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LARGE AMPLITUDE FREE VIBRATION ANALYSIS OF CROSS-PLY COMPOSITE AND SANDWICH LAMINATES WITH A REFINED THEORY AND C^0 FINITE ELEMENTS

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Abstract—A refined higher-order shear deformation theory has been developed for large amplitude, in the sense of von Kármán, free vibration analysis of fibre reinforced cross-ply composite and sandwich laminates by assuming cubic variations of in-plane displacement components and a constant transverse displacement component through the thickness of the laminate. The theory accounts for warping of the transverse cross-section, which cannot be modelled with the Reissner–Mindlin first-order shear deformation theory. The displacement-based finite element method of analysis using C^0 isoparametric nine-node quadrilateral elements of the Lagrangian family is adopted. A special mass matrix diagonalization scheme is employed which conserves the total mass of the element and includes the effects due to rotary inertia terms. The validity and efficiency of the present development is then established by obtaining the solutions to a wide range of problems and comparing them with the available two- and three-dimensional closed-form and finite element solutions. Some new results are also generated in a non-linear context for future comparisons.

INTRODUCTION

The advent of new stiff, strong and lightweight composite materials, consisting of high performance fibres, unified by advanced binders, has played a key role in the success of the aerospace and aircraft industries. And also the optimum specific stiffness and good damping characteristics along with design versatility, aerodynamically smooth surfaces and minimum fatigue resistance gives sandwich laminates a wide application in the aerospace industry. However, the analysis of composite and sandwich structures is more complex when compared to metallic structures, because composite structures are anisotropic and are characterized by bending–stretching coupling. Very often these structures are subjected to severe environmental conditions which necessitates the study of their vibration behaviour in the non-linear domain. This topic has attracted many researchers and a number of approximate methods have been developed.

A comprehensive study of large amplitude free vibrations of plates using approximate analytical and numerical methods (finite element) has been presented by Sathyamoorthy [1]. A wealth of information on the non-linear response of structures is available in a standard book by Chia [2]. Large-amplitude free vibration analysis of orthotropic plates was studied by Ambartsumyan [3] and Hassert and Nowinski [4]; Wu and Vinson [5, 6] evaluated the non-linear frequencies of orthotropic and symmetric laminates using Berger's [7] approach. Whitney and Leissa [8] recognized the effects of bending extension coupling is non-linear dynamic plate theory. Chandra [9] and Chandra and Raju [10] studied the large amplitude flexural vibrations of cross-ply plates. Their study is based on the two-term perturbation solution technique for non-symmetric laminates. Reddy and Chao [11, 12] presented the finite element solution for the large amplitude free vibration analysis of anisotropic composite plates by using the first-order shear deformation theory. All of these studies were based on either the classical (Kirchhoff) plate theory (CPT) or the first-order shear deformation (Mindlin/Reissner) theory (FOST). Owing to the low transverse shear modulus relative to the in-plane modulii, in the case of composite and sandwich laminates, especially in thick zones, a reliable prediction of the response characteristics of high modulus composite and sandwich laminates requires the use of higher-order shear deformable theories.

To the best of the authors' knowledge there is no published work on large amplitude free flexural vibration analysis of composite and sandwich laminates based on a higher-order shear deformtion theory. As an attempt to fill this gap, a third-order shear deformation theory including the non-linear effects in the sense of von Kármán is presented here for large amplitude free vibration analysis of cross-ply composite and sandwich laminates.

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THEORY

A composite laminate consisting of laminas with isotropic/orthotropic material properties oriented arbitrarily and having a total thickness of h (such that $h_1, h_2, h_3, \ldots$, etc. are thicknesses of individual layers making $h = h_1 + h_2 + \cdots$) is considered. The xy plane coincides with the middle plane of the laminate with the z axis oriented in the thickness direction such that x, y and z form a right-handed screw coordinate system. In the present theory, the displacement components of a generic point in the laminate are assumed to be of the form used earlier by Kant and Kommineni [13], and is given as follows:

$$
\begin{aligned}
u(x, y, z, t) &= u_0(x, y, t) + z\theta_x(x, y, t) \\
&\quad + z^2 u_0^*(x, y, t) + z^3 \theta_x^*(x, y, t) \\
v(x, y, z, t) &= v_0(x, y, t) + z\theta_y(x, y, t) \\
&\quad + z^2 v_0^*(x, y, t) + z^3 \theta_y^*(x, y, t) \\
w(x, y, z, t) &= w_0(x, y, t),
\end{aligned}
\tag{1}
$$

where t denotes the time, u_0, v_0 and w_0 are the components of mid-plane displacements of a generic point having displacements u, v and w in the x, y and z directions, respectively. The parameters θ_x and θ_y are rotations of the transverse normal cross-section in the xz and yz planes, respectively. The parameters u_0^*, v_0^*, θ_x^* and θ_y^* are the higher-order terms in the Taylor's series expansion and are also defined at the mid-plane. A total Lagrangian approach is adopted and the stress and strain descriptions used are those of Piola–Kirchhoff and Green (see [14]), respectively. In the present context, large displacements, in the sense of von Kármán, are considered here. Both isotropic and anisotropic situations can be accommodated with arbitrary thicknesses for different layers. By invoking von Kármán large deflection assumptions, which in particular imply that the first derivatives of u, v with respect to x, y and z are small, so that their particular products can be neglected (see [15]).

LAMINA STRAIN–DISPLACEMENT RELATIONS

The relationship between the strains at any point within the laminate and the corresponding deformations are functions of the assumed displacement field. The following are the Green–Lagrangian strain–displacement relations

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.
\end{aligned}
\tag{2}
$$

The in-plane non-linear strains are now linearized by assuming

$$
f_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right) \quad \text{and} \quad f_y = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)
$$

as follows [15]:

$$
\begin{aligned}
\epsilon_x^{nl} &= f_x\left(\frac{\partial w}{\partial x}\right) \\
\epsilon_y^{nl} &= f_y\left(\frac{\partial w}{\partial y}\right) \\
\gamma_{xy}^{nl} &= f_y\frac{\partial w}{\partial x} + f_x\frac{\partial w}{\partial y}.
\end{aligned}
\tag{3}
$$

By expressing the strains in terms of the mid-plane components and that in terms of mid-plane displacements, the following can be written

$$
\bar{\epsilon}^l = (\bar{\epsilon}_m^l, \bar{\epsilon}_b^l, \bar{\epsilon}_s^l)
$$

$$
\bar{\epsilon}_m = \begin{bmatrix}
\dfrac{\partial u_0}{\partial x} + f_x \cdot \left(\dfrac{\partial w_0}{\partial x}\right) \\
\dfrac{\partial v_0}{\partial y} + f_y\left(\dfrac{\partial w_0}{\partial y}\right) \\
\dfrac{\partial v_0}{\partial x} + \dfrac{\partial u_0}{\partial y} + f_y\dfrac{\partial w_0}{\partial x} + f_x\dfrac{\partial w_0}{\partial y} \\
\dfrac{\partial u_0^*}{\partial x} \\
\dfrac{\partial v_0^*}{\partial y} \\
\dfrac{\partial v_0^*}{\partial x} + \dfrac{\partial u_0^*}{\partial y}
\end{bmatrix}
$$

$$
\bar{\epsilon}_b = \begin{bmatrix}
\dfrac{\partial \theta_x}{\partial x} \\
\dfrac{\partial \theta_y}{\partial y} \\
\dfrac{\partial \theta_y}{\partial x} + \dfrac{\partial \theta_x}{\partial y} \\
\dfrac{\partial \theta_x^*}{\partial x} \\
\dfrac{\partial \theta_y^*}{\partial y} \\
\dfrac{\partial \theta_y^*}{\partial x} + \dfrac{\partial \theta_x^*}{\partial y}
\end{bmatrix}
$$

$$\bar{\epsilon}_s = \begin{bmatrix} \theta_x + \dfrac{\partial w_0}{\partial x} \\ \theta_y + \dfrac{\partial w_0}{\partial y} \\ 3\theta_x^* \\ 3\theta_y^* \\ 2u_0^* \\ 2v_0^* \end{bmatrix}. \tag{4}$$

LAMINATE CONSTITUTIVE RELATIONS

The membrane, flexure and shear stress-resultants for the differential element of the laminate will be expressed in terms of the mid-plane stretching, curvature and shear rotation terms, respectively. The resulting equations are referred to as the laminate constitutive relations. The strain expressions given by eqn (4) are substituted in the strain energy expression and an explicit integration through the laminate thickness is then carried out to obtain the following two-dimensional strain energy expression

$$U = \frac{1}{2}\int_A \bar{\epsilon}^t \bar{\sigma}\, dA. \tag{5}$$

The mid-plane strain vector $\bar{\epsilon}$ is defined in eqn (4) and the stress-resultant vector $\bar{\sigma}$ is defined as follows:

$$\bar{\sigma} = (N_x, N_y, N_{xy}, N_x^*, N_y^*, N_{xy}^*, M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*, Q_x, Q_y, Q_x^*, Q_y^*, S_x, S_y)^t \tag{6a}$$

in which

$$\begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} [1, z^2]\, dz$$

$$\begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} [z, z^3]\, dz$$

$$\begin{bmatrix} Q_x & Q_x^* & S_x \\ Q_y & Q_y^* & S_y \end{bmatrix} = \sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} [1, z^2, z]\, dz, \tag{6b}$$

where

$$\sigma^t = [\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}]$$

and

$$\epsilon^t = [\epsilon_x, \epsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]$$

are respectively vectors of stress and strain components with respect to laminate axes (see Fig. 1).

The stress strain relations of Lth lamina in the laminate coordinates (x, y, z) can be written as

$$\sigma = Q\epsilon \tag{6c}$$

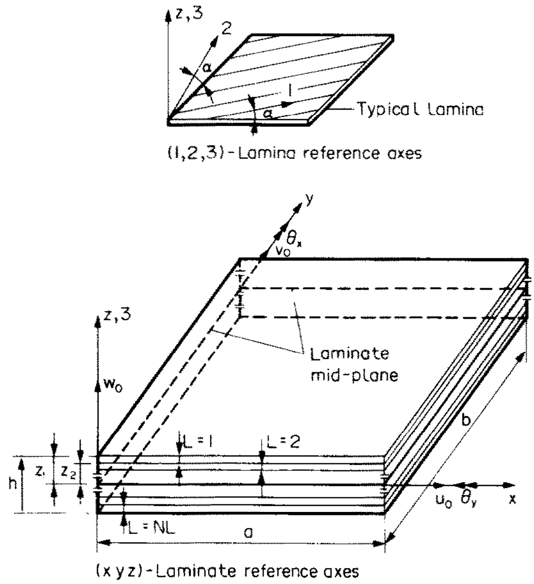

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

which in the expanded form appears as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ & Q_{22} & 0 & 0 & 0 \\ & & Q_{33} & 0 & 0 \\ & & & Q_{44} & 0 \\ \text{Symmetric} & & & & Q_{55} \end{bmatrix} \times \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{6d}$$

in which the non-zero coefficients Q_{ij} are defined as follows:

$$\begin{aligned} Q_{11} &= C_{11}c^4 + C_{22}s^4 \\ Q_{12} &= C_{12}(c^4 + s^4) \\ Q_{22} &= C_{11}s^4 + C_{22}c^4 \\ Q_{33} &= C_{33}(c^4 + s^4) \\ Q_{44} &= C_{44}c^2 + C_{55}s^2 \\ Q_{55} &= C_{44}s^2 + C_{55}c^2, \end{aligned} \tag{6e}$$

where $c = \cos\theta$, $s = \sin\theta$, θ is the angle between fibre direction of the lamina and the x axis of the laminate, and the coefficients c_{mn} are elements of the composite material stiffness matrix giving the stress–strain relations with respect to lamina axes (1, 2, 3) [16].

The laminate constitutive relations can now be obtained in compact form as

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_m & \mathbf{D}_c & \mathbf{0} \\ \mathbf{D}_c^t & \mathbf{D}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_s \end{bmatrix} \begin{bmatrix} \bar{\epsilon}_m \\ \bar{\epsilon}_b \\ \bar{\epsilon}_s \end{bmatrix} \tag{7a}$$

or symbolically

$$\bar{\sigma} = \mathbf{D}\bar{\epsilon}, \tag{7b}$$

where

$$\mathbf{N}^t = [N_x, N_y, N_{xy}, N_x^*, N_y^*, N_{xy}^*],$$

$$\mathbf{M}^t = [M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*],$$

$$\mathbf{Q}^t = [Q_x, Q_y, Q_x^*, Q_y^*, S_x, S_y],$$

$$\bar{\sigma}^t = [N_x, N_y, N_{xy}, N_x^*, N_y^*, N_{xy}^*, M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*, Q_x, Q_y, Q_x^*, Q_y^*, S_x, S_y]$$

and the stiffness coefficient matrices $\mathbf{D}_m$, $\mathbf{D}_c$, $\mathbf{D}_b$ and $\mathbf{D}_s$ corresponding to in-plane, coupling between in-plane and bending, bending and shear terms respectively and are defined as follows:

$$\mathbf{D}_m = \sum_{L=1}^{NL} \begin{bmatrix} \mathbf{Q}_{ij}H_1 & \mathbf{Q}_{ij}H_3 \\ \mathbf{Q}_{ij}H_3 & \mathbf{Q}_{ij}H_5 \end{bmatrix}$$

$$\mathbf{D}_c = \sum_{L=1}^{NL} \begin{bmatrix} \mathbf{Q}_{ij}H_2 & \mathbf{Q}_{ij}H_4 \\ \mathbf{Q}_{ij}H_4 & \mathbf{Q}_{ij}H_6 \end{bmatrix}$$

$$\mathbf{D}_b = \sum_{L=1}^{NL} \begin{bmatrix} \mathbf{Q}_{ij}H_3 & \mathbf{Q}_{ij}H_5 \\ \mathbf{Q}_{ij}H_5 & \mathbf{Q}_{ij}H_7 \end{bmatrix}$$

$$\mathbf{D}_s = \sum_{L=1}^{NL} \begin{bmatrix} \mathbf{Q}_{lm}H_1 & \mathbf{Q}_{lm}H_3 & \mathbf{Q}_{lm}H_2 \\ \mathbf{Q}_{lm}H_3 & \mathbf{Q}_{lm}H_5 & \mathbf{Q}_{lm}H_4 \\ \mathbf{Q}_{lm}H_2 & \mathbf{Q}_{lm}H_4 & \mathbf{Q}_{lm}H_3 \end{bmatrix}.$$

In the above relations $i, j = 1, 2, 3$ and $l, m = 4, 5$ and

$$H_k = \frac{1}{k}(z_{L+1}^k - z_L^k), \quad k = 1, 2, 3, 4, 5, 6, 7$$

and NL is the number of layers and $\bar{\epsilon} = (\bar{\epsilon}_m^t, \bar{\epsilon}_b^t, \bar{\epsilon}_s^t)^t$. These mid-surface strain components represent the mid-plane membrane, bending and shear strain components, respectively.

C^0 FINITE ELEMENT FORMULATION

The finite element used here is a nine-noded isoparametric quadrilateral element. The laminate displacement field in the element can be expresssed in terms of nodal variables, such that

$$\mathbf{d}(\xi, \eta) = \sum_{i=1}^{NN} N_i(\xi, \eta)\mathbf{d}_i, \tag{8}$$

where NN represents the number of nodes in the element, $N_i(\xi, \eta)$ defines the interpolation function associated with node i in terms of the normalized coordinates ξ, η, and $\mathbf{d}_i$ is generalized displacement vector of the mid-plane at node i, such that

$$\mathbf{d}_i^t = (u_{0i}, v_{0i}, w_{0i}, \theta_{xi}, \theta_{yi}, u_{0i}^*, v_{0i}^*, \theta_{xi}^*, \theta_{yi}^*). \tag{9}$$

As defined earlier, the in-plane, flexural and shear strain contributions are denoted by subscripts m, b and s, respectively and if the linear and non-linear contributions are denoted by superscripts l and nl, respectively, then the non-linear strain displacement relations can be written in partitioned form as [15]

$$\begin{bmatrix} \bar{\epsilon}_m \\ \bar{\epsilon}_b \\ \bar{\epsilon}_s \end{bmatrix} = \sum_{i=1}^{NN} \begin{bmatrix} \mathbf{B}_{mi}^l + \mathbf{B}_{mi}^{nl} \\ \mathbf{B}_{bi}^l \\ \mathbf{B}_{si}^l \end{bmatrix} \mathbf{d}_i. \tag{10}$$

The non-zero terms of the **B** matrices can be as follows. The linear membrane and flexure terms

$$B_{1,1} = B_{3,2} = B_{4,6} = B_{6,7} = B_{7,4} = B_{9,5} = B_{10,8} = B_{12,9} = \frac{\partial N_i}{\partial x}$$

$$B_{2,2} = B_{3,1} = B_{5,7} = B_{6,6} = B_{8,5} = B_{9,4} = B_{11,9} = B_{12,8} = \frac{\partial N_i}{\partial y}.$$

The non-linear (linearized) membrane terms

$$B_{1,3} = f_x \frac{\partial N_i}{\partial x}, \qquad B_{2,3} = f_y \frac{\partial N_i}{\partial y},$$

$$B_{3,3} = f_x \frac{\partial N_i}{\partial y} + f_y \frac{\partial N_i}{\partial x}.$$

The linear shear terms

$$B_{1,3} = \frac{\partial N_i}{\partial x}, \qquad B_{2,3} = \frac{\partial N_i}{\partial y},$$

$$B_{1,4} = B_{2,5} = N_i, \qquad B_{3,8} = B_{4,9} = 3N_i,$$

$$B_{5,6} = B_{6,7} = 2N_i. \tag{11}$$

SOLUTION TECHNIQUE

Hamilton's variational principle is employed here to derive the equations of motion. The functional of interest is

$$F = \int_{t_1}^{t_2} (E - \Pi)\,\mathrm{d}t, \tag{12}$$

where t is time, E is the total kinetic energy of the system and Π is the potential energy of the system.

including both strain energy V and potential energy W of the conservative external forces. Since the primary interest here is free vibration analysis, the potential energy due to applied loads is taken as zero.

Using $a_1, a_2, a_3, \ldots, a_r$ as the generalized displacements such that $\mathbf{a}^t = (\mathbf{d}_1^t, \mathbf{d}_2^t, \ldots, \mathbf{d}_{NN}^t)$ and assuming that they are independent, the Euler–Lagrange equations then yield the well-known Lagrange equations of motion as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial a_r}\right) - \frac{\partial F}{\partial a_r} = 0, \quad r = 1, 2, \ldots, R, \tag{13}$$

where R is the total number of degrees of freedom of the system.

When the space is discretized with the usual finite element method, the above Lagrangian equations of motion in matrix form can be written as

$$\mathbf{Ka} + \mathbf{M\ddot{a}} = 0, \tag{14}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global stiffness and mass matrices, respectively, obtained by the assembly of the corresponding element matrices, $\mathbf{a}$ is the nodal displacement vector and $\mathbf{\ddot{a}}$ is the second derivative of the displacements of the structure with respect to the time.

The above relation is the global discrete equation for free vibration. We now assume a solution for $\mathbf{a}$ of the form

$$\mathbf{a} = \mathbf{\bar{a}}\, \mathrm{e}^{i\omega t}, \tag{15}$$

where $\mathbf{\bar{a}}$ is the vector of unknown amplitudes at time $t = 0$ at the nodes (modal vector), and ω is the circular natural frequency of the system. When the eqn (15) is substituted into relation (14), one gets

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{\bar{a}} = 0 \quad \text{or} \quad (\mathbf{K} - \lambda\mathbf{M})\mathbf{\bar{a}} = 0. \tag{16}$$

A subspace iteration method [17] is used to obtain the eigenvalues λ and associated eigenvectors $\mathbf{\bar{a}}$.

The sequence of steps in the iterative procedure to evaluate the non-linear frequencies can be summarized as follows:

1. Assume f_x and f_y are equal to zero (i.e. the linear case).
2. Evaluate each element mass matrix and assemble into $\mathbf{M}$.
3. Evaluate each element stiffness matrix and assemble into $\mathbf{K}$.
4. Use the subspace iterative technique to find the lowest eigenvalue λ and associated eigenvector $\mathbf{\bar{a}}$.
5. By appropriately scaling the eigenvector $\mathbf{\bar{a}}$ ensure that the maximum displacement equals to the desired amplitude, say c.
6. Calculate f_x and f_y at all the element integration points.
7. If the solution has converged, i.e. the non-linear frequency obtained during two consecutive iterations differ by some small number (say 10^{-3}) stop; otherwise return to step 3.

NUMERICAL RESULTS

In the present study a nine-noded quadrilateral isoparametric element is employed. A convergence study was first undertaken with a view to assess the type and nature of discretization required for reliable converged results. It was seen that with a nine-node Lagrangian quadrilateral element, a 2×2 mesh (four elements) in a quarter laminate and a 4×4 mesh (16 elements) in a full laminate were sufficient to get converged solutions for all the geometrical configurations, boundary and loading conditions considered in this paper. Due to the biaxial symmetry of the problems discussed only one quadrant of the laminate is analysed with a 2×2 mesh except for angle-ply laminates which are analysed by considering full laminates with a 4×4 mesh. In all the numerical computations, the selective integration rule is employed. A 3×3 Gaussian rule is used to compute in-plane, coupling between in-plane and bending and bending deformations, while a 2×2 rule is used to evaluate the terms associated with transverse shear deformation. The element mass matrix is evaluated using a 3×3 Gauss quadrature rule. For numerical computations two programs, FOST and HOST with five and nine degrees of freedom per node, respectively, are developed. All the computations were carried out in a single precision on CDC Cyber 180/840 computer at Indian Institute of Technology, Bombay. The shear correction coefficient used in first order shear deformation theory is assumed as 5/6.

In order to test the accuracy and efficiency of developed algorithm, and to investigate effects of transverse shear deformations, the following material property sets were used in obtaining the numerical results.

Material set 1. The material properties are taken from Noor and Burton [19]

$$E_1/E_2 = \text{open}, \qquad G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.35E_2, \qquad E_2 = 1.0,$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0.3 \quad \text{and} \quad \rho = 1.$$

Material set 2. The material properties are taken from Reddy and Kuppusamy [20] and Ganapathi *et al.* [21]

$$E_1 = 25E_2, \qquad G_{12} = G_{31} = 0.5E_2,$$
$$G_{23} = 0.2E_2, \qquad E_2 = 1.0,$$
$$\rho = 1.0, \qquad \nu_{12} = \nu_{13} = \nu_{23} = 0.25.$$

Material set 3. The material properties are taken from Reddy and Kuppusamy [20]

$$E_1 = 40E_2, \quad G_{12} = G_{31} = 0.6E_2,$$
$$G_{23} = 0.2E_2, \quad E_2 = 1.0,$$
$$\rho = 1.0, \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25.$$

Material set 4. The material properties are taken from Reddy and Chao [12] and Chia and Prabhakara [18]

$$E_1 = 40E_2, \quad G_{12} = G_{31} = G_{23} = 0.5E_2,$$
$$E_2 = 1.0, \quad \rho = 1.0,$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25.$$

Material set 5. The material properties are taken from Ganapathi *et al.* [21] and Reddy and Chao [11]: same as material set 4 except $G_{23} = 0.5E_2$.

Material set 6. The material properties are taken from Reddy and Chao [12]

$$E_1 = 3E_2, \quad G_{12} = G_{31} = G_{23} = 0.5E_2,$$
$$\rho = 1.0, \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25.$$

Material set 7. The material properties are taken from Chandra and Raju [10]

$$E_1 = 7.07 \times 10^6 \text{ psi}, \quad E_2 = 3.58 \times 10^6 \text{ psi},$$
$$G_{12} = G_{23} = G_{13} = 1.41 \times 10^6 \text{ psi},$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25 \quad \text{and} \quad \rho = 1.0.$$

Material set 8. The material properties are taken from Mallikarjuna and Kant [22]. Face sheets (graphite/epoxy prepreg system)

$$E_1 = 1.308 \times 10^7 \text{ N/cm}^2, \quad E_2 = 1.06 \times 10^6 \text{ N/cm}^2,$$
$$G_{12} = G_{13} = 6 \times 10^5 \text{ N/cm}^2, \quad G_{23} = 3.9 \times 10^5 \text{ N/cm}^2,$$
$$\rho = 1.58 \times 10^{-5} \text{ N s}^2/\text{cm}^4, \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.28.$$

Thickness of each stiff layer $= 0.05h$; core (U.S. commercial aluminium honeycomb, 1/4 in cell size, 0.003 in foil). Thickness of core $= 0.8h$.

Core 1

$$G_{23} = 1.772 \times 10^4 \text{ N/cm}^2,$$
$$G_{13} = 5.206 \times 10^4 \text{ N/cm}^2,$$
$$\rho = 1.009 \times 10^{-6} \text{ N s}^2/\text{cm}^4.$$

Core 2

$$G_{23} = 1.772 \times 10^3 \text{ N/cm}^2,$$
$$G_{13} = 5.206 \times 10^3 \text{ N/cm}^2,$$
$$\rho = 1.009 \times 10^{-6} \text{ N s}^2/\text{cm}^4.$$

Core 3

$$G_{23} = 1.772 \times 10^2 \text{ N/cm}^2,$$
$$G_{13} = 5.206 \times 10^2 \text{ N/cm}^2,$$
$$\rho = 1.009 \times 10^{-6} \text{ N s}^2/\text{cm}^4.$$

The finite element displacement formulation developed in this paper is based entirely on assumed displacement functions and thus, only displacement boundary conditions are required to be specified. The boundary conditions corresponding to the present higher-order formulation are specified in Table 1 for different types of supports used in the present investigation.

The corresponding boundary conditions for the FOST are simply obtained by omitting the higher-order displacement quantities marked with an asterisk. For example, there are nine displacement quantities required to be specified at $x = 0, a$ for a C1 type of boundary condition in this higher-order formulation (HOST), whereas in first-order formulation (FOST) the corresponding boundary conditions shall be only five. The boundary condition

Table 1. Boundary conditions

Type	$x = 0, x = a$		$x = a/2$		$y = 0, y = b$		$y = b/2$	
S1	$v_0 = 0$ $\theta_y = 0$ $w_0 = 0$	$v_0^* = 0$ $\theta_y^* = 0$	$u_0 = 0$ $\theta_x = 0$	$u_0^* = 0$ $\theta_x^* = 0$	$u_0 = 0$ $\theta_x = 0$ $w_0 = 0$	$u_0^* = 0$ $\theta_x^* = 0$	$v_0 = 0$ $\theta_y = 0$	$v_0^* = 0$ $\theta_y^* = 0$
S2	$u_0 = 0$ $v_0 = 0$ $\theta_y = 0$ $w_0 = 0$	$u_0^* = 0$ $v_0^* = 0$ $\theta_y^* = 0$	$u_0 = 0$ $\theta_x = 0$	$u_0^* = 0$ $\theta_x^* = 0$	$u_0 = 0$ $v_0 = 0$ $\theta_x = 0$ $w_0 = 0$	$u_0^* = 0$ $v_0^* = 0$ $\theta_x^* = 0$	$v_0 = 0$ $\theta_y = 0$	$v_0^* = 0$ $\theta_y^* = 0$
S3	$\theta_y = 0$ $w_0 = 0$	$\theta_y^* = 0$	$u_0 = 0$ $\theta_x = 0$	$u_0^* = 0$ $\theta_x^* = 0$	$\theta_x = 0$ $w_0 = 0$	$\theta_x^* = 0$	$v_0 = 0$ $\theta_y = 0$	$v_0^* = 0$ $\theta_y^* = 0$
C1	$u_0 = 0$ $v_0 = 0$ $\theta_x = 0$ $\theta_y = 0$ $w_0 = 0$	$u_0^* = 0$ $v_0^* = 0$ $\theta_x^* = 0$ $\theta_y^* = 0$	$u_0 = 0$ $\theta_x = 0$	$u_0^* = 0$ $\theta_x^* = 0$	$u_0 = 0$ $v_0 = 0$ $\theta_x = 0$ $\theta_y = 0$ $w_0 = 0$	$u_0^* = 0$ $v_0^* = 0$ $\theta_x^* = 0$ $\theta_y^* = 0$	$v_0 = 0$ $\theta_y = 0$	$v_0^* = 0$ $\theta_y^* = 0$
C2	$\theta_x = 0$ $\theta_y = 0$ $w_0 = 0$	$\theta_x^* = 0$ $\theta_y^* = 0$	$u_0 = 0$ $\theta_x = 0$	$u_0^* = 0$ $\theta_x^* = 0$	$\theta_x = 0$ $\theta_y = 0$ $w_0 = 0$	$\theta_x^* = 0$ $\theta_y^* = 0$	$v_0 = 0$ $\theta_y = 0$	$v_0^* = 0$ $\theta_y^* = 0$

types S1, S2, S3 and C2 have been specially chosen in order to compare our results with those of other authors. Incidentally the S1-type condition corresponds to the usual diaphragm type of simple support. The edge conditions, which have been derived in a variationally consistent manner in the present higher-order theory may not appear so (except in the case of fully clamped edge specified by C1), because, in any way, the natural boundary conditions cannot be prescribed in the displacement-based finite element method.

The results are grouped into two categories, namely (1) linear analysis and (2) non-linear analysis.

1. *Linear analysis*

Since there are no exact solutions available in the non-linear context, the superiority of the present higher-order displacement model is proved by comparing the present FOST and HOST results with the available exact three-dimensional elasticity solutions in the linear context.

Example 1. *Skew-symmetric and symmetric cross-ply laminates*. A square, simply supported (S1) cross-ply laminate of $a/h = 5$ having skew-symmetric and symmetric laminations with respect to the mid-plane is considered. The fibre orientations of the different laminae alternate between 0° and 90° with respect to the x axis. The material characteristics of the individual layers are as the material set 1.

Two parameters were varied, namely the degree of orthotropy of the individual layers (E_1/E_2) between 3 and 40 and the number of layers between 2 and 10. In Table 2, the fundamental frequencies obtained by the present FOST and HOST are compared with the three-dimensional elasticity results given by Noor and Burton [19]. Noor and Burton [19] obtained the exact three-dimensional elasticity solutions by using the method presented by Srinivas and Rao [23].

In the whole range the present results are in excellent agreement with the elasticity results; the present higher-order results, in particular, are very close to the three-dimensional elasticity results. Thus, it proves the effectiveness of the present higher-order shear deformation theory over first-order shear deformation theory.

The non-dimensional frequency parameter adopted is as follows:

$$\bar{\omega} = \omega h \sqrt{\rho / E_2}. \tag{17}$$

Example 2. *A* 10-*layer skew-symmetric cross-ply laminate*. A simply supported (S1) 10-layer cross-ply (0°/90°/0°/⋯/90°) laminate with $E_1/E_2 = 15$ and the material properties as material set 1 is considered. This example is considered here to find the effect of thickness ratio on the accuracy of the vibration frequency obtained by both the methods, i.e. FOST and HOST, by comparing the present results with the three-dimensional elasticity solutions presented by Noor and Burton [19]. These are given in Table 3.

As expected, the gross response characteristics (vibration frequency) predicted by the first-order shear deformation theory, are reasonably accurate

Table 2. Comparison of the present results with the exact three-dimensional elasticity solutions given by Noor and Burton for a simply supported (S1) cross-ply (0°/90°/0°/⋯) laminate of skew-symmetric and symmetric laminations with $a/h = 5$ and $\bar{\omega} = \omega h\sqrt{\rho/E_2}$

			E_1/E_2				
Lamination	Source	NL	3	10	15	30	40
Skew-symmetric	†		0.2392	0.2671	0.2815	0.3117	0.3256
	FOST	2	0.2379	0.2653	0.2806	0.3165	0.3348
	HOST		0.2388	0.2675	0.2809	0.3117	0.3236
	†		0.2493	0.3063	0.3307	0.3726	0.3887
	FOST	4	0.2496	0.3133	0.3427	0.3960	0.4172
	HOST		0.2495	0.3002	0.3306	0.3725	0.3899
	†		0.2517	0.3164	0.3441	0.3914	0.4092
	FOST	6	0.2516	0.3204	0.3509	0.4046	0.4253
	HOST		0.2517	0.3171	0.3442	0.3918	0.4100
	†		0.2530	0.3220	0.3518	0.4027	0.4220
	FOST	10	0.2527	0.3238	0.3548	0.4086	0.4290
	HOST		0.2531	0.3224	0.3519	0.4028	0.4220
Symmetric	†		0.2529	0.3195	0.3470	0.3931	0.4102
	FOST	5	0.2528	0.3202	0.3481	0.3971	0.4163
	HOST		0.2528	0.3201	0.3470	0.3935	0.4121
	†		0.2533	0.3222	0.3514	0.4005	0.4190
	FOST	7	0.2530	0.3227	0.3525	0.4040	0.4239
	HOST		0.2534	0.3224	0.3520	0.4004	0.4204
	†		0.2535	0.3234	0.3533	0.4040	0.4231
	FOST	9	0.2531	0.3238	0.3542	0.4067	0.4268
	HOST		0.2536	0.3248	0.3535	0.4047	0.4237

† The corresponding values are the exact three-dimensional elasticity results given by Noor and Burton [19].
NL = number of layers.

Table 3. Effect of the thickness ratio on the accuracy of lowest vibration frequency of simply supported (S1) 10-layer cross-ply ($0°/90°/0°/\cdots$) composite laminate [non-dimensional fundamental frequency $\bar{\omega} = 10^4\omega^2(\rho h^2/E_2)$]

a/h	†	FOST	Percentage difference	HOST	Percentage difference
100	0.0148	0.01477	0.0680	0.01477	0.0680
20	8.770	8.7889	0.2155	8.78078	0.1229
10	120.300	121.1703	0.7235	120.91444	0.5107
5	1237.0	1259.1750	1.7926	1241.21	0.3233
10/3	3966	4053.275	2.2006	4005.22	0.9889
10/4	8332	8509.7673	2.1335	8471.127	1.6698

† The corresponding values are the exact three-dimensional results given by Noor and Burton [19].

for thin and medium thick composite plates with $a/h \geqslant 10$. As the thickness ratio reduces the error in the predictions of the first-order shear deformation theory increases. Even though the results given by all the theories are almost same for thin laminates, for the medium thick and thick composite laminates, i.e. $a/h < 10$, the present HOST results are very close to the exact elasticity results.

The non-dimensional frequency parameter used is

$$\bar{\omega} = 10^4\omega^2\left(\frac{\rho h^2}{E_2}\right). \tag{18}$$

Example 3. *Symmetric laminate.* A simply supported (S1) cross-ply (0°/90°/0°) laminate with $h_1 = h_3 = h/4$ and $h_2 = h/2$ and the material properties as material sets 2 and 3 is considered. The non-dimensional fundamental frequencies for square and rectangular laminates with different material properties are computed and the results are compared with the three-dimensional finite element results given by Reddy and Kuppusamy [20]. These are given in Table 4. The non-dimensional quantity used for representing the frequency is

$$\bar{\omega} = (\omega a^2/h)\sqrt{\rho/E_2}. \tag{19}$$

The present HOST results are in excellent agreement with the three-dimensional finite element results, whereas the difference in results predicted by FOST with respect to both three-dimensional FEM and HOST increases as a/h ratio reduces.

Hence, wherever shear deformation effects are predominant the results predicted by present HOST are more reliable than FOST.

2. *Non-linear analysis*

To the best of the authors' knowledge only approximate solutions exist for predicting the non-linear vibration responses of thin and moderately thick laminates where the shear deformation effects are not pronounced. The present results in the following examples establish the validity of the HOST.

Example 4. *Symmetric laminate.* A simply supported (S2) cross-ply (0°/90°/90°/0°) laminate with $a/h = 10$ and 1000 and the material properties as material sets 2 and 4 is considered. The present results for the ratio of non-linear to linear frequency against the amplitude ratio (w_0/h) are compared with Ganapathi *et al.* [21] and are presented in Table 5. The results show that there generally is a good agreement with the existing results.

Example 5. *Unsymmetric laminates.* A simply supported (S3) cross-ply (0°/90°) laminate of $a/h = 1000$ with the material properties as material sets 4 and 6 is considered. The present results are compared with the finite element results based on first-order shear deformation theory presented by Reddy and Chao [12] and are plotted in Fig. 2(a). There is good agreement between various results as expected for a very thin laminate with $a/h = 1000$.

Table 4. Non-dimensional fundamental frequency $\bar{\omega} = (\omega a^2/h)\sqrt{\rho/E_2}$, of a simply supported (S1) cross-ply (0°/90°/0°) laminate with $h_1 = h_3 = h/4$; $h_2 = h/2$

		Material 2			Material 3		
b/a	a/h	3D FEM	FOST	HOST	3D FEM	FOST	HOST
1	5	8.317	8.7620 (5.350)	8.2862 (0.370)	9.1190	9.8271 (7.765)	9.1449 (0.284)
1	10	11.805	12.2236 (3.546)	11.7345 (0.597)	13.3700	14.2841 (6.837)	13.4323 (0.466)
1	100	15.473	15.1837 (1.869)	15.4714 (0.010)	18.9590	18.8162 (0.753)	18.8880 (0.375)
3	10	—	11.1317	10.7080	11.4380	12.4205 (8.590)	11.4828 (0.392)

The values in the parantheses are percentage difference with respect to the three-dimensional FEM results given by Reddy and Kuppusamy [20].

Table 5. Frequency ratio ω_{NL}/ω_L of non-linear vibration of a simply supported (S2) cross-ply (0°/90°/90°/0°) laminate

		Material 3			Material 5		
a/h	w_0/h	FEM†	FOST	HOST	FEM†	FOST	HOST
	0.2	1.04125	1.02844	1.02843	1.04108	1.02808	1.028084
	0.4	1.15093	1.14007	1.14575	1.15029	1.13437	1.134369
1000	0.6	1.31825	1.29166	1.29166	1.31653	1.28324	1.283241
	0.8	1.51495	1.48372	1.48372	1.51394	1.47889	1.478904
	1.0	1.73820	1.70091	1.70091	1.73650	1.68399	1.683998
	0.2	1.06453	1.04903	1.04843	1.06016	1.04840	1.040821
	0.4	1.22915	1.21572	1.21575	1.21973	1.20933	1.200577
10	0.6	1.44215	1.42991	1.42617	1.43125	1.41436	1.404013
	0.8	1.66125	1.63855	1.63372	1.65078	1.61057	1.601058
	1.0	1.85671	1.83342	1.82091	1.85126	1.80214	1.814492

† First-order shear deformation theory result given by Ganapathi *et al.* [21].

A square four-layer cross-ply (0°/90°/0°/90°) laminate of $a/h = 1000$ with the material properties as material set 4 is considered next. The present results are compared with the closed-form plate solution results given by Chia and Prabhakara [18] by using the Fourier series solution technique. The boundary conditions specified are simply supported (S3) and loosely clamped (C2). The results are presented in Fig. 2(b). It is observed that the frequency ratio is more for a simply supported laminate, however, the magnitude of frequency is higher with clamped boundaries. The present results are slightly higher than that given by Chia and Prabhakara [18].

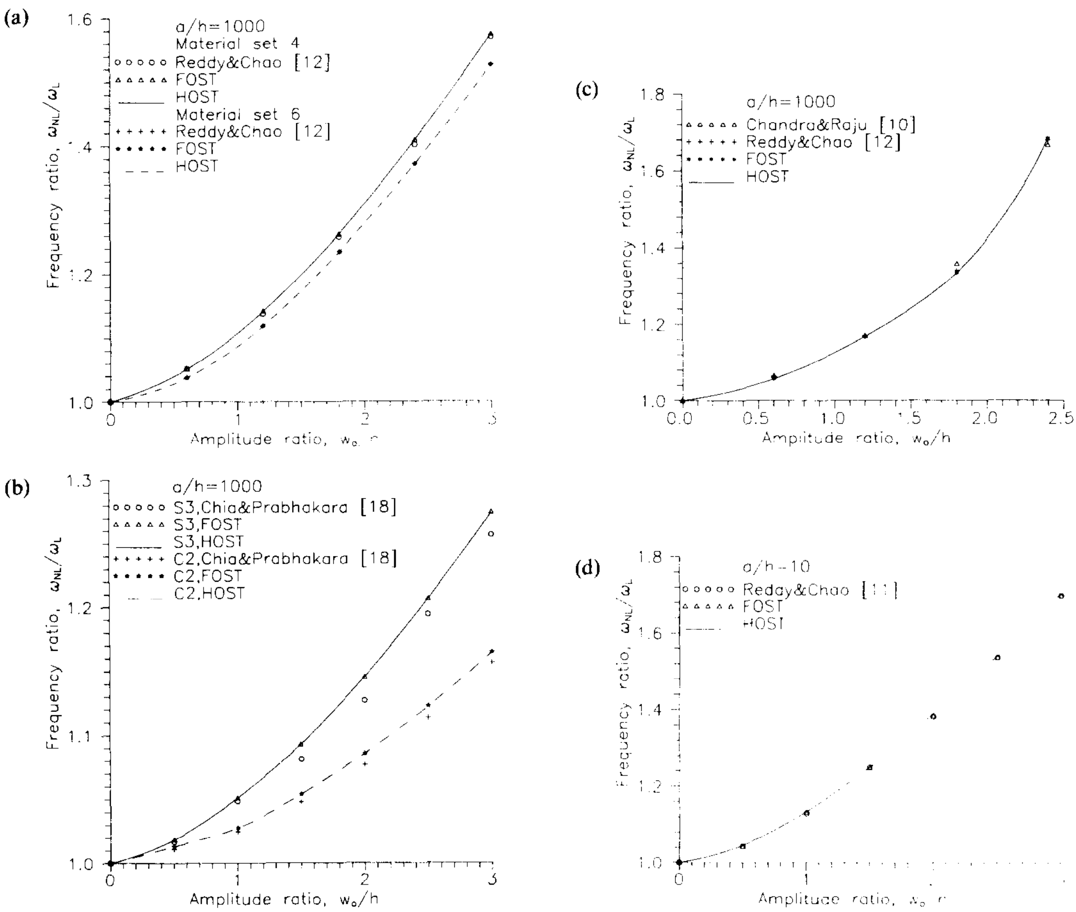

Fig. 2. (a) Ratio of non-linear to linear frequency vs amplitude ratio for a simply supported (S3) cross-ply (0°/90°) laminate. (b) Ratio of non-linear to linear frequency vs amplitude ratio for a square cross-ply (0°/90°/0°/90°) laminate. (c) Ratio of non-linear to linear frequency vs amplitude ratio for a square simply supported (S1) cross-ply (0°/90°) laminate. (d) Ratio of non-linear to linear frequency vs amplitude ratio for a square simply supported (S1) cross-ply (0°/90°) laminate.

Table 6. Non-linear to linear frequency ratio against a/h ratio of a simply supported (S1) cross-ply (0°/90°/core/90°/0°) sandwich laminate for an amplitude ratio of 1

	ω_{NL}/ω_L	
a/h	FOST	HOST
5	1.067305	1.12301
10	1.047534	1.06607
100	1.040280	1.04364
1000	1.040145	1.04018

A simply supported (S1) cross-ply (0°/90°) laminate of $a/h = 1000$ with the material properties as material set 7 is further considered. The present results are compared with the finite element result of Reddy and Chao [12] and the Galerkin perturbation solution presented by Chandra and Raju [10] and these are plotted in Fig. 2(c). The results exactly match the existing results.

A square cross-ply (0°/90°) laminate of $a/h = 10$ with the material properties as material set 5 is considered. The results are compared with Reddy and Chao [11] and are plotted in Fig. 2(d). Good agreement between the results is observed.

These comparisons establish the validity of the present models. However, the limitation of this comparison is that the laminates considered are geometrically thin with negligible shear deformation effects. This comparison has certainly proved the validity of the present formulation in the non-linear context.

Example 6. *Sandwich laminates.* A symmetric cross-ply (0°/90°/core/90°/0°) sandwich laminate with the thickness of each facing as $0.05h$ and that of the core as $0.8h$ is considered. The material properties set 8 is used and the boundary conditions are simply supported (S1).

The present HOST results are compared with the FOST results. The results presented in Table 6 show that there is a considerable difference in the results predicted by the two theories at low a/h ratios, whereas at high a/h ratios there is no significant difference in the results. This was expected due to significant shear deformation effects in the case of thick laminates.

Table 7 shows the comparison of the results predicted by FOST and HOST with different core properties. As core properties change there is not much change in results predicted by FOST, whereas there is a considerable difference in the predictions of HOST. This is because of the assumptions made in FOST, i.e. the transverse shear strains are constant through the thickness of the laminate. Thus, the

Table 7. Non-linear frequency ratio for a simply supported (S1) cross-ply (0°/90°/core/90°/0°) sandwich laminate with different core material properties for an amplitude ratio of 1

	ω_{NL}/ω_L	
Material	FOST	HOST
Core 1	1.067305	1.12301
Core 2	1.076148	1.42248
Core 3	1.077459	1.67281

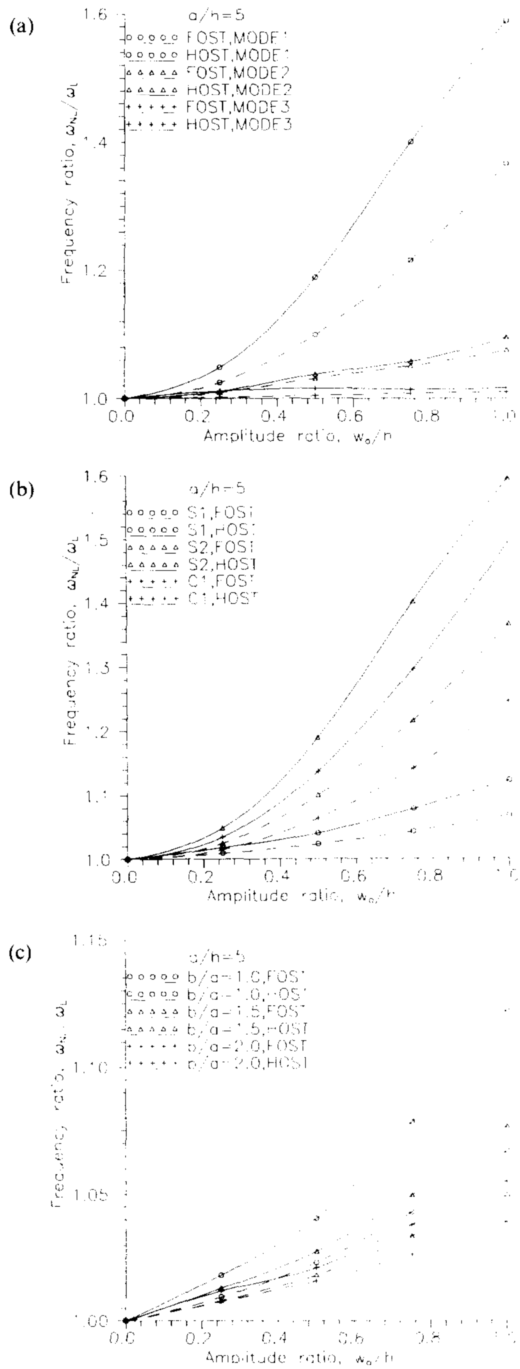

Fig. 3. (a) Non-dimensional frequency vs amplitude ratio for a simply supported (S2) cross-ply (0°/90°/core/90°/0°) sandwich laminate for different modes. (b) Effect of support conditions on non-linear frequency of square cross-ply (0°/90°/core/90°/0°) sandwich laminate. (c) Effect of aspect ratio on non-linear frequency of simply supported (S1) cross-ply (0°/90°/core/90°/0°) rectangular sandwich laminate.

predictions for thick composite and moderately thick to thick sandwich laminates, where the shear deformation effects are pronounced, the present HOST theory predictions are more reliable.

Figure 3(a) shows the variation of non-linear to linear frequency ratio with amplitude ratio of vibration for different modes of simply supported (S2) sandwich laminate. It is observed that the frequency ratio always remains higher than 1.0 and increases with the amplitude ratio. However, this increase is less pronounced for higher modes at any amplitude ratio. Similarly the difference between FOST and HOST predictions is much more pronounced in first mode than higher modes. It is interesting to note that the non-linear frequency is more than 1.6 times the linear frequency ratio at an amplitude ratio of 1.0, indicating that the behaviour of a sandwich laminate is highly non-linear as compared to composite laminates. Further the FOST predicts a frequency roughly 15% lower than that of HOST values at the amplitude ratio of 1.0.

Figure 3(b) shows that the effect of boundary conditions on the variation of non-linear to linear frequency ratio with respect to amplitude ratio. It is observed that the in-plane displacement normal to the boundary has a significant effect. Further the laminate with a simple support (S2 preventing the in-plane displacements) would yield a higher frequency ratio than clamped (C1) and simply supported (S1 diaphragm type) boundary conditions. However, there is not much variation in the prediction of linear frequencies with S1 and S2 boundaries. The frequency ratios are in between the two simple support boundary conditions for clamped boundaries.

Figure 3(c) shows that the variation of non-linear to linear frequency with respect to the amplitude ratio for different aspect ratios. It is observed that the frequency ratio is higher in square laminates than in rectangular laminates.

CONCLUSIONS

The geometric non-linear free vibration analysis of square/rectangular cross-ply composite and sandwich laminates, carried out with a refined theory and C^0 finite elements, is reported in this paper. A parametric study was carried out by varying aspect ratio, amplitude ratio and core stiffness for different boundary conditions. Results for higher modes are also included. It is seen that the behaviour of sandwich laminates is highly non-linear compared to the composite laminates. The results using the present theory show considerable warping of the cross-section for composite sandwich laminates. The usefulness of the higher-order shear deformation theory in the non-linear context is established.

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