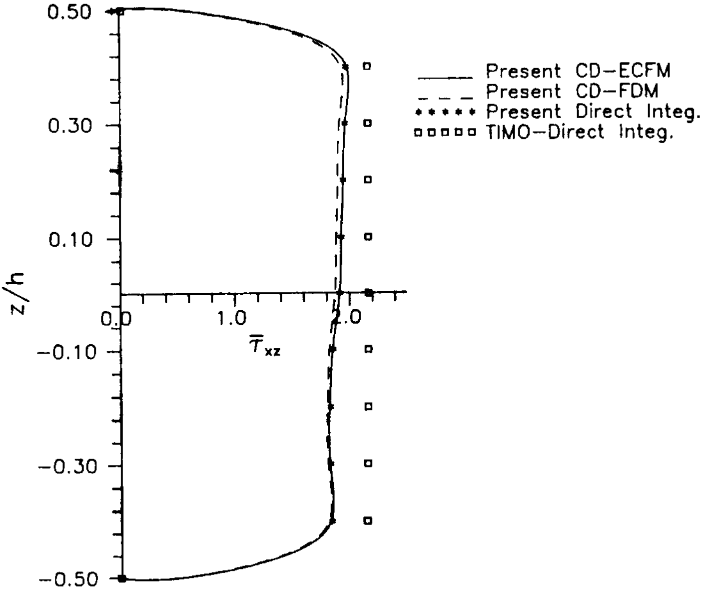

Figure 15. *Variation of transverse shear stress ($\bar{\tau}_{xz}$) through the thickness of a simply supported unsymmetrical sandwich laminate under uniformly distributed loading* (a/h = *4) (HOSTB8).*

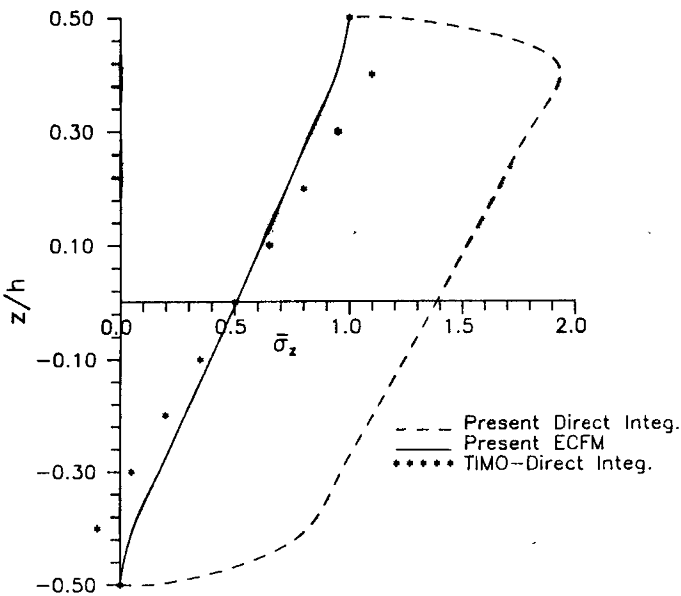

Figure 16. *Variation of transverse normal stress ($\bar{\sigma}_z$) through the thickness of a simply supported unsymmetrical sandwich laminate under uniformly distributed loading* (a/h = *4) (HOSTB8).*

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theories do not require the use of shear correction coefficient(s) which is/are generally used in the Timoshenko theory. The results obtained by the present theory show excellent agreement with the elasticity and other closed-form solutions for thick-to-thin beams. In the case of sandwich beams, large difference in results were obtained between present higher-order theory and Timoshenko theory which uses an arbitrary shear correction coefficient with a linear longitudinal displacement variation through the beam thickness. While here the discussion is limited to a particular type of loading and boundary conditions, these theories can be used to tackle any type of loading and boundary conditions. The numerical estimate of transverse normal stress, which is of paramount importance in the design of composite and sandwich laminate is presented and compared with the available elasticity solution. But this requires the use of higher-order numerical differentiation (third derivative) in the longitudinal direction associated with the integration of the elasticity equilibrium equations. The use of the proposed new methods with cubic C° elements seems to have given fairly accurate estimates of this stress.

The proposed forward difference exact curve fitting method can be efficiently employed for evaluating the transverse shear stresses, and central difference exact curve fitting method is recommended for evaluation of transverse normal stress, as the results obtained by these methods are close to the available elasticity and closed-form solutions when compared to direct integration and direct finite difference methods. The results of transverse normal stress are also presented for new problems where elasticity solutions are not available. These can be used for future reference.

The models HOSTB3 and HOSTB5 (with the inbuilt condition that the transverse normal strain in the z-direction is negligible) yield encouraging results. However, these models cannot be used for the accurate modelling of composite and sandwich laminates by one-dimensional beam model. This is because, one has to necessarily use only isotropic stress-strain constitutive relation [Equation (7)] with these models. We are not in a position to use two-dimensional constitutive relation in x-z plane and therefore the orthotropic properties with reference to x- and z-directions cannot be incorporated. But generally in composite and sandwich laminates, the material properties are different in different directions (orthotropic or anisotropic). Thus, to account for the variation in the directional properties, two-dimensional stress-strain constitutive relation must be used. This problem does not arise in the modelling of laminates by two-dimensional plate models.

From the numerical study, it can be seen that the results obtained by HOSTB4 are close to the results of elasticity solution for symmetric composite and sandwich beams compared to other models. Thus, HOSTB4 model is recommended for one-dimensional symmetric composite and sandwich laminates.

In case of unsymmetric composite and sandwich beams, the results of HOSTB8 match well with the elasticity and other closed-form solutions compared to other models, as the transverse displacement and its higher-order modes also play a paramount role in the modelling. Thus, this model should be used to tackle unsymmetric composite and sandwich beams.

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