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Composite Structures 49 (2000) 65–75

COMPOSITE STRUCTURES

www.elsevier.com/locate/compstruc

Estimation of transverse/interlaminar stresses in laminated composites – a selective review and survey of current developments

T. Kant *, K. Swaminathan

Department of Civil Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India

Abstract

A review is made on the different methods used for the estimation of transverse/interlaminar stresses in laminated composite plates and shells. Both analytical and numerical methods are considered. In numerical methods while the emphasis is given on finite element methods, other methods like the finite difference method is also briefly discussed. Aspects considered are: effects of variation in geometric and material parameters, transverse shear and normal deformation, interface stress continuity and the interfacial bonding on the accuracy of prediction of transverse/interlaminar stresses. Finally some general conclusions are presented along with future directions of research on the analysis of multilayered composite plates and shells for free-edge effects. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Composite laminates; Transverse stresses; Free-edge effects; Analytical methods; Finite element methods

1. Introduction

It is a well established fact that at free-edges in composite laminates, interlaminar stresses arise due to a mismatch in elastic properties between plies. Thus in this region near the free edge known as boundary layer, it has been shown that the stress state is three-dimensional in nature and not predictable accurately by *classical lamination theory* (CLT) [1,2]. Over the past 25 yrs, numerous investigators have used a variety of methods to attempt to calculate these stresses at straight free-edges. These include analytical and numerical methods. Review of literature with many citations up to the year 1989 can be found in [3–5]. A complete review of various shear deformation theories for the analysis of multilayered composite plates and shells is available in the review articles by Noor and Burton [6,7]. Later Reddy and Robbins [8] presented a review of various equivalent-single-layer and layerwise laminated plate theories and their finite element models. The purpose of the survey herein is to provide a brief but concise review of the current state-of-the-art in various methods of evaluation of interlaminar stresses in composite laminates.

The present paper deals with developments in the following sequence: analytical methods, numerical methods which include finite difference method, two-dimensional finite element method, three-dimensional finite element method, two-dimensional (2D) to three-dimensional (3D) global-local method.

It is felt that the present survey paper will be of interest to researchers and engineers already involved in the analysis and design of composite structures.

2. Analytical methods

In order to evaluate the 3D stress-field and the nature of stress concentration that occurs in composite laminates that have edge boundaries, a 3D elasticity boundary value problem must be solved. Unfortunately exact solutions to this problem are, as yet, unavailable. Thus numerous investigators have presented a variety of approximate methods to calculate the transverse/interlaminar stresses at straight free-edges.

The first approximate solution of finite-width composite laminates was proposed by Puppo and Evensen [9] based on a laminate model containing anisotropic laminae and isotropic shear layers with interlaminar normal stress being neglected throughout the laminate. Other approximate methods were also attempted to examine the problem such as extension of the

* Corresponding author.
E-mail addresses: tkant@civil.iitb.ernet.in (T. Kant), rsswami@civil.iitb.ernet.in (K. Swaminathan).

0263-8223/00/$ - see front matter
PII: S0263-8223(99)00126-9

higher-order plate theory [10] by Pagano [11], the perturbation method by Hsu and Herakovich [12], a boundary layer theory by Tang [13] and Tang and Levy [14], and approximate elasticity solutions by Pipes and Pagano [15]. Later, based on assumed in-plane stresses and the use of Reissner's variational principle, Pagano [16,17] developed an approximate theory. Even though there is no stress singularities involved in the formulation, the approach has certain features significantly important in objectively determining detailed laminate stress fields. In most methods the laminate is assumed to be sufficiently long. Hence, due to principle of Saint–Venant, the influence of the loading point on other remote regions is negligible. The validity of this principle was also assumed by Wang and Choi [18–21]. They used the Lekhnitskii's [22] stress potential and the theory of anisotropic elasticity and were able to determine the order of stress singularities at the laminate free-edges. The eigenfunction method developed by them involves the solution of a complicated and tedious eigenvalue problem and requires the use of a collocation technique at every ply interface in order to satisfy traction continuity. This limits the application of this technique to relatively thin laminate.

Stein [23] developed a 2D theory wherein the displacements are expressed by trigonometric series. In addition to the usual algebraic through-the-thickness terms assumed for the displacements, trigonometric through-the-thickness terms are added to give more accurate results. Later Stein and Jegley [24] using this theory studied the effects of transverse shear on cylindrical bending of laminated composite plates and proved that this theory predicts the stresses more accurately than other theories. A simple technique to analyse symmetric laminates under tension or compression based on assumed stress distributions using the principle of minimum complimentary energy and the force balance method was presented by Kassapogolou and Lagace [25,26]. Later Kassapogolou [27] generalised this approach for general unsymmetric laminates under combined in-plane and out-of-plane (moment and shear) loads. The formulation, although accurate for plates that are homogeneously anisotropic, does not adequately model the mismatches in Poisson's ratios and the coefficients of mutual influence that exist between different plies in the through thickness direction.

An accurate theory for interlaminar stress analysis should consider the transverse shear effect and continuity requirements for both displacements and interlaminar stresses on the composite interface. It is also advantageous if the formulation is variationally consistent so that it can also be used for finite element formulation. In view of the importance of satisfying the above conditions and obtaining the interlaminar shear stress directly from the constitutive equations, Lu and Liu [28] developed an *Interlaminar Shear Stress Continuity Theory* (ISSCT). Using this theory they were able to determine the interlaminar shear stress directly from the constitutive equations. However, due to the neglecting deformation in the thickness direction, they could not calculate the interlaminar normal stress directly from the constitutive equations. In addition, a small discrepancy between their results and Pagano's elasticity solution [29] in the interlaminar shear stress for composite laminates with small aspect ratios has also been reported. Touratier [30] proposed a theory in which the shear stress is represented by a certain sinusoidal function. Numerical results are presented for the bending of sandwich plates and compared with the results obtained from other theories to show that this theory was more accurate than both first-order shear deformation and some higher-order shear deformation theories. In conventional analysis for laminated composite materials the composite interface is always assumed to be rigidly bonded. However due to the low shear modulus and poor bonding, the composite interface can be non-rigid. Based on this understanding, Lu and Liu [31] in a continuation of the ISSCT, later developed the so-called *Interlayer Shear Slip Theory* (ISST) based on a multilayer approach to investigate interfacial bonding on the behavior of composite laminates. They used Hermite cubic shape functions as the interpolation function for composite layer assembly in the thickness direction and obtained closed-form solution for the cases of cylindrical bending of cross-ply laminates with non-rigid interfaces. From the results it was concluded that at some special locations, namely singular points, the transverse shear stress or in-plane normal stress remains insensitive to the condition of interfacial bonding. Later using the ISST, Lee and Liu [32] derived closed-form solution for the complete analysis of interlaminar stresses for both thin and thick composite laminates subjected to sinusoidal distributed loading. From the results it was shown that this theory could exactly satisfy the continuity of both interlaminar shear stress and interlaminar normal stress at the composite interface and also the interlaminar stresses could be determined directly from the constitutive equations.

Rohwer [33] presented a comparative study of various higher-order shear deformation theories for the bending analysis of multilayer composite plates. The advantages and disadvantages of the various theories were highlighted with the analysis carried out on a rectangular plate with varying slenderness ratio, layer number and thicknesses, edge ratios and material property relations. Using a double Fourier series approach Kabir [34,35] presented the results of the variations of transverse displacements and moments for various parametric effects for antisymmetric angle-ply $(45^\circ/-45^\circ)$ and symmetric angle-ply $(45^\circ/-45^\circ)_s$ laminated plate with simply supported boundary conditions at all edges. Later Kabir [36] using the same analytical approach and Kirchhoff's theory analysed a simply supported laminated plate with arbitrary laminations $(0^\circ/45^\circ)$ and compared the results with the available first-order shear deformation based finite element solutions. Ko and Lin [37] used boundary layer theory in conjunction with the method proposed by Kassapogolou and Lagace [25,26] to analyse the 3D stress distribution around a circular hole in symmetric laminate subjected to far-field in-plane stresses. All the boundary conditions for each ply and the interface traction continuity were exactly satisfied. The laminate was subdivided into interior region and boundary layer region and each stress component is determined by superposition of the interior stress and boundary layer stress. The Lekhnitskii's [22] theory of 2D anisotropic elasticity in conjunction with CLT [1,2] was used for the interior region and the stress function approach with principle of minimum complimentary energy in the vicinity of free-edge around the hole namely the boundary layer region. Later Ko and Lin [38] extended the same approach to analyse complete state of stress around a circular hole in symmetric cross-ply laminates under bending/torsion.

Wang and Li [39] used 3D anisotropic elasticity and the method of separation of variables to derive the equilibrium equations with unknown displacements for each cylindrical lamina of a multilayered shell subjected to axisymmetrically distributed mechanical and thermal load with various end boundary conditions. Then making the displacements and stress expressions satisfy the boundary conditions at the interfaces of the plies, they were able to determine the interlaminar stresses exactly.

Wu and Kuo [40] proposed a local higher-order lamination theory to evaluate the interlaminar stresses. They derived the equilibrium equation by introducing the displacement continuity constraints at the interface between layers into the potential energy functional of the laminates by Lagrange multiplier method and defining the Lagrange multipliers as the interlaminar stresses $(\tau_{xz}, \tau_{yz}, \sigma_z)$ at the interface between the layers. They used the Fourier series expansion method to analyse the problem. Since they introduced the interlaminar stresses as the primary variables, they could avoid the tedious integration operation in the equilibrium equations method as well as the discontinuities in interlaminar stresses at the interface in constitutive equations method.

Becker [41] made use of warp deformation mode in the form of a cos-function for v displacement and sine-function deformation mode for w displacement and developed a new closed-form higher-order laminated plate theory.

Using the approach similar to that of Kassapoglou and Lagace (see, for example, Refs. [25,26]), an analytical method was presented by Mortan and Webber [42] to determine the free-edge stresses due to thermal effects.

To study the interlaminar stresses in cylindrical shells under static and dynamic transverse loads and to determine the dynamic magnification factors (DMF), (i.e. the ratio of the maximum dynamic response to the corresponding static response) Bhaskar and Varadan [43] used the combination of Navier's approach and a Laplace transform technique to solve the dynamic equations of equilibrium. The analysis has been carried out within the purview of a Mindlin type first-order shear deformation theory (FSDT). From the results they observed that the DMF for the deflection and the in-plane stress remain close to 2.0, and for the interlaminar stresses can reach higher values depending on the geometry of the shell and the localised nature of loading.

An approximate method based on equilibrated stress representations and using the principle of minimum complimentary energy to investigate the interlaminar stresses near straight free-edges of beam-type composite laminate structures under out-of-plane shear/bending was developed by Kim and Atluri [44]. The analysis is different from the previous assumed stress method in that it includes longitudinal degrees-of-freedom (dof) in the stress distribution. The unknowns in the resulting stress expressions are obtained by solving an eigenvalue problem whose coefficients are not constants but depend on the shear loading location. The stress equilibrium, compatability and all of the boundary conditions are satisfied.

Interlaminar stresses arise in order to satisfy equilibrium at locations with in-plane stress gradients. One such case of stress gradients arises when there is a material discontinuity within a structure. To evaluate the interlaminar stresses at material discontinuities Bhat and Lagace [45] proposed an analytical method.The laminate under investigation was subdivided into two regions with different layups joined together. They expressed the stress in each region in terms of eigenfunctions which satisfy equilibrium and used the principle of minimum complimentary energy to obtain the differential equations of the problem and thus solve the eigenfunctions in each region. An approximate analytical method based on the variational principle of complimentary virtual work and using Lekhnitskii's stress functions in each layer was proposed by Yin [46] for free-edge stresses due to thermal and mechanical loads. The method, though not rigorous and accurate, was applicable to free-edge problems involving non-linear and inelastic material behavior.

Connolly [47] obtained simplified equations for determination of interlaminar normal stress based on the simplification of the more general solution provided by Kassapoglou [27]. The equation takes into account the influence of material properties and geometry on the maximum values of normal interlaminar stresses at layer interfaces.

For most designs of composite laminates, symmetric layup about the midplane are often desirable in order to avoid coupling effects between bending and extension. However some practical applications require unsymmetric laminates to achieve specific design requirements. In order to study the influence of bending-extension coupling on the interlaminar stress distribution in unsymmetric laminates Lin et al. [48] extended the method proposed by Kassapoglou and Lagace [25] and showed that the stress variations for each ply in unsymmetric laminates are more complex than those assumed by Kassapoglou and Lagace.

He [49] proposed a refined shear deformation laminated shell theory with discrete layer modeling based on the assumption that the transverse shear strain across any two different layers are linearly dependent on each other. The theory contains only five dependent variables as in FSDT [50] for laminated shells but the set of governing equations is of the twelfth-order, i.e. two orders higher than FSDT. The restriction on the application of this theory is that the thickness of the shell must be small compared to the principal radii of curvature. Thus, the analytical solutions can be obtained for only a few cases. Later He and Zhang [51] using the above theory obtained closed-form solutions for the bending analysis of rectangular simply supported antisymmetric angle-ply laminated plates subjected to sinusoidal transverse loads. From the results it was shown that this theory could give better estimates of stresses and displacements as compared to FSDT and CLT.

Ramalingeswara and Ganesan [52] made a comparative study of the interlaminar stresses in shells of revolution using FSDT, higher-order shear deformation theory with thickness stretch (HSDT7) and a higher-order shear deformation theory with higher-order inplane displacement terms (HSDT9). The interlaminar stresses were evaluated using equilibrium equations. Cross-ply parabolic and hyperbolic caps subjected to uniform external pressure and a simply supported cylindrical shell subjected to an internal sinusoidal pressure were considered in their study. Later using the above three models, Ramalingeswara and Ganesan [53] compared the results of interlaminar stresses in a cross-ply spherical shell subjected to uniform transverse pressure. Recently a comparison of the analytical solutions of a few laminated plate theories for the analysis of multilayer composite plates were presented by Idlbi et al. [54] Carvelli and Savoia, [55] Bose and Reddy [56].

3. Numerical methods

The approximate analytical methods discussed in Section 2 are often inadequate for evaluation of local stress concentrations, and the procedure becomes extremely tedious when a multilayer laminate is involved. A common feature of all analytical methods is that they can only be used for the simplistic geometric cases, since for thick realistic structural laminates, the solution to the full 3D problem is extremely complex. Thus a variety of numerical methods, e.g., finite difference and finite element, have been developed to calculate these interlaminar stresses at straight free edges. These methods not only provide the option of placing a refined mesh near regions of possible stress concentration but can also be used with ease for the accurate analysis of laminated composite structures having complicated geometry and/or loading.

3.1. Finite difference method

The first theoretical attempt to solve the free-edge problems of anisotropic elasticity in conjunction with a numerical method to solve the governing partial differential equation was given by Pipes and Pagano [57]. They employed the finite difference method for their solution. This method of solution has been well established as a technique for obtaining numerical solutions for elliptical partial differential equations [58]. A four layer finite-width composite laminate under uniform axial strain was studied. In their investigation the first hint for possible stress singularities at the free-edge was given. Moreover since they used a relatively coarse mesh in the finite difference method, the exact nature of the stress singularities at the free edges could not be ascertained. Following the approach used by Pipes and Pagano [57] the interlaminar stress distribution in a four layer composite laminate in bending was studied by Salamon [59]. He predicted that the magnitudes of the interlaminar normal and shear stresses, although in general relatively small, rise sharply near the free-edges. This distinguishing feature was observed over a boundary region of the order of one laminate thickness inward from the free-edge. Later Atlus et al. [60] presented a 3D finite difference solution for the free edge effects in angle-ply laminates. It was shown that the 3D finite difference method gave improved results as compared to 2D analytical or numerical methods used earlier. They were able to conclude that the peeling stress σ_{zz} and the longitudinal stress σ_{xx} have a dominant effect on interlaminar failure characteristics.

3.2. 2D finite element method

A considerable body of literature exists on the modeling and analysis of laminated composite plates and shells using a 2D finite element. Reddy [61] gave a complete review of the literature on finite element modeling of laminated composite plates. But in his paper only investigations up to the year 1985 were included. Later Kapania [62] presented a review of literature up to the year 1989 on the analysis of laminated shells. For this reason and in view of the recent diverse advances in finite element analyses of laminated composite plates and shells, it is timely to consider the subject again and represent it in detail. A review of the recent literature on the 2D finite element modeling of laminated composite plates and shells is given in this section.

Engblom et al. [63] developed a shear deformable isoparametric plate and shell element which includes the shear effects by allowing mid surface displacements to be independent of the rotations. The formulation is based on eight-noded quadrilateral geometries with four corner nodes and four midside nodes located at the mid surface of the element with six dof per node. The displacement field is expressed in the following form:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z[n_z\theta_x(x,y) - n_y\theta_z(x,y)],\\ v(x,y,z) &= v_0(x,y) + z[n_z\theta_y(x,y) + n_x\theta_z(x,y)],\\ w(x,y,z) &= w_0(x,y) - z[n_x\theta_x(x,y) + n_y\theta_y(x,y)] \end{aligned} \tag{1}$$

in which $u_0,\ v_0,\ w_0$, respectively represents the midplane displacements and $\theta_x,\ \theta_y,\ \theta_z$ are the surface rotations. A bi-quadratic interpolation (shape) function is utilised to specify location of nodal points and to specify displacement variations. They used Guass quadrature to perform the integration within each layer and the transverse stresses are calculated using equilibrium equations.

An eight-noded quadrilateral plate element with five dof at each of the midside and corner nodes was formulated by Hamdallah and Engblom [64]. The plate element developed includes shear effects. For the purpose of analysing 3D structures they introduced a sixth dof to represent rotations normal to the plane of the element. The displacement field of the element can be written as:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y),\\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y),\\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{2}$$

where the total displacements are represented by u, v and w whereas the mid surface displacements are given by $u_0,\ v_0$ and w_0. The rotations about y and x axes are represented by θ_x and θ_y, respectively. They used the equilibrium equations for calculating the transverse stresses.

Manjunatha and Kant [65] formulated C^0 finite elements based on a set of higher-order theories which take into account the effect of non-linear variations of in-plane displacements, transverse shear deformation, transverse normal strain thus eliminating the need for shear correction coefficients. Sixteen and nine noded Lagrangian isoparametric elements are used for comparison. The various refined higher-order theories used in their study are summarized below separately for symmetric and unsymmetric laminates in the increasing order of their dof.

Symmetric laminates:

1. Higher-order shear deformation theory (HOST 5A), 5 dof/node

$$\begin{aligned} u(x,y,z) &= z\theta_x(x,y) + z^3\theta_x^*(x,y),\\ v(x,y,z) &= z\theta_y(x,y) + z^3\theta_y^*(x,y),\\ w(x,y,z) &= w_0(x,y). \end{aligned} \tag{3}$$

2. Higher-order shear deformation theory (HOST 6A), 6 dof/node

$$\begin{aligned} u(x,y,z) &= z\theta_x(x,y) + z^3\theta_x^*(x,y),\\ v(x,y,z) &= z\theta_y(x,y) + z^3\theta_y^*(x,y),\\ w(x,y,z) &= w_0(x,y) + z^2 w_0^*(x,y). \end{aligned} \tag{4}$$

Unsymmetric laminates:

1. Higher-order shear deformation theory (HOST 7A), 7 dof/node

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^3\theta_x^*(x,y),\\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^3\theta_y^*(x,y),\\ w(x,y,z) &= w_0(x,y). \end{aligned} \tag{5}$$

2. Higher-order shear deformation theory (HOST 9), 9 dof/node

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y),\\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y),\\ w(x,y,z) &= w_0(x,y). \end{aligned} \tag{6}$$

3. Higher-order shear deformation theory (HOST 11), 11 dof/node

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y),\\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y),\\ u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y). \end{aligned} \tag{7}$$

4. Higher-order shear deformation theory (HOST 12), 12 dof/node

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y),\\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y),\\ w(x,y,z) &= w_0(x,y) + z\theta_z(x,y) + z^2 w_0^*(x,y) + z^3\theta_z^*(x,y), \end{aligned} \tag{8}$$

where u, v and w are the displacements of a general point $(x,\ y,\ z)$ in the laminate in the x, y and z directions, respectively. The parameters $u_0,\ v_0,\ w_0,\ \theta_x,\ \theta_y$ and θ_z are the appropriate 2D terms in the Taylor series and are defined in x–y plane at $z = 0$. The parameters $u_0^*, v_0^*, w_0^*, \theta_y^*$ and θ_z^* are higher-order terms in the Taylor's series expansion that are difficult to interpret in physical terms, except that they represent higher-order transverse cross-sectional deformation modes. Later Kant and Manjunatha [66] used the same formulation to study the transverse stresses in multilayer laminates but different methods to integrate the equilibrium equations. They used the exact surface fitting method, direct integration method and forward and central direct finite difference methods. Their study shows that the exact surface fitting method gave an accurate estimate of the transverse stresses compared to other methods. Kant and Menon [67] extending the same C^0 based finite element formulation presented a higher-order displacement model for the analysis of symmetric and unsymmetric laminated composite sandwich cylindrical shells. Two shell theories, namely the geometrically thin shell theory with the shell thickness to radius is less than unity and a geometrically thick shell theory with the square of the ratio of the shell thickness to radius is less than unity, were developed. Results are compared with available results in the literature to show the accuracy of the above model. Using the same formulation Kant and Menon [68] analysed a symmetric and asymmetric laminated cylindrical shell for interlaminar stresses. A finite difference scheme maintaining the continuity of interlaminar stresses across the shell thickness was developed and used. From the results it was concluded that the geometrically thick shell theory gives more accurate and reliable solutions than those of shell theory for both thin and moderately thick shells.

An interlaminar stress mixed finite element method based on the local higher-order lamination theory was presented by Wu and Kuo [69] to analyse thick symmetric laminated composite plates. In their theory, the displacement continuity at the interface between layers are introduced in to the potential energy functional of the considered laminates using Lagrange multipliers and these Lagrange multipliers are defined to be the interlaminar stresses (τ_{xy}, τ_{yz} and σ_z) at the interface between layers. The modified potential energy functional is given by

$$\begin{aligned}\pi_{\mathrm{mp}} = {} & \sum_{i=1}^{NL} \int\!\!\int_A \int_{-h_i/2}^{h_i/2} \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy}\right.\\ & \left. + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right]_i \mathrm{d}z\,\mathrm{d}A - \int\!\!\int_A \left[T_x u^* + T_y v^*\right.\\ & \left. + T_z w^*\right] \mathrm{d}A + \sum_{i=1}^{NL-1} \int\!\!\int_A \left[(\lambda_x)_i (f_x)_i\right.\\ & \left. + (\lambda_y)_i (f_y)_i + (\lambda_z)_i (f_z)_i\right] \mathrm{d}A, \end{aligned} \tag{9}$$

where T_x, T_y and T_z are the tractions applied at upper surfaces. u^*, v^* and w^* are the displacement components at upper surface. $(\lambda_x)_i(\lambda_y)_i$ and $(\lambda_z)_i$ are defined as the tractions (τ_{xz}, τ_{yz} and σ_z, respectively) at the interface between ith and $(i+1)$th layers. A nine-noded quadrilateral C^0 isoparametric element is used in their formulation. The nodal unknowns assumed are the three displacements, three rotations and five higher-order functions as the dof in the mid surface of each layer and three interlaminar stress function as the dof at the interface between the layers. The local displacement fields are expressed by:

$$\begin{aligned} u_i(x,y,z) &= u_0(x,y) + z_i[\theta_x(x,y)] + z_i^2[\psi_x(x,y)] + z_i^3[\phi_x(x,y)],\\ v_i(x,y,z) &= v_0(x,y) + z_i[\theta_y(x,y)] + z_i^2[\psi_y(x,y)] + z_i^3[\phi_y(x,y)],\\ w_i(x,y,z) &= w_0(x,y) + z_i[\theta_z(x,y)] + z_i^2[\psi_z(x,y)], \end{aligned} \tag{10}$$

where $u_0, v_0,$ and w_0 are three mid-surface displacement functions, θ_x, θ_y and θ_z are three rotation function and $\psi_x, \psi_y, \psi_z, \phi_x$ and ϕ_y are the other higher-order functions. z_i is measured from the middle surface of the ith layer. The main advantage of this formulation is that since interlaminar stresses are treated as the primary parameters, the interlaminar stresses at the interface between the layers can be uniquely and accurately determined. Later Wu and Yen [70] extended the same formulation to analyse unsymmetrically laminated composite plates.

Di Sciuva [71] developed a four noded quadrilateral plate element with 10 dof/node using improved zig-zag model proposed by him earlier [72]. The displacement field allows a non-linear variation of the in-plane displacements through the laminate thickness and fulfils a priori geometric and transverse stress continuity condition at interfaces, and the static condition of zero transverse shear stresses on the top and bottom surfaces for symmetric laminates. The plate model used is based on the following representation of the displacement field across the plate thickness.

$$\begin{aligned} u &= u^0 - z\,w^0_{,x} + f(z)g_x + \sum_{k=1}^{N-1}\phi_k(z-z_k)Y_k,\\ v &= v^0 - z\,w^0_{,y} + f(z)g_y + \sum_{k=1}^{N-1}\psi_k(z-z_k)Y_k,\\ w &= w_0, \end{aligned} \tag{11}$$

where

$$f(z) = z\left(\delta_F - \delta_T \frac{4}{3h^2} z^2\right).$$

$u^0, v^0, w^0, g_x, g_y, \phi_k, \psi_k$ are unknown functions of the x and y; z_k are the coordinates of the $N-1$ interfaces; Y_k is the Heaviside unit function; it takes a value of 0 for $z < z_k$ and the value 1 for $z \geqslant z_k$. The rotation $(\cdot)_{,\alpha}$ stands for partial derivative $\partial(\cdot)/\partial\alpha$, δ_F and δ_T are tracers which identify the contribution brought by the various plate models.

A computational model based on FSDT was presented by Noor et al. [73] for accurate determination of transverse shear stresses and their sensitivity coefficients in flat multilayered composite panels subjected to mechanical and thermal loads. The sensitivity coefficients measure the sensitivity of the transverse shear stresses to variation in different lamination and material parameters of the panel. The panel discretization is done by using either a three-field mixed finite element based on a 2D first-order shear deformation plate theory or a two-field degenerate solid element with each of the displacement components having a linear variation through the thickness of the laminate. They evaluated the transverse shear stresses in two phases. The first-phase consists of using a superconvergent recovery technique for evaluating the in-plane stresses in different layers. In the second-phase transverse shear stresses are evaluated through piecewise integration of the 3D equilibrium equation in the thickness direction. The same procedure is used for evaluating the sensitivity coefficients of the transverse shear stresses. They have presented extensive numerical results for multilayered cross-ply panels and made comparison with those of 3D finite element models and an exact solution of the 3D thermo-elasticity equation of the panel.

A three-noded axisymmetric shell element in curvilinear coordinates with 10 dof was proposed by Touratier and Faye [74] to analyse the edge effects in axisymmetric shells. The formulation does not take in to account the transverse normal strain. The element proposed is of C^1 continuity for the transverse displacements and C^0 continuity for the membrane displacement and the membrane-shear rotation. The displacement field assumed is of the form

$$\begin{aligned}\overline{U}_1^{\beta}(\xi_1,\xi_2,\zeta,t) &= \frac{L_\beta}{\alpha_\beta}\overline{u}_\beta(\xi_1,\xi_2,t) - \frac{\zeta}{\alpha_\beta}\,\overline{w}_{,\beta}(\xi_1,\xi_2,t)\\ &\quad + \frac{c}{\pi}\,\sin\frac{\pi\xi}{e}\overline{\gamma}_\beta^0(\xi_1,\xi_2,t), \qquad (12)\\ \overline{U}_\zeta^{a}(\xi_1,\xi_2,\zeta,t) &= \overline{w}(\xi_1,\xi_2,t),\end{aligned}$$

where

$$\overline{w},\beta(\xi_1,\xi_2,t) = \frac{\partial\overline{w}}{\partial\xi_\beta}, \quad \beta = 1 \text{ or } 2.$$

t is the time, e the thickness of the shell, $\overline{U}_1^a, \overline{U}_2^a, \overline{U}_\xi^a$ the appropriate displacement components in curvilinear coordinates at an arbitrary points (ξ_1,ξ_2,ζ) and in the direction of ξ_1,ξ_2,ζ, $(\overline{u}_1,\overline{u}_2,\overline{w})$ are the displacements of a point on the middle surface, $\overline{\gamma}_0^1$ and $\overline{\gamma}_0^2$ are the transverse shear strains at $\zeta = 0$. The formulation fully avoids transverse shear locking and membrane locking is avoided by using the assumed covariant strain method [75]. Bose and Reddy [76] presented finite element models of various shear deformation theories for the analysis of composite plates and compared transverse displacements and through the thickness distributions of in-plane and transverse stresses. A new method which reduces the order of differentiation by one as compared to the standard equilibrium approach was presented by Rolfes et al. [77]. The method requires only quadratic shape functions for evaluating the required derivatives at the element level and also the computational effort is low since it requires only C^0continuity shape functions in the finite element code. The accuracy of this method is established by comparing the results of symmetric cross-ply and antisymmetric angle-ply laminates with exact 3D elasticity solution.

3.3. 3D finite element method

There have been several studies in the literature using three-dimensional finite elements to estimate interlaminar stresses in the critical regions of laminates. Usually the analysis results in a large sparse system of equations, which requires a vast amount of computer storage space and thus makes 3D finite element modeling impracticable and possibly formidable. In view of the above fact only a few publications are available in this category. In this section only the developments that have taken place over the past five years are presented. For earlier work the reader may consult Refs. [78–82].

Kim and Hong [83] used a 16 noded curved isoparametric element without a midside node in the thickness direction and 48 dof. They used the substructure technique and analysed a laminate with and without hole. The effect of laminate thickness and stacking sequence on the interlaminar stress near the free edges in the case of a solid laminate and near the hole boundary in the case of a laminate with a hole were studied.

Wanthal and Yang [84] developed three finite elements for the analysis of thick laminates where the effect of transverse shear deformation was very severe. The first layer quadrilateral element (LQ1) is of 16 nodes with 40 dof with zero transverse normal strain and constant transverse shear strain. The second layer quadrilateral element (LQ2) is of 16 nodes with 48 dof and allows for a constant transverse normal strain and one of the two terms in the expression for transverse shear strain is allowed to vary linearly through the thickness. The third layer quadrilateral element (LQ3) is of 24 nodes with 64 dof and improves upon the LQ2 element by allowing both terms in the transverse shear displacement expression to vary linearly through the layer thickness. Later Yang and He [85] used the LQ3 element for the analysis of free-edge stresses in cross-ply and angle-ply laminate. They used the preconditioned conjugate method to solve the system of linear equations.

Wei and Zhao [86] used a eight-noded linear element to analyse the stresses of the symmetric cross-ply and angle-ply laminate loaded with uniform axial strain. A quasi 3D finite element analysis with an initial iteration stress method was used by Wu [87] for the elasto-plastic analysis of metal matrix angle-ply composite for their thermo-mechanical free-edge effects. The Hill yield criterion for anisotropic materials is used in the analysis.

Icardi and Bertetto [88] studied the stress singularity at the free edge using a 20 noded quadratic interpolation, isoparametric brick element and a 15 noded quadratic interpolation singular wedge element generated from the 20 noded brick element. The nodal parameters assumed for both elements are the three displacement components u, v, w in the x, y, z directions. A predictor–corrector procedure is used to fulfil the stress contact and traction free condition. The effect of material properties and layer orientations, the slope of inclined edges and corner angles were studied.

Chen et al. [89] used finite element least-square extrapolation and smoothing technique to evaluate the interfacial stress distributions in composite laminates. A quasi 3D technique and complete 3D analysis were both used to investigate the stress distribution in a graphite-epoxy laminate. Linear and quadratic least-square fits using two-point and three-point Gaussian integration in eight noded parabolic quadrilaterals and 20 noded solid isoparametric elements were used. From the results obtained for symmetric laminate, it was concluded that the use of above technique offers better estimates of stress distributions and interfacial stresses in composite laminates.

In order to overcome the difficulties encountered by the fully 3D model Lessard et al. [90] developed the 'Slice Model' thereby reducing the number of elements required by the fully 3D model, while at the same time retaining the use of 20 noded quadratic brick element. A cross-ply laminate subjected to uniaxial tensile strain was used in the analysis. Through the analysis, it was shown that the slice model minimizes the number of elements far from the anticipated singularity and allows for a very fine mesh area near the critical high stress gradient regions of a composite laminate. It was also demonstrated that the computer run time also reduced drastically. Soutis et al. [91] presented the results of interlaminar stress distributions around a circular hole in symmetric composite laminates under in-plane tensile loading using 3D finite element analysis and compared the results with those estimated by Ko and Lin [37] analytical approach. From the results it was shown that the finite element analysis results are considerably higher than those predicted by Ko and Lin analysis.

3.4. 2D–3D global/local finite element method

The initial approach adopted to analyse composite laminates of finite size subjected to external loads was of a 2D finite element method. Though the 2D elements can yield accurate results at locations away from the traction free-edges and discontinuities, it cannot predict accurately the complex stress state near any geometric or material discontinuities or near a traction free-edge [92]. As limitations of the 2D technique became known and more powerful computers became available, 3D finite elements became increasingly used [78,92]. For many applications however a full 3D analysis can be a waste of resources. In view of the above facts many investigators attempted global/local finite element analysis [93–97] that perform separate analyses on the global and local region. Thomson and Griffin [98] extended the same approach and proposed a 2D–3D global/local finite element analysis. They subdivided the entire laminate into local and global regions, the local region being the traction free-edges, and the area around geometric or material discontinuities and the global region is location far away from local region. They used a simplified 2D finite element analysis on a global region and a more detailed 3D finite element analysis on a local region. Later they extended this approach to study the stress state around a hole in a cross-ply and general symmetric laminated plate with a central hole [99]. They were able to demonstrate that the global/local analysis technique yields a reasonably economical solution by achieving considerable savings in computer time and storage as compared to a complete 3D finite element method.

4. Future directions of research

The occurence of interlaminar stresses at the geometric boundaries such as free-edges, cut-outs, notches, and holes of structural components madeup of composite laminate is an important phenomenon since high a concentration of these stresses may result in delamination cracks at these locations which reduce the strength and stiffness and thus limit structural life. Therefore there is a need to undertake the following studies and extensions:

1. A detailed study of the effects of anisotropy and discontinuities in the plate/shell topology on the significance of transverse shear and transverse normal strains and the extent of edge zone or boundary layer.
2. A comparative study not only to validate the accuracy but also to highlight the advantages and drawbacks of many existing computational models based on various laminate theories (equivalent single layer theories, discrete layerwise theories, complete layerwise theories) and to understand the physical phenomenon associated with the transverse/interlaminar stresses on the plate/shell behavior under complex geometry, loading and boundary conditions.

5. Conclusion

A review is made of recent developments in different methods used for the estimation of transverse/interlaminar stresses in multilayered plates and shells. The literature devoted to analytical and numerical methods is reviewed. Discussion focuses on the accuracy of various analytical and numerical models. The effects of variations in geometric and material parameters, transverse shear and normal deformation, aspects of interface stress continuity and interfacial bonding on the accuracy of prediction of transverse/interlaminar stresses in laminated composite plates and shells are also discussed.

On the basis of current literature survey, the following general conclusions seem to be justified.

1. The CLT and first-order shear deformation theory (FOST) generally provide an acceptable compromise between accuracy and economy in predicting the global responses of thin and relatively thin composite laminates. But these theories fail to give accurate results for the through-the-thickness stress response in regions of discontinuity such as cut-outs, holes, and boundaries. Moreover these theories require shear correction coefficients to rectify unrealistic variations of the shear strain/stress through the thickness.
2. 3D theories, in which each layer is treated as a homogeneous anisotropic material, predicts the 3D stress state at the boundaries more accurately than CLT and FOST, but their storage requirements due to the large number of variables and computer costs make them impracticable.
3. Because of the complexities, analytical solutions for the prediction of transverse/interlaminar stresses exist for composite laminates with simple geometry, loading and boundary conditions. Therefore more emphasis has been placed on the use of numerical methods when the composite laminate problem involves complicated geometry, loading and boundary conditions. Among the various numerical techniques available, it is seen that the finite element method is not only simple but straight forward for efficient programming and also versatile enough to cover all types of problems relevant to practical situations. Using this technique it has been possible to incorporate the effect of moisture and temperature on the interlaminar stresses.
4. Laminate analyses using displacement theories are preferred to stress based theories which are seldom used in practice because of the difficulty in developing reliable finite element methods.
5. In the finite element method C^0 continuity elements are preferred to C^1 continuity elements as the latter complicates development of conforming elements and inhibits their use with other commonly used finite elements.
6. For many applications a full 3D finite element analysis is not cost effective as it needs large computer core storage and running times in comparison with 2D finite element analysis. Thus, a 2D–3D global/local finite element method can drastically improve the efficiency of computerised analysis and provide sizable savings by circumventing the need to perform expensive analyses near critical regions.
7. In the evaluation of transverse/interlaminar stresses, the stresses are most commonly obtained using post processing technique by integrating the equilibrium equations of 3D elasticity rather than using the constitutive relations as the latter method leads to discontinuities of stresses at the interface of two adjacent layers of a laminate and thus violates the equilibrium equations.

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