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FREE VIBRATION ANALYSIS OF FIBER REINFORCED COMPOSITE BEAMS USING HIGHER ORDER THEORIES AND FINITE ELEMENT MODELLING

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Three higher order refined displacement models are proposed for the free vibration analysis of sandwich and composite beam fabrications. These theories model the warping of the cross-section by taking the cubic variation of axial strain and they eliminate the need for a shear correction coefficient by assuming a quadratic shear strain variation across the depth of the cross-section. Numerical experiments for various lamination schemes, boundary conditions and aspect ratios are carried out to compare these models with the first order shear deformation theory, earlier investigations and also among themselves to ascertain the most efficient one. Numerical results for deep sandwich and composite beams are also presented for future references.

1. INTRODUCTION

Sandwich and composite constructions are much in use for the design and development of nuclear structures, robots and aerospace applications, to name a few, due to their outstanding characteristics like high resistance to fatigue failure induced by acoustic pressures [1–4], high strength and stiffness for the given weight etc., in spite of the presence of complexities like warping of the entire cross-section including that of the core, predominant manifestation of transverse shear effects (due to the high ratio of Young's and shear modulus of the facings and core), delamination at the free edges created by the interlaminar normal stresses. Hence, any theory employed to analyze such fabrications, has to have the capability to address these problems, in order to realistically predict their behaviour.

The classical Euler-Bernoulli theory [5] neglects the transverse shear deformation completely, which restricts its applications to thin sections only. The first order shear deformation theory of Timoshenko [6] assumes a constant shear strain across the cross-section, necessitating a problem dependent shear correction factor. This shear correction factor had been derived for various types of cross-sections [7–9]. A refined first order theory was proposed by Cowper [10] where the average of transverse displacements of all points of cross-section is considered as w and the ratio of the first moment of

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transverse displacement of all points of the cross-section and the second moment of inertia of the cross-section is taken to be the face rotation. Sandwich beams were analyzed by using the Kirchhoff-Love theory and C^1 continuous elements for ascertaining the flexural vibrations by Ahmed [11]. His subsequent work [12] included the effects of transverse shear deformation, for the free vibration analysis of sandwich beams. Natural frequencies of fibre reinforced beams were analyzed by Abarcar and Cunniff [13] incorporating the secondary effects. Khatua and Cheung [14] reported a finite element analysis of multilayer sandwich beams and plates using cubic and linear polynomials for the transverse and axial displacements respectively of the stiff layers and taking the shear deformation of the core into consideration. Based on the first order shear deformation theory, the free vibration analysis of fiber reinforced composite beams were carried out with the secondary effects [15]. Teh and Huang [16] studied the vibration of orthotropic beams using Timoshenko's theory and finite element approximations. Chen and Yang [17] obtained the natural frequencies of symmetric laminated beam using finite element approach. Chandhrasekara *et al*. [18] analyzed the composite beam for its natural frequencies due its rotary inertia and shear deformation. Later on, the study was extended [19] to the vibration analysis of symmetric laminated composite beams, with a mass at the free end.

Finite elements were also developed by Archer [20], Kapur [21] and Nickel and Secor [22] using the first order shear deformation theory with all the secondary effects.

It has been observed [23] that the discrepancies between the results of this theory, even after refining the values of the shear correction factor and those of the theory of elasticity are quite large for built-up beams. Moreover, it has been established that for thick (aspect ratio of four) and moderately thick (aspect ratio of ten) sandwich and composite laminates, the predictions of deflections and stresses by first order theory are grossly in error [24].

This led to the development of a second-order theory by Stephen and Levinson [25] which models the bending behaviour using constants dependent on the cross-sectional warping and the transverse direct stresses.

A third order theory by Heyliger and Reddy [26] takes a quadratic variation of the shear strain across the cross-section and ensures a stress-free condition at the top and bottom surfaces. An improvement over this theory, by the incorporation of transverse normal stresses in the formulation was proposed by Soldatos and Elishakoff [27]. But the major disadvantage with this theory is the presence of a higher order derivative of the transverse deflection, making it a C^1 continuous formulation.

A fourth order theory of Levinson [28, 29] considers the transverse shear deformation and the cross-sectional warping without the need for a shear correction factor. The inability of this theory to model two-dimensional displacement patterns has been overcome by Rychter [30] in his work by incorporating the 2-d theory of elasticity.

Bickford [31] published a consistent higher order theory based on Hamilton's principle and Levinson's theory. Another higher order theory was proposed by Reddy [33], which also retained the C^1 continuity without the consideration of transverse normal strain.

Elasticity solutions of the bending problem of composite beams also are available in the literature [32].

While the classical theory is very much restricted in its scope, the first order theory requires a problem dependent shear correction factor. Moreover for low aspect ratios, its performance was found out to be [24] poorer even than that of the classical theory. Also, this theory does not consider the effects of cross-sectional warping, which plays an important role in the case of deep sandwich constructions with stiff facings and weak cores and neglects the transverse normal strains too! As many of the successful commercial finite element packages are also based on this theory, their results too become unreliable where the first order theory fails.

Though the second order theory [25] takes shear curvature and transverse direct stresses into account, it requires the evaluation of two coefficients based on the shape of the cross-section under consideration—the first one dependent on the cross-sectional warping and the second on the transverse direct stresses, thus making this theory also a problem dependent one. The third order theory [26] has the inherent disadvantage of C^1 continuity. The fourth order theory [28] has been formulated only for beams with narrow rectangular cross-sections—thus rendering itself directly inapplicable to general beam problems with arbitrary cross-sections. The higher order theory [33] has the same problem of C^1 continuity as the third order theory.

While the exact/closed form analytical solutions could be obtained only for problems with simple geometry, loading and boundary conditions, the practical problems with complex geometries, loadings and boundaries could be solved with ease, by using numerical solution techniques like finite element method.

Thus the process of development of theories is a clear pointer to the sustained interest of researchers in this area over such a long period of time and a study of them brings out two salient points clearly: first, the inadequacies/deficiencies of each theory have necessitated the development of a new theory; second, the present need to formulate a refined higher order theory which must be quite general in its scope without shortcomings such as problem dependent factors, unconventional boundary conditions, etc., and must retain the ease of formulation and coding and finally must increase the accuracy of the analysis of deep composite-sandwich structures.

The higher order theory proposed by Kant and Gupta [23] clearly met all these requirements; it is based on C^0 finite elements, it assumes a cubic, quadratic and linear variation for the axial, transverse shear and normal strain components across the thickness of the cross-section, thereby catering for all the possible secondary effects and also eliminating the need for a shear correction coefficient.

As it has become clear that the analysis of deep fiber reinforced composite constructions requires a tool better than the first order theory and as sophisticated as the higher order theory, three higher order models are proposed here and compared for their relative ability to predict the natural frequencies of such constructions.

2. HIGHER ORDER BEAM THEORIES (HOBT)

The higher order displacement models, based on the Taylor's series expansion [34] of the displacement components are defined as follows:
HOBT5 (5 DOF per node),

$$u(x, z, t) = u_0(x, t) + z\theta_x(x, t) + z^2 u_0^*(x, t) + z^3 \theta_x^*(x, t), \tag{1}$$

$$w(x, z, t) = w_0(x, t); \tag{1a}$$

HOBT4a (4 DOF per node),

$$u(x, z, t) = u_0(x, t) + z\theta_x(x, t) + z^2 u_0^*(x, t), \tag{2}$$

$$w(x, z, t) = w_0(x, t); \tag{2a}$$

HOBT4b (4 DOF per node),

$$u(x, z, t) = u_0(x, t) + z\theta_x(x, t) + z^3 \theta_x^*(x, t), \tag{3}$$

$$w(x, z, t) = w_0(x, t). \tag{3a}$$

Here u and w are the axial and transverse displacements in the x–z plane at time t, θ_x is the rotation of the cross-section about the y-axis and u_0^* and θ_x^* are the higher order terms arising out of the Taylor series expansion and defined at the neutral axis. This presentation is based on HOBT5, as the other two models are special cases of it.

The total energy of a system, in the absence of external and damping forces can be given by

$$L = T - \Pi \tag{4}$$

where $\Pi = U_s$. U_s is the internal strain energy, and T is the kinetic energy. Equation (4) can be rewritten as

$$L = \frac{1}{2}\int \dot{\mathbf{u}}^t \rho \dot{\mathbf{u}}\, \mathrm{d}v - \left[\frac{1}{2}\int \boldsymbol{\epsilon}^t \sigma\, \mathrm{d}v\right], \tag{5}$$

where

$$\mathbf{u} = [u \quad w]^t, \qquad \dot{\mathbf{u}} = [\dot{u} \quad \dot{w}]^t, \qquad \boldsymbol{\epsilon} = [\epsilon_x \quad \gamma_{xz}]^t, \qquad \boldsymbol{\sigma} = [\sigma_x \quad \tau_{xz}]^t.$$

The displacements can be written as

$$\mathbf{u} = \mathbf{Z}_d \mathbf{d}, \tag{6}$$

where

$$\mathbf{u} = [u \quad w]^t, \qquad \mathbf{d} = [u_0 \quad w_0 \quad \theta_x \quad u_0^* \quad \theta_x^*]^t, \tag{6a, b}$$

$$\mathbf{Z}_d = \begin{bmatrix} 1 & 0 & z & z^2 & z^3 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{6c}$$

The strains are written as

$$\epsilon_x = \epsilon_{x0} + z^2\epsilon_{x0}^* + zK_x + z^3K_x^*, \tag{7}$$

$$\gamma_{xz} = \phi + z^2\phi^* + zK_{xz}, \tag{8}$$

where

$$[\epsilon_{x0} \quad \epsilon_{x0}^* \quad K_x \quad K_x^*] = [u_{0,x} \quad u_{0,x}^* \quad \theta_{x,x} \quad \theta_{x,x}^*], \tag{8a}$$

$$[\phi \quad \phi^* \quad K_{xz}] = [(w_{0,x} + \theta_{x,x}) \quad 3\theta_x^* \quad 2u_0^*], \tag{8b}$$

and can be expressed in the matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Z}_s \bar{\boldsymbol{\epsilon}}, \tag{9}$$

where

$$\bar{\boldsymbol{\epsilon}} = [\epsilon_{x0} \quad \epsilon_{x0}^* \quad K_x \quad K_x^* \quad | \quad \phi \quad \phi^* \quad K_{xz}], \tag{9a}$$

$$\mathbf{Z}_s = \left[\begin{array}{cccc|ccc} 1 & z^2 & z & z^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z^2 & z \end{array}\right]. \tag{9b}$$

In this formulation for planar beam deformations, only the axial, flexural and transverse shear strain/stress components in the x–z plane are taken into consideration. As the torsional effects are negligible in a planar formulation, the bending-twisting coupling effects are also neglected.

The constitutive relation of a typical lamina is given by

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}, \tag{10}$$

where

$$\mathbf{D} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}, \qquad \boldsymbol{\sigma} = [\sigma_x \tau_{xz}]^t. \tag{10a, b}$$

The internal strain energy, after carrying out the integration across the cross-section, becomes

$$U_s = \frac{b}{2}\int_l \bar{\boldsymbol{\epsilon}}^t \bar{\boldsymbol{\sigma}}\,\mathrm{d}x, \tag{11}$$

where

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{D}}\bar{\boldsymbol{\epsilon}}. \tag{11a}$$

The stress resultants are given by

$$\bar{\boldsymbol{\sigma}} = [N_x \quad N_x^* \quad M_x \quad M_x^* \quad | \quad Q \quad Q^* \quad S]^t, \tag{12}$$

and

$$\bar{\mathbf{D}} = \int_z \mathbf{Z}_s^t \mathbf{D} \mathbf{Z}_s\,\mathrm{d}z = \sum_{L=1}^{NL} \int_{h_{L-1}}^{h_L} \mathbf{Z}_s^t \mathbf{D} \mathbf{Z}_s\,\mathrm{d}z, \tag{13}$$

which in matrix form is

$$\bar{\mathbf{D}} = \left[\begin{array}{cc|c} \mathbf{D}_m & \mathbf{D}_c & 0 \\ \mathbf{D}_c^t & \mathbf{D}_b & \\ \hline \multicolumn{2}{c|}{0} & \mathbf{D}_s \end{array}\right] \tag{14}$$

where NL denotes the (total) Number of Layers of the cross-section,

$$\mathbf{D}_m = \sum_{L=1}^{NL} E_L \begin{bmatrix} H_1 & H_3 \\ H_3 & H_5 \end{bmatrix}, \qquad \mathbf{D}_c = \sum_{L=1}^{NL} E_L \begin{bmatrix} H_2 & H_4 \\ H_4 & H_6 \end{bmatrix},$$

$$\mathbf{D}_b = \sum_{L=1}^{NL} E_L \begin{bmatrix} H_3 & H_5 \\ H_5 & H_7 \end{bmatrix}, \qquad \mathbf{D}_s = \sum_{L=1}^{NL} G_L \begin{bmatrix} H_1 & H_3 & H_2 \\ & H_5 & H_4 \\ & & H_3 \end{bmatrix}, \tag{14a}$$

$$H_k = (h_L^k - h_{L-1}^k)/k, \qquad k = 1, \ldots, 7. \tag{14b}$$

The kinetic energy can be expressed by using equation (6) as

$$T = \frac{1}{2}\int_l \dot{\mathbf{d}}^t \bar{\mathbf{m}} \dot{\mathbf{d}}\,\mathrm{d}x, \tag{15}$$

where

$$\bar{\mathbf{m}} = b \int_z \mathbf{Z}_d^t \rho_L \mathbf{Z}_d \, \mathrm{d}z = b \sum_{L=1}^{NL} \int_{h_{L-1}}^{h_L} \mathbf{Z}_d^t \rho_L \mathbf{Z}_d \, \mathrm{d}z \tag{15a}$$

where ρ_L is the mass density of a particular layer. The diagonal elements of the matrix given by equation (15a) corresponding to any node i can be expressed by

$$\bar{\mathbf{m}}_{ii} = b \sum_{L=1}^{NL} \int_{h_{L-1}}^{h_L} [1 \quad 1 \quad z^2 \quad z^4 \quad z^6] \rho_L \, \mathrm{d}z. \tag{16}$$

The total energy can thus be expressed, by using equations (11) and (15), as

$$L = \frac{1}{2} \int_l \dot{\mathbf{d}}^t \bar{\mathbf{m}} \dot{\mathbf{d}} \, \mathrm{d}x - \frac{b}{2} \int_l \bar{\boldsymbol{\epsilon}}^t \bar{\boldsymbol{\sigma}} \, \mathrm{d}x. \tag{17}$$

3. FINITE ELEMENT MODELLING

In isoparametric formulations, the displacements within an element can be expressed in terms of the nodal displacements as

$$\mathbf{d} = \mathbf{N}\mathbf{a}_e, \tag{18}$$

where $\mathbf{a}_e$ is a vector containing nodal displacement vectors of an element and is given by

$$\mathbf{a}_e = [\mathbf{d}_1^t \quad \mathbf{d}_2^t \quad \mathbf{d}_3^t \quad \ldots \quad \mathbf{d}_n^t]^t \tag{18a}$$

and $\mathbf{N}$ is the shape function matrix.

Similarly, the strain within an element can be written as

$$\bar{\boldsymbol{\epsilon}} = \mathbf{B}\mathbf{a}_e, \tag{19}$$

where $\mathbf{B}$ is the strain displacement matrix. The non-zero elements of $\mathbf{B}$ corresponding to a particular node i can be expressed as

$$B_{11} = B_{24} = B_{33} = B_{45} = B_{52} = N_{i,x}, \qquad B_{53} = N_i, \qquad B_{65} = 3N_i, \qquad B_{74} = 2N_i. \tag{20}$$

By using equations (11a), (18) and (19), the total energy can be written as

$$L = \frac{1}{2} \dot{\mathbf{a}}_e^t \int_l \mathbf{N}^t \bar{\mathbf{m}} \mathbf{N} \, \mathrm{d}x \, \dot{\mathbf{a}}_e - \frac{b}{2} \mathbf{a}_e^t \int_l \mathbf{B}^t \bar{\boldsymbol{\sigma}} \, \mathrm{d}x$$

$$= \frac{1}{2} \dot{\mathbf{a}}_e^t \int_1 \mathbf{N}^t \bar{\mathbf{m}} \mathbf{N} \, \mathrm{d}x \, \dot{\mathbf{a}}_e - \frac{b}{2} \mathbf{a}_e^t \int_1 \mathbf{B}^t \bar{\mathbf{D}} \mathbf{B} \, \mathrm{d}x \, \mathbf{a}_e. \tag{21}$$

Applying Hamilton's principle to L, one obtains the equation of motion as

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{0} \tag{22}$$

where

$$\mathbf{M} = \int_l \mathbf{N}^t \bar{\mathbf{m}} \mathbf{N} \, \mathrm{d}x, \qquad \mathbf{K} = b \int_1 \mathbf{B}^t \bar{\mathbf{D}} \mathbf{B} \, \mathrm{d}x. \tag{22a, b}$$

As it is recognized that the consistent mass formulation yields a better rate of convergence for higher order differential equations [35] and also that the lumping of masses leads to a poor approximation of the mass of the element [36, 37], the consistent mass formulation is considered here. The consistent mass matrix is evaluated as

$$\mathbf{M}_e = \sum_{g=1}^{NG} W_g \mathbf{N}^t \bar{\mathbf{m}} \mathbf{N} |J|, \tag{23}$$

where NG is the (total) Number of Gauss points (four in this case), W_g is the weighing coefficient and $|J|$ is the determinant of the Jacobian.

The stiffness matrix can be evaluated as

$$\mathbf{K}_e = b \sum_{g=1}^{NG} W_g \mathbf{B}^t \bar{\mathbf{D}} \mathbf{B} |J|, \tag{24}$$

where the total number of Gauss points is four for bending and three for shear term evaluation.

The governing equation of motion (22) is solved by expressing the displacement vector as

$$\mathbf{d} = \bar{\mathbf{d}}\, e^{i\omega t} = \bar{\mathbf{d}}(\cos \omega t + i \sin \omega t), \tag{25}$$

where $\bar{\mathbf{d}}$ is the modal vector and ω is the natural frequency. From equation (25) one can derive the acceleration as

$$\ddot{\mathbf{d}} = -\omega^2 \bar{\mathbf{d}}\, e^{i\omega t}. \tag{26}$$

Substituting equations (25) and (26) into the equation of motion (22) yields

$$(\mathbf{K} - \omega^2 \mathbf{M})\bar{\mathbf{d}} = \mathbf{0}, \tag{27}$$

or, in a more generalised form,

$$(\mathbf{K} - \lambda \mathbf{M})\bar{\mathbf{d}} = \mathbf{0}, \qquad \lambda = \omega^2. \tag{28, 28a}$$

When equation (28) is solved by using the standard eigenvalue solution schemes, after the imposition of boundary conditions, the natural frequency eigenvalues (λ) and the corresponding eigenvectors ($\bar{\mathbf{d}}$) are obtained.

4. NUMERICAL EXPERIMENTS

Numerical experiments with these higher order theories and isoparametric elements were carried out on an IBM compatible 386 platform with DOS in double precision. Details of the boundary conditions and material properties used are given in Table 1. In all these experiments frequencies are classified based on their mode shapes as axial, flexural and shear frequencies. In Tables 2–9 values given in parantheses for each frequency correspond to the actual mode of beam vibration.

4.1. THIN SYMMETRIC BEAMS

First, thin symmetrical sandwich and composite beams were analyzed using higher order models, and their performances validated by comparison with results from earlier investigations. Numerical results of these models for a sandwich beam ($L/D = 67$), a cantilever ($L/D = 52$), another sandwich beam ($L/D = 23$) and a clamped-free composite beam ($L/D = 15$) are presented in Tables 2–5.

In the classical sandwich beam/plate theories [12, 14] the facings are assumed to resist only the bending stresses while the core is considered to resist only the transverse shear stresses. This results in the energy expression of the face emerging purely from flexure as $\int \boldsymbol{\epsilon}_x^t \boldsymbol{\sigma}_x \, dv$ while that of the core from shear as $\int \boldsymbol{\gamma}_{xz}^t \boldsymbol{\tau}_{xz} \, dv$. Hence the shear stiffness of facings

TABLE 1

Details of boundary conditions and properties of materials used in the numerical experiments

1.1	Boundary conditions Type	$x = 0$	$x = L$
	Simply supported	$u_0 = w_0 = u_0^* = 0$	$u_0 = w_0 = u_0^* = 0$
	Clamped-free	$u_0 = w_0 = \theta_x = u_0^* = \theta_x^* = 0$ at the fixed end.	
1.2	Material types		
1 DATA-1 Ref [12]	Details $L = 36$in, $b = 1$in, $d = 0{\cdot}536$in Face properties; $(t_f)_{inner} = (t_f)_{outer} = 0{\cdot}018$in, $E_f = 10^7$psi, $\rho_f = 2{\cdot}5098\text{E} - 4$ lbs^2/in^4 Core properties; $t_c = 0{\cdot}5$in, $E_c = 0$, $G_c = 12000$psi, $\rho_c = 3{\cdot}0717\text{E} - 6$ lbs^2/in^4 No. of layers of $c/s = 8$, No. of elements used = 5 cubic, B.C. simply supported		
2 DATA-2 Ref [12]	Details $L = 28$in Rest are same as DATA1 B.C. Clamped-free		
3 DATA-3 Ref [14]	Details $L = 20$in, $b = 1$in, $d = 0{\cdot}86$in Face properties (Bot/Mid/Top); $t_f = 0{\cdot}02$in, $E_f = 10^7$psi, $\rho_f = 1{\cdot}0$ lbs^2/in^4 Core properties (Bot/Top); $t_c = 0{\cdot}4$in, $E_c = 0$, $G_c = 5000$psi, $\rho_c = 0{\cdot}25$ lbs^2/in^4 No. of layers of $c/s = 5$, No. of elements used = 5 cubic, B.C. simply supported		
4 DATA-4 Ref [19]	Details Mat. AS4/3501-6/Graphite/Epoxy; Lamination scheme 0/90/90/0; $L/d = 15$, $b = 1$m, $E_1 = 147{\cdot}454$E8 kg/m^2, $E_2 = 9{\cdot}8269$E8 kg/m^2, $G_{12} = 4{\cdot}2159$E8 kg/m^2, $\rho = 141{\cdot}45$ kgs^2/m^4 No. of layers of $c/s = 8$, No. of elements used = 5 cubic, B.C. clamped-free		
5 DATA-5 Ref [38] Ref [39]	Details Mat. Graphite/Epoxy $L/d = 5$, $b = 1$in Symmetric 0/90/core/90/0, Unsymmetric 0/90/core/0/90 Face properties; $E_{fx} = 0{\cdot}1742$E8 psi, $E_{fz} = 0{\cdot}1147$E7 psi, $G_{fxz} = 0{\cdot}7983$E6 psi, $\rho_f = 0{\cdot}1433\text{E} - 3$ lbs^2/in^4, $t_f = 0{\cdot}6$in Mat. US commercial aluminium, Honey comb 0·25in cell size, 0·007in foil Core properties; $E_x = E_z = 0$, $G_{cxz} = 0{\cdot}2042$E5 psi, $\rho_c = 0{\cdot}3098$E-5 lbs^2/in^4, $t_c = 4{\cdot}8$in, $t_c/t_f = 8$ No. of layers of $c/s = 6$, No. of elements used = 5 cubic, B.C. simply supported		
6 DATA-6 Ref [40]	Details $L/d = 5$, $b = 1$in Symmetric 0/0/90/90/0/0, Unsymmetric 0/90/0/90/0/90 $E_x = 0{\cdot}7620$E8 psi, $E_z = 0{\cdot}3048$E7 psi, $G_{xz} = 0{\cdot}1524$E7 psi, $\rho = 0{\cdot}7257\text{E} - 4$ lbs^2/in^4, $t_1 = 1{\cdot}0$in No. of layers of $c/s = 6$, No. of elements used = 5 cubic, B.C. simply supported		

TABLE 2

Comparison of natural frequencies (Hz) *of a sandwich beam* (DATA-1)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5	Ref [12]	Ref [41]	Exact [42]
1a. Axial frequencies							
1	2562(10)	2562(9)	2562(9)	2562(9)	2510	2594	2549·5
1b. Bending frequencies							
1	57(1)	57(1)	57(1)	57(1)	55·5	57·5	56·028
2	216(2)	219(2)	218(2)	218(2)	–	–	–
3	452(3)	461(3)	461(3)	461(3)	451	467	457·12
4	736(4)	760(4)	759(4)	759(4)	–	–	–
5	1054(5)	1099(5)	1097(5)	1097(5)	1073	1111	1090·26
6	1388(6)	1460(6)	1457(6)	1457(6)	–	–	–
7	1749(7)	1853(7)	1849(7)	1849(7)	1779	1842	1809·80
8	2139(8)	2282(8)	2276*(8)	2276*(8)	–	–	–

Note: *Values correspond to first θ_x^* mode.

and the flexural stiffness of the core are considered to be quite negligible and are disregarded in the experiments with DATA-1, DATA-2 and DATA-3 of Table 1.

Axial frequencies produced by HOBT4a are reduced by the term u_0^* compared to the Timoshenko ones while θ_x^* in HOBT4b does not participate in frequency evaluation, which results in identical predictions by Timoshenko and HOBT4b. Active and passive presence of u_0^* and θ_0^* respectively in HOBT5 makes it equivalent to HOBT4a.

In all these experiments, it can be observed that flexural frequencies given by higher order theories are higher than those given by first order theory. Due to the symmetry in the lamination, u_0^* remains ineffective and thus HOBT5 and HOBT4b compute identical frequencies. The term θ_x^* reduces frequencies of HOBT4b compared to those of HOBT4a.

HOBT4a is seen to compute higher frequencies than Timoshenko, while it may be expected to yield frequencies equal to those of first order theory as u_0^* is passive due to the symmetric lamination scheme. This apparent contradiction can be resolved when HOBT3 (with only u_0, w_0 and θ_x and without a shear correction factor) results (not presented here) are compared with those of HOBT4a. Through such an exercise, it has been observed that HOBT4a and HOBT3 results are equal and u_0^* is indeed ineffective.

TABLE 3

Comparison of natural frequencies (Hz) *of a sandwich cantilever* (DATA-2)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5	Ref [12]	Ref [41]	Ref [42]
3a. Axial frequencies							
1	1648(6)	1648(6)	1648(6)	1648(6)	–	–	–
2	4943(13)	4941(13)	4943(13)	4941(13)	–	–	–
3b. Bending frequencies							
1	33·6(1)	33·7(1)	33·7(1)	33·7(1)	32·79	33·97	34·242
2	195(2)	198(2)	197·5(2)	197·5(2)	193·5	200·5	201·85
3	492(3)	506(3)	505·5(3)	505·5(3)	499	517	520·85
4	857(4)	892(4)	890·5(4)	890·5(4)	886	918	925·4
5	1260(5)	1323(5)	1321(5)	1321(5)	1320	1368	1381·3
6	1690(7)	1790(7)	1786(7)	1786(7)	1779	1844	1867
7	2136(8)	2276(8)	2271*(8)	2271*(8)	2249	2331	2374
8	2610(9)	2798(9)	2792*(9)	2792*(9)	2723	2824	2905

Note: *Values correspond to first and second θ_x^* modes.

TABLE 4

Comparison of natural frequencies of a simply supported sandwich beam (DATA-3)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5	Ref [14]	Ref [14] from Ref [43]
4a. Axial frequencies						
1	238·60(12)	234·80(11)	238·60(11)	234·80(11)	–	–
4b. Bending frequencies						
1	10·41(1)	10·73(1)	10·72(1)	10·72(1)	10·91	10·89
2	29·24(2)	31·04(2)	31·04(2)	31·04(2)	32·21	32·02
3	48·45(3)	52·17(3)	52·15(3)	52·15(3)	54·66	54·24
4	67·31(4)	72·95(4)	72·93(4)	72·93(4)	76·75	76·10
5	86·31(5)	93·88(5)	93·85(5)	93·85(5)	98·48	97·59
6	105·00(6)	114·50(6)	114·40(6)	114·40(6)	119·96	118·85
7	124·60(7)	136·10(7)	136·00(7)	136·00(7)	141·33	139·99
8	145·40(8)	158·90(8)	158·90(8)	158·90(8)	162·83	160·97
9	167·30(9)	183·00(9)	182·90(9)	182·90(9)	184·02	181·93
10	182·30(10)	199·40(10)	199·30(10)	199·30(10)	–	–

Thus the reason for the prediction by Timoshenko to be less than those of HOBT4a is its constant shear strain variation along the thickness and the subsequent usage of the shear correction factor.

4.2. THICK SYMMETRIC BEAMS

A deep symmetric sandwich beam with properties of DATA-5 of Table 1 is studied here. The natural frequencies for all these models are compared with those of first order theory and presented in Table 6. The frequencies are non-dimensionalized by using the expression $\bar{\omega} = \omega L^2[\rho_f/(E_{fx}d^2)]^{1/2}$.

The higher order term u_0^* contributes towards the reduction of axial frequencies given by HOBT4a and HOBT5, while θ_x^* does not, resulting in HOBT4b performing like first order theory and HOBT5 predictions very similar to those of HOBT4a.

The symmetry of lamination nullifies the influence of u_0^* due to which HOBT4b and HOBT5 compute identical flexural and shear frequencies. In this case, the contribution of θ_x^* towards the reduction of frequencies given by HOBT4b and HOBT5 is strikingly significant—the order of reduction in flexure being around 50% and in shear ranging from 58% to 15%.

In this case also, the computation by HOBT4a gives higher values than those given by Timoshenko, for the same reason as discussed earlier.

Next a composite beam as described by DATA-6 of Table 1 is considered. Table 7

TABLE 5

Comparison of non-dimensional natural frequencies of a symmetrically laminated clamped-free beam (DATA-4)

Bending mode	Timo.	HOBT4a	HOBT4b	HOBT5	Ref [19]
1	0·923(1)	0·927(1)	0·924(1)	0·924(1)	0·923
2	4·941(2)	5·073(2)	4·985(2)	4·985(2)	4·888
3	11·656(3)	12·159(3)	11·832(3)	11·832(3)	11·433
4	19·180(4)	20·262(4)	19·573(4)	19·573(4)	18·689
5	27·038(5)	28·820(5)	27·720(5)	27·720(5)	26·203

TABLE 6

Comparison of non-dimensional natural frequencies of a simply supported symmetric sandwich beam (DATA-5)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5
6a. Axial frequencies				
1	11·001(4)	10·971(4)	11·001(8)	10·971(8)
2	22·001(11)	21·752(10)	22·001(14)	21·752(14)
6b. Bending frequencies				
1	2·226(1)	2·327(1)	1·293(1)	1·293(1)
2	5·558(2)	5·971(2)	2·787(2)	2·787(2)
3	8·824(3)	9·564(3)	4·315(3)	4·315(3)
4	12·029(6)	13·096(6)	5·933(5)	5·933(5)
5	15·277(7)	16·667(7)	7·697(6)	7·697(6)
6	18·482(9)	20·190(9)	9·602(7)	9·602(7)
7	21·859(10)	23·899(11)	11·792(9)	11·792(9)
8	25·443(12)	27·831(13)	14·374(11)	14·374(11)
9	29·216(14)	31·974(14)	17·497(12)	17·497(12)
6c. Shear frequencies				
1	11·250(5)	12·321(5)	4·853(4)	4·853(4)
2	16·499(8)	17·291(8)	12·287(10)	12·287(10)
3	26·420(13)	26·949(12)	22·991(15)	22·991(15)

presents the results obtained by using these theories. The frequencies are non-dimensionalized by using the relationship $\bar{\omega} = \omega L^2[\rho/(E_x d^2)]^{1/2}$.

The axial frequency pattern is the same as for the sandwich construction: u_0^* reducing the frequencies of HOBT4a and HOBT5, θ_x^* remaining passive in HOBT4b and HOBT5

TABLE 7

Comparison of non-dimensional natural frequencies of a simply supported symmetric composite beam (DATA-6)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5
7a. Axial frequencies				
1	12·953(8)	12·636(7)	12·953(7)	12·636(7)
2	25·910(14)	23·597(14)	25·910(14)	23·597(14)
7b. Bending frequencies				
1	1·639(1)	1·736(1)	1·656(1)	1·656(1)
2	3·810(2)	4·125(2)	3·923(2)	3·923(2)
3	5·912(3)	6·439(3)	6·191(3)	6·191(3)
4	7·988(4)	8·722(4)	8·470(4)	8·470(4)
5	10·100(5)	11·042(5)	10·803(5)	10·803(5)
6	12·188(7)	13·333(8)	13·117(8)	13·117(8)
7	14·392(9)	15·751(9)	15·561(9)	15·561(9)
8	16·732(10)	18·313(10)	18·151(10)	18·151(10)
9	19·205(12)	21·021(12)	20·889(12)	20·889(12)
10	20·874(13)	22·865(13)	22·763(13)	22·763(13)
7c. Shear frequencies				
1	11·181(6)	12·248(6)	11·111(6)	11·111(6)
2	19·088(11)	19·747(11)	18·927(11)	18·927(11)

TABLE 8

Comparison of non-dimensional natural frequencies of a simply supported unsymmetric sandwich beam (DATA-5)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5
8a. Axial frequencies				
1	10·988(4)	10·988(4)	10·519(8)	10·515(8)
2	21·980(11)	21·950(10)	21·218(14)	21·188(14)
8b. Bending frequencies				
1	2·174(1)	2·266(1)	1·242(1)	1·241(1)
2	5·503(2)	5·894(2)	2·660(2)	2·659(2)
3	8·776(3)	9·495(3)	4·133(3)	4·132(3)
4	11·990(6)	13·031(6)	5·636(5)	5·632(5)
5	15·247(7)	16·602(7)	7·348(6)	7·348(6)
6	18·456(9)	20·130(9)	8·944(7)	8·940(7)
7	21·838(10)	23·838(11)	11·212(9)	11·207(9)
8	25·426(12)	27·771(13)	13·866(11)	13·857(11)
9	29·203(14)	31·914(14)	16·421(12)	16·413(12)
8c. Shear frequencies				
1	11·250(5)	12·321(5)	4·853(4)	4·853(4)
2	16·120(8)	16·921(8)	12·278(10)	12·266(10)
3	25·460(13)	25·929(12)	23·051(15)	22·957(15)

and thus making Timoshenko and HOBT4b results equal, and HOBT4a and HOBT5 results equal.

The flexural and shear frequencies given by HOBT4b and HOBT5 are equal due to symmetry while θ_x^* in both these models reduces their values compared to those of HOBT4a. Again, frequencies of HOBT4a are higher than those of Timoshenko.

It is important to note here that the order of difference between HOBT4b and HOBT5, and the rest, is more pronounced for sandwiches than for composites.

4.3. THICK UNSYMMETRIC BEAMS

The same DATA-5 sandwich beam but with an unsymmetric configuration is considered now and the results are presented in Table 8.

The axial frequencies are reduced by u_0^* in HOBT4a, by θ_x^* in HOBT4b and by both in HOBT5 compared to those given by Timoshenko. Incidentally, in the second axial mode the reduction in HOBT5 can be quantified as the sum of reduction from the Timoshenko value given by HOBT4a and HOBT4b, respectively.

In this case too, HOBT4a computes the highest flexural frequencies. The higher order rotation term reduces the frequencies of HOBT4b and HOBT5 quite significantly—around 50%—from those given by HOBT4a. As u_0^* becomes active due to the unsymmetric lamination scheme, HOBT5 yields the lowest frequencies of all the models.

Shear frequencies given by HOBT4b and HOBT5 are considerably less than those given by HOBT4a while HOBT4a yields the highest and HOBT5 computes the lowest.

While u_0^* reduces frequencies given by HOBT5 compared to those from HOBT4b, it is seen to make those given by HOBT4a higher than the Timoshenko ones. As in symmetric laminates, when a comparison was made with HOBT3, it was found that u_0^* in fact reduces frequencies of HOBT4a compared to those of HOBT3. The differences between HOBT4a and Timoshenko frequencies are due to the reduction in frequencies of the latter caused by the shear correction factor.

Next an unsymmetric composite beam of DATA-6 was analyzed and the results are presented in Table 9.

Axial frequencies are reduced by u_0^* and θ_x^* and hence frequency predictions given by HOBT5 are less than those given by other models.

Flexural frequencies given by HOBT4a are the highest. While θ_x^* contributes towards the reduction of HOBT4b frequencies, higher order terms u_0^* and θ_x^* together reduce the frequencies given by HOBT5 still further. A similar pattern can also be observed for shear frequencies.

5. CONCLUSIONS

Three higher order displacement models have been proposed and tested for free vibration analysis of sandwich and composite beams with various boundary conditions and aspect ratios.

All higher order models are found to compute frequencies which are numerically higher than those of first order theory for the thin beams considered. In the case of thick sandwiches, on the other hand, higher order theories give quite significantly lower frequencies than does Timoshenko theory.

For symmetric laminates, only u_0^* reduces the axial frequencies while for unsymmetric laminates both u_0^* and θ_x^* contribute to frequency reduction.

In the case of flexural and shear modes, HOBT4a computes the highest frequencies of all the other models for all cases considered and the increase above first order theory results was traced to the shear correction factor of the latter theory.

The higher order rotation term θ_x^* reduces frequencies given by HOBT4b compared to those given by HOBT4a for both symmetric and unsymmetric laminates. The term u_0^* remains ineffective for symmetric laminates and hence, for such cases, HOBT4b and HOBT5 predict identical results. As u_0^* as well as θ_x^* contribute towards the reduction in

TABLE 9

Comparison of non-dimensional natural frequencies of a simply supported unsymmetric composite beam (DATA-6)

Mode	Timo.	HOBT4a	HOBT4b	HOBT5
9a. Axial frequencies				
1	10·932(6)	10·935(6)	10·762(6)	10·668(6)
2	21·855(14)	21·709(13)	21·430(13)	21·108(13)
9b. Bending frequencies				
1	1·432(1)	1·483(1)	1·434(1)	1·416(1)
2	3·597(2)	3·806(2)	3·614(2)	3·531(2)
3	5·750(3)	6·153(3)	5·870(3)	5·675(3)
4	7·856(4)	8·457(4)	8·114(4)	7·795(4)
5	9·994(5)	10·809(5)	10·462(5)	10·021(5)
6	12·104(8)	13·132(8)	12·807(8)	12·285(8)
7	14·319(9)	15·575(9)	15·253(9)	14·633(9)
8	16·673(11)	18·166(11)	17·873(11)	17·244(11)
9	19·147(12)	20·889(12)	20·611(12)	19·981(12)
9c. Shear frequencies				
1	11·181(7)	12·248(7)	11·110(7)	11·110(7)
2	15·868(10)	16·468(10)	15·839(10)	15·663(10)
3	24·958(15)	23·670(15)	24·973(15)	23·216(15)

frequencies for unsymmetric laminates, the predictions of HOBT5 remain the lowest of all the frequencies given by the higher order theories for bending and shear modes.

While u_0^* is alone effective for axial frequencies and only θ_x^* is effective for flexural and shear modes of symmetric laminates, both these terms together influence the behaviour predicted for all modes of unsymmetric laminates and hence one can sum up by identifying HOBT5 as the most effective model for the analysis of composite and sandwich beam constructions.

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