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Analysis of Orthotropic Plates Based on Three Theories by Segmentation Method

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ABSTRACT

The scope of the segmentation method is extended to the solution of orthotropic rectangular plates, simply supported on two opposite edges and with any other boundary conditions on the remaining two edges. Governing first-order ordinary differential equations are derived for each of three theories, Kirchhoff, Reissner-Mindlin, and a higher-order theory. A wide range of problems of plates are solved, and the solutions obtained are compared with three-dimensional (3-D) elasticity solutions wherever available. New results of orthotropic plates with different boundary conditions are presented. It is shown that for geometrically thin plates, solutions from all three plate theories converge to the classical Kirchhoff solution, while for thick plates, solutions from only the higher-order theory are found to be close to the 3-D elasticity solution.

The present work is next in a series of articles dealing with the static analysis of thin/thick plates by the segmentation method [1–4]. In the earlier articles, the problem of bending of thin/thick isotropic plates, simply supported on two opposite edges and with any other boundary conditions on the remaining two edges, was attempted using the segmentation method. The formulations were based on Kirchhoff [2], Reissner-Mindlin [3], and higher-order [4] theories. In these articles, isotropic homogeneous material was considered in the formulation, requiring only two material constants (E and ν). In the present work the earlier formulations are extended to tackle the problem of geometrically thin/thick orthotropic plates. The number of required material constants in the constitutive relations increases to 4, 6, and 9 for the Kirchhoff, Reissner-Mindlin, and higher-order plate theories, respectively. Complete formulations suitable for the segmentation method are presented for each of these theories, which in final form are a set of first-order ordinary differential equations with constant coefficients. The set of ordinary differential equations thus obtained is integrated numerically by the segmentation method, from one edge of the plate to the other. Both the force and displacement boundary conditions are satisfied at all the edges. The solution technique employed is found to be numerically very efficient and accurate for the class of problems considered here. Results obtained here are compared with three-dimensional (3-D)

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elasticity analytical solutions wherever available. Excellent agreement of results obtained in the present work is seen with the 3-D elasticity analytical solution. It is also seen that as the span/depth ratio increases, results from all the theories converge to the classical thin orthotropic plate solution.

THEORETICAL FORMULATION

Governing equations of three theories, Kirchhoff, Reissner-Mindlin, and higher-order, which define boundary-value problems, are summarized in Appendixes A, B, and C, respectively. Numerical integration of such boundary-value problems by the segmentation method, which was originally due to Goldberg et al. [5], involves reduction of the two-dimensional (2-D) plate problem to a one-dimensional (1-D) problem by assuming a solution in one direction, which is chosen here as the y direction. The governing equations are manipulated algebraically to obtain a set of first-order differential equations involving only some dependent variables which appear naturally on an edge $x =$ constant. These dependent variables are called *intrinsic dependent variables*, and the corresponding differential equations with x as independent variable are termed *intrinsic equations*. The number of such intrinsic variables equals the order of the governing partial differential equation of the theory (fourth, sixth, and twelfth order for Kirchhoff, Reissner-Mindlin, and higher-order theories, respectively). Intrinsic equations consist of a set of first-order partial differential equations, each of which necessarily contains the first derivative with respect to x of one of the so-called intrinsic dependent variables. These intrinsic dependent variables appear naturally on the edge $x =$ constant. In the present formulation these dependent variables, stored in vector **Y**, are w, θ_x, V_x, and M_x for Kirchhoff theory, w, θ_x, θ_y, Q_x, M_x, M_{xy} for Reissner-Mindlin theory, and w, θ_x, θ_y, w^*, θ_x^*, θ_y^*, Q_x, M_x, M_{xy}, Q_x^*, M_x^*, M_{xy}^* for higher-order theory. The system of equations obtained after the required manipulations for each of the three theories, along with the basic assumptions in the respective theories, are presented below.

Kirchhoff thin-plate theory

The formulation for the Kirchhoff thin-plate theory is based on the following assumptions:

1. The plate is thin, i.e., $h/a \ll 1$, where h and a are the thickness of the plate and the width of the plate, respectively.
2. In-plane displacements vary linearly through the plate thickness.
3. Normals to the mid-surface remain straight and normal to it, and their lengths remain unchanged during deformation.

Algebraic manipulation of the governing equations, given in Appendix A, gives rise to the following system of equations:

$$\frac{\partial w}{\partial x} = -\theta_x \tag{1a}$$

$$\frac{\partial \theta_x}{\partial x} = \frac{D_1}{D_x}\frac{\partial^2 w}{\partial y^2} + \frac{M_x}{D_x} \tag{1b}$$

$$\frac{\partial V_x}{\partial x} = \left(D_y - \frac{D_1^2}{D_x}\right)\frac{\partial^4 w}{\partial y^4} - \frac{D_1}{D_x}\frac{\partial^2 M_x}{\partial y^2} - (p_z^+ + p_z^- + \rho h) \tag{1c}$$

$$\frac{\partial M_x}{\partial x} = -4D_{xy}\frac{\partial^2 \theta_x}{\partial y^2} + V_x \tag{1d}$$

The other dependent variables are expressed as functions of intrinsic variables, by simple algebraic relations called *auxiliary relations*, in the following form:

$$M_y = -\left(D_y - \frac{D_1^2}{D_x}\right)\frac{\partial^2 w}{\partial y^2} + \frac{D_1}{D_x}M_x \tag{2a}$$

$$M_{xy} = 2D_{xy}\frac{\partial \theta_x}{\partial y} \tag{2b}$$

$$Q_x = 2D_{xy}\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial M_x}{\partial x} \tag{2c}$$

$$Q_y = \left(\frac{D_x D_y - 2D_{xy}D_1 - D_1^2}{D_x}\right)\cdot\frac{\partial^3 w}{\partial y^3} + \left(\frac{2D_{xy} + D_1}{D_x}\right)\frac{\partial M_x}{\partial y} \tag{2d}$$

$$V_y = \left(\frac{D_x D_y - 4D_{xy}D_1 - D_1^2}{D_x}\right)\cdot\frac{\partial^3 w}{\partial y^3} + \left(\frac{4D_{xy} + D_1}{D_x}\right)\frac{\partial M_x}{\partial y} \tag{2e}$$

The generalized displacement components and the corresponding stress resultants which form the vector **Y** of the intrinsic variables are functions of x and y. For a plate with two opposite edges $y = 0$ and $y = b$ simply supported, these may be represented in the form of a Fourier series, which automatically satisfies both the displacement and the force boundary conditions along these edges to any desired degree of accuracy, as follows:

$$w(x, y) = \sum_{m=1,3,5} w_m(x)\sin\frac{m\pi y}{b} \tag{3a}$$

$$\theta_x(x, y) = \sum_{m=1,3,5} \theta_{xm}(x)\sin\frac{m\pi y}{b} \tag{3b}$$

$$V_x(x, y) = \sum_{m=1,3,5} V_{xm}(x)\sin\frac{m\pi y}{b} \tag{3c}$$

$$M_x(x, y) = \sum_{m=1,3,5} M_{xm}(x)\sin\frac{m\pi y}{b} \tag{3d}$$

Substitution of the Eqs. (3*a*)–(3*d*) in the system of Eqs. (1*a*)–(1*d*) and analytic integration of these equations with respect to the independent variable y, coupled with the use of orthogonality conditions of the basic beam functions used in the y direction in the aforesaid expansions, reduces the set of partial differential Eqs. (1*a*)–(1*d*) into the following set of simultaneous first-order ordinary differential equations (say, for the mth harmonic) involving only four intrinsic variables. The series uncouples with respect to the harmonic m, leading to a term-by-term analysis which enables storage of only the final discrete values of the intrinsic and auxiliary dependent variables corresponding to a particular harmonic analysis to be added to the values of subsequent harmonic analysis:

$$\frac{dw_m}{dx} = -\theta_{xm} \tag{4a}$$

$$\frac{d\theta_{xm}}{dx} = -\frac{D_1}{D_x}m^2\frac{\pi^2}{b^2}w_m + \frac{M_{xm}}{D_x} \tag{4b}$$

$$\frac{dV_{xm}}{dx} = \left(D_y - \frac{D_1^2}{D_x}\right)\frac{m^4\pi^4}{b^4}w_{mx} + \frac{D_1}{D_x}\frac{m^2\pi^2}{b^2}M_{xm} - \frac{4}{m\pi}(p_z^+ + p_z^- + \rho h) \quad (4c)$$

$$\frac{dM_{xm}}{dx} = 4D_{xy}\frac{m^2\pi^2}{b^2}\theta_{xm} + V_{xm} \quad (4d)$$

and the auxiliary relations (2*a*)–(2*e*) take the form

$$M_y = \sum_{m=1,3,5}\left[\left(D_y - \frac{D_1^2}{D_x}\right)\frac{m^2\pi^2}{b^2}w_{mx} + \frac{D_1}{D_x}M_{xm}\right]\sin\frac{m\pi y}{b} \quad (5a)$$

$$M_{xy} = \sum_{m=1,3,5} 2D_{xy}\theta_{xm}\frac{m\pi}{b}\cos\frac{m\pi y}{b} \quad (5b)$$

$$Q_x = \sum_{m=1,3,5}\left(2D_{xy}\frac{m^2\pi^2}{b^2}\theta_{xm} + V_{xm}\right)\sin\frac{m\pi y}{b} \quad (5c)$$

$$Q_y = \sum_{m=1,3,5}\left[-\left(\frac{D_1^2 + 2D_{xy}D_1 - D_xD_y}{D_x}\right)\frac{m^2\pi^2}{b^2}w_m + \left(\frac{2D_{xy}+D_1}{D_x}\right)M_{xm}\right] \times\frac{m\pi}{b}\cos\frac{m\pi y}{b} \quad (5d)$$

$$V_y = \sum_{m=1,3,5}\left[-\left(\frac{D_1^2 + 4D_{xy}D_1 - D_xD_y}{D_x}\right)\frac{m^2\pi^2}{b^2}w_m + \left(\frac{4D_{xy}+D_1}{D_x}\right)M_{xm}\right] \times\frac{m\pi}{b}\cos\frac{m\pi y}{b} \quad (5e)$$

Reissner-Mindlin plate theory

In the Reissner-Mindlin plate theory the restriction on transverse normal (which is assumed to be perpendicular to middle surface) in the Kirchhoff theory is relaxed. Thus it is assumed that the transverse normal does not remain normal to the (deformed) middle surface of the plate. This amounts to including transverse shear strain energy in the theory.

Algebraic manipulation of the governing equations, given in Appendix B, gives rise to the following system of equations:

$$\frac{\partial w}{\partial x} = -\theta_x + \frac{Q_x}{G_{xz}hk_s} \quad (6a)$$

$$\frac{\partial \theta_x}{\partial x} = -\frac{D_1}{D_x}\frac{\partial \theta_y}{\partial y} + \frac{M_x}{D_x} \quad (6b)$$

$$\frac{\partial \theta_y}{\partial x} = -\frac{\partial \theta_x}{\partial y} + \frac{M_{xy}}{D_{xy}} \quad (6c)$$

$$\frac{\partial Q_x}{\partial x} = -\frac{G_{yz}h}{k_s}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_y}{\partial y}\right) - (p_z^+ + p_z^- + \rho h) \quad (6d)$$

$$\frac{\partial M_x}{\partial x} = Q_x - \frac{\partial M_{xy}}{\partial y} \quad (6e)$$

$$\frac{\partial M_{xy}}{\partial x} = \frac{G_{yz}}{k_s}h\left(\frac{\partial w}{\partial y} + \theta_y\right) - \left(D_y - \frac{D_1^2}{D_x}\right)\frac{\partial^2 \theta_y}{\partial y^2} - \frac{D_1}{D_x}\frac{\partial M_x}{\partial y} \quad (6f)$$

Other dependent variables for Reissner-Mindlin theory are given by

$$Q_y = \frac{G_{yz}h}{k_s}\left(\frac{\partial w}{\partial y} + \theta_y\right) \tag{7a}$$

$$M_y = \left(D_y - \frac{D_1^2}{D_x}\right)\frac{\partial \theta_y}{\partial y} + \frac{D_1}{D_x}M_x \tag{7b}$$

For a plate with two opposite edges $y=0$ and $y=b$ simply supported, intrinsic variables may be represented by means of a Fourier series as described earlier in Eqs. (3*a*)–(3*d*). Only those variables not defined earlier are given below.

$$\theta_y(x, y) = \sum_{m=1,3,5} \theta_{ym}(x) \cos \frac{m\pi y}{b} \tag{8a}$$

$$Q_x(x, y) = \sum_{m=1,3,5} Q_{xm}(x) \sin \frac{m\pi y}{b} \tag{8b}$$

$$M_{xy}(x, y) = \sum_{m=1,3,5} M_{xym}(x) \cos \frac{m\pi y}{b} \tag{8c}$$

Substitution of Eqs. (3*a*)–(3*d*) and (8*a*)–(8*c*) into Eqs. (6*a*)–(6*f*) and following the earlier procedure reduces this set of partial differential equations to a set of simultaneous first-order ordinary differential equations (say, for the *m*th harmonic) involving only six intrinsic variables as follows:

$$\frac{dw_m}{dx} = -\theta_{xm} + \frac{Q_{xm}}{G_{xz}hk_s} \tag{9a}$$

$$\frac{d\theta_x}{dx} = -\frac{D_1}{D_x}\frac{m\pi}{b}\theta_{ym} + \frac{M_{xm}}{D_x} \tag{9b}$$

$$\frac{d\theta_{ym}}{dx} = -\frac{m\pi}{b}\theta_{xm} + \frac{M_{xym}}{D_{xy}} \tag{9c}$$

$$\frac{dQ_{xm}}{dx} = -\frac{G_{yz}h}{k_s}\left(\frac{m^2\pi^2}{b^2}w_m - \frac{m\pi}{b}\theta_{ym}\right) - \frac{4}{m\pi}(p_z^+ + p_z^- + \rho h) \tag{9d}$$

$$\frac{dM_{xm}}{dx} = Q_{xm} - \frac{m\pi}{b}M_{xym} \tag{9e}$$

$$\frac{dM_{xym}}{dx} = -\frac{G_{yz}h}{k_s}\left(\frac{m\pi}{b}w_m\right) + \left[\frac{G_{yz}h}{k_s} + \left(D_y - \frac{D_1^2}{D_x}\right)\frac{m^2\pi^2}{b^2}\right]\theta_{ym} + \frac{D_1}{D_x}\frac{m\pi}{b}M_{xm} \tag{9f}$$

Other auxiliary variables are given by

$$Q_y = \frac{G_{yz}h}{k_s}\sum_{m=1,3,5}\left(\frac{m\pi}{b}w_m - \theta_{ym}\right)\cos\frac{m\pi y}{b} \tag{10a}$$

$$M_y = \left[\sum_{m=1,3,5}\left(D_y - \frac{D_1^2}{D_x}\right)\frac{m\pi}{b}\theta_{ym} + \frac{D_1}{D_x}M_{xm}\right]\sin\frac{m\pi y}{b} \tag{10b}$$

Higher-order plate theory

The present formulation is based on a theory incorporating a higher-order displacement model, quadratic variation of the transverse shearing strains (γ_{xz} and γ_{yz}) through the plate thickness, linear variation of the transverse normal strain (ε_{zz}) through the plate thickness, and consideration of the three-dimensional Hooke's law.

Algebraic manipulation of the governing equations, given in Appendix C, gives rise to the following system of equations, similar to Eqs. (1*a*)–(1*d*) and (6*a*)–(6*f*) obtained earlier for other theories:

$$\frac{\partial w}{\partial x} = -\theta_x + \frac{K_{2x}^* Q_x - Q_x^* K_{2x}}{K_{1x}K_{2x}^* - K_{1x}K_{2x}} \tag{11a}$$

$$\frac{\partial \theta_x}{\partial x} = -2w^* \frac{D_{3x}D_{4x}^* - D_{3x}^* D_{4x}}{D_{1x}D_{4x}^* - D_{1x}^* D_{4x}} - \frac{\partial \theta_y}{\partial y}\frac{(D_{2x}D_{4x}^* - D_{2x}^* D_{4x})}{D_{1x}D_{4x}^* - D_{1x}^* D_{4x}} + \frac{D_{4x}^* M_x - D_{4x}M_x^*}{D_{1x}D_{4x}^* - D_{1x}^* D_{4x}} \tag{11b}$$

$$\frac{\partial \theta_y}{\partial x} = -\frac{\partial \theta_x}{\partial y} + \frac{D_{2xy}^* M_{xy} - D_{2xy}M_{xy}^*}{D_{1xy}D_{2xy}^* - D_{1xy}^* D_{2xy}} \tag{11c}$$

$$\frac{\partial w^*}{\partial x} = -3\theta_x^* + \frac{K_{1x}^* Q_x - K_{1x}Q_x^*}{K_{2x}K_{1x}^* - K_{2x}^* K_{1x}} \tag{11d}$$

$$\frac{\partial \theta_x^*}{\partial x} = \frac{-(D_{5x}D_{1x}^* - D_{5x}^* D_{1x})}{D_{4x}D_{1x}^* - D_{4x}^* D_{1x}}\frac{\partial \theta_y^*}{\partial y} + \frac{D_{1x}^* M_x - D_{1x}M_x^*}{D_{4x}D_{1x}^* - D_{4x}^* D_{1x}} \tag{11e}$$

$$\frac{\partial \theta_y^*}{\partial y} = -\frac{\partial \theta_x^*}{\partial y} + \frac{M_{xy}D_{1xy}^* - M_{xy}^* D_{1xy}}{D_{2xy}D_{1xy}^* - D_{2xy}^* D_{1xy}} \tag{11f}$$

$$\frac{\partial Q_x}{\partial x} = -K_{1y}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_y}{\partial y}\right) - K_{2y}\left(\frac{\partial^2 w^*}{\partial y^2} + \frac{3\partial \theta_y^*}{\partial y}\right) - (p_z^+ + p_z^- + \rho h) \tag{11g}$$

$$\frac{\partial M_x}{\partial x} = Q_x - \frac{\partial M_{xy}}{\partial y} \tag{11h}$$

$$\frac{\partial M_{xy}}{\partial x} = -K_{1y}\left(\frac{\partial w}{\partial y} + \theta_y\right) + K_{2y}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right) - \left[D_{1y}\frac{\partial}{\partial x}\left(\frac{\partial \theta_x}{\partial y}\right) + D_{2y}\frac{\partial^2 \theta_y}{\partial y^2} + D_{3y}\frac{\partial w^*}{\partial y} + D_{4y}\frac{\partial}{\partial x}\left(\frac{\partial \theta_x^*}{\partial y}\right) + D_{5y}\frac{\partial^2 \theta_y^*}{\partial y^2}\right] \tag{11i}$$

$$\frac{\partial Q_x^*}{\partial x} = -K_{1y}^*\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_y}{\partial y}\right) - K_{2y}^*\left(\frac{\partial^2 w}{\partial y^2} + \frac{3\partial \theta_y^*}{\partial y}\right) - \left(2D_{1z}\frac{\partial \theta_x}{\partial x} + 2D_{2z}\frac{\partial \theta_y}{\partial y} + 2D_{3z}2w^* + 2D_{4z}\frac{\partial \theta_x^*}{\partial x} + 2D_{5z}\frac{\partial \theta_y^*}{\partial y}\right) - \left(p_z^+ + p_z^- + \frac{\rho h}{3}\right)\frac{h^2}{4} \tag{11j}$$

$$\frac{\partial M_x^*}{\partial x} = 3Q_x^* - \frac{\partial M_{xy}^*}{\partial x} \tag{11k}$$

$$\frac{\partial M_{xy}^*}{\partial x} = 3K_{1y}^*\left(\frac{\partial w}{\partial y} + \theta_y\right) + 3K_{2y}^*\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right)$$
$$- \left[D_{1y}^* \frac{\partial}{\partial x}\left(\frac{\partial \theta_x}{\partial y}\right) + D_{2y}^* \frac{\partial^2 \theta_y}{\partial y^2} + D_{3y}^* \frac{2\partial w^*}{\partial y} + D_{4y}^* \frac{\partial}{\partial x}\frac{\partial \theta_x^*}{\partial y} + D_{5x}^* \frac{\partial^2 \theta_y^*}{\partial y^2}\right] \tag{11l}$$

Other dependent variables are

$$Q_y = K_{1y}\left(\frac{\partial w}{\partial y} + \theta_y\right) + K_{2y}\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right) \tag{12a}$$

$$M_y = D_{3y} 2w^* + D_{1y}\frac{\partial \theta_x}{\partial x} + D_{2y}\frac{\partial \theta_y}{\partial y} + D_{4y}\frac{\partial \theta_x^*}{\partial x} + D_{5y}\frac{\partial \theta_y^*}{\partial y} \tag{12b}$$

$$Q_y^* = K_{2y}\left(\frac{\partial w}{\partial y} + \theta_y\right) + K_{2y}^*\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right) \tag{12c}$$

$$M_y^* = \left(D_{1y}^*\frac{\partial \theta_x}{\partial x} + D_{2y}^*\frac{\partial \theta_y}{\partial y} + D_{3y}^* 2w^* + D_{4y}^*\frac{\partial \theta_x^*}{\partial x} + D_{5y}^*\frac{\partial \theta_y^*}{\partial y}\right) \tag{12d}$$

$$M_z = D_{1z}\frac{\partial \theta_x}{\partial x} + D_{2z}\frac{\partial \theta_y}{\partial y} + D_{3z} 2w^* + D_{4z}\frac{\partial \theta_x^*}{\partial x} + D_{5z}\frac{\partial \theta_y^*}{\partial y} \tag{12e}$$

For a plate with two opposite edges $y = 0$, b simply supported, intrinsic variables are represented in the form of a Fourier series as described earlier in Eqs. (3*a*)–(3*d*) and (8*a*)–(8*c*). The variables not described earlier are given below.

$$w^*(x, y) = \sum_{m=1,3,5} w_m^*(x) \sin\frac{m\pi y}{b} \tag{13a}$$

$$\theta_x^*(x, y) = \sum_{m=1,3,5} \theta_{xm}^*(x) \sin\frac{m\pi y}{b} \tag{13b}$$

$$\theta_y^*(x, y) = \sum_{m=1,3,5} \theta_{ym}^*(x) \sin\frac{m\pi y}{b} \tag{13c}$$

$$Q_x^*(x, y) = \sum_{m=1,3,5} Q_{xm}^*(x) \sin\frac{m\pi y}{b} \tag{13d}$$

$$M_x^*(x, y) = \sum_{m=1,3,5} M_x^*(x) \sin\frac{m\pi y}{b} \tag{13e}$$

Following the earlier procedure, the partial differential Eqs. (11*a*)–(11*k*) are transformed into the following set of first-order ordinary differential equations for the *m*th harmonic:

$$\frac{dw_m}{dx} = -\theta_{xm} + \frac{K_{2x}^* Q_x - Q_x^* K_{2x}}{K_{1x}K_{2x}^* - K_{1x}^* K_{2x}} \tag{14a}$$

$$\frac{d\theta_{xm}}{\partial x} = -2w^*\frac{D_{3x}D_{4x}^* - D_{3x}^* D_{4x}}{D_{1x}D_{4x}^* - D_{1x}^* D_{4x}} - \frac{(D_{2x}D_{4x}^* - D_{2x}^* D_{4x})}{D_{1x}D_{4x}^* - D_{1x}^* D_{4x}}\frac{m\pi}{b}\theta_y$$
$$+ \frac{D_{4x}^* M_x - D_{4x} M_x^*}{D_{1x}D_{4x}^* - D_{1x}^* D_{4x}} \tag{14b}$$

Table 4

Square orthotropic plate under uniformly distributed load, simply supported along four edges ($\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.3$, $k_s = 6/5$, a/h = 5): parametric study with various E_x/E_y ratios (converged values with n = 20)

E_x/E_y	$wE_yh^3/p_z^+a^4$			$M_x/p_z^+a^2$			$M_y/p_z^+a^2$			$M_{xy}/p_z^+a^2$			Q_x/p_z^+a			Q_y/p_z^+a		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
1	.0384	.0448	.0441	.0382	.0385	.0393	.0382	.0379	.0386	.0443	.0324	.0414	.3229	.3296	.3301	.3278	.3229	.3285
1.25	.0367	.0431	.0425	.0438	.0439	.0445	.0361	.0359	.0366	.0423	.0421	.0397	.3360	.3420	.3423	.3177	.3179	.3184
1.50	.0352	.0416	.0410	.0488	.0487	.0493	.0342	.0342	.0348	.0405	.0403	.0381	.3477	.3528	.3529	.3083	.3086	.3092
2	.0324	.0388	.0383	.0577	.0571	.0574	.0309	.0313	.0319	.0372	.0373	.0352	.3675	.3711	.3709	.2912	.2924	.2930
3	.0278	.0344	.0340/ .03445**	.0717	.0698	.0699	.0259	.0270	.0277	.0321	.0327	.0308	.3975	.3982	.3975	.2636	.2670	.2679
10.0	.0137	.0220	.0217/ .02215**	.1121	.1044	.1039	.0112	.0154	.0162	.0170	.0197	.0184	.4812	.4706	.4688	.1768	.1938	.1959
40.0	.0041	.0142	.0138/ .01472**	.1326	.1242	.1237	.0020	.0083	.0092	.0065	.0116	.0103	.5180	.5115	.5096	.1102	.1463	.1490

(Continued on next page)

Table 4

Square orthotropic plate under uniformly distributed load, simply supported along four edges ($\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.3$, $k_s = 6/5$, $a/h = 5$): parametric study with various E_x/E_y ratios (converged values with n = 20) (Continued)

	$\sigma_x(p_z^+ a^2/h^2)$			$\sigma_y(p_z^+ a^2/h^2)$			$\tau_{xy}(p_z^+ a^2/h^2)$			$\tau_{xz}(p_z^+ a^2/h^2)$			$\tau_{xz}(p_z^+ a^2/h^2)$		
E_x/E_y	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
1.0	.2296	.2310	.2404	.2296	.2270	.2367	.2662	.2640	.2727	—	.3300	.4665	—	.3280	.4652
1.25	.2592	.2740	.2729	.1622	.2150	.2243	.2540	.2520	.2622	—	.3420	.4827	—	.3180	.4501
1.5	.2866	.3130	.3024	.1197	.2050	.2137	.2428	.2420	.2524	—	.3530	.4970	—	.3090	.4363
2.0	.3351	.3800	.3536	.0700	.1880	.1961	.2232	.2240	.2342	—	.3710	.5820	—	.2920	.4122
3.0	.4117	.4840	.4337	.0273	.1620	.1701	.1926	.1960	.2082	—	.3980	.5560	—	.2670	.3755
10.0	.6346	.8000	.6836	.0080	.0920	.1001	.1020	.1190	.1361	—	.4710	.6432	—	.1940	.2666
40.0	.7469	.1188	1.003	.0046	.0500	.0568	.0392	.0690	.0808	—	.5070	.6648		.1460	.1962

**Solution obtained using MIF [7].
Boundary conditions along four edges: K, $w = M_n = 0$; R-M, $w = \theta_t = M_n = 0$; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.
K, Kirchhoff theory; R-M, Reissner-Mindlin theory; H, higher-order theory.

Table 5

Square orthotropic plate under uniformly distributed load, simply supported on all edges: ($\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.25$, $k_s = 6/5$) (a/h = 2–100, $E_x/E_y = 3.0$) *(converged values with* n = 20*)*

	$wE_yh^3/p_z^+a^4$			$M_x/p_z^+a^2$			$M_y/p_z^+a^2$			$M_{xy}/p_z^+a^2$			Q_x/p_z^+a			Q_y/p_z^+a		
a/h	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
2	.0278	.0690	.0638	.0717	.0639	.0652	.0259	.0307	.0346	.0321	.0340	.0253	.3976	.3838	.3800	.2635	.2808	.2850
5	.0278	.0344	.0340	.0717	.0698	.0699	.0259	.0269	.0277	.0321	.0327	.0308	.3976	.3982	.3975	.2635	.2669	.2679
10	.0278	.0294	.0294	.0717	.0711	.0712	.0259	.0262	.0263	.0321	.0323	.0318	.3976	.4012	.4012	.2635	.2643	.2646
20	.0278	.0282	.0282	.0717	.0715	.0715	.0259	.0259	.0260	.0321	.0322	.0320	.3976	.4021	.4022	.2635	.2637	.2638
25	.0278	.0280	.0280	.0717	.0716	.0715	.0259	.0259	.0259	.0321	.0322	.0321	.3976	.4021	.4023	.2635	.2636	.2637
50	.0278	.0278	.0278	.0717	.0716	.0716	.0259	.0259	.0259	.0321	.0321	.0321	.3976	.4023	.4025	.2635	.2634	.2636
100	.0278	.0278	.0278	.0717	.0716	.0716	.0259	.0259	.0259	.0321	.0321	.0321	.3976	.4024	.4024	.2635	.2635	.2635

K, $w = M_n = 0$; R-M, $w = \theta_t = M_n = 0$; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.
K, Kirchhoff theory; R-M, Reissner-Mindlin theory; H, higher-order theory.

Table 6

Square orthotropic plate under uniformly distributed load, simply supported along four edges ($\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.25$, $k_s = 6/5$), (a/h = 2–100, E_x/E_y = 1–40) (converged values with n = 20)

		$wD_y/p_z^+a^4$			$M_x/p_z^+a^2$			$M_y/p_z^+a^2$			$M_{xy}/p_z^+a^2$			Q_x/p_z^+a			Q_y/p_z^+a		
a/h	E_x/E_y	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
2	1.00	.0033	.0071	.0064	.0395	.0412	.0465	.0395	.0376	.0430	.0428	.0419	.0275	.3279	.3305	.3368	.3276	.3259	.3274
	1.25	.0032	.0068	.0062	.0448	.0457	.0502	.0372	.0363	.0414	.0410	.0403	.0269	.3407	.3402	.3454	.3176	.3171	.3188
	1.50	.0030	.0066	.0060	.0496	.0495	.0534	.0352	.0352	.0401	.0393	.0389	.0262	.3520	.3484	.3525	.3082	.3097	.3118
	2.00	.0029	.0063	.0057	.0582	.0557	.0586	.0318	.0335	.0383	.0363	.0365	.0249	.3714	.3612	.3638	.2914	.2976	.3008
	3.00	.0025	.0059	.0054	.0717	.0643	.0659	.0265	.0313	.0360	.0315	.0332	.0228	.4014	.3785	.3790	.2641	.2810	.2859
	10.00	.0012	.0051	.0046	.1118	.0832	.0829	.0117	.0265	.0311	.0168	.0255	.0171	.4840	.4166	.4141	.1776	.2432	.2516
	40.00	.0004	.0047	.0041	.1324	.0921	.0931	.0022	.0243	.0277	.0065	.0215	.0135	.5223	.4347	.4345	.1105	.2246	.2295
5	1.00	.0033	.0039	.0039	.0395	.0397	.0406	.0395	.0391	.0400	.0428	.0423	.0392	.3279	.3245	.3252	.3276	.3278	.3285
	1.25	.0032	.0038	.0037	.0448	.0449	.0457	.0372	.0369	.0378	.0410	.0410	.0378	.3407	.3367	.3420	.3176	.3178	.3184
	1.50	.0030	.0036	.0035	.0496	.0496	.0502	.0352	.0351	.0360	.0393	.0393	.0364	.3520	.3474	.3525	.3082	.3086	.3093
	2.00	.0029	.0033	.0033	.0582	.0576	.0581	.0318	.0321	.0330	.0363	.0363	.0338	.3714	.3655	.3701	.2914	.2925	.2933
	3.00	.0025	.0029	.0028	.0717	.0700	.0702	.0265	.0277	.0286	.0315	.0315	.0298	.4014	.3924	.3965	.2641	.2674	.2685
	10.00	.0012	.0018	.0018	.1118	.1042	.1037	.0117	.0158	.0168	.0168	.0168	.0179	.4840	.4647	.4675	.1776	.1944	.1969
	40.00	.0004	.0012	.0012	.1324	.1241	.1235	.0022	.0085	.0095	.0065	.0065	.0100	.5223	.5061	.5087	.1105	.1465	.1049

10	1.00	.0033	.0035	.0035	.0395	.0395	.0398	.0395	.0394	.0396	.0428	.0426	.0417	.3279	.3231	.3283	.3276	.3276	.3279
	1.25	.0032	.0033	.0033	.0448	.0448	.0450	.0372	.0371	.0373	.0410	.0408	.0400	.3407	.3358	.3410	.3176	.3175	.3178
	1.50	.0030	.0032	.0031	.0496	.0496	.0498	.0352	.0351	.0354	.0393	.0390	.0384	.3520	.3451	.3522	.3082	.3082	.3085
	2.00	.0029	.0029	.0029	.0582	.0580	.0582	.0318	.0319	.0321	.0363	.0362	.0356	.3714	.3662	.3712	.2914	.2916	.2919
	3.00	.0025	.0025	.0025	.0717	.0713	.0714	.0265	.0269	.0272	.0315	.0316	.0310	.4014	.3952	.4000	.2641	.2648	.2652
	10.00	.0012	.0013	.0013	.1118	.1096	.1096	.0117	.0128	.0131	.0168	.0177	.0173	.4840	.4759	.4805	.1776	.1776	.1825
	40.00	.0004	.0005	.0005	.1324	.1313	.1313	.0022	.0038	.0041	.0065	.0102	.0077	.5223	.5188	.5234	.1105	.1105	.1221
100	1.00	.0033	.0034	.0034	.0395	.0394	.0395	.0395	.0395	.0395	.0428	.0427	.0428	.3279	.3245	.3276	.3276	.3278	.3258
	1.25	.0032	.0032	.0032	.0448	.0447	.0447	.0372	.0371	.0372	.0410	.0409	.0410	.3407	.3356	.3404	.3176	.3169	.3239
	1.50	.0030	.0030	.0030	.0496	.0496	.0496	.0352	.0351	.0352	.0393	.0393	.0393	.3520	.3471	.3520	.3082	.3084	.2950
	2.00	.0029	.0027	.0027	.0582	.0581	.0582	.0318	.0318	.0318	.0363	.0362	.0363	.3714	.3665	.3714	.2914	.2902	.2869
	3.00	.0025	.0023	.0023	.0717	.0717	.0718	.0265	.0266	.0267	.0315	.0314	.0315	.4014	.3963	.4012	.2641	.2617	.2529
	10.00	.0012	.0011	.0011	.1118	.1117	.1118	.0117	.0117	.0117	.0168	.0168	.0169	.4840	.4797	.4848	.1776	.1780	.1867
	40.00	.0004	.0003	.0003	.1324	.1324	.1324	.0022	.0022	.0022	.0065	.0065	.0065	.5223	.5176	.5227	.1105	.1105	.1107

K, $w = M_n = 0$; R-M, $w = \theta_t = M_n = 0$; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.
K, Kirchhoff theory; R-M, Reissner-Mindlin theory; H, higher-order theory.

Table 7

Square isotropic plate under uniformly distributed load, simply supported along edges y = *0,* b, *and just supported along edges* x = *0,* a *(ν = 0.3,* $k_s = 6/5$*),* (a/h = *2–100)* *(converged values with* n = *20)*

	$wD_y/p_z^+a^4$		$M_x/p_z^+a^2$		$M_y/p_z^+a^2$		$M_{xy}/p_z^+a^2$		Q_x/p_z^+a		Q_y/p_z^+a	
a/h	R-M	H	R-M	H	R-M	H	R-M	H	R-M	H	R-M	H
2	.0101	.0091	.0558	.0574	.0534	.0556	.0031	.0042	.3570	.3504	.3509	.3572
5	.0053	.0051	.0507	.0512	.0515	.0519	.0056	.0052	.3915	.3823	.3481	.3474
	.0053#	.0051	.0507	.0512	.0515	.0519			.4030	.4010	.3480	.3470
10	.0044	.0044	.0491	.0493	.0498	.0499	.0096	.0095	.3955	.3825	.3405	.3405
	.0044#	.0044	.0491	.0493	.0498	.0499			.4130	.4130	.3400	.3400
20	.0042	.0042	.0484	.0485	.0488	.0489	.0158	.0159	.3900	.3670	.2264	.3365
25	.0042	.0042	.0483	.0484	.0486	.0487	.0183	.0184	.3850	.3614	.3356	.3357
50	.0041	.0041	.0481	.0481	.0482	.0483	.0262	.0262	.3681	.3427	.3339	.3344
	.0041#	.0041	.0481	.0481	.0482	.0482			.4190	.4190	.3340	.3340
100	.0041	.0041	.0479	.0480	.0481	.0481	.0313	.0313	.3484	.3301	.3332	.3302
	.0041#	.0041	.0480	.0481	.0480	.0480			.4200	.4200	.3330	.3330

Boundary conditions on simply supported edges $y = 0, b$: K, $w = M_n = 0$; R-M, $w = \theta_t =$ Mn = 0; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.

Boundary conditions on just supported edges $x = 0, a$: R-M, $w = M_n = M_{nt} = 0$; H, $w = M_n = M_{nt} = w^* = M_n^* = M_{nt}^* = 0$.

#Results published in [4].

K, Kirchhoff theory, R-M, Reissner-Mindlin theory; H, higher-order theory.

Example 7: Square isotropic plate with S boundary conditions along edges y = *0 and* b *and free (F) boundary conditions along edges* x = *0 and* a

In this example, the free edge conditions, implying vanishing of transverse shear, normal bending moment, and twisting moment along $x = 0, a$, are assumed (F). Exact satisfaction of such boundary conditions is quite simple in the present formulation. Results for w, M_x, $M_x(-ve)$, M_y, Q_x, and Q_y are shown in Table 11. These results are compared with the results published earlier [2–4].

Example 8: Square orthotropic plate with S boundary conditions along edges y = *0 and* b *and free (F) boundary conditions along edges* x = *0 and* a

In this example, the free edge conditions, implying vanishing of transverse shear, normal bending moment, and twisting moment along $x = 0, a$, are assumed (F). Exact satisfaction of such boundary conditions is quite simple in the present formulation. Relevant results are presented in Table 12 for maximum central deflection w and stress resultants such as M_x, M_y, etc.

Example 9: Square orthotropic plate with S boundary conditions along edges y = *0 and* b *and unsymmetric boundary conditions along edges* x = *0 and* a

Different boundary conditions are considered next along edges $x = 0$ and a, to generate data for future comparisons. These boundary conditions are C-S, C-F, C-S*, S-F, S-S*, F-S*. The results for these boundary conditions are presented in Tables 13–18.

Table 8

Square orthotropic plate under uniformly distributed load, simply supported along edges y = 0, b, *just supported along edges* x = 0, a ($\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.3$, $k_s = 6/5$), (a/h = 2–100, E_x/E_y = 1–40) *(converged values with* n = 20*)*

		$wD_y/p_z^+a^4$		$M_x/p_z^+a^2$		$M_y/p_z^+a^2$		$M_{xy}/p_z^+a^2$		Q_x/p_z^+a		Q_x/p_z^+a	
a/h	E_x/E_y	R-M	H	R-M	H	R-M	H	R-M	H	R-M	H	R-M	H
2	1.00	.0081	.0072	.0516	.0542	.0457	.0489	.0071	.0072	.3732	.3715	.3542	.3476
	1.25	.0077	.0069	.0569	.0585	.0436	.0468	.0068	.0042	.3836	.3810	.3417	.3367
	1.50	.0074	.0066	.0613	.0622	.0420	.0452	.0065	.0040	.3885	.3885	.3313	.3276
	2.00	.0070	.0063	.0681	.0680	.0394	.0428	.0060	.0038	.3997	.3997	.3147	.3133
	3.00	.0064	.0058	.0770	.0757	.0361	.0397	.0053	.0035	.4140	.4140	.2926	.2944
	10.00	.0053	.0048	.0950	.0922	.0293	.0332	.0039	.0025	.4437	.4437	.2458	.2534
	40.00	.0049	.0042	.1028	.1012	.0263	.0029	.0032	.0020	.4603	.4603	.2242	.2291
5	1.00	.0044	.0043	.0431	.0438	.0436	.0443	.0129	.0121	.3959	.4063	.3500	.3494
	1.25	.0042	.0041	.0487	.0493	.0411	.0417	.0123	.0116	.4607	.4166	.3387	.3382
	1.50	.0040	.0039	.0537	.0542	.0389	.0396	.0117	.0109	.4157	.4251	.3283	.3279
	2.00	.0036	.0036	.0624	.0626	.0353	.0366	.0108	.0102	.4300	.4387	.3098	.3096
	3.00	.0031	.0031	.0753	.0752	.0301	.0309	.0093	.0088	.4504	.4578	.2807	.2810
	10.00	.0019	.0019	.1089	.1082	.0165	.0175	.0055	.0052	.4998	.5042	.1989	.2019
	40.00	.0012	.0012	.1270	.1262	.0086	.0096	.0032	.0029	.5247	.5277	.1497	.1509

(Continued on next page)

CONCLUSIONS

The segmentation method is used in conjunction with Kirchhoff, Reissner-Mindlin, and higher-order theories for linear elastic analysis of orthotropic plates. It is well known that the effect of transverse shear deformation and normal stress in the thickness direction becomes important above a certain value of thickness/length ratio. One significant feature of the present formulation is its ability to satisfy exactly both the displacement and the force boundary conditions along an edge. This is in contrast with the popular displacement-based finite-element formulation, where only displacement boundary conditions can be satisfied exactly. However, the formulation is not general, as all equations in the present formulation are derived for specific boundary conditions on edges $y =$ constant. It is proposed to extend the formulation to the solution of layered, cross-ply composite and sandwich orthotropic plates simply supported on two opposite edges.

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APPENDIX A

Basic equations of Kirchhoff plate theory

A complete set of equations defining a fourth-order boundary-value problem for an isotropic material plate is available in [2]. This formulation is extended here for an orthotropic material.

Displacement model (see Figure A1)

$$u(x, y, z) = z\theta_x(x, y) \tag{A1a}$$

$$v(x, y, z) = z\theta_y(x, y) \tag{A1b}$$

$$w(x, y, z) = w(x, y) \tag{A1c}$$

Strain displacement relations

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \tag{A2a}$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \tag{A2b}$$

CONCLUSIONS

The segmentation method is used in conjunction with Kirchhoff, Reissner-Mindlin, and higher-order theories for linear elastic analysis of orthotropic plates. It is well known that the effect of transverse shear deformation and normal stress in the thickness direction becomes important above a certain value of thickness/length ratio. One significant feature of the present formulation is its ability to satisfy exactly both the displacement and the force boundary conditions along an edge. This is in contrast with the popular displacement-based finite-element formulation, where only displacement boundary conditions can be satisfied exactly. However, the formulation is not general, as all equations in the present formulation are derived for specific boundary conditions on edges $y = \text{constant}$. It is proposed to extend the formulation to the solution of layered, cross-ply composite and sandwich orthotropic plates simply supported on two opposite edges.

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$$w(x, y, z) = w(x, y) \tag{A1c}$$

Strain displacement relations

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \tag{A2a}$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \tag{A2b}$$

$$\frac{d\theta_{ym}}{dx} = -\frac{m\pi}{b}\theta_{xm} + \frac{D^*_{2xy}M_{xym} - D_{2xy}M^*_{xym}}{D_{1xy}D^*_{2xy} - D^*_{1xy}D_{2xy}} \tag{14c}$$

$$\frac{dw^*_m}{dx} = -3\theta^*_{xm} + \frac{K^*_{1x}Q_{xm} - K_{1x}Q^*_{xm}}{K_{2x}K^*_{1x} - K^*_{2x}K_{1x}} \tag{14d}$$

$$\frac{d\theta^*_{xm}}{dx} = \frac{m\pi}{b}\frac{(D_{5x}D^*_{1x} - D^*_{5x}D_{1x})}{(D_{4x}D^*_{1x} - D^*_{4x}D_{1x})}\theta^*_{ym} + \frac{D^*_{1x}M_{xm} - D_{1x}M^*_{xm}}{D_{4x}D^*_{1x} - D^*_{4x}D_{1x}} \tag{14e}$$

$$\frac{d\theta^*_{ym}}{dx} = -\frac{m\pi}{b}\theta^*_{xm} + \frac{M_{xym}D^*_{1xy} - M^*_{xym}D_{1xy}}{D^*_{1xy}D_{2xy} - D^*_{2xy}D_{1xy}} \tag{14f}$$

$$\frac{dQ_{xm}}{dx} = K_{1y}\frac{m\pi}{b}\left(\frac{m\pi}{b}w_m + \theta_{ym}\right) + K_{2y}\frac{m\pi}{b}\left(\frac{m\pi}{b}w^*_m + 3\theta^*_{ym}\right) - \frac{4}{m\pi}[p^+_z + p^-_z + \rho h] \tag{14g}$$

$$\frac{dM_{xm}}{dx} = Q_{xm} + \frac{m\pi}{b}M_{xym} \tag{14h}$$

$$\frac{dM_{xym}}{dx} = K_{1y}\left(\frac{m\pi}{b}w_m + \theta_{ym}\right) + K_{2y}\left(\frac{m\pi}{b}w^*_m + 3\theta^*_{ym}\right) + \frac{m\pi}{b} \times\left(D_{1y}\frac{d\theta_{xm}}{dx} - \frac{m\pi}{b}D_{2y}\theta_{ym} + D_{3y}2w^*_m + D_{4y}\frac{d\theta^*_{xm}}{dx} + \frac{m\pi}{b}D_{5y}\theta^*_{ym}\right) \tag{14i}$$

$$\frac{dQ^*_{xm}}{dx} = K^*_{1y}\frac{m\pi}{b}\left(\frac{m\pi}{b}w_m + \theta_{ym}\right) + k^*_{2y}\frac{m\pi}{b}\left(\frac{m\pi}{b}w^*_m + 3\theta^*_{ym}\right) + \left(2D_{1z}\frac{d\theta_{xm}}{dx} - 2D_{2z}\frac{m\pi}{b}\theta_{ym} + 2D_{3z}2w^* + 2D_{4z}\frac{d\theta^*_{xm}}{dx} - 2D_{5z}\frac{m\pi}{b}\theta^*_{ym}\right) - \frac{4}{m\pi}\frac{h^2}{4}\left(p^+_z + p^-_z + \frac{\rho h}{3}\right) \tag{14j}$$

$$\frac{dM^*_{xm}}{dx} = 3Q^*_{xm} + \frac{m\pi}{b}M^*_{xym} \tag{14k}$$

$$\frac{dM^*_{xy}}{dx} = 3K^*_{1y}\left(\frac{m\pi}{b}w_m + \theta_{ym}\right) + 3K^*_{2y}\left(\frac{m\pi}{b}w^*_m + 3\theta^*_{ym}\right) + \frac{m\pi}{b} \times\left(-D^*_{1y}\frac{d\theta_x}{dx} + D^*_{2y}\frac{m\pi}{b}\theta_{ym} - D^*_{3y}2w^*_m - D^*_{4y}\frac{d\theta^*_x}{dx} + D^*_{5y}\frac{m\pi}{b}\theta^*_{ym}\right) \tag{14l}$$

Auxiliary variables defined in Eqs. (12a)–(12e) now take the form:

$$Q_y = \sum_{m=1,3,5} K_{1y}\left[\left(\frac{m\pi}{b}w_m + \theta_y\right) + K_{2y}\left(\frac{m\pi}{b}w^*_m + 3\theta^*_y\right)\right]\cos\frac{m\pi y}{b} \tag{15a}$$

$$M_y = \sum_{m=1,3,5}\left(D_{3y}2w_m + D_{1y}\frac{d\theta_{xm}}{dx} - D_{2y}\frac{m\pi}{b}\theta_{ym} + D_{4y}\frac{\partial\theta^*_{xm}}{\partial x} - D_{5y}\frac{m\pi}{b}\theta^*_{ym}\right) \times \sin\frac{m\pi y}{b} \tag{15b}$$

$$Q^*_y = \sum_{m=1,3,5}\left[K_{2y}\left(\frac{m\pi}{b}w_m + \theta_{ym}\right) + K^*_{1xz}\left(\frac{m\pi}{b}w^*_m + 3\theta^*_{ym}\right)\right]\cos\frac{m\pi y}{b} \tag{15c}$$

$$M_y^* = \sum_{m=1,3,5} \left(D_{1y}^* \frac{d\theta_{xm}}{dx} - D_{2y}^* \frac{m\pi}{b} \cdot \theta_{ym} + D_{3y}^* 2w_m^* + D_{4y}^* \frac{d\theta_x^*}{dx} - D_{5y}^* \frac{m\pi}{b} \cdot \theta_{ym}^* \right) \sin \frac{m\pi y}{b} \quad (15d)$$

$$M_z = \sum_{m=1,3,5} \left(D_{1z} \frac{d\theta_{xm}}{dx} - \frac{m\pi}{b} D_{2z}\theta_{ym} + D_{3z} 2w_m^* + D_{4z} \frac{d\theta_{xm}^*}{dx} - D_{5z} \frac{m\pi}{b} \theta_{ym}^* \right) \sin \frac{m\pi y}{b} \quad (15e)$$

NUMERICAL EXAMPLES

A square plate of side a, thickness h, and loaded with a uniformly distributed transverse load p is considered throughout. The two opposite edges, $y = 0$ and $y = b$, are always assumed to be simply supported (S), implying specific boundary conditions for each of the three theories, Kirchhoff, Reissner-Mindlin, and the higher-order theory. However, any boundary conditions are considered along the edges $x = 0$ and $x = a$. These boundary conditions are summarized below.

a. Along edges $y = 0$ and $y = b$
Kirchhoff: $w = M_y = 0$
Reissner-Mindlin: $w = \theta_x = M_y = 0$
Higher-order: $w = \theta_x = w^* = \theta_x^* = M_y = M_y^* = 0$

b. Boundary conditions along edges x = constant

Theory	Boundary conditions: Simply supported (S)	Clamped (C)	Free (F)	Just supported (S*)
Kirchhoff	$w = M_x = 0$	$w = \theta_x = 0$	$V_x = M_x = 0$	--
Reissner-Mindlin	$w = \theta_y = M_x = 0$	$w = \theta_x = \theta_y = 0$	$Q_x = M_x = M_{xy} = 0$	$w = M_x = M_{xy} = 0$
Higher order	$w = \theta_y = M_x = w^* = \theta_y^* = M_x^* = 0$	$w = \theta_x = \theta_y = w^* = \theta_x^* = \theta_y^* = 0$	$Q_x = M_x = M_{xy} = Q_x^* = M_x^* = M_{xy}^* = 0$	$w = M_x = M_{xy} = w^* = M_x^* = M_{xy}^* = 0$

The x dimension of the plate is divided into a number of segments and each segment into a number of subdivisions for numerical integration [1]. Proper length of the segment was decided after taking several trials for various segment lengths and number of subdivisions in each of the segments. Numerical convergence was achieved by taking number of segments/number of subdivisions in each segment as 10/5, 50/5, and 100/5 for Kirchhoff, Reissner-Mindlin, and the higher-order formulations, respectively.

The following parametric variations are considered in the present study:

a. Geometric parameters: set of a/h values = 2, 5, 10, 20, 50, 100

b. Material parameters: set of E_x/E_y values = 1.0, 1.25, 1.5, 2.0, 3.0, 10.0, 40.0, and $\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.3$, $G_{xy} = G_{xz} = 0.5E_y$, $G_{yz} = 0.2E_y$

Discrete numerical values of dependent variables are presented in nondimensional form as follows:

$$\bar{w} = w \Big/ \left(\frac{p_z^+ a^4}{D_y}\right) \tag{16a}$$

$$\overline{M_x} = M_x/(p_z^+ a^2) \tag{16b}$$

$$\overline{M_y} = M_y/(p_z^+ a^2) \tag{16c}$$

$$\overline{Q_x} = Q_x/(p_z^+ a) \tag{16d}$$

$$\overline{Q_y} = Q_y/(p_z^+ a) \tag{16e}$$

$$\overline{V_x} = V_x/(p_z^+ a) \tag{16f}$$

$$\overline{V_y} = V_y/(p_z^+ a) \tag{16g}$$

$$\overline{\sigma_x} = \sigma_x \Big/ \left(\frac{p_z^+ a^2}{h^2}\right) \tag{16h}$$

$$\overline{\sigma_y} = \sigma_y \Big/ \left(\frac{p_z^+ a^2}{h^2}\right) \tag{16i}$$

$$\overline{\tau_{xy}} = \tau_{xy} \Big/ \left(\frac{p_z^+ a^2}{h^2}\right) \tag{16j}$$

$$\overline{\tau_{xz}} = \tau_{xz} \Big/ \left(\frac{p_z^+ a^2}{h^2}\right) \tag{16k}$$

$$\overline{\tau_{yz}} = \tau_{yz} \Big/ \left(\frac{p_z^+ a^2}{h^2}\right) \tag{16l}$$

Here p_z^+ is the intensity of load on face $z = +h/2$.

Example 1: Square isotropic plate with S boundary conditions along all four edges

3-D exact solution of this problem is obtained in [6]. Values in Table 1 clearly indicate that for w, M_x, M_y, M_{xy}, σ_x, σ_y, and τ_{xy}, convergence is achieved quite early, with only five harmonics. The same is not true for transverse forces/stresses, for which convergence is seen only after 15 harmonics.

Results in Table 2 show the effect of a/h ratio on the deflection, stress resultants, and stresses as predicted by different theories. As seen from the results in Table 2, Kirchhoff theory gives results which are independent of a/h ratio, while the other two theories take into account the shear deformation effects and hence for thick plates ($a/h < 20$) there is a large difference between results predicted by Kirchhoff theory and those predicted by Reissner-Mindlin and the higher-order theories. It is also seen that as the ratio a/h is increased, results of all theories converge to the exact solution given by classical (Kirchhoff) theory.

Example 2: Square orthotropic plate with S boundary conditions along all four edges

The second example is a square, simply supported orthotropic plate subjected to uniform transverse load. Key parameters of the plate are given in Table 3. Convergence of results from various theories is shown in Table 3. For $a/h = 5$ and $E_x/E_y = 3.0$, it is seen that the Reissner-Mindlin and higher-order theories give excellent agreement with theoretical solutions for transverse deflection. Numerical values of M_x and M_y predicted by

Table 1

Square isotropic plate under uniformly distributed load, simply supported along four edges ($\nu = 0.3$, $k_s = 6/5$, a/h = 10): convergence study

Σn	$wE/p_z^+a^4$			$M_x/p_z^+a^2$			$M_y/p_z^+a^2$			$M_{xy}/p_z^+a^2$			Q_x/p_z^+a			Q_y/p_z^+a		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
1	.0449	.0472	.0470	.0492	.0491	.0494	.0517	.0516	.0518	.0301	.0301	.0299	.3591	.3651	.3654	.2438	.2436	.2437
2	.0443	.0465	.0463	.0492	.0476	.0478	.0471	.0516	.0473	.0318	.0318	.0341	.3182	.3222	.3225	.2880	.2879	.2880
5	.0444	.0466	.0464	.0476	.0479	.0480	.0479	.0471	.0480	.0323	.0323	.0318	.3292	.3341	.3344	.3175	.3173	.3174
10	.0444	.0466	.0464	.0479	.0479	.0480	.0479	.0479	.0480	.0324	.0324	.0318	.3275	.3321	.3323	.3275	.3274	.3275
15	.0444	.0466	.0464	.0479	.0478	.0480	.0479	.0479	.0480	.0324	.0324	.0318	.3278	.3326	.3328	.3278	.3307	.3309
20	.0444	.0466	.0464	.0479	.0478	.0480	.0479	.0479	.0480	.0324	.0324	.0318	.3278	.3324	.3326	.3277	.3324	.3326
**		0.0464																

Σn	$\sigma_x/(p_z^+a^2/h^2)$			$\sigma_y/(p_z^+a^2/h^2)$			$\tau_{xy}/(p_z^+a^2/h^2)$			$\tau_{xz}/(p_z^+a^2/h^2)$			$\tau_{yz}/(p_z^+a^2/h^2)$		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
1	.2950	.2950	.2992	.3100	.3100	.3138	.1808	.1810	.1807	—	.3590	.5333	—	.2440	.3652
2	.2860	.2860	.2886	.2887	.2830	.2853	.1906	.1910	.1906	—	.3220	.4645	—	.2880	.4312
5	.2873	.2870	.2907	.2889	.2880	.2909	.1940	.1940	.1940		.3290	.4892		.3170	.4740
10	.2873	.2870	.2903	.2873	.2870	.2902	.1942	.1940	.1946		.3270	.4877		.3270	.4866
15	.2873	.2870	.2903	.2873	.2870	.2904	.1958	.1940	.1947		.3270	.4874		.3310	.4893
20	.2873	.2870	.2903	.2873	.2870	.2903	.1946	.1940	.1948		.3270	.4873		.3320	.4897
**		.2901			.2901						.4882			0.4882	

**Elasticity solution [6].
Boundary conditions along four edges: K, $w = M_n = 0$; R-M, $w = \theta_t = M_n = 0$; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.
K, Kirchhoff theory; R-M, Reissner-Mindlin theory; H, higher-order theory.

Table 2

Square isotropic plate under uniformly distributed load, simply supported along four edges (a/h = 2–100, ν = 0.3, $k_s = 6/5$): parametric study with various a/h ratios (converged values with n = 20)

a/h	$wD/p_z^+a^4$			$M_x/p_z^+a^2$			$M_y/p_z^+a^2$			$M_{xy}/p_z^+a^2$			Q_x/p_z^+a			Q_y/p_z^+a		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
2	.00406	.00931	.00852	.0479	.0478	.0516	.0479	.0478	.0516	.0324	.0324	.0216	.3278	.3275	.3326	.3277	.3324	.3325
5	.00406	.00499	.00480	.0479	.0478	.0485	.0479	.0478	.0485	.0324	.0324	.0299	.3278	.3275	.3326	.3277	.3324	.3325
10	.00406	.00427	.00425	.0479	.0478	.0480	.0479	.0478	.0480	.0324	.0324	.0317	.3278	.3275	.3326	.3277	.3324	.3326
20	.00406	.00411	.00411	.0479	.0478	.0479	.0479	.0478	.0479	.0324	.0324	.0323	.3278	.3275	.3326	.3277	.3324	.3326
25	.00406	.00409	.00407	.0479	.0478	.0479	.0479	.0478	.0479	.0324	.0324	.0323	.3278	.3275	.3326	.3277	.3324	.3326
50	.00406	.00406	.00406	.0479	.0478	.0479	.0479	.0478	.0479	.0324	.0324	.0323	.3278	.3275	.3326	.3277	.3324	.3323
100	.00406	.00406	.00406	.0479	.0479	.0479	.0479	.0478	.0479	.0324	.0324	.0323	.3278	.3275	.3326	.3277	.3324	.3327

a/h	$\sigma_x/(p_z^+a^2/h^2)$			$\sigma_y/(p_z^+a^2/h^2)$			$\tau_{xy}/(p_z^+a^2/h^2)$			$\tau_{xz}/(p_z^+a^2/h^2)$			$\tau_{yz}/(p_z^+a^2/h^2)$		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
2	.2950	.2870	.3631	.2873	.2870	.3630	.1946	.1946	.2115	—	.3270	.4075	—	.3327	.4081
5	.2860	.2870	.2994	.2873	.2870	.2993	.1946	.1946	.1971	—	.3270	.4716	—	.3320	.4707
10	.2873	.2870	.2903	.2873	.2870	.2903	.1946	.1946	.1948	—	.3270	.4873	—	.3320	.4897
20	.2873	.2870	.2881	.2873	.2870	.2881	.1946	.1946	.1948	—	.3270	.4968	—	.3320	.4965
25	.2873	.2870	.2878	.2873	.2870	.2874	.1946	.1946	.1948	—	.3270	.4979	—	.3320	.4975
50	.2873	.2870	.2874	.2873	.2870	.2874	.1946	.1946	.1948	—	.3270	.4989	—	.3320	.4986
100	.2873	.2870	.2870	.2873	.2870	.2874	.1946	.1946	.1948	—	.3270	.4989	—	.3330	.5042

K, Kirchhoff theory; R-M, Reissner-Mindlin theory; H, higher-order theory.
Boundary conditions along four edges: K, $w = M_n = 0$; R-M, $w = \theta_t = M_n = 0$; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.

Table 3

Square orthotropic plate under uniformly distributed load, simply supported along four edges (a/h = 5, $\nu_{xy} = \nu_{xz} = \nu_{yz} = 0.25$, k = 6/5, $E_x/E_y = 3.0$, $k_s = 6/5$): convergence study

Σn	$wE_yh^3/p_z^+a^4$			$M_x/p_z^+a^2$			$M_y/p_z^+a^2$			$M_{xy}/p_z^+a^2$			Q_x/p_z^+a			Q_y/p_z^+a		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
1	.0283	.0353	.0348	.0733	.0714	.0717	.0294	.0305	.0313	.0290	.0296	.0287	.4442	.4379	.4368	.1753	.1789	.1798
2	.0278	.0343	.0339	.0715	.0696	.0696	.0251	.0262	.0269	.0313	.0318	.0306	.3896	.3863	.3858	.2189	.2224	.2233
5	.0278	.0345	.0340	.0717	.0698	.0700	.0259	.0271	.0278	.0320	.0325	.0309	.4048	.4002	.3995	.2484	.2519	.2528
10	.0278	.0345	.0340	.0717	.0698	.0700	.0259	.0270	.0278	.0321	.0326	.0309	.4021	.3979	.3972	.2585	.2619	.2629
15	.0278	.0345	.0340	.0717	.0698	.0700	.0259	.0270	.0278	.0321	.0327	.0309	.4028	.3985	.3978	.2619	.2653	.2662
20	.0278	.0345	.0340	.0717	.0698	.0700	.0259	.0270	.0278	.0321	.0327	.0309	.4025	.3985	.3975	.2635	.2670	.2677
**		0.03445																

Σn	$\sigma_x/(p_z^+a^2/h^2)$			$\sigma_y(p_z^+a^2/h^2)$			$\tau_{xy}(p_z^+a^2/h^2)$			$\tau_{xz}(p_z^+a^2/h^2)$			$\tau_{yz}(p_z^+a^2/h^2)$		
	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H	K	R-M	H
1	.4210	.5040	.4456	.0338	.1830	.1938	.1742	.1780	.1817	—	.4380	.6039	—	.1790	.2686
2	.4105	.4800	.4311	.0259	.1570	.1642	.1876	.1910	.1966	—	.3860	.5441	—	.2220	.3326
5	.4118	.4800	.4344	.0275	.1620	.1714	.1876	.1950	.2040	—	.4000	.5564	—	.2520	.3717
10	.4118	.4800	.4335	.0275	.1620	.1698	.1926	.1960	.2069	—	.3980	.5562	—	.2620	.3775
15	.4118	.4800	.4338	.0275	.1620	.1704	.1926	.1960	.2078	—	.3980	.5557	—	.2650	.3761
20	.4118	.4800	.4337	.0275	.1620	.1701	.1926	.1960	.2082	—	.3980	.5560	—	.2670	.3745

**Method of initial function [7].
Boundary conditions along four edges: K, $w = M_n = 0$; R-M, $w = \theta_t = M_n = 0$; H, $w = \theta_t = M_n = w^* = \theta_t^* = M_n^* = 0$.
K, Kirchhoff theory; R-M, Reissner-Mindlin theory; H, higher-order theory.

Reissner-Mindlin and the higher-order theories are very close to each other. The predictions of Kirchhoff theory for M_x and M_y are seen to be 2.5% higher and 4% lower, respectively, compared to those given by the higher-order/Reissner-Mindlin theories.

Results of a parametric study for varying E_x/E_y ratios are shown in Table 4. It is seen from the results that for $a/h = 5$, predictions of the Kirchhoff theory are very much in error for orthotropic plates with E_x/E_y more than 2. For ratios of E_x/E_y as high as 40, predictions of the Reissner-Mindlin and higher-order theories are in excellent agreement with exact results [7]. Results of stress resultants M_x, M_y, M_{xy}, Q_x, and Q_y and stresses σ_x, σ_y, τ_{xy}, τ_{xz}, and τ_{yz} are also shown in Table 4. It is seen from Table 4 that moments M_x and M_y are vastly influenced by the rigidity ratio E_x/E_y, while maximum shear force, Q_x and Q_y, though affected, do not change to the same extent as M_x and M_y. Thus, for $E_x/E_y = 10$, $M_x/M_y = 6.4$, while $Q_x/Q_y = 2.45$. No analytical solution is available in the literature for these problems.

Further parametric results, for a square orthotropic plate, simply supported on all four edges, shown in Table 5, show the effect of varying a/h ratios on the maximum deflection and stress resultants. It is seen from the results shown in Table 5 that as the a/h ratio approaches 100, all the theories predict almost identical results. It is also seen from Table 5 that Kirchhoff theory always gives results of thin plates without regard of the ratio a/h, while Reissner-Mindlin theory and the higher-order theory are seen to take into account the effect of low a/h ratio in calculating transverse deflection and stress resultants.

Compilation of effects of both ratios, a/h (ranging from 2 to 100) and E_x/E_y (ranging from 1.0 to 40.0) is shown in Table 6. It is seen from the results shown in Table 6 that as the ratio E_x/E_y increases, the plate starts behaving like a one-way plate, so that for E_x/E_y equal to 3 or more, the moment M_x is much larger than M_y though the plate is square and simply supported. It is also seen that for thin plates ($a/h > 10$), all theories predict more or less identical results.

Example 3: Square isotropic plate with S boundary conditions along edges y = 0 and b and just supported (S^*) along edges x = 0 and a

Results for w, M_x, M_{xy}, M_y, Q_x, and Q_y are shown in Table 7. These results are compared with results published earlier [2–4].

Example 4: Square orthotropic plate with S boundary conditions along edges y = 0 and b and just supported (S^*) boundary conditions along edges x = 0 and a

In this example, edges along $x = 0$, a are taken to be just supported, implying vanishing of w (w^*), M_x (M_x^*), M_y (M_y^*). Relevant results are shown in Table 8.

Example 5: Square isotropic plate with S boundary conditions along edges y = 0 and b and clamped (C) along edges x = 0 and a

Results for w, M_x, $M_x(-ve)$, M_y, Q_x, and Q_y are shown in Table 9. These results are compared with the results published earlier [2–4].

Example 6: Square orthotropic plate with S boundary conditions along edges y = 0 and b and clamped (C) boundary conditions along edges x = 0 and a

Results for w, M_x $(+ve)$, M_x $(-ve)$, M_y, Q_x, and Q_y are shown in Table 10. These are the maximum values of stress resultants and deflection. M_x $(-ve)$ and Q_x are calculated at $x = 0$, $y = b/2$, and other values are calculated at $x = a/2$, $y = b/2$. As seen from Table 10, for high modular ratio E_x/E_y, the moment M_x $(+ve)$ is much larger than M_y, as indicated by all theories, and Q_x is also larger than Q_y by a reasonably large margin.

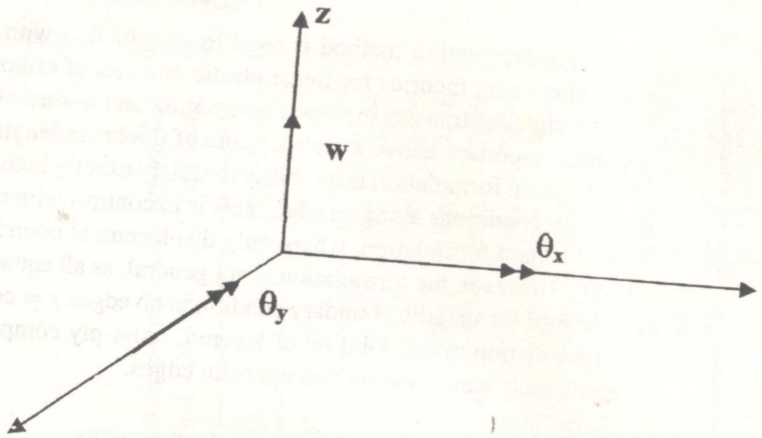
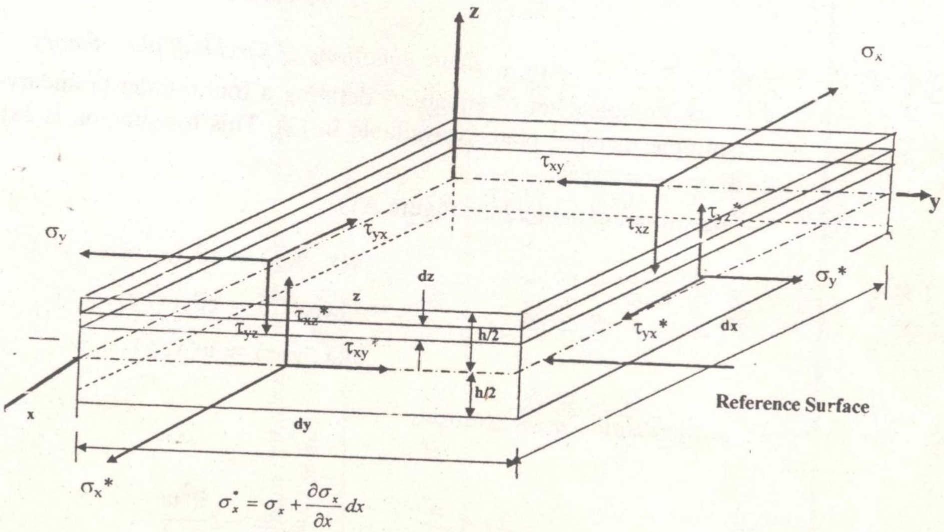


Figure A1. *Positive set of displacement components at mid-surface.*

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y} \tag{A2c}$$

$$\theta_x = -\frac{\partial w}{\partial x} \tag{A2d}$$

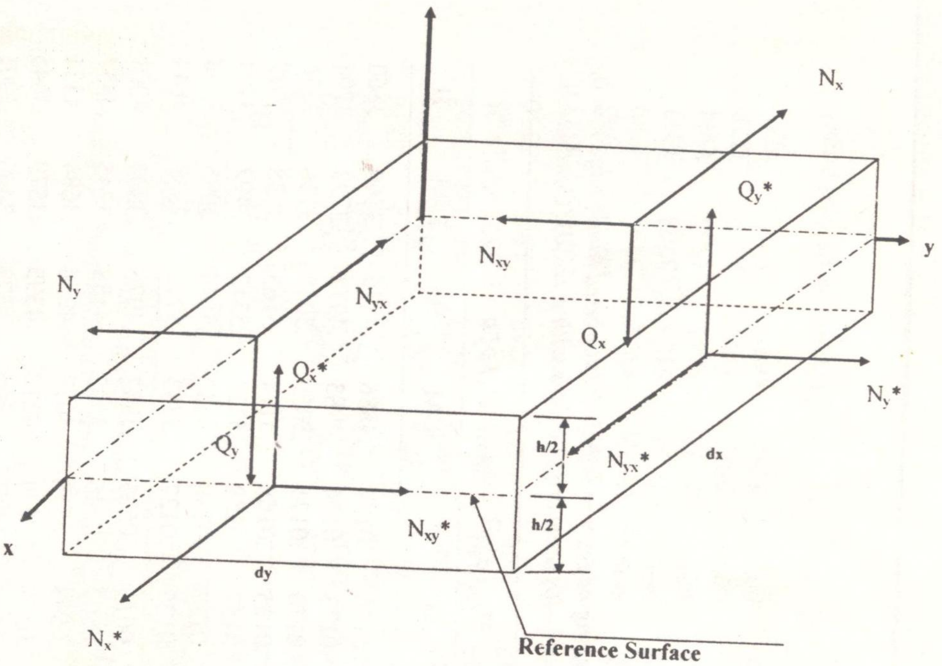
$$\theta_y = -\frac{\partial w}{\partial y} \tag{A2e}$$



$$\sigma_x^* = \sigma_x + \frac{\partial \sigma_x}{\partial x}dx$$

$$\tau_{xy}^* = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx \text{ etc.}$$

Figure A2. *Positive set of stress components.*



$$Q_x^* = Q_x + \frac{\partial Q_x}{\partial x}\,dx$$

$$Q_y^* = Q_y + \frac{\partial Q_y}{\partial y}\,dy \text{ etc.}$$

Figure A3. *Positive set of stress resultants—forces.*

Equilibrium equations

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ + p_z^- + \rho h = 0 \qquad \text{(A3}a\text{)}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \qquad \text{(A3}b\text{)}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \qquad \text{(A3}c\text{)}$$

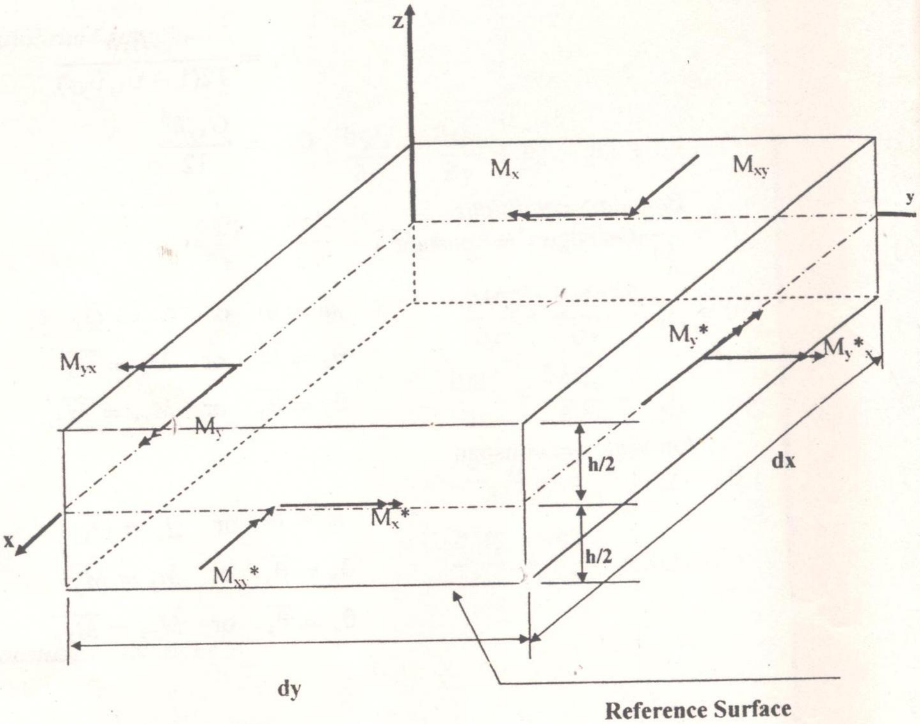
in which ρ is the unit weight of the material and h is the total plate thickness; p_z^+ and p_z^- are surface tractions on the top and bottom surfaces of the plate, respectively.

Constitutive relations (see Figure A2)

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\upsilon_{yx}\sigma_y}{E_y} \qquad \text{(A4}a\text{)}$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \frac{\upsilon_{xy}\sigma_x}{E_x} \qquad \text{(A4}b\text{)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \text{(A4}c\text{)}$$



$$M_x^* = M_x + \frac{\partial M_x}{\partial x} dx$$

$$M_{xy}^* = M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \text{ etc}$$

Figure A4. *Positive stress of resultants—couples.*

and the inverse relations are

$$\sigma_x = \frac{E_x}{1 - \nu_{yx}\nu_{xy}}(\varepsilon_x + \nu_{yx}\varepsilon_y) \tag{A4d}$$

$$\sigma_y = \frac{E_y}{1 - \nu_{yx}\nu_{xy}}(\varepsilon_y + \nu_{xy}\varepsilon_x) \tag{A4e}$$

$$\tau_{xy} = G_{xy}\gamma_{xy} \tag{A4f}$$

Force displacement relations (see Figures A3 and A4)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z\, dz = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right) \tag{A5a}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z\, dz = -\left(D_1 \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right) \tag{A5b}$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz = -2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \tag{A5c}$$

where

$$D_x = \frac{E_x h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})} \tag{A5d}$$

$$D_y = \frac{E_y h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})} \tag{A5e}$$

$$D_1 = \frac{\upsilon_{yx} E_x h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})} \tag{A5f}$$

$$D_{xy} = \frac{G_{xy} h^3}{12} \tag{A5g}$$

Boundary conditions

On edge x = constant

$$w = \bar{w} \quad \text{or} \quad V_x = \overline{V_x} \tag{A6a}$$

$$\theta_x = \overline{\theta_x} \quad \text{or} \quad M_x = \overline{M_x} \tag{A6b}$$

On edge y = constant

$$w = \bar{w} \quad \text{or} \quad V_y = \overline{V_y} \tag{A6c}$$

$$\theta_y = \overline{\theta_y} \quad \text{or} \quad M_y = \overline{M_y} \tag{A6d}$$

APPENDIX B

Basic equations of Reissner-Mindlin plate theory

A complete set of equations defining a sixth-order boundary-value problem for an isotropic material is available in [3]. This formulation is extended here for an orthotropic material.

Displacement model (see Figure A1)

$$u(x, y, z) = z\theta_x(x, y) \tag{B1a}$$

$$v(x, y, z) = z\theta_y(x, y) \tag{B1b}$$

$$w(x, y, z) = w(x, y) \tag{B1c}$$

Strain displacement relations

$$\varepsilon_x = z\frac{\partial \theta_x}{\partial x} \tag{B2a}$$

$$\varepsilon_y = z\frac{\partial \theta_y}{\partial y} \tag{B2b}$$

$$\gamma_{xy} = z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \tag{B2c}$$

$$\gamma_{xz} = \frac{1}{k_s}\left(\frac{\partial w}{\partial x} + \theta_x\right) \tag{B2d}$$

$$\gamma_{yz} = \frac{1}{k_s}\left(\frac{\partial w}{\partial y} + \theta_y\right) \tag{B2e}$$

Equations of equilibrium

These equations are the same as Eqs. (A3*a*)–(A3*c*)

Constitutive relations

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\upsilon_{yx}\sigma_y}{E_y} \tag{B3a}$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \frac{\upsilon_{xy}\sigma_x}{E_x} \tag{B3b}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \tag{B3c}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G_{xz}} \tag{B3d}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \tag{B3e}$$

and the inverse relations are

$$\sigma_x = \frac{E_x}{1 - \nu_{yx}\nu_{xy}}[\varepsilon_x + \nu_{yx}\varepsilon_y] \tag{B3f}$$

$$\sigma_y = \frac{E_y}{1 - \nu_{yx}\nu_{xy}}[\varepsilon_y + \nu_{xy}\varepsilon_x] \tag{B3g}$$

$$\tau_{xy} = G_{xy}\gamma_{xy} \tag{B3h}$$

$$\tau_{xz} = G_{xz}\gamma_{xz} \tag{B3i}$$

$$\tau_{yz} = G_{yz}\gamma_{yz} \tag{B3j}$$

Force displacement relations (see Figures A3 and A4)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z\, dz = D_x \frac{\partial\theta_x}{\partial x} + D_1 \frac{\partial\theta_y}{\partial y} \tag{B4a}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z\, dz = D_1 \frac{\partial\theta_x}{\partial x} + D_y \frac{\partial\theta_y}{\partial y} \tag{B4b}$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz = 2D_{xy}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) \tag{B4c}$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz = \frac{G_{xz}h}{k_s}\left(\frac{\partial w}{\partial x} + \theta_x\right) \tag{B4d}$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz = \frac{G_{yz}h}{k_s}\left(\frac{\partial w}{\partial y} + \theta_y\right) \tag{B4e}$$

where

$$D_x = \frac{E_x h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})} \tag{B4f}$$

$$D_y = \frac{E_y h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})} \tag{B4g}$$

$$D_1 = \frac{\upsilon_{yx} E_x h^3}{12(1 - \upsilon_{xy}\upsilon_{yx})} \tag{B4h}$$

$$D_{xy} = \frac{G_{xy} h^3}{12} \tag{B4i}$$

Boundary conditions

On edge $x =$ constant

$$w = \bar{w} \quad \text{or} \quad Q_x = \overline{Q_x} \tag{B5a}$$
$$\theta_x = \overline{\theta_x} \quad \text{or} \quad M_x = \overline{M_x} \tag{B5b}$$
$$\theta_y = \overline{\theta_y} \quad \text{or} \quad M_{xy} = \overline{M_{xy}} \tag{B5c}$$

On edge $y =$ constant

$$w = \bar{w} \quad \text{or} \quad Q_y = \overline{Q_y} \tag{B5d}$$
$$\theta_y = \overline{\theta_y} \quad \text{or} \quad M_y = \overline{M_y} \tag{B5e}$$
$$\theta_x = \overline{\theta_x} \quad \text{or} \quad M_{xy} = \overline{M_{xy}} \tag{B5f}$$

APPENDIX C

Basic equations of higher-order plate theory

A complete set of equations defining a twelth-order boundary-value problem for an isotropic material is available in [4]. This formulation is extended here for an orthotropic material.

Displacement model (see Figure A1)

$$u(x, y, z) = z\theta_x(x, y) + z^3\theta_x^*(x, y) \tag{C1a}$$
$$v(x, y, z) = z\theta_y(x, y) + z^3\theta_y^*(x, y) \tag{C1b}$$
$$w(x, y, z) = w(x, y) + z^2 w^*(x, y) \tag{C1c}$$

Strain displacement relations

$$\varepsilon_x = z\frac{\partial \theta_x}{\partial x} + z^3\frac{\partial \theta_x^*}{\partial x} \tag{C2a}$$
$$\varepsilon_y = z\frac{\partial \theta_y}{\partial y} + z^3\frac{\partial \theta_y^*}{\partial y} \tag{C2b}$$
$$\varepsilon_z = 2zw^* \tag{C2c}$$
$$\gamma_{xy} = z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) + z^3\left(\frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) \tag{C2d}$$
$$\gamma_{xz} = \left(\frac{\partial w}{\partial x} + \theta_x\right) + z^2\left(\frac{\partial w^*}{\partial w} + 3\theta_x^*\right) \tag{C2e}$$
$$\gamma_{yz} = \left(\frac{\partial w}{\partial y} + \theta_y\right) + z^2\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right) \tag{C2f}$$

Equilibrium equations

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ + p_z^- + \rho h = 0 \quad (C3a)$$

$$\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z + \frac{h^2}{4}(p_z^+ + p_z^-) + \frac{\rho h^3}{12} = 0 \quad (C3b)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad (C3c)$$

$$\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = 0 \quad (C3d)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad (C3e)$$

$$\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - 3Q_y^* = 0 \quad (C3f)$$

Constitutive relations

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} - \frac{\nu_{zx}\sigma_z}{E_z} \quad (C4a)$$

$$\varepsilon_y = -\frac{\nu_{xy}\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \frac{\nu_{zy}\sigma_z}{E_z} \quad (C4b)$$

$$\varepsilon_z = -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} + \frac{\sigma_z}{E_z} \quad (C4c)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \quad (C4d)$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G_{xz}} \quad (C4e)$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \quad (C4f)$$

and the inverse relations are

$$\sigma_x = \frac{E_x}{\Delta}[(1 - \nu_{yz}\nu_{zy})\varepsilon_x + (\nu_{yx} + \nu_{zx}\nu_{yz})\varepsilon_y + (\nu_{zx} + \nu_{yx}\nu_{zy})\varepsilon_z] \quad (C5a)$$

$$\sigma_y = \frac{E_y}{\Delta}[(\nu_{xy} + \nu_{zy}\nu_{xz})\varepsilon_x + (1 - \nu_{xz}\nu_{zx})\varepsilon_y + (\nu_{zy} + \nu_{xy}\nu_{zx})\varepsilon_z] \quad (C5b)$$

$$\sigma_z = \frac{E_z}{\Delta}[(\nu_{xz} + \nu_{xy}\nu_{yz})\varepsilon_x + (\nu_{yz} + \nu_{yx}\nu_{xz})\varepsilon_y + (1 - \nu_{xy}\nu_{yx})\varepsilon_z] \quad (C5c)$$

$$\tau_{xy} = G_{xy}\gamma_{xy} \quad (C5d)$$

$$\tau_{yz} = G_{yz}\gamma_{yz} \quad (C5e)$$

$$\tau_{zx} = G_{xz}\gamma_{xz} \quad (C5f)$$

The paper in this publication meets the requirements of the ANSI Standard Z39.48-1984 (Permanence of Paper), effective with Volume 21.

Force displacement relations (see Figures A3 and A4)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz = D_{1x}\frac{\partial \theta_x}{\partial x} + D_{2x}\frac{\partial \theta_y}{\partial y} + D_{3x}2w^* + D_{4x}\frac{\partial \theta_x^*}{\partial x} + D_{5x}\frac{\partial \theta_y^*}{\partial y} \quad \text{(C6}a\text{)}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z \, dz = D_{1y}\frac{\partial \theta_x}{\partial x} + D_{2y}\frac{\partial \theta_y}{\partial y} + D_{3y}2w^* + D_{4y}\frac{\partial \theta_x^*}{\partial x} + D_{5y}\frac{\partial \theta_y^*}{\partial y} \quad \text{(C6}b\text{)}$$

$$M_z = \int_{-h/2}^{h/2} \sigma_z z \, dz = D_{1z}\frac{\partial \theta_x}{\partial x} + D_{2z}\frac{\partial \theta_y}{\partial y} + D_{3z}2w^* + D_{4z}\frac{\partial \theta_x^*}{\partial x} + D_{5z}\frac{\partial \theta_y^*}{\partial y} \quad \text{(C6}c\text{)}$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz = D_{1xy}\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) + D_{2xy}\left(\frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) \quad \text{(C6}d\text{)}$$

$$M_x^* = \int_{-h/2}^{h/2} \sigma_x z^3 \, dz = D_{1x}^*\frac{\partial \theta_x}{\partial x} + D_{2x}^*\frac{\partial \theta_y}{\partial y} + D_{3x}^*2w^* + D_{4x}^*\frac{\partial \theta_x^*}{\partial x} + D_{5x}^*\frac{\partial \theta_y^*}{\partial y} \quad \text{(C6}e\text{)}$$

$$M_y^* = \int_{-h/2}^{h/2} \sigma_y z^3 \, dz = D_{1y}^*\frac{\partial \theta_x}{\partial x} + D_{2y}^*\frac{\partial \theta_y}{\partial y} + D_{3y}^*2w^* + D_{4y}^*\frac{\partial \theta_x^*}{\partial x} + D_{5y}^*\frac{\partial \theta_y^*}{\partial y} \quad \text{(C6}f\text{)}$$

$$M_{xy}^* = \int_{-h/2}^{h/2} \tau_{xy} z^3 \, dz = D_{1xy}^*\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) + D_{2xy}^*\left(\frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) \quad \text{(C6}g\text{)}$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} \, dz = K_{1x}\left(\frac{\partial w}{\partial x} + \theta_x\right) + K_{2x}\left(\frac{\partial w^*}{\partial x} + 3\theta_x^*\right) \quad \text{(C6}h\text{)}$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} \, dz = K_{1y}\left(\frac{\partial w}{\partial x} + \theta_y\right) + K_{2y}\left(\frac{\partial w^*}{\partial x} + 3\theta_y^*\right) \quad \text{(C6}i\text{)}$$

$$Q_x^* = \int_{-h/2}^{h/2} \tau_{xz} z^2 \, dz = K_{1x}^*\left(\frac{\partial w}{\partial x} + \theta_x\right) + K_{2x}^*\left(\frac{\partial w^*}{\partial x} + 3\theta_x^*\right) \quad \text{(C6}j\text{)}$$

$$Q_y^* = \int_{-h/2}^{h/2} \tau_{yz} z^2 \, dz = K_{1y}^*\left(\frac{\partial w}{\partial y} + \theta_y\right) + K_{2y}^*\left(\frac{\partial w^*}{\partial y} + 3\theta_y^*\right) \quad \text{(C6}k\text{)}$$

where

$$D_{1x} = \frac{E_x h^3}{12\Delta}(1 - \upsilon_{yz}\upsilon_{zy}) \quad \text{(C7}a\text{)}$$

$$D_{2x} = \frac{E_x h^3}{12\Delta}(\upsilon_{yx} + \upsilon_{zx}\upsilon_{yz}) \quad \text{(C7}b\text{)}$$

$$D_{3x} = \frac{E_x h^3}{12\Delta}(\upsilon_{zx} + \upsilon_{yx}\upsilon_{zy}) \quad \text{(C7}c\text{)}$$

$$D_{4x} = \frac{E_x h^5}{80\Delta}(1 - \upsilon_{yz}\upsilon_{zy}) \quad \text{(C7}d\text{)}$$

$$D_{5x} = \frac{E_x h^3}{80\Delta}(\upsilon_{yx} + \upsilon_{zx}\upsilon_{yz}) \quad \text{(C7}e\text{)}$$

$$D_{1x}^* = \frac{E_x h^5}{80\Delta}(1 - \upsilon_{yz}\upsilon_{zy}) \quad \text{(C7}f\text{)}$$

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$$D^*_{2x} = \frac{E_x h^5}{80\Delta}(\upsilon_{yx} + \upsilon_{zx}\upsilon_{yz}) \tag{C7g}$$

$$D^*_{3x} = \frac{E_x h^5}{80\Delta}(\upsilon_{zx} + \upsilon_{yx}\upsilon_{zy}) \tag{C7h}$$

$$D^*_{4x} = \frac{E_x h^7}{448\Delta}(1 - \upsilon_{yz}\upsilon_{zy}) \tag{C7i}$$

$$D^*_{5x} = \frac{E_x h^7}{448\Delta}(\upsilon_{yx} + \upsilon_{zx}\upsilon_{yz}) \tag{C7j}$$

$$\Delta = 1 - \nu_{xy}\nu_{yx} - \nu_{yz}\nu_{zy} - \nu_{zx}\nu_{xz} - 2\nu_{yx}.\nu_{zy}.\nu_{xz}$$

Cyclic expressions are obtained for other constants for the y and z directions.

$$D_{1xy} = \frac{G_{xy}h^3}{12} \tag{C7k}$$

$$D_{2xy} = \frac{G_{xy}h^5}{80} \tag{C7l}$$

$$D^*_{1xy} = \frac{G_{xy}h^5}{80} \tag{C7m}$$

$$D^*_{2xy} = \frac{G_{xy}h^7}{448} \tag{C7n}$$

$$K_{1x} = G_{xz}h \tag{C7o}$$

$$K_{2x} = \frac{G_{xz}h^3}{12} \tag{C7p}$$

$$K_{1y} = G_{yz}h \tag{C7q}$$

$$K_{2y} = \frac{G_{yz}h^3}{12} \tag{C7r}$$

$$K^*_{1x} = \frac{G_{xz}h^3}{12} \tag{C7s}$$

$$K^*_{2x} = \frac{G_{xz}h^5}{80} \tag{C7t}$$

$$K^*_{1y} = \frac{G_{yz}h^3}{12} \tag{C7u}$$

$$K^*_{2y} = \frac{G_{yz}h^5}{80} \tag{C7v}$$

Boundary conditions

On edge $x =$ constant

$$w = \bar{w} \quad \text{or} \quad Q_x = \overline{Q_x} \qquad w^* = \overline{w^*} \quad \text{or} \quad Q^*_x = \overline{Q^*_x} \tag{C8a}$$

$$\theta_x = \overline{\theta_x} \quad \text{or} \quad M_x = \overline{M_x} \qquad \theta^*_x = \overline{\theta^*_x} \quad \text{or} \quad M^*_x = \overline{M^*_x} \tag{C8b}$$

$$\theta_y = \overline{\theta_y} \quad \text{or} \quad M_{xy} = \overline{M_{xy}} \qquad \theta^*_y = \overline{\theta^*_y} \quad \text{or} \quad M^*_{xy} = \overline{M^*_{xy}} \tag{C8c}$$

On edge $y =$ constant

$$w = \bar{w} \quad \text{or} \quad Q_y = \overline{Q_y} \qquad w^* = \overline{w^*} \quad \text{or} \quad Q_x^* = \overline{Q_x^*} \tag{C8d}$$

$$\theta_x = \overline{\theta_x} \quad \text{or} \quad M_{yx} = \overline{M_{yx}} \qquad \theta_x^* = \overline{\theta_x^*} \quad \text{or} \quad M_x^* = \overline{M_x^*} \tag{C8e}$$

$$\theta_y = \overline{\theta_y} \quad \text{or} \quad M_y = \overline{M_y} \qquad \theta_y^* = \overline{\theta_y^*} \quad \text{or} \quad M_y^* = \overline{M_y^*} \tag{C8f}$$