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Comparisons of displacement-based theories for waves and vibrations in laminated and sandwich composite plates

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Abstract

A semi-analytical method incorporating various displacement-based formulations has been developed to investigate propagation of time harmonic waves and vibrations in fiber reinforced polymer composite laminated and sandwich plates. Various displacement-based models starting from the first order shear deformation theory to the fourth order theory have been developed using combinations of linear, quadratic, cubic and quartic variation of axial and transverse displacements through the thickness of a lamina or a mathematical sub-layer. These displacement-based formulations have been validated by comparing their results with the analytical solutions available in the literature. Results of all the displacement models have been compared with those obtained by displacement model using quartic variation of in-plane and transverse displacements for vibration problem. Higher order displacement-based theory using cubic variation of in-plane and transverse displacements through the thickness of sub-layer has been found to yield converging results for wave propagation in laminated composite plates as well as for vibration problems. All the investigations performed indicate the importance of higher order theories for analysis of wave propagation and vibrations in composite laminated and sandwich plates.

1. Introduction

Composite laminated plates are being increasingly used in many engineering applications like components of space structures, bridges, automobile vehicles, nuclear reactors etc. Therefore analysis of sandwich and composite laminated plates subjected to harmonic waves and natural vibrations has become a topic of major concern for researchers in this field. The complexities, attendant with the dynamics of composite laminated plates, are so many that except for a few special cases, exact solutions do not exist. The anisotropic properties of the composite lamina

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along-with the through thickness warping of cross-section make the analysis of such structures a difficult task. Propagation of waves through laminated composite plates has been investigated extensively in the past by many researchers.

A few exact solutions have been reported on wave propagation in isotropic plates (e.g., Refs. [1,2]). Dong and Nelson [3], on the other hand, presented a method to analyze natural vibrations of laminated orthotropic plates in which a displacement field was assumed for each lamina. The displacements were characterized by a discrete number of generalized co-ordinates at the lamina bounding planes and their mid-surfaces. Subsequently, Dong and Pauley [4] presented a Ritz method for determination of frequencies and modal patterns of vibrations and waves in an infinite anisotropic plate. Dong and Huang [5] investigated plane-strain edge vibrations in laminated composite plates by using finite element method in which anisotropic laminate properties were considered. All these studies were based on a parabolic variation of displacement field through thickness.

Shah and Datta [6] investigated harmonic wave propagation through a periodically laminated infinite medium by using a stiffness method. Each lamina was divided into several sub-layers and the displacement distribution through the thickness of each layer was approximated by cubic polynomial interpolation functions, involving a number of discrete generalized co-ordinates, which were the displacements and tractions at the interfaces of the adjoining sub-layers. Datta et al. [7] presented a similar technique for investigating dispersion of waves in a laminated plate and an improvement over the work of Dong and Huang [5] was claimed. Each lamina was modelled as a homogeneous transversely isotropic medium with the symmetry axis parallel to the fibers. The overall effective elastic properties of a lamina were calculated from the fiber and matrix properties by using an effective modulus theory developed by Datta et al. [8]. Karunasena et al. [9] used a stiffness method and an analytical method to investigate the dispersion characteristics of guided waves in laminated composite plates. A Rayleigh–Ritz type of approximation of the through-thickness variation of the displacements that maintain continuity of displacements and tractions at the interfaces between the layers had been used in the stiffness method. The analytical method used Muller's method to obtain exact dispersion relation of the laminated plate with initial guesses obtained through the stiffness method.

Various techniques for the free vibration analysis of composite laminated plates have been reported in the literature. Reddy and Khedeir [10] presented analytical and finite element solutions for vibration and buckling of laminated composite plates using various plate theories to prove necessity of shear deformation theories to predict the behavior of composite laminates. Khedeir and Reddy [11] obtained a complete set of linear equations of the second order theory to analyze the free vibration behavior of cross-ply and antisymmetric angle-ply laminated plates. The exact analytical solutions were obtained for thick and moderately thick plates as well as for thin plates and plate strips. Cho et al. [12], for example, used a higher order plate theory in each individual layer of a simply supported rectangular laminated plate to determine the natural frequencies and the relative stress and deflection distributions through the thickness of plate. The theory approximated the in-plane and normal displacements by employing third and second order functions of the thickness co-ordinate, respectively. Dawe and Wang [13], on the other hand, utilized B-spline functions to define the displacement field in analysis of composite laminated rectangular plates by Rayleigh–Ritz method. Taylor and Nayfeh [14] obtained solutions for the individual layers which relate the field variables at the upper and lower layer surfaces and used linear transformations to refer to the anisotropy of each layer to a global co-ordinate system. Wang and Lin [15], on the other hand, presented a finite strip method based on higher order plate theory for determining the natural frequencies of laminated plate. Chen et al. [16] investigated free vibration analysis of symmetrically laminated thick rectangular plates with various combinations of free, simply supported and clamped boundary conditions. The p-Ritz method was employed in which uniquely defined polynomials for displacement and rotation functions were used. Liew et al. [17] examined the sensitivity of the vibration responses to variations in the lamination, boundary constraints and thickness effects and also their interactions using Ritz procedure and first order shear deformable plate theory. Srinivas and Rao [18] presented an exact analysis for the statics and dynamics of various thick laminates. A three-dimensional, linear, small deformation theory of elasticity solution was developed for the bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates.

Liew [19] employed a global p-Ritz method for vibration analysis of thick rectangular laminates with various combinations of boundary conditions. First order Reissner/Mindlin plate theory was utilized to incorporate the effects of transverse shear deformation and rotary inertia. Chen et al. [20] analyzed free vibrations of symmetrically laminated thick plates with rounded corners. The plate perimeter has been defined by a super elliptic function with a power defining the shape ranging from an ellipse to a rectangle. The Reddy third order plate theory has been employed to incorporate the transverse shear deformation. Liew et al. [21] reviewed existing literature on the vibration analysis of thick plates. Most of the works covered were based on the Mindlin theory and the modified Mindlin plate theories of laminated plates, while some papers using higher order shear deformation plate theories were also included.

Karunasena et al. [22] proposed a hybrid method for analysis of lamb wave reflection by a crack at the fixed edge of a composite plate by combining finite element formulation in a bounded interior region of the plate with a wave function expansion representation in an unbounded exterior region. Karunasena et al. [23] also employed an approximate method based on the wave function expansion procedure to solve the reflection of plate waves at the fixed edge of a composite plate. The amplitudes of reflected waves have been determined by satisfying the fixed edge condition through the application of variational principle. Lim et al. [24] performed free vibration analysis of pre-twisted, cantilevered composite shallow conical shells, wherein an extremum energy principle was employed to derive the eigenvalue equation and a flexible global admissible function was developed to account for the geometric boundary conditions.

Various displacement models have been developed in the present work, by considering combinations of displacement fields for in-plane and transverse displacements inside a mathematical sub-layer to investigate the phenomenon of wave propagation as well as vibrations in laminated composite plates. Numerical evaluations obtained for wave propagation and vibrations in isotropic, orthotropic and composite laminated plates have been used to determine the efficient displacement field for economic analysis of wave propagation and vibrations in laminated composite plate. The numerical method developed follows a semi-analytical approach with analytical field applied in longitudinal direction and layer-wise displacement field employed in transverse direction. The present work aims at developing a simple numerical technique, which can produce very accurate results in comparison with the available analytical solution and also to decide upon the level of refinement in higher order theory that is needed for accurate and efficient analysis.

2. Formulation

A semi-analytical method has been presented for analyzing plane-strain propagation of waves and vibrations in composite laminated plate. Two types of problems have been investigated, viz., (a) wave propagation through composite laminated plate; and (b) free vibrations of simply supported composite laminated plate.

The laminated composite plate shown in Fig. 1 consists of a number of layers. The local co-ordinate system (x_i, y_i, z_i) for ith layer is selected parallel to the global system (x, y, z). The origin of the local system is located at the mid-plane of a lamina of thickness $2h_i$. In the sequel, h_i will be denoted as h.

By assuming different combinations of variation of displacements through a lamina of a laminate, the time-dependent axial and transverse displacements of any point lying in the $x-z$ plane for various models can be expressed as

$$u(x,z,t) = \sum_{i=0}^{n} z^i a_i(x,t),$$
$$w(x,z,t) = \sum_{j=0}^{m} z^j b_j(x,t). \tag{1}$$

The models are classified as follows:

1. First order shear deformation theory (FOST, $n=1$, $m=0$).
2. Higher order shear deformation theory 1 (HOST1, $n=1$, $m=1$).

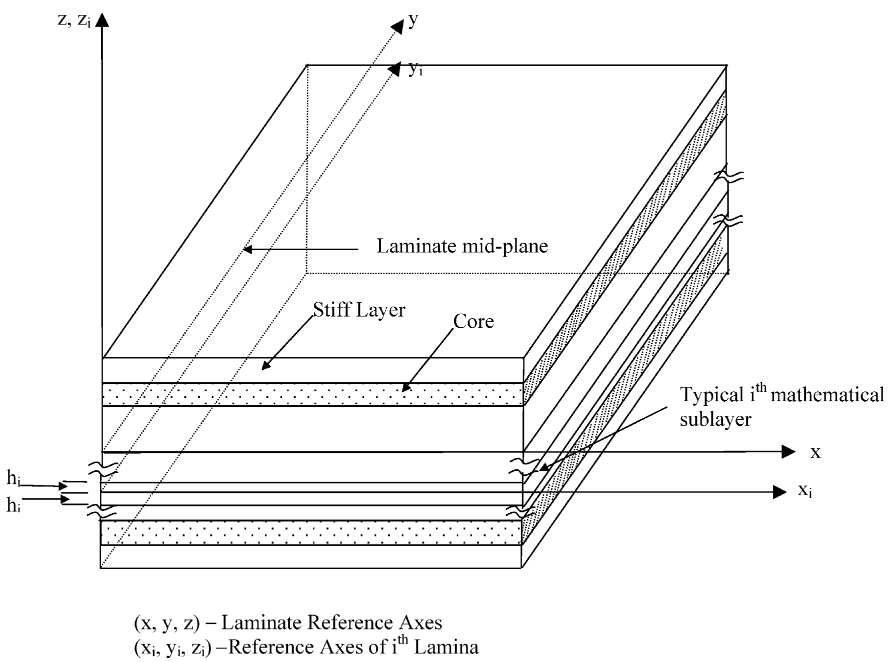

Fig. 1. Laminate geometry with positive set of laminate reference axes.

3. Higher order shear deformation theory 2 (HOST2, $n = 2$, $m = 1$).
4. Higher order shear deformation theory 3 (HOST3, $n = 2$, $m = 2$).
5. Higher order shear deformation theory 4 (HOST4, $n = 3$, $m = 2$).
6. Higher order shear deformation theory 5 (HOST5, $n = 3$, $m = 3$).
7. Higher order shear deformation theory 6 (HOST6, $n = 4$, $m = 3$).
8. Higher order shear deformation theory 7 (HOST7, $n = 4$, $m = 4$).

Here, $a_i, i = 0, 1, \ldots, n, b_j, j = 0, 1, \ldots, m$ are the generalized parameters. By expressing a_i and b_j in terms of the generalized displacements (at $z = -h, +h, 0$), rotations and normal strains (at $z = \pm h$), the following equations are obtained:

$$\begin{Bmatrix} u(x,z,t) \\ w(x,z,t) \end{Bmatrix} = [X(z)]\{q(x,t)\}, \tag{2}$$

where

$$[X] = \begin{bmatrix} [X_n] & [0] \\ [0] & [X_m] \end{bmatrix}, \quad \{q\}^{\mathrm{t}} = [\,[p_n] \quad [q_m]\,]. \tag{3a}$$

Constituent matrices $[X]$ and vectors $[p]$ and $[q]$ are defined below:

$$[X_0] = [X_{01}], \quad [X_1] = [\,X_{11} \quad X_{12}\,], \quad [X_2] = [\,X_{21} \quad X_{22} \quad X_{23}\,],$$
$$[X_3] = [\,X_{31} \quad X_{32} \quad X_{33} \quad X_{34}\,], \quad \text{and} \quad [X_4] = [\,X_{41} \quad X_{42} \quad X_{43} \quad X_{44} \quad X_{45}\,].$$

$$\begin{aligned} &[p_1] = [\,p_{11} \quad p_{12}\,], \quad [p_2] = [\,p_{21} \quad p_{22} \quad p_{23}\,], \quad [p_3] = [\,p_{31} \quad p_{32} \quad p_{33} \quad p_{34}\,], \\ &[p_4] = [\,p_{41} \quad p_{42} \quad p_{43} \quad p_{44} \quad p_{45}\,], \quad [q_0] = [q_{01}], \quad [q_1] = [\,q_{11} \quad q_{12}\,], \\ &[q_2] = [\,q_{21} \quad q_{22} \quad q_{23}\,], \\ &[q_3] = [\,q_{31} \quad q_{32} \quad q_{33} \quad q_{34}\,] \quad \text{and} \quad [q_4] = [\,q_{41} \quad q_{42} \quad q_{43} \quad q_{44} \quad q_{45}\,]. \end{aligned} \tag{3b}$$

The X_{ij}, $i = 0, 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$ appearing in Eq. (3b) are the shape functions for various diaplacement models given by

$$\begin{aligned} &X_{01} = 1, \quad X_{11} = \frac{(1-\xi)}{2}, \quad X_{12} = \frac{(1+\xi)}{2}, \\ &X_{21} = \frac{-\xi(1-\xi)}{2}, \quad X_{22} = 1 - \xi^2 \quad \text{and} \quad X_{23} = \frac{\xi(1+\xi)}{2}, \\ &X_{31} = \tfrac{1}{4}(2 - 3\xi + \xi^3), \quad X_{32} = \tfrac{1}{4}(2 + 3\xi + \xi^3), \\ &X_{33} = \frac{h}{4}(1 - \xi - \xi^2 + \xi^3) \quad \text{and} \quad X_{34} = \frac{h}{4}(1 - \xi + \xi^2 + \xi^3), \\ &X_{41} = \tfrac{1}{4}(-3\xi + 4\xi^2 + \xi^3 - 2\xi^4), \quad X_{42} = (1 - 2\xi^2 + \xi^4), \quad X_{43} = \tfrac{1}{4}(3\xi + 4\xi^2 - \xi^3 - 2\xi^4), \\ &X_{44} = \frac{h}{4}(-\xi + \xi^2 + \xi^3 - \xi^4) \quad \text{and} \quad X_{45} = \frac{h}{4}(-\xi - \xi^2 + \xi^3 + \xi^4), \end{aligned} \tag{4}$$

where $\xi = z/h$.

Furthermore, p_{ij}, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$ and q_{ij}, $i = 0, 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$ are

$$p_{11} = p_{21} = p_{31} = p_{41} = u_1, \quad p_{12} = p_{23} = p_{32} = p_{43} = u_2, \quad p_{22} = p_{42} = u_3,$$
$$p_{33} = p_{44} = \theta_{x1}, \quad p_{34} = p_{45} = \theta_{x2}, \quad q_{01} = w_0, \quad q_{11} = q_{21} = q_{31} = q_{41} = w_1,$$
$$q_{12} = q_{23} = q_{32} = q_{43} = w_2, \quad q_{22} = q_{42} = w_3, \quad q_{33} = q_{44} = \theta_{z1}, \quad \text{and} \quad q_{34} = q_{45} = \theta_{z2}.$$

u_i and w_i, $i = 1, 2, 3$ are the displacements at interfaces along the x and z directions, at $z = -h$, $+h$ *and 0*, respectivety. Displacement w_0 indicates constant displacement through the thickness. The θ_{xi}, θ_{zi}, $i = 1, 2$, on the other hand, are termed rotation and normal strain, respectively, in the text and are defined at $z = (-1)^i h$ as $\theta_x = \partial u/\partial z$ and $\theta_z = \partial w/\partial z$. The strain–displacement relations for a lamina or a mathematical sub-layer are

$$\varepsilon_x = \frac{\partial u}{\partial x} = \sum_{j=1}^{n+1} X_{nj} p'_{nj}, \tag{5}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = \sum_{j=1}^{m+1} \bar{X}_{mj} q_{mj}, \tag{6}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \sum_{j=1}^{n+1} \bar{X}_{nj} p_{nj} + \sum_{j=1}^{m+1} X_{mj} q'_{mj}, \tag{7}$$

where the primes denote the partial derivative with respect to x whereas the overbars denote the partial derivative with respect to z.

Eqs. (4)–(7) can be written in a concise matrix form

$$\{\varepsilon\} = [B_1]\{q\} + \{B_2\}\{q'\} \tag{8}$$

with

$$\begin{aligned} \{\varepsilon\}^{\mathrm{t}} &= [\,\varepsilon_x \quad \gamma_{xz}\,] \quad \text{(for model FOST)}, \\ \{\varepsilon\}^{\mathrm{t}} &= [\,\varepsilon_x \quad \varepsilon_z \quad \gamma_{xz}\,] \quad \text{(for models HOST1}-\text{HOST7)}, \end{aligned} \tag{9}$$

$$[B_1] = \begin{bmatrix} [0] & [\bar{X}_0] \\ [\bar{X}_1] & [0] \end{bmatrix}, \quad [B_2] = \begin{bmatrix} [X'_1] & [0] \\ [0] & [X'_2] \end{bmatrix} \quad \text{(for model FOST)},$$

$$[B_1] = \begin{bmatrix} [0] & [0] \\ [0] & [\bar{X}_m] \\ [\bar{X}_n] & [0] \end{bmatrix}, \quad [B_2] = \begin{bmatrix} [X'_n] & [0] \\ [0] & [0] \\ [0] & [X'_m] \end{bmatrix} \quad \text{(for models HOST1}-\text{HOST7)}, \tag{10}$$

where $[\bar{X}_i] = [\,\bar{X}_{i1} \quad \bar{X}_{i2} \quad \dots \quad \bar{X}_{ii}\,]$ and $[X'_i] = [\,X'_{i1} \quad X'_{i2} \quad \dots \quad X'_{ii}\,]$.

The stress–strain relationships of a lamina are

$$\{\sigma\} = [C]\{\varepsilon\}, \tag{11}$$

where

$$\begin{aligned} \{\sigma\}^{\mathrm{t}} &= [\,\sigma_x \quad \tau_{xz}\,] \quad \text{(for model FOST)}, \\ \{\sigma\}^{\mathrm{t}} &= [\,\sigma_x \quad \sigma_z \quad \tau_{xz}\,] \quad \text{(for model HOST1}-\text{HOST7)}. \end{aligned} \tag{12}$$

Here, $[C]$ is the elasticity matrix constituted by Young's moduli E_x, E_z, the Poisson ratios v_{xz} and v_{zx} and the shear modulus of elasticity G_{xz}.

The equation of motion for a lamina can be obtained by using the variational principle

$$\int_{t_1}^{t_2} \bar{\delta} \sum_k (T_k - U_k)\, \mathrm{d}t = 0, \tag{13}$$

where T_k and U_k are, respectively, the kinetic and the strain energies of the kth lamina and $\bar{\delta}$ is the first variation. The kinetic energy can be expressed as

$$T_k = \frac{1}{2} \int_V \rho \{\dot{u}\}^{\mathrm{t}} \{\dot{u}\}\, \mathrm{d}V, \tag{14}$$

where

$$\{\dot{u}\} = [\, \dot{u}(x, z, t) \quad \dot{w}(x, z, t)\,]^{\mathrm{t}} = [X(z)]\{\dot{q}(x, t)\}. \tag{15}$$

Here, the dot indicates derivative with respect to time 't'.

By substituting Eq. (15) in Eq. (14),

$$T_k = \frac{1}{2} \int [\{\dot{q}\}^{\mathrm{t}} [m]\{\dot{q}\}]\, \mathrm{d}x, \tag{16}$$

is obtained where

$$[m] = \int_{-h}^{h} ([X]^{\mathrm{t}} \rho [X])\, \mathrm{d}z. \tag{17}$$

The explicit form of $[m]$ has been presented in the appendix for various displacement models discussed above.

The internal strain energy, U_k, of a lamina can be computed from

$$U_k = \frac{1}{2} \int_V \{\varepsilon\}^{\mathrm{t}} \{\sigma\}\, \mathrm{d}v. \tag{18}$$

The strain energy per unit width of lamina can be derived by substituting Eqs. (8) and (11) into Eq. (18) as

$$U_k = \frac{1}{2} \int (\{q\}^{\mathrm{t}} [k_{11}]\{q\} + \{q\}^{\mathrm{t}} [k_{12}]\{q'\} + \{q'\}^{\mathrm{t}} [k_{12}]^{\mathrm{t}}\{q\} + \{q'\}^{\mathrm{t}} [k_{22}]\{q'\})\, \mathrm{d}x, \tag{19}$$

where

$$[k_{\alpha\beta}] = \int_{-h}^{h} ([B_\alpha]^{\mathrm{t}}[C][B_\beta])\, \mathrm{d}z, \quad \alpha, \beta = 1, 2. \tag{20}$$

Matrix $[k]$ in Eq. (20) has been evaluated explicitly and is presented in the appendix for various displacement models discussed above for ready reference and also for ease in computer implementation.

By substituting Eqs. (16) and (19) into Eq. (13), performing variation and collecting terms, the equation of motion for a laminated plate is obtained as

$$[K_{11}]\{q\} + [[K_{12}] - [K_{12}]^{\mathrm{t}}]\{q'\} - [K_{22}]\{q''\} + [M]\{\ddot{q}\} = \{0\}, \tag{21}$$

where the global matrices are evaluated as

$$[K_{\alpha\beta}] = \sum_k [k_{\alpha\beta}] \quad \text{and} \quad [M] = \sum_k [m].$$

Eq. (21) can be shown to be

$$[[K_1] + \mathrm{i}\lambda([K_2] - [K_2]^{\mathrm{t}}) + \lambda^2[K_3] - \omega^2[M]]\{q_0\} = \{0\} \tag{22}$$

for wave propagation problem, by assuming a general solution

$$\{q\} = \{q_0\} \exp \mathrm{i}(\lambda x - \omega t), \tag{23}$$

where

$$[K_1] = [K_{11}], \quad [K_2] = [K_{22}] \quad \text{and} \quad [K_3] = [K_{33}].$$

Here, $\{q_0\}$ is the amplitude vector, ω is circular frequency and λ is the wave number. On the other hand, following homogeneous equation can be obtained for vibration problem:

$$[[K_1] + \lambda([K_2^*] - [K_2^*]^{\mathrm{t}}) + \lambda^2[K_3] - \omega^2[M]]\{q_0\} = 0, \tag{24}$$

by assuming a general solution

$$\{q_0\} = \{q_{0,1}\}\sin(\lambda x)\exp(-\mathrm{i}\omega t) + \{q_{0,2}\}\cos(\lambda x)\exp(-\mathrm{i}\omega t). \tag{25}$$

Here, $\lambda = 1/L_s$ and L_s is the span length (half-wavelength). The superscript * in Eq. (24) indicates the modified nature of stiffness matrix $[K_2]$ after substitution of Eq. (25) into Eq. (21). Here, $\{q_{0,1}\}$ and $\{q_{0,2}\}$ are the amplitude vectors defined as follows:

$$\{q_{0,1}\} = \{\, [A]_{(1xn)} \quad [0]_{(1xm)} \,\}^{\mathrm{t}} \quad \text{and} \quad \{q_{0,2}\} = \{\, [0]_{(1xn)} \quad [B]_{(1xm)} \,\}^{\mathrm{t}}, \tag{26}$$

where

$$[A] = [\, A_1 \quad A_2 \quad \ldots \quad A_n \,] \quad \text{and} \quad [B] = [\, B_1 \quad B_2 \quad \ldots \quad B_m \,].$$

Equation for wave propagation and vibrations in a lamina is written in a compact form as

$$[[K] - \omega^2[M]]\{q_0\} = \{0\}, \tag{27}$$

where

$$[K] = [K_1] + \mathrm{i}\lambda([K_2] - [K_2]^{\mathrm{t}}) + \lambda^2[K_3] \quad \text{for wave propagation problem,}$$
$$[K] = [K_1] + \lambda([K_2^*] - [K_2^*]^{\mathrm{t}}) + \lambda^2[K_3] \quad \text{for vibration problem,} \tag{28}$$

from which the frequency ω can be computed for given λ.

Both the mass and stiffness matrices presented in the appendix have been derived, by performing explicit integration of individual terms with respect to thickness direction. By using such explicitly derived matrices in program, approximate numerical integration procedures have been avoided.

The stiffness and mass matrices thus calculated for all laminae are assembled to form global matrices by enforcing the compatibility of generalized displacements, rotations and normal strains at the interfaces of laminae.

The determinant of the coefficient matrix in Eq. (27) must be zero for a non-trivial solution. This results in a generalized eigenvalue problem when λ is specified. The wave travelling in the positive x direction must correspond to a complex wave number, λ, having a form $\lambda = \lambda_R + \lambda_I$ where λ_R and $\lambda_I \geqslant 0$ for a bounded solution. In contrast, if $\lambda_R =$ zero and $\lambda_I \neq 0$, the mode is evanescent or non-propagating.

3. Numerical examples

Various displacement-based formulations have been encoded into a general purpose FORTRAN-90 program which can evaluate the frequencies for free vibration as well as for wave propagation problems for different values of wave number, λ. Both real and imaginary wave numbers have been considered in wave propagation. The frequencies and the wave number have been normalized to facilitate comparative study. The normalized frequency, Ω, and the normalized complex wave number, ζ,, have been defined as

$$\Omega = \omega \frac{H_p}{\pi} \sqrt{\left(\frac{\rho}{C_{55}}\right)_{0^\circ}} \quad \text{and} \quad \zeta = \frac{\lambda H_p}{\pi},$$

where H_p is the total thickness and ρ is the mass density. The natural frequencies are normalized with respect to the reference frequency (ω_{ref}) which has been specified as the third lowest frequency near the cut off for a wave number $\eta = 0.001$ for natural vibrations of composite laminated plates. Similar normalization procedure was followed in Ref. [3], using a rational argument that, it is not possible to have a normalization factor for a laminated plate similar to $\omega_{ref} = \pi/H\sqrt{G/\rho}$ used for isotropic plate. The real wave number has been expressed as $\eta = H_p/L_s$.

Four examples involving analysis of isotropic, orthotropic and composite laminated and sandwich plates have been considered. Number of sub-layers have been chosen such that total number of degrees of freedoms (d.o.f.'s) remain same for all the displacement models in all the examples. A convergence study has been performed in each example by considering different number of sub-layers with HOST7 model. The total number of d.o.f.s of HOST7 model, which provided converging solution, has been taken as a reference value based on which number of sub-layers for all other displacement models have been decided. This approach of considering same number of total d.o.f.'s for various displacement models seems more logical as compared to some of the earlier works in which identical number of layers were considered to investigate vibrations in plate by using different displacement models.

A summary of material properties such as thickness, mass density and elasticity coefficients for the illustrative examples considered has been presented in Table 1. Comparison and brief discussion of the results obtained by various displacement models are presented next.

Karunasena et al. [9] has provided an analytical method of analysis to analyze wave propagation in composite laminated plates. The method employs Muller's algorithm for getting exact frequencies of vibration using the frequencies obtained with a numerical technique making use of parabolic displacement model as an initial guess. These results have been used in the following three examples for comparison of results obtained with various displacement models

Table 1
Material properties of various plates considered for the investigation

Data for Example No.	Type of plate	Thickness of plate H (10^{-3} m)	Mass density ρ (kg/m^3)	C_{11} (GPa)	C_{13} (GPa)	C_{33} (GPa)	C_{55} (GPa)
1	Isotropic	25.4	2.7668×10^4	955.9200	429.4700	955.9200	263.2000
2	(0°/90°/90°/0°)						
	0° Lamina	12.7	2.7668×10^4	1468.3000	40.8300	159.9100	58.6200
	90° Lamina	12.7	2.7668×10^4	159.9100	40.8300	1468.3000	58.6200
3	Sandwich						
	Face Sheet	0.9144	2.6831×10^3	928.3400	397.890	928.3400	265.2600
	Core	12.7	32.83810	2.8970	1.2410	2.8970	0.8270
4	Orthotropic	25.4	2.7668×10^4	2069.1000	0.8870	22.7390	0.0796

of the proposed method applied to an isotropic, cross-ply laminated and sandwich composite plates.

Example 1. Wave propagation through an isotropic plate having properties defined in Table 1 has been analyzed by employing all the displacement models presented here. By and large, 304 d.o.f.'s have been considered to model the isotropic plate with various displacement formulations. Results obtained by using different displacement models and the analytical solution [9] have been compared in Fig. 2(a)–(h). Both symmetric and antisymmteric modes obtained using analytical method are shown separately in these graphs. It can be observed that the displacement models after HOST4 have yielded equally accurate results. The close agreement of results obtained by higher order displacement models with the analytical solution demonstrates accuracy and applicability of the displacement-based formulations developed in this paper.

Example 2. A cross-ply laminate with lamina stacking sequence (0°/90°/90°/0°) and material properties as presented in Table 1 was investigated for plane-strain condition. The normalized frequencies given by various displacement models for real and imaginary wave numbers have been superimposed on the symmetric and antisymmetric dispersion curves obtained using analytical method. In total, 484 d.o.f.'s were utilized to model the laminated plate for each of these seven displacement models. It can be observed from Fig. 3(a–h) that all the higher order models after HOST4 have produced results exactly coinciding with the analytical solution. On the other hand, the lower order models like FOST, HOST1 completely fail in analyzing wave propagation in composite laminated plate.

Example 3. A sandwich plate with two stiff layers and a soft core in between having properties as defined in Table 1 was analyzed for wave propagation problem using different displacement models. The sandwich plate has been modelled using 304 d.o.f.'s for all the displacement models. Normalized frequencies obtained using different displacement models have been compared in Fig. 4 with the dispersion curves obtained using analytical method [9] for real as well as imaginary wave numbers. The results obtained with all the higher order models after HOST4 have shown excellent agreement with the analytical solution whereas the results of HOST1, HOST2, HOST3

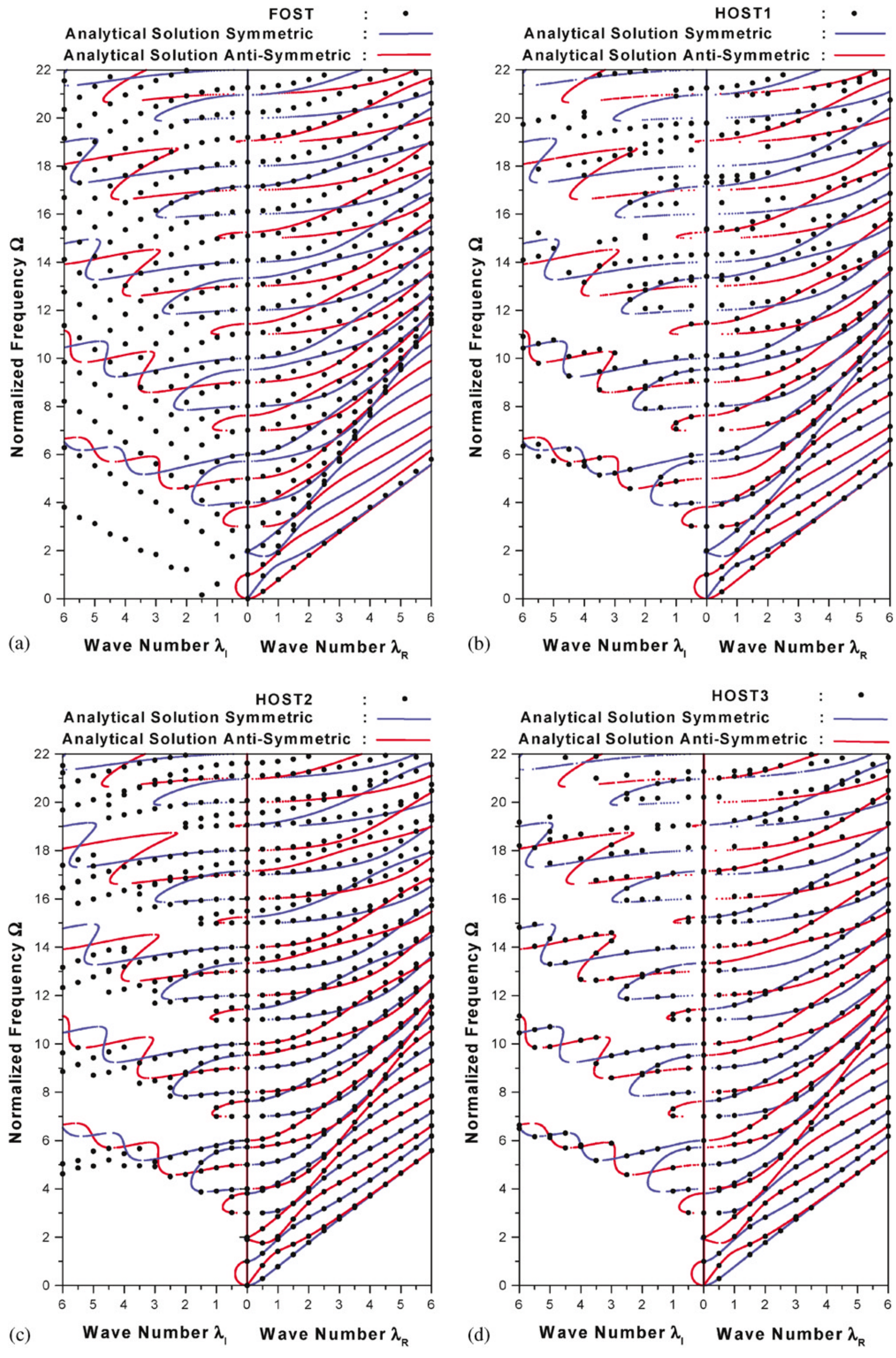

Fig. 2. Comparison of results obtained by using analytical method and various displacement models for an isotropic plate of Example 1.

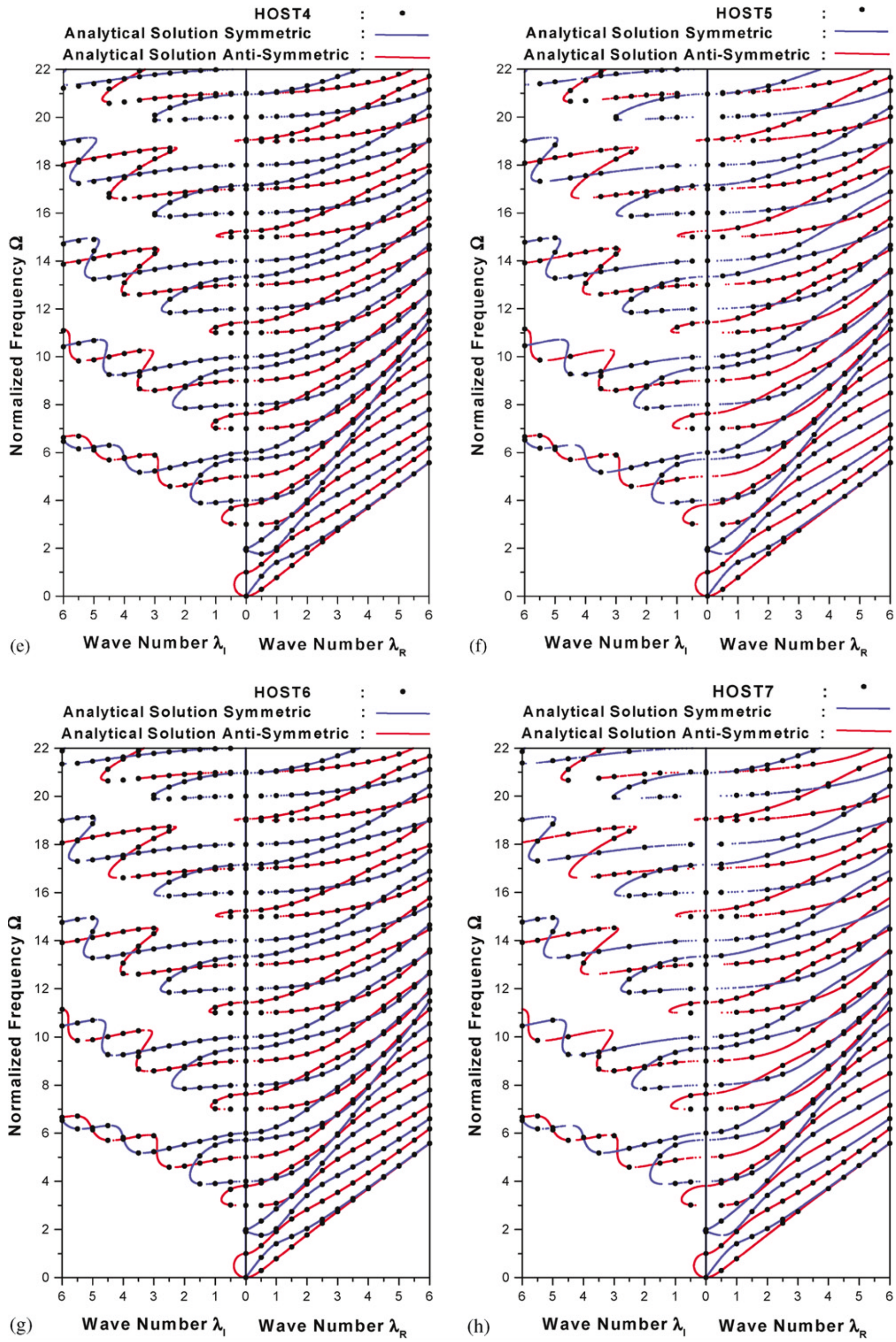

Fig. 2 (*continued*).

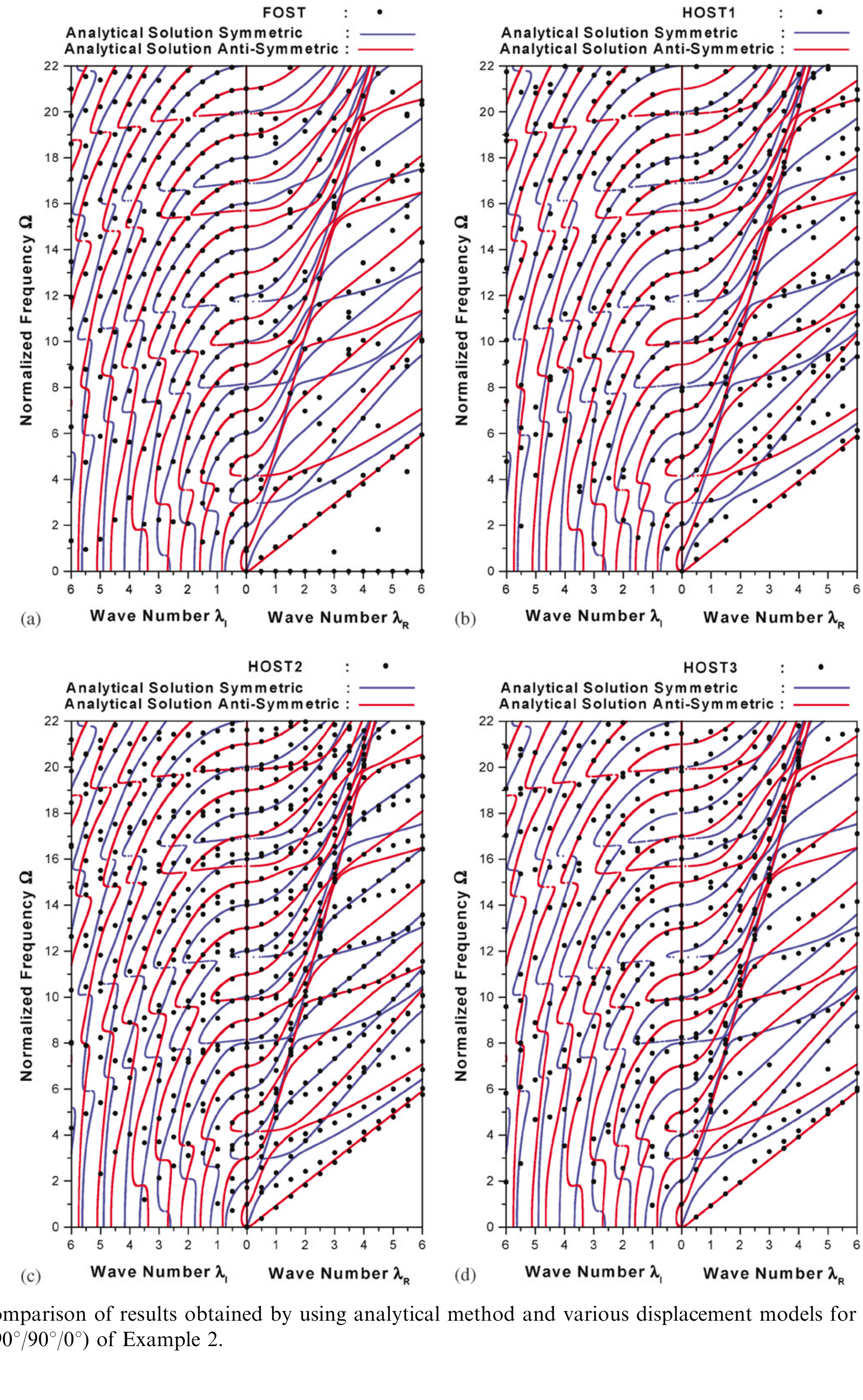

Fig. 3. Comparison of results obtained by using analytical method and various displacement models for a laminated plate (0°/90°/90°/0°) of Example 2.

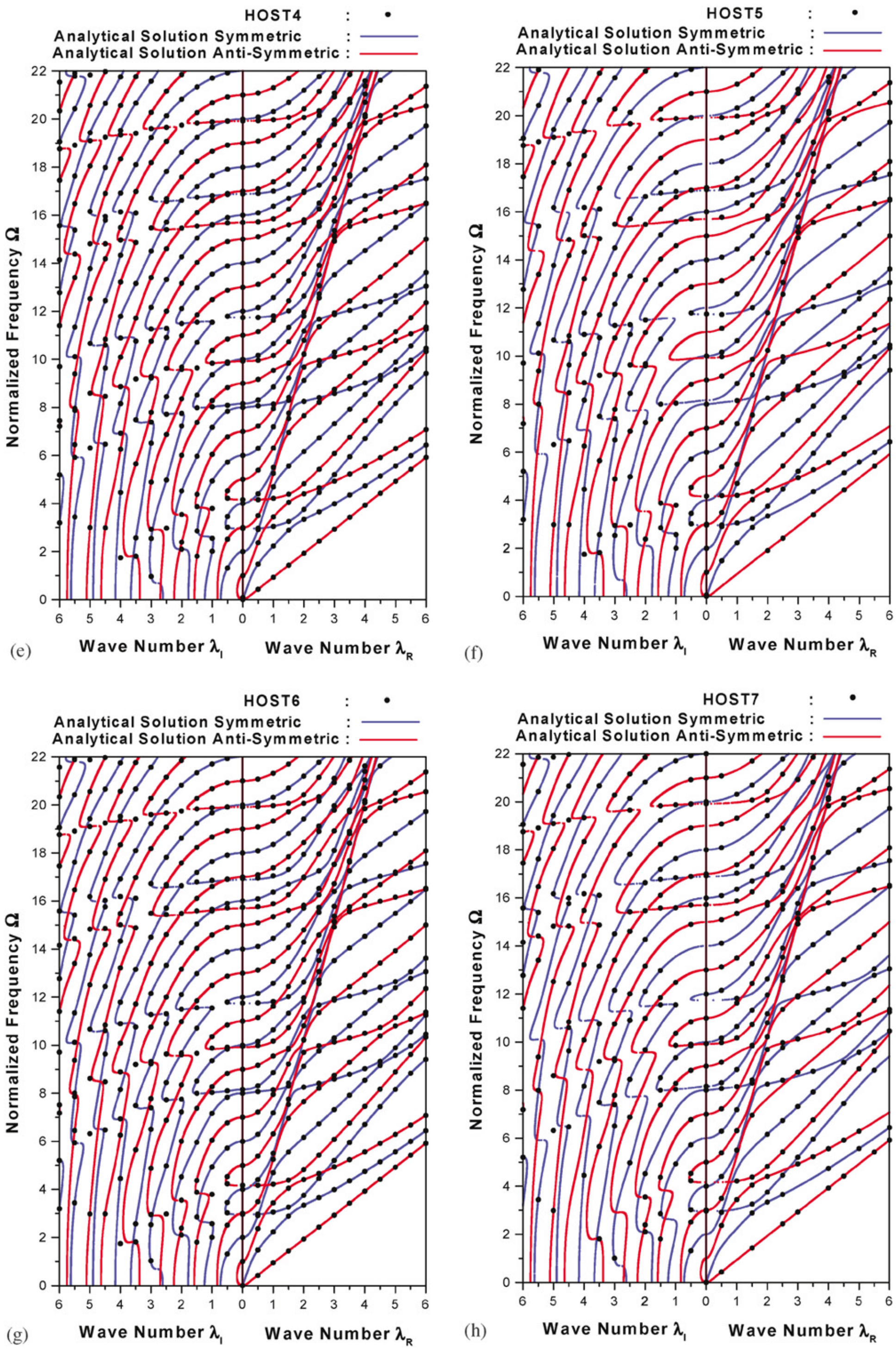

Fig. 3 (*continued*).

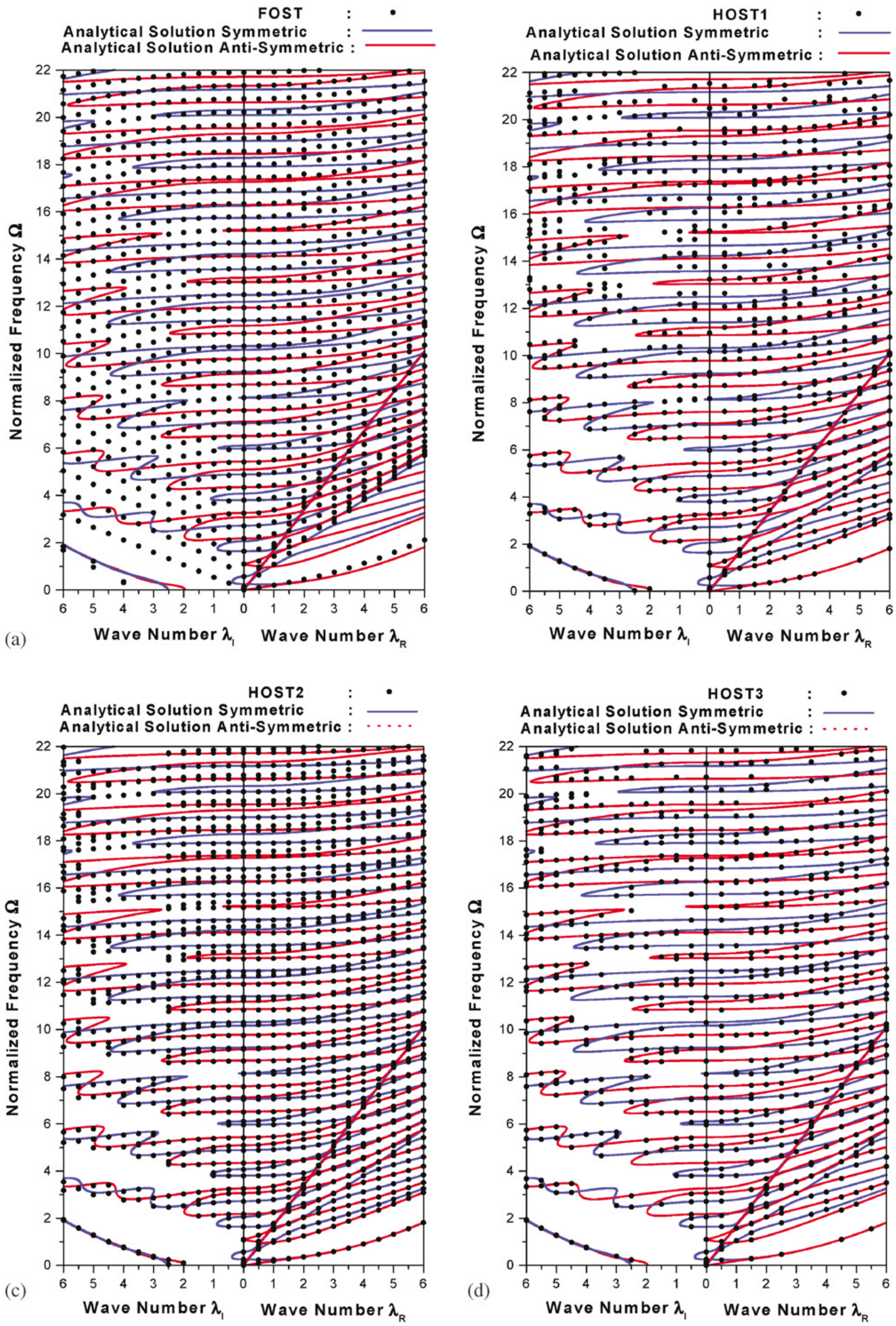

Fig. 4. Comparison of results obtained by using analytical method and various models for sandwich plate of Example 3.

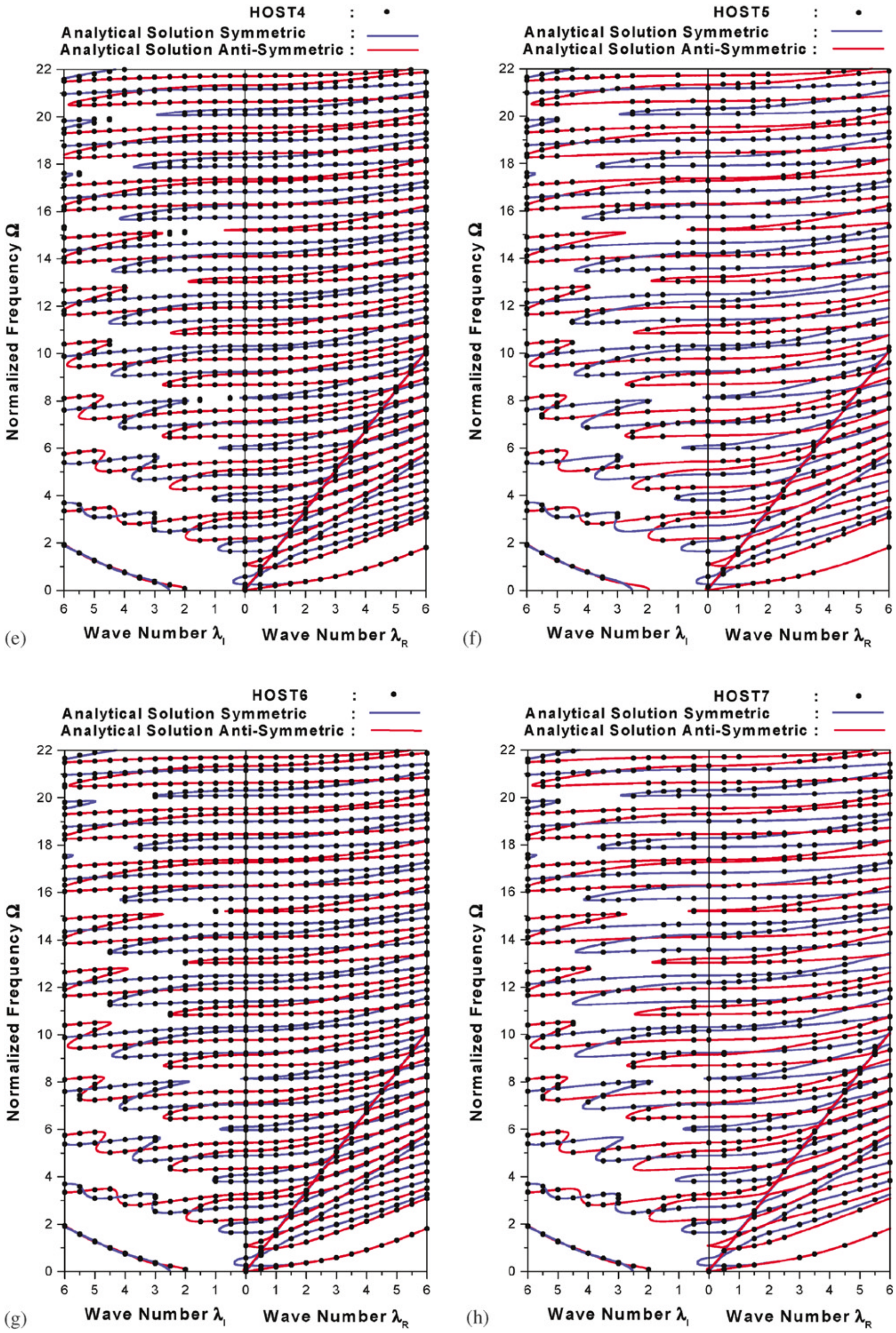

Fig. 4 (*continued*).

show some variations at higher frequencies. It can be observed from Fig. 4(a) that FOST completely fails to predict the complex phenomenon of wave propagation in sandwich plate.

These investigations for wave propagation phenomenon in a variety of plates from isotropic to cross-ply laminated and composite sandwich plates indicate that higher order displacement-based theories are indeed very effective in analyzing this particular problem. A close comparison with the results obtained by analytical method [9] reveals that displacement formulation making use of cubic variation of in-plane and transverse displacements through the thickness of an individual lamina or sub-layer of a plate proves to be the most effective model in terms of accuracy of results and number of degrees of freedom consumed in modelling the plate thickness.

The exact solution provided by Srinivas and Rao [18] has been utilized to solve the problem of vibrations in an orthotropic plate. This analytical solution is used as a bench mark solution for comparing the results of various displacement-based models for the vibration analysis of orthotropic plate considered in next example.

Example 4. An orthotropic plate with material properties as presented in Table 1 was investigated to illustrate the validity of various displacement models developed, for plane-strain vibrations in plate. A collective comparison of results obtained with different displacement models with the analytical solution presented in Table 2 demonstrates the usefulness of higher order displacement formulations for vibration analysis of plane-strain vibrations in an orthotropic plate. It can be observed that FOST yields totally erroneous results. The marginal differences in accuracy achieved with various displacement models can be appreciated from the numerical values of frequencies presented in Table 2. The results of all models beyond HOST3 match very closely with analytical results. Therefore, it appears that the displacement model with cubic variation of in-plane and transverse displacement across the thickness can be adopted for accurate analysis. It can be concluded that displacement-based formulation employing cubic variation through thickness for in-plane as well as transverse displacements appears to be most effective model even for the problem of analysis of plane-strain vibrations in composite plates.

4. Conclusions

Higher order displacement-based formulations assuming linear, parabolic, cubic and quartic displacement fields have been developed and incorporated into a semi-analytical numerical technique for solution of wave propagation and natural vibrations in fiber reinforced polymer composite laminated plates. Results obtained by using different displacement models for analysis of isotropic, orthotropic, sandwich and cross-ply composite laminated plates with equal number of degrees of freedom used in modelling the plate have been compared. Comparison of results obtained by various displacement models with the analytical solution for wave propagation in isotropic, cross-ply laminated and sandwich plates validated the accuracy of all the displacement formulations developed. On the basis of the numerical investigations, it can be concluded that higher order displacement-based theory using cubic variation of in-plane as well as transverse displacements through thickness of lamina provides the most accurate and economical

Table 2

Model	Frequency									
	Mode1	Mode2	Mode3	Mode4	Mode5	Mode6	Mode7	Mode8	Mode9	Mode10
(a) Normalized frequencies of vibrations in an orthotropic plate obtained by employing various displacement models for a wave number of $\lambda = 1.0$ and 2.0										
$\lambda = 1.0$										
FOST	0.98549	159.17990	159.18290	159.19210	159.20740	159.22880	159.25640	159.29010	159.32990	159.37580
HOST1	0.98584	16.71544	33.38918	50.07622	66.77113	83.47423	100.18680	116.91030	133.64570	150.38880
HOST2	0.98551	16.71608	33.39281	50.08740	66.79671	83.52338	100.27100	117.04320	133.84340	150.66940
HOST3	0.98567	16.71541	33.38733	50.06889	66.75283	83.43770	100.12300	116.80830	133.49300	150.17140
HOST4	0.98539	16.71538	33.38731	50.06887	66.75282	83.43769	100.12300	116.80830	133.49300	150.17140
HOST5	0.98539	16.71538	33.38731	50.06887	66.75278	83.43758	100.12270	116.80770	133.49190	150.16930
HOST6	0.98533	16.71537	33.38731	50.06886	66.75278	83.43758	100.12270	116.80770	133.49190	150.16930
HOST7	0.98536	16.71537	33.38731	50.06886	66.75278	83.43758	100.12270	116.80770	133.49190	150.16930
Analytical	0.98536	16.71537	33.38731	50.06886	66.75278	83.43758	100.12270	116.80770	133.49190	150.16930
$\lambda = 2.0$										
FOST	1.97309	318.3597	318.3612	318.3658	318.3735	318.3842	318.3980	318.4148	318.4347	318.4577
HOST1	1.97357	16.80263	33.43293	50.10544	66.79311	83.49198	100.2019	116.9240	133.6595	150.4100
HOST2	1.97321	16.80322	33.43653	50.11659	66.81868	83.54112	100.2861	117.0569	133.8572	150.6910
HOST3	1.97344	16.80259	33.43107	50.09810	66.77483	83.45546	100.1381	116.8220	133.5068	150.1924
HOST4	1.97302	16.80249	33.43102	50.09806	66.7748	83.45544	100.1381	116.8220	133.5068	150.1924
HOST5	1.97302	16.80249	33.43102	50.09806	66.77476	83.45533	100.1378	116.8214	133.5057	150.1903
HOST6	1.97281	16.80244	33.43100	50.09804	66.77475	83.45532	100.1378	116.8214	133.5057	150.1903
HOST7	1.97294	16.80247	33.43101	50.09805	66.77476	83.45532	100.1378	116.8214	133.5056	150.1903
Analytical	1.97294	16.80247	33.43101	50.09805	66.77476	83.45532	100.1378	116.8214	133.5056	150.1903
(b) Normalized frequencies of vibrations in an orthotropic plate obtained by employing various displacement models for a wave number of $\lambda = 3.0$ and 4.0										
$\lambda = 3.0$										
FOST	2.96063	477.5396	477.5406	477.5436	477.5487	477.5559	477.5651	477.5763	477.5896	477.6049
HOST1	2.96106	16.94701	33.50573	50.15405	66.8296	83.52119	100.2263	116.9449	133.6779	150.4265
HOST2	2.96078	16.94757	33.50930	50.16518	66.85515	83.57031	100.3105	117.0778	133.8756	150.7074
HOST3	2.96099	16.94696	33.50387	50.14671	66.81132	83.48469	100.1625	116.8430	133.5252	150.2089
HOST4	2.96060	16.94683	33.50380	50.14667	66.81128	83.48466	100.1625	116.8430	133.5252	150.2089
HOST5	2.96060	16.94683	33.50380	50.14666	66.81125	83.48455	100.1622	116.8424	133.5241	150.2068
HOST6	2.96036	16.94674	33.50376	50.14663	66.81123	83.48454	100.1622	116.8424	133.5241	150.2068
HOST7	2.96052	16.94680	33.50379	50.14665	66.81124	83.48454	100.1622	116.8424	133.5241	150.2068
Analytical	2.96052	16.94680	33.50379	50.14665	66.81124	83.48454	100.1622	116.8424	133.5241	150.2068
$\lambda = 4.0$										
FOST	3.94809	636.7194	636.7202	636.7225	636.7263	636.7316	636.7385	636.7470	636.7569	636.7684
HOST1	3.94844	17.14711	33.60738	50.22202	66.88063	83.56204	100.2604	116.9741	133.7034	150.4493
HOST2	3.94824	17.14766	33.61094	50.23313	66.90616	83.61113	100.3445	117.1070	133.9011	150.7301
HOST3	3.94841	17.14707	33.60553	50.21469	66.86236	83.52555	100.1966	116.8722	133.5508	150.2317
HOST4	3.94808	17.14692	33.60546	50.21464	66.86232	83.52552	100.1966	116.8722	133.5508	150.2316
HOST5	3.94808	17.14692	33.60545	50.21463	66.86228	83.52541	100.1963	116.8716	133.5497	150.2296
HOST6	3.94785	17.14681	33.6054	50.2146	66.86226	83.52538	100.1963	116.8716	133.5497	150.2296
HOST7	3.94800	17.14688	33.60543	50.21462	66.86227	83.5254	100.1963	116.8716	133.5496	150.2296
Analytical	3.94800	17.14688	33.60543	50.21462	66.86227	83.5254	100.1963	116.8716	133.5496	150.2296

Table 2 (*continued*)

Model	Frequency									
	Mode1	Mode2	Mode3	Mode4	Mode5	Mode6	Mode7	Mode8	Mode9	Mode10
(c) Normalized frequencies of vibrations in an orthotropic plate obtained by employing various displacement models for a wave number of $\lambda = 5.0$ and 6.0										
$\lambda = 5.0$										
FOST	4.93548	795.8992	795.8998	795.9017	795.9048	795.9091	795.9146	795.9213	795.9293	795.9385
HOST1	4.93577	17.401	33.73763	50.30927	66.94617	83.61452	100.3041	117.0116	133.7362	150.4784
HOST2	4.93562	17.40154	33.74118	50.32037	66.97168	83.66357	100.3882	117.1444	133.9339	150.7593
HOST3	4.93577	17.40097	33.73579	50.30196	66.92793	83.57805	100.2404	116.9097	133.5837	150.2609
HOST4	4.93549	17.40081	33.73571	50.30191	66.92789	83.57801	100.2403	116.9097	133.5837	150.2609
HOST5	4.93549	17.40081	33.73571	50.3019	66.92785	83.5779	100.2401	116.9091	133.5825	150.2588
HOST6	4.93529	17.4007	33.73565	50.30186	66.92782	83.57788	100.2400	116.9091	133.5825	150.2588
HOST7	4.93542	17.40077	33.73569	50.30189	66.92784	83.57790	100.2400	116.9091	133.5825	150.2588
Analytical	4.93542	17.40077	33.73569	50.30189	66.92784	83.57790	100.2400	116.9091	133.5825	150.2588
$\lambda = 6.0$										
FOST	5.92284	955.0791	955.0796	955.0811	955.0837	955.0873	955.0919	955.0975	955.1041	955.1118
HOST1	5.92306	17.70637	33.89614	50.41571	67.02620	83.67860	100.3575	117.0574	133.7763	150.5141
HOST2	5.92297	17.70692	33.89968	50.42678	67.05167	83.72762	100.4416	117.1902	133.9739	150.7948
HOST3	5.92310	17.70635	33.89432	50.40842	67.00797	83.64217	100.2938	116.9556	133.6238	150.2966
HOST4	5.92286	17.70619	33.89424	50.40836	67.00793	83.64213	100.2938	116.9556	133.6238	150.2966
HOST5	5.92286	17.70619	33.89423	50.40836	67.00790	83.64202	100.2935	116.9550	133.6227	150.2945
HOST6	5.92267	17.70607	33.89417	50.40831	67.00787	83.64200	100.2935	116.9550	133.6226	150.2945
HOST7	5.92280	17.70615	33.89421	50.40834	67.00789	83.64201	100.2935	116.9550	133.6226	150.2945
Analytical	5.92280	17.70615	33.89421	50.40834	67.00789	83.64201	100.2935	116.9550	133.6226	150.2945

displacement model for analyzing wave propagation as well as vibrations in composite laminated plates. Performance of displacement model with parabolic through thickness variation also produced good results in comparison with the analytical solution for the analysis of natural vibrations in an orthotropic plate. However, the solution provided by displacement model with cubic variation of displacements through thickness yielded results accurate up to fourth decimal. The whole range of investigations performed justifies the use of higher order theories for analyzing waves and vibrations in laminated composite plates.

Appendix. A

The mass $[M]$ and stiffness $[K]$ matrices for a lamina of laminated plate, explicitly derived based on various displacement theories are presented here. The superscript $1, 2, \ldots, 8$ denote the displacement model number. For example, $[K^3]$ represents the stiffness matrix for a single lamina of a laminated plate derived on the basis of displacement model 3 (HOST2). Furthermore, $\delta = \Delta = i$ for wave propagation problem and $\delta = -1$, $\Delta = 1$ for vibration problem.

A.1. Displacement model No. 1 [FOST]

$$[m^1] = \frac{\rho h}{3}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad [K^1] = \begin{bmatrix} (A+2B) & (-A+B) & -\delta\lambda C_{33} \\ (-A+B) & (A+2B) & \delta\lambda C_{33} \\ \Delta\lambda C_{33} & -\Delta\lambda C_{33} & 2\lambda^2\delta C_{33}h \end{bmatrix}, \tag{A.1}$$

$$A = \frac{C_{33}}{2h}, \quad B = \frac{\lambda^2 2C_{11}h}{3}.$$

A.2. Displacement model No. 2 [HOST1]

$$[m^2] = \begin{bmatrix} [m_1^2] & 0 \\ 0 & [m_1^2] \end{bmatrix}, \quad \text{where} \quad [m_1^2] = \frac{\rho h}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$[K^2] = \begin{bmatrix} [K_1^2] & [K_2^2] \\ [K_3^2] & [K_4^2] \end{bmatrix}, \tag{A.2}$$

where

$$[K_1^2] = \begin{bmatrix} (A+2B) & (-A+B) \\ (-A+B) & (A+2B) \end{bmatrix}, \quad [K_2^2] = \delta\begin{bmatrix} (-E+F) & (-E-F) \\ (E+F) & (E-F) \end{bmatrix},$$

$$[K_3^2] = \Delta\begin{bmatrix} (E-F) & (-E-F) \\ (E+F) & (-E+F) \end{bmatrix} \quad \text{and} \quad [K_4^2] = \begin{bmatrix} (C+2D) & (-C+D) \\ (-C+D) & (C+2D) \end{bmatrix},$$

where

$$C = \frac{C_{22}}{2h}, \quad D = \frac{\lambda^2 2C_{33}h}{3}, \quad E = \frac{\lambda C_{33}}{2}, \quad F = \frac{\lambda C_{12}}{2}.$$

A.3. Displacement model No. 3 [HOST2]

$$[m^3] = \begin{bmatrix} [m_1^3] & 0 \\ 0 & [m_1^2] \end{bmatrix} \quad \text{where} \quad [m_1^3] = \frac{\rho h}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$[K^3] = \begin{bmatrix} [K_2^3] & [K_2^3] \\ [K_3^3] & [K_4^2] \end{bmatrix}, \tag{A.3}$$

where

$$[K_1^3] = \begin{bmatrix} 7G+4H & -8G+2H & G-H \\ -8G+2H & 16G+16H & -8G+2H \\ G-H & -8G+2H & 7G+4\delta H \end{bmatrix},$$

$$[K_2^3] = \delta \begin{bmatrix} -5I+J & -I-J \\ 4I+4J & -4I-4J \\ I+J & 5I-J \end{bmatrix}$$

and

$$[K_3^3] = \Delta \begin{bmatrix} 5I-J & -4I-4J & -I-J \\ I+J & 4I+4J & -5I+J \end{bmatrix},$$

$$G = \frac{C_{33}}{6h}, \quad H = \frac{\lambda^2 C_{11} h}{15} \quad I = \frac{\lambda C_{33}}{6} \quad J = \frac{\lambda C_{12}}{6}.$$

A.4. Displacement model No. 4 [HOST3]

$$[m^4] = \begin{bmatrix} [m_1^3] & 0 \\ 0 & [m_1^3] \end{bmatrix} \quad \text{and} \quad [K^4] = \begin{bmatrix} [K_1^3] & [K_2^4] \\ [K_3^4] & [K_4^4] \end{bmatrix}, \tag{A.4}$$

where

$$[K_2^4] = \delta \begin{bmatrix} -3I+3J & -4I-4J & I+3J \\ 4I+4J & 0 & -4I-4J \\ -I-3J & 4I+4J & 3I-3J \end{bmatrix},$$

$$[K_3^4] = \Delta \begin{bmatrix} 3I-3J & -4I-4J & I+3J \\ 4I+4J & 0 & -4I-4J \\ -I-3J & 4I+4J & -3I+3J \end{bmatrix}$$

and

$$[K_4^4] = \begin{bmatrix} 7K+4L & -8K+2L & K-L \\ -8K+2L & 16K+16L & -8K+2L \\ K-L & -8K+2L & 7K+4L \end{bmatrix},$$

$$K = \frac{C_{22}}{6h}, \quad L = \frac{\lambda^2 C_{33} h}{15}.$$

A.5. Displacement model No. 5 [HOST4]

$$[m^5] = \begin{bmatrix} [m_1^5] & 0 \\ 0 & [m_1^3] \end{bmatrix} \quad \text{and} \quad [m_1^5] = \frac{\rho h}{105}\begin{bmatrix} 78 & 27 & 22h & -13h \\ 27 & 78 & 13h & -22h \\ 22h & 13h & 8h^2 & -6h^2 \\ -13h & -22h & -6h^2 & 8h^2 \end{bmatrix},$$

$$[K^5] = \begin{bmatrix} [K_1^5] & [K_2^5] \\ [K_3^5] & [K_4^4] \end{bmatrix}, \tag{A.5}$$

where

$$[K_1^5] = \begin{bmatrix} 18M + 78N & -18M + 27N & 3Mh + 22Nh & 3Mh - 13Nh \\ 18M + 27N & 18M + 78N & -3Mh + 13Nh & -3Mh - 22Nh \\ -3Mh + 22Nh & 3Mh + 13Nh & 8Mh^2 + 8Nh^2 & -2Mh^2 - 6Nh^2 \\ -3Mh - 13Nh & 3Mh - 22Nh & 2Mh^2 - 6Nh^2 & 8Mh^2 + 8Nh^2 \end{bmatrix},$$

$$[K_2^5] = \delta\begin{bmatrix} 27P - 3Q & -24P - 24Q & -3P - 3Q \\ 3P + 3Q & 24P + 24Q & -27P + 3Q \\ 7Ph + 7Qh & -4Ph - 4Qh & -3Ph - 3Qh \\ -3Ph - 3Qh & 4Ph - 4Qh & 7Ph + 7Qh \end{bmatrix}$$

and

$$[K_3^5] = \Delta\begin{bmatrix} -27P + 3Q & -3P - 3Q & -7Ph - 7Qh & 3Ph + 3Qh \\ 24P + 24Q & -24P - 24Q & 4Ph + 4Qh & -4Ph + 4Qh \\ 3P + 3Q & 27P - 3Q & 3Ph + 3Qh & -7Ph - 7Qh \end{bmatrix},$$

$$M = \frac{C_{33}}{30h}, \quad N = \frac{\lambda^2 C_{11} h}{105}, \quad P = \frac{\lambda C_{33}}{30} \quad \text{and} \quad Q = \frac{\lambda C_{12}}{30}.$$

A.6. Displacement model No. 6 [HOST5]

$$[m^6] = \begin{bmatrix} [m_1^5] & 0 \\ 0 & [m_1^5] \end{bmatrix} \quad \text{and} \quad [K^6] = \begin{bmatrix} [K_1^5] & [K_2^6] \\ [K_3^6] & [K_4^6] \end{bmatrix}, \tag{A.6}$$

where

$$[K_2^6] = \delta \begin{bmatrix} -15P+15Q & -15P-15Q & -6Ph-6Qh & 6Ph+6Qh \\ 15P+15Q & 15P-15Q & 6Ph+6Qh & -6Ph-6Qh \\ 6Ph+6Qh & -6Ph-6Qh & 0 & 2Ph^2+2Qh^2 \\ -6Ph-6Qh & 6Ph+6Qh & -2Ph^2-2Qh^2 & 0 \end{bmatrix},$$

$$[K_3^6] = \Delta \begin{bmatrix} 15P-15Q & -15P-15Q & -6Ph-6Qh & 6Ph+6Qh \\ 15P+15Q & -15P+15Q & 6Ph+6Qh & -6Ph-6Qh \\ 6Ph+6Qh & -6Ph-6Qh & 0 & 2Ph^2+2Qh^2 \\ -6Ph-6Qh & 6Ph+6Qh & -2Ph^2-2Qh^2 & 0 \end{bmatrix}$$

and

$$[K_4^6] = \begin{bmatrix} 18R+78S & -18R+27S & 3Rh+22Sh & 3Rh-13Sh \\ 18R+27S & 18R+78S & -3Rh+13Sh & -3Rh-22Sh \\ -3Rh+22Sh & 3Rh+13Sh & 8Rh^2+8Sh^2 & -2Rh^2-6Sh^2 \\ -3Rh-13Sh & 3Rh-22Sh & 2Rh^2-6Sh^2 & 8Rh^2+8Sh^2 \end{bmatrix},$$

$$R = \frac{C_{22}}{30h}, \quad S = \frac{\lambda^2 C_{33} h}{105}.$$

A.7. Displacement model No. 7 [HOST6]

$$[m^7] = \begin{bmatrix} [m_1^7] & 0 \\ 0 & [m_1^5] \end{bmatrix}, \quad \text{where} \quad [m_1^7] = \frac{\rho h}{315} \begin{bmatrix} 130 & 40 & -23 & 20h & 7h \\ 40 & 256 & 40 & 8h & -8h \\ -23 & 40 & 130 & -7h & -20h \\ 20h & 8h & -7h & 4h^2 & 2h^2 \\ 7h & -8h & -20h & 2h^2 & 4h^2 \end{bmatrix},$$

$$[K^7] = \begin{bmatrix} [K_1^7] & [K_2^7] \\ [K_3^7] & [K_4^6] \end{bmatrix}, \tag{A.7}$$

where

$$[K_1^7] = \begin{bmatrix} 254T+130U & -256T+40U & 2T-23U & 29Th+20Uh & 13Th+7Uh \\ -256T+40U & 512T+256U & -256T+40U & -16Th+8Uh & 16Th-8Uh \\ 2T-23U & -256T+40U & 254T+130U & -13Th-7Uh & -29Th-20Uh \\ 29Th+20Uh & -16Th+8Uh & -13Th-7Uh & 32Th^2+4Uh^2 & 10Th^2+2Uh^2 \\ 13Th+7Uh & 16Th-8Uh & -29Th-20Uh & 10Th^2+2Uh^2 & 32Th^2+4Uh^2 \end{bmatrix},$$

$$[K_2^7] = \delta \begin{bmatrix} -177V+33W & -33V-33W & -58Vh-58Wh & 26Vh+26Wh \\ -144V+144W & -144V-144W & 32Vh+32Wh & 32Vh+32Wh \\ 33V+33W & 177V-33W & 26Vh+26Wh & -58Vh-58Wh \\ 6Vh+6Wh & -6Vh-6Wh & -8Vh^2-8Wh^2 & 6Vh^2+6Wh^2 \\ -6Vh-6Wh & 6Vh+6Wh & -6Vh^2-6Wh^2 & 8Vh^2+8Wh^2 \end{bmatrix},$$

$$[K_3^7] = \Delta \begin{bmatrix} 177V-33W & 144V-144W & -33V-33W & -6Vh-6Wh & 6Vh+6Wh \\ 33V+33W & 144V+144W & -177V+33W & 6Vh+6Wh & -6Vh-6Wh \\ 58Vh+58Wh & -32Vh-32Wh & -26Vh-26Wh & 8Vh^2+8Wh^2 & 6Vh^2+6Wh^2 \\ -26Vh-26Wh & -32Vh-32Wh & 58Vh+58Wh & -6Vh^2-6Wh^2 & -8Vh^2-8Wh^2 \end{bmatrix},$$

$$T = \frac{C_{33}}{210h}, \quad U = \frac{\lambda^2 C_{33} h}{315}, \quad V = \frac{\lambda C_{33}}{210}, \quad \text{and} \quad W = \frac{\lambda C_{12}}{210}.$$

A.8. Displacement model No. 8 [HOST7]

$$[m^8] = \begin{bmatrix} [m_1^7] & 0 \\ 0 & [m_1^7] \end{bmatrix},$$

$$[K^8] = \begin{bmatrix} [K_1^7] & [K_2^8] \\ [K_3^8] & [K_4^8] \end{bmatrix}, \tag{A.8}$$

where

$$[K_2^8] = \delta \begin{bmatrix} -105V+105W & -144V-144W & 39V+39W & -22Vh-22Wh & -10Vh-10Wh \\ 144V+144W & 0 & -144V-144W & 32Vh+32Wh & 32Vh+32Wh \\ -39V-39W & 144V+144W & 105V-105W & -10Vh-10Wh & -22Vh-22Wh \\ 22Vh+22Wh & -32Vh-32Wh & 10Vh+10Wh & 0 & -2Vh^2-2Wh^2 \\ 10Vh+10Wh & -32Vh-32Wh & 22Vh+22Wh & 2Vh^2+2Wh^2 & 0 \end{bmatrix},$$

$$[K_3^8] = \Delta \begin{bmatrix} 105V-105W \leftrightarrows & -144V-144W & 39V+39W & -22Vh-22Wh & -10Vh-10Wh \\ 144V+144W & 0 & -144V-144W & 32Vh+32Wh & 32Vh+32Wh \\ V \quad -39V-39W & 144V+144W & -105V+105W & -10Vh-10Wh & -22Vh-22Wh \\ 22Vh+22Wh & -32Vh-32Wh & 10Vh+10Wh & 0 & -2Vh^2-2Wh^2 \\ 10Vh+10Wh & -32Vh-32Wh & 22Vh+22Wh & 2Vh^2+2Wh^2 & 0 \end{bmatrix}$$

and

$$[K_4^8] = \begin{bmatrix} 254X + 130Y & -256X + 40Y & 2X - 23Y & 29Xh + 20Yh & 13Xh + 7Yh \\ -256X + 40Y & 512X + 256Y & -256X + 40Y & -16Xh + 8Yh & 16Xh - 8Yh \\ 2X - 23Y & -256X + 40Y & 254X + 130Y & -13Xh - 7Yh & -29Xh - 20Yh \\ 29Xh + 20Yh & -16Xh + 8Yh & -13Xh - 7Yh & 32Xh^2 + 4Yh^2 & 10Xh^2 + 2Yh^2 \\ 13Xh + 7Yh & 16Xh - 8Yh & -29Xh - 20Yh & 10Xh^2 + 2Yh^2 & 32Xh^2 + 4Yh^2 \end{bmatrix},$$

$$X = \frac{C_{22}}{210h}, \quad Y = \frac{\lambda^2 C_{33}}{315}.$$

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