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A Higher Order Finite Element Model for the Vibration Analysis of Laminated Beams

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A higher order displacement model based on a cubic axial strain, cubic transverse shear strain and quadratic transverse normal strain across the thickness of the beam, to model exactly the warping of the cross section is proposed which maintains zero stress at the top and bottom of the beam with out the aid of any shear correction factor. Numerical experiments carried out clearly bring out the efficacy of this model over the first order theory for laminated beams.

1 Introduction

The necessity of a flexural theory—which is not dependent on any shear correction factor, is capable of modelling transverse shear and normal strain effects and also the warping of cross section and finally is accurate enough for deeper sandwich and composite laminates with the ease of coding—becomes much more in the wake of requirement for the analysis of fiber reinforced structures.

The higher order theory by Kant and Gupta (1988) could meet all these demands with C^0 elements for isotropic beams. This theory was later applied to vibration studies of sandwich and composite beams by Marur and Kant (1996), considering each layer of cross section to be isotropic in its plane in one dimensional state of stress with out transverse normal strain effects. Later, an analytical solution to higher order beam dynamics with transverse normal strain was proposed by Kant et al. (1997). A finite element solution with both transverse shear and normal strains which assumes each layer to be orthotropic and in a state of two-dimensional plane stress is presented here for the laminated beam vibration analysis.

2 Higher Order Formulation

The Higher Order Beam Theory (HOBT) based on Taylor's series expansion in a x-z plane can be expressed as,

$$u(x, z, t) = u_o(x, t) + z\theta_x(x, t) + z^2u_o^*(x, t) + z^3\theta_x^*(x, t) \quad (1)$$

$$w(x, z, t) = w_o(x, t) + z\theta_z(x, t) + z^2w_o^*(x, t) + z^3\theta_z^*(x, t) \quad (2)$$

where u_o and w_o are axial and transverse displacements defined at the neutral axis of the beam, θ_x is the face rotation and u_o^*, θ_x^*, θ_z, w_o^* and θ_z^* are higher order terms arising out of Taylor's series.

Total energy of an undamped free system is expressed by,

$$L = \frac{1}{2}\int \dot{u}'\rho\dot{u}dv - \frac{1}{2}\int \epsilon'\sigma dv \quad (3)$$

where

$$\epsilon = [\epsilon_x \quad \epsilon_z \quad \gamma_{xz}]', \quad \sigma = [\sigma_x \quad \sigma_z \quad \tau_{xz}]' \quad (3a)$$

The field-variables can be expressed interms of nodal degrees of freedom as,

$$u = \underline{Z}_d d \quad (4)$$

where

$$u = [u \quad w]' \quad (4a)$$

$$d = [u_o \quad w_o \quad \theta_x \quad u_o^* \quad \theta_x^* \quad \theta_z \quad w_o^* \quad \theta_z^*]' \quad (4b)$$

and

$$\underline{Z}_d = \begin{bmatrix} 1 & 0 & z & z^2 & z^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & z & z^2 & z^3 \end{bmatrix} \quad (4c)$$

The strains are given by,

$$\epsilon_x = u_{o,x} + z\theta_{x,x} + z^2u_{o,x}^* + z^3\theta_{x,x}^* \quad (5)$$

$$\epsilon_z = \theta_z + z(2w_o^*) + z^2(3\,\theta_z^*) \quad (6)$$

$$\gamma_{xz} = (w_{o,x} + \theta_x) + z(2u_o^* + \theta_{z,x}) + z^2(3\theta_x^* + w_{o,x}^*) + z^3\theta_{z,x}^* \quad (7)$$

or expressed in matrix form as,

$$\epsilon_x = Z_x'\epsilon^* \quad (8)$$

$$\epsilon_z = Z_z'\epsilon^* \quad (9)$$

$$\gamma_{xz} = Z_{xz}'\epsilon_{xz} \quad (10)$$

where

$$\epsilon^* = [u_{o,x} \quad 0 \quad \theta_{x,x} \quad u_{o,x}^* \quad \theta_{x,x}^* \quad \theta_z \quad 2w_o^* \quad 3\theta_z^*]' \quad (10a)$$

$$\epsilon_{xz} = [0 \quad w_{o,x} \quad \theta_x \quad 2u_o^* \quad 3\theta_x^* \quad \theta_{z,x} \quad w_{o,x}^* \quad \theta_{z,x}^*]' \quad (10b)$$

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$$Z_x = [1 \quad 0 \quad z \quad z^2 \quad z^3 \quad 0 \quad 0 \quad 0]' \tag{10c}$$

$$Z_z = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad z \quad z^2]' \tag{10d}$$

$$Z_{xz} = [0 \quad 1 \quad 1 \quad z \quad z^2 \quad z \quad z^2 \quad z^3]' \tag{10e}$$

Each layer of the cross section is assumed to be orthotropic and in a two dimensional state of plane stress and is given by,

$$\sigma = \underline{C}\epsilon \tag{11}$$

where $\underline{C}$ is given in ref. by Kant et al. (1997).

The internal strain energy can be evaluated as,

$$U = \frac{b}{2}\iint (\epsilon_x'\sigma_x + \epsilon_z'\sigma_z + \gamma_{xz}'\tau_{xz})dzdx \tag{12}$$

Substituting values from Eqs. (8–11), Eq. (12) can be expressed as,

$$U = \frac{b}{2}\int (\epsilon^{*\prime}\underline{D}^*\epsilon^* + \epsilon_{xz}'\underline{D}_{xz}\epsilon_{xz})dx \tag{13}$$

where,

$$\underline{D}^* = \int Z_x C_{11} Z_x' dz + \int Z_x C_{12} Z_z' dz + \int Z_z C_{12} Z_x' dz + \int Z_z C_{22} Z_z' dz \tag{13a}$$

and

$$\underline{D}_{xz} = \int Z_{xz} G Z_{xz}' dz \tag{13b}$$

The kinetic energy can be expressed as,

$$T = 1/2 \int \dot{d}'\underline{\bar{m}}\dot{d}dx \tag{14}$$

where

$$\underline{\bar{m}} = b \sum_{L=1}^{NL} \int_{h_{L-1}}^{h_L} Z_d' \rho_L \underline{Z}_d dz \tag{14a}$$

where ρ_L is the mass density of a particular layer.

The displacements within an element can be expressed interms of its nodal displacements in isoparametric formulations as,

Table 1 Comparison of nondimensional natural frequencies of a thick symmetric sandwich beam

n	FOBT	HOBT4	HOBT5	HOBT
a. Axial Frequencies				
Axial Mode #				
1	11.035(4)	11.005(4)	11.005(8)	5.989(10)
b. Bending Frequencies				
Bending Mode #				
1	2.229(1)	2.330(1)	1.293(1)	1.293(1)
2	5.563(2)	5.976(2)	2.787(2)	2.787(2)
3	8.824(3)	9.568(3)	4.315(3)	4.311(3)
4	12.016(6)	13.083(6)	5.924(5)	5.920(9)
5	15.174(7)	16.555(7)	7.645(6)	7.632(13)
6	18.314(9)	20.005(9)	9.491(7)	9.465(15)
c. Shear Frequencies				
Shear Mode #				
1	11.250(5)	12.321(5)	4.853(4)	4.853(4)
2	16.525(8)	17.312(8)	12.317(10)	12.309(20)

Table 2 Comparison of nondimensional natural frequencies of a deep unsymmetric composite beam

n	FOBT	HOBT4	HOBT5	HOBT
a. Axial Frequencies				
Axial Mode #				
1	10.951(6)	10.954(6)	10.686(6)	10.682(6)
2	21.884(14)	21.738(13)	21.079(13)	21.079(13)
b. Bending Frequencies				
Bending Mode #				
1	1.434(1)	1.484(1)	1.418(1)	1.417(1)
2	3.598(2)	3.807(2)	3.532(2)	3.532(2)
3	5.750(3)	6.153(3)	5.675(3)	5.674(3)
4	7.849(4)	8.449(4)	7.788(4)	7.785(4)
5	9.928(5)	10.736(5)	9.953(5)	9.945(5)
6	11.995(8)	13.009(8)	12.157(8)	12.147(8)
c. Shear Frequencies				
Shear Mode #				
1	11.181(7)	12.248(7)	11.111(7)	11.111(7)
2	15.883(10)	16.483(10)	15.663(10)	15.663(10)

$$d = N\mathbf{a}_e \tag{15}$$

where $\mathbf{a}_e$ is a vector containing nodal displacement vectors of an element with n nodes and N is the shape function matrix.

The strain with in an element can be written as,

$$\epsilon^* = \underline{B}^*\mathbf{a}_e \tag{16a}$$

$$\epsilon_{xz} = \underline{B}_{xz}\mathbf{a}_e \tag{16b}$$

where $\underline{B}^*$ and $\underline{B}_{xz}$ are strain displacement matrices corresponding to combined axial bending and transverse normal strains and transverse shear strain components respectively.

The non-zero elements of $\underline{B}^*$ corresponding to a particular node i can be given as,

$$B_{11} = B_{33} = B_{44} = B_{55} = N_{i,x};$$

$$B_{66} = N_i; \quad B_{77} = 2N_i; \quad B_{88} = 3N_i \tag{17}$$

and the non-zero elements of $\underline{B}_{xz}$ corresponding to a particular node i can be expressed as,

$$B_{22} = B_{66} = B_{77} = B_{88} = N_{i,x};$$

$$B_{33} = N_i; \quad B_{44} = 2N_i; \quad B_{55} = 3N_i \tag{18}$$

With Eqs. (15) and (16), the total energy can be written as,

$$L = \tfrac{1}{2}\dot{a}_e' \int \underline{N'\bar{m}N}dx\dot{a}_e - \tfrac{1}{2}a_e'[b\int \underline{B}^{*\prime}\underline{D}^*\underline{B}^*dx + b\int \underline{B}_{xz}'\underline{D}_{xz}\underline{B}_{xz}dx]a_e \tag{19}$$

Applying Hamilton's principle on L, we get the governing equation of motion as,

$$\underline{M}\ddot{d} + \underline{K}d = 0 \tag{20}$$

where

$$\underline{M} = \int \underline{N}'\underline{\bar{m}}\,\underline{N}dx, \tag{20a}$$

$$\underline{K} = b\int \underline{B}^{*\prime}\underline{D}^*\underline{B}^*dx + b\int \underline{B}_{xz}'\underline{D}_{xz}\underline{B}_{xz}dx \tag{20b}$$

By solving Eq. (20), using standard eigen value solvers, the natural frequencies and corresponding eigen vectors are directly obtained.

3 Numerical Experiments

In Tables 1 and 2, values in paranthesis against each frequency correspond to the actual mode of beam vibration. HOBT frequencies are compared with those of First Order Beam Theory of Timoshenko (FOBT) and two other models from Eqs. (1) and (2) – HOBT4 (u_o, w_o, θ_x, u_o^*) and HOBT5 (u_o, w_o, θ_x, u_o^*, θ_x^*). Material data, frequency normalization factors and boundary conditions are as shown in the work of Kant et al. (1997).

It can be observed that the flexural and shear frequencies of each model are successively getting reduced with HOBT yielding lowest values. This is due to the softening effect of higher order terms making the beam more flexible.

Going by the same logic, HOBT4 predictions must be lower than those of FOBT, but on the contrary their results are higher in magnitude. This apparent contradiction can be resolved (Marur and Kant, 1996) by a comparison of results of HOBT3 (u_o, w_o, θ_x with out shear correction factor) with HOBT4. It was observed that HOBT4 and HOBT3 were equal for symmetric laminates, as u_o^* would be passive for such cases and HOBT4 was lower than HOBT3 for unsymmetric laminates, confirming the reduction by u_o^*. Thus in both these cases, the reason for lower predictions of FOBT compared to HOBT4 was then traced to its shear correction factor.

In the case of axial frequencies also, HOBT predictions are low—significantly for sandwiches and marginally for composites—compared to FOBT due to the influence of higher order terms.

4 Conclusions

It can be summarized that the proposed HOBT is quite effective, particularly for thick sandwiches, that its predictions are almost half of those by FOBT and for composites too, it is efficient, albeit marginally and thus positions itself as an effective tool for laminated beam analysis.

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A Contribution to the Moving Mass Problem

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Introduction

The problem of the dynamic response of a distributed parameter elastic system due to a moving mass arises in many engineering applications. In a recently published paper, Lee (1996) presents a new method for solving this problem. The authors believe that Lee is the first to take into account the effect of separation between the moving mass and beam, which can occur for large speeds of the moving mass. In this case, solution of the problem reduces to the subsequent solution of two related problems. In the first, the conventional moving mass problem is solved until the moment when the interaction force equals zero, which implies the onset of separation between the mass and beam. In the second, during separation, the equations governing free vibration of the beam and motion of the mass due to gravity are solved separately until the moving mass recontacts the structure.

The purpose of this Tech Brief is to suggest some improvements to the method described in Lee (1996). The first (and main) suggestion is concerned with the method for solving the conventional moving mass problem. Lee expands the solution to the problem in terms of eigenfunctions of the beam and derives a system of second-order ordinary differential equations (ODE's) governing the time-dependent coefficients of the expansion. This system is obtained in a form unresolved with respect to higher derivatives; that is, their coefficients are time-dependent functions. Moreover, the question of existence of the inverse of the matrix of coefficients is not discussed in the paper.

We will establish that this matrix of coefficients can be inverted analytically, resulting in a system of ODE's in standard form that is resolved with respect to higher derivatives. At the same time, the invertibility of the matrix for all values of time will be proved. We will also demonstrate that, while the method described in Lee (1996) gives the solution for the case of constant mass speed, it can readily be extended to the case of varying mass speed. Finally, it will be shown that the equation of free beam vibration (during separation) can be presented in terms of the solution of the moving mass problem solved in the time interval immediately preceding the separation.

Reduction of the System of Differential Equations

The system under consideration consists of an Euler-Bernoulli beam of length l and a concentrated mass M moving along the beam at a constant speed v. We will use the same notation as in Lee (1996). Let ω_n and $\phi_n(x)$ be natural frequencies and corresponding eigenfunctions of the beam. For simplicity, let $\phi_n(x)$ be normalized to unitary mass

$$\int_0^l m(x)\phi_n(x)\phi_m(x)dx = \delta_{mn}, \tag{1}$$

where δ_{mn} is the Kronecker delta and $m(x)$ is the mass of the beam per unit length. As shown in Lee (1996), the transverse beam response $w(x, t)$ can be represented in terms of the $\phi_n(x)$ as

$$w(x, t) = \sum_{n=1}^{N} \phi_n(x)q_n(t). \tag{2}$$

The generalized coordinates $q_n(t)$ satisfy the system of ODE's [Eqs. (10) in Lee, 1996, taking Eq. (1) into account]

$$\ddot{q}_n + \omega_n^2 q_n = M\phi_n(\zeta)\{g - \sum \phi_m(\zeta)\ddot{q}_m - 2v \sum \phi'_m(\zeta)\dot{q}_m - v^2 \sum \phi''_m(\zeta)q_m\}, \quad n = 1, \ldots, N, \tag{3}$$

or in matrix form [Eqs. (12) in Lee, 1996]

$$[M]\{\ddot{\mathbf{q}}\} + [C]\{\dot{\mathbf{q}}\} + [K]\{\mathbf{q}\} = \{\mathbf{P}\}, \tag{4}$$

where $\zeta = vt$ is the current position of the moving mass in time.

Clearly, matrix $[M]$ depends on time, and the application of conventional procedures for numerical integration requires the inversion of the matrix at each time step, but the fact that this

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