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Free Vibration of Skew Fiber-reinforced Composite and Sandwich Laminates using a Shear Deformable Finite Element Model

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ABSTRACT: A simple C^o isoparametric finite element model, based on a higher-order shear deformation theory is presented for the free vibration analysis of isotropic, orthotropic, and layered anisotropic composite and sandwich skew laminates. This theory incorporates a realistic nonlinear variation of displacements through the laminate thickness, and eliminates the use of shear correction coefficients. The boundary conditions on the skew edges are imposed through the suitable transformation of element matrices. The accuracy of the present model is demonstrated by comparing with alternative solutions available in the literature. Extensive numerical results are presented for natural frequencies of cross-ply and angle-ply skew laminates with various lamination parameters, skew angles, boundary conditions, and width-to-thickness ratios. New results on skew sandwiches are obtained for different lamination parameters and skew angles.

KEY WORDS: higher-order shear deformation theory, skew laminates, free vibration.

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INTRODUCTION

FIBER-REINFORCED LAMINATED COMPOSITE and sandwich materials are being used more extensively as structural components in aerospace, automobile, civil, marine, and other related weight-sensitive engineering applications requiring high strength-to-weight and stiffness-to-weight ratios. It is well-known that skew plates made of these materials are important structural components of ship hulls and swept wings of aeroplanes. The free vibration characteristics of such plates are of interest to the designer in consideration of dynamic response.

The simplifying assumptions, made in classical and first-order theories, are reflected by the high percentage error in the results of thick composite and sandwich plates with highly stiff facings. In contrast to the first-order shear deformation theories (FOSTs), higher-order shear deformation theories (HOSTs) do not require a shear correction coefficient, owing to the more realistic representation of the cross-sectional deformation. Thus, the use of HOST is very important for the vibration analysis of laminated composite skew plates, especially for thick sandwich laminates.

A considerable amount of literature exists on the free vibration of isotropic and orthotropic skew plates. Barton [1], Kaul and Cadambe [2], Hasegawa [3], Durvasula [4], Chopra and Durvasula [5], Nair and Durvasula [6], and Mizusawa et al. [7] studied the free vibrational characteristics of isotropic skew plates using the Rayleigh–Ritz method (RRM), while Conway and Farnham [8] employed a point-matching method to study the free vibration of triangular, rhombic, and parallelogram plates. Laura and Grosson [9] and Durvasula [10] studied the free vibration of simply supported and clamped isotropic skew plates, respectively using Galerkin's method. Bardell [11] adopted the hierarchical finite element method to determine the frequencies and mode shapes of isotropic skew plates. Liew and Wang [12] presented a comprehensive literature survey on the vibration of thin skew plates.

Nair and Durvasula [13], Srinivasan and Ramchandran [14], and Sakata [15] studied the free vibrational characteristics of orthotropic skew plates. Babu and Reddy [16] and Srinivasan and Munaswamy [17] employed the finite strip method to study the natural frequencies of orthotropic skew plates. McGee et al. [18] used a higher-order shear deformable thick plate theory to analyze the natural vibrations of thin and thick skewed plates with clamped and completely free edges. McGee and Leissa [19,20] presented three-dimensional elasticity-based Ritz analyses to predict the frequencies for skewed cantilevered plates.

Compared with the literature on isotropic and orthotropic skew plates, very little is reported on free vibration analysis of skew composite and sandwich laminates. Reddy and Palaninathan [21] employed high-precision

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triangular plate bending elements to the free vibration analysis of laminated skew plates. The results are presented in a graphical form. Raju and Hinton [22] reported natural frequencies and mode shapes of thin and thick skew plates using finite element method based on the first-order shear deformation plate theory. Wang [23] presented the B-spline RRM based on first-order shear deformation plate theory to study the vibration problem of skew composite laminates. Han and Dickinson [24] employed the Ritz approach for the free vibration of symmetrically laminated composite, skew plates. Recently, Anlas and Goker [25] used the Ritz method to study the vibration of skew laminated composite plates with simply supported and clamped edges, while Wang et al. [26] adopted the Ritz method for the free vibration analysis of skew sandwich plates.

It is observed that free vibration analysis of skew plates using HOST has rarely been attempted so far. Kant et al. [27] are the first to present a finite element formulation of a higher-order flexure theory. This theory considers three-dimensional Hooke's law and incorporates the effect of transverse normal strain in addition to transverse shear deformations. Reddy [28] later proposed a HOST utilizing a displacement field with cubic variations with respect to the thickness direction. Kant and Mallikarjuna [29] streamlined the HOST by allowing the displacement in the thickness direction to be quadratic with respect to the thickness coordinate, and developed a simple C^o finite element formulation and presented solutions for the free vibration analysis of general laminated composite and sandwich plate problems.

In the present work, a C^o continuous shear deformable finite element formulation based on a higher-order displacement model is presented, which does not require the use of a shear correction coefficient. The formulation is used for the free vibration analysis of skew composite and sandwich laminates. Accuracy of the present formulation is verified against the literature values for isotropic and laminated composite skew plates. New results are presented for the skew sandwich laminates using the standard material properties available in the literature.

THEORY AND FORMULATION

The higher-order shear deformation theory considered for investigation in the present work is based on the assumption of the displacement field in the form:

$$
\begin{aligned}
u(x, y, z, t) &= u_o(x, y, t) + z\theta_y(x, y, t) + z^2 u_o^*(x, y, t) + z^3\theta_y^*(x, y, t),\\
v(x, y, z, t) &= v_o(x, y, t) - z\theta_x(x, y, t) + z^2 v_o^*(x, y, t) - z^3\theta_x^*(x, y, t),\\
w(x, y, z, t) &= w_o(x, y, t)
\end{aligned}
\tag{1}
$$

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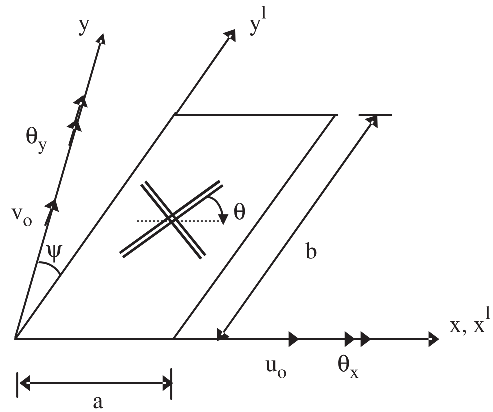

Figure 1. *The geometry of a skew laminate.*

where t is the time, u, v, and w are the displacements of a general point (x, y, z) in the plate space in the x-, y-, and z-directions, respectively. The parameters u_o, v_o are the in-plane displacements and w_o is the transverse displacement of a point (x, y) on the laminate middle plane (Figure 1). The functions, θ_x and θ_y are the rotations of the normal to the laminate middle plane about x- and y-axes, respectively. The parameters u_o^*, v_o^*, θ_x^*, θ_y^* are the higher-order terms in the Taylor's series expansion and are also defined at the midsurface. The continuum displacement vector at the midplane can thus be defined as:

$$\bar{\boldsymbol{d}} = \left\{u_o, v_o, w_o, \theta_x, \theta_y, u_o^*, v_o^*, \theta_x^*, \theta_y^*\right\}^T \tag{2}$$

Substituting Equation (1) into the general linear strain–displacement relations, the following relations are obtained:

$$\begin{aligned}
\varepsilon_x &= \varepsilon_{xo} + z\chi_x + z^2\varepsilon_{xo}^* + z^3\chi_x^* \\
\varepsilon_y &= \varepsilon_{yo} + z\chi_y + z^2\varepsilon_{yo}^* + z^3\chi_y^* \\
\gamma_{xy} &= \varepsilon_{xyo} + z\chi_{xy} + z^2\varepsilon_{xyo}^* + z^3\chi_{xy}^* \\
\gamma_{xz} &= \varphi_x + z\chi_{xz} + z^2\varphi_x^* \\
\gamma_{yz} &= \varphi_y + z\chi_{yz} + z^2\varphi_y^*
\end{aligned} \tag{3a}$$

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where

$$
\begin{aligned}
&\varepsilon_{xo} = \frac{\partial u_o}{\partial x}, \quad \varepsilon_{yo} = \frac{\partial v_o}{\partial y}, \quad \varepsilon_{xyo} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}, \\
&\varepsilon^*_{xo} = \frac{\partial u^*_o}{\partial x}, \quad \varepsilon^*_{yo} = \frac{\partial v^*_o}{\partial y}, \quad \varepsilon^*_{xyo} = \frac{\partial u^*_o}{\partial y} + \frac{\partial v^*_o}{\partial x}, \\
&\chi_x = \frac{\partial \theta_y}{\partial x}, \quad \chi_y = -\frac{\partial \theta_x}{\partial y}, \quad \chi_{xy} = \frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}, \\
&\chi^*_x = \frac{\partial \theta^*_y}{\partial x}, \quad \chi^*_y = -\frac{\partial \theta^*_x}{\partial y}, \quad \chi^*_{xy} = \frac{\partial \theta^*_y}{\partial y} - \frac{\partial \theta^*_x}{\partial x}, \\
&\varphi_x = \theta_y + \frac{\partial w_o}{\partial x}, \quad \varphi_y = -\theta_x + \frac{\partial w_o}{\partial y}, \quad \chi_{xz} = 2u^*_o, \\
&\varphi^*_x = 3\theta_y, \quad \varphi^*_y = -3\theta^*_x, \quad \chi_{yz} = 2v^*_o
\end{aligned}
\tag{3b}
$$

The stress–strain relations for the Lth lamina in the element coordinates (x, y, z) are written as:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^L \tag{4}
$$

or in short form

$$
\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\varepsilon} \tag{5}
$$

in which $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^{\mathrm{T}}$ and $\boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}\}^{\mathrm{T}}$ are the stress and strain vectors, respectively. The $\mathbf{Q}_{ij}$s are the plane stress reduced stiffness coefficients. The transformation rule of stresses/strains between the lamina and laminate coordinate systems follows the usual stress tensor transformation rule.

The governing differential equations of motion can be derived using Hamilton's principle;

$$
\delta \int_{t_1}^{t_2} (\Pi - E)\,\mathrm{d}t = 0 \tag{6}
$$

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where t is the time, E is the total kinetic energy of the system, and Π is the potential energy of the system, including both strain energy and potential of conservative external forces. For the ideal case in which the system has no damping and no external forcing function, the mathematical statement of Hamilton's principle can be written as:

$$\delta \int_{t_1}^{t_2} \left[\frac{1}{2} \int_V \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}} \bar{\boldsymbol{\sigma}} \,\mathrm{d}v - \frac{1}{2} \int_V \dot{\boldsymbol{u}}^{\mathrm{T}} \rho \dot{\boldsymbol{u}} \,\mathrm{d}v \right] \mathrm{d}t = 0 \tag{7}$$

where, ρ is the mass density of the material, $\dot{\boldsymbol{u}}$ defines the particle velocity vector, and

$$\bar{\boldsymbol{\sigma}} = \left\{ N^{\mathrm{T}}, M^{\mathrm{T}}, Q^{\mathrm{T}} \right\}^{\mathrm{T}}. \tag{8}$$

The stress resultants in Equation (8) for the laminate with NL number of layers are defined as follows:

$$\begin{bmatrix} N_x & N_x^* & M_x & M_x^* \\ N_y & N_y^* & M_y & M_y^* \\ N_{xy} & N_{xy}^* & M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} (1, z^2, z, z^3) \,\mathrm{d}z \tag{9}$$

$$\begin{bmatrix} Q_x & Q_x^* & S_x \\ Q_y & Q_y^* & S_y \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} (1, z^2, z) \,\mathrm{d}z \tag{10}$$

After integration, these stress resultants can be written in matrix form as:

$$\bar{\boldsymbol{\sigma}} = \mathbf{D}\, \bar{\boldsymbol{\varepsilon}} \tag{11}$$

where,

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{\mathrm{m}} & \mathbf{D}_{\mathrm{c}} & 0 \\ \mathbf{D}_{\mathrm{c}}^{\mathrm{T}} & \mathbf{D}_{\mathrm{b}} & 0 \\ 0 & 0 & \mathbf{D}_{\mathrm{m}} \end{bmatrix} \tag{12}$$

in which $\mathbf{D}_{\mathrm{m}}$, $\mathbf{D}_{\mathrm{b}}$, $\mathbf{D}_{\mathrm{c}}$, and $\mathbf{D}_{\mathrm{s}}$ are the membrane, bending, membrane–bending coupling, and shear rigidity matrices, respectively, and are defined in Appendix A.

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FINITE ELEMENT FORMULATION

Let the region of the plate be divided into a finite number of quadrilateral elements. The continuum displacement vector within an element is discretized such that

$$\boldsymbol{d} = \sum_{i=1}^{NN} \mathbf{N}_i \boldsymbol{d}_i \tag{13}$$

where $\boldsymbol{d}_i$ is the displacement vector corresponding to node i, $\mathbf{N}_i$ is the interpolating or shape function associated with node i, and NN is the total number of nodes per element.

The generalized midsurface strains at any point given by Equation (3) can be expressed in terms of nodal displacements in matrix form as:

$$\bar{\boldsymbol{\varepsilon}} = \sum_{i=1}^{NN} \mathbf{B}_i \boldsymbol{d}_i \tag{14}$$

where $\mathbf{B}_i$ is a differential operator matrix of shape functions. Substituting for $\bar{\boldsymbol{\varepsilon}}$, $\bar{\boldsymbol{\sigma}}$, and velocity vector $\dot{\boldsymbol{u}}$ in Equation (7), we get

$$\int_{t_1}^{t_2} \sum_{e=1}^{NE} \delta \boldsymbol{d}_e^t [\mathbf{K}^e \boldsymbol{d}_e + \mathbf{M}^e \ddot{\boldsymbol{d}}_e] \, \mathrm{d}t = 0 \tag{15}$$

in which $\mathbf{K}^e$ is the stiffness matrix for an element 'e' which includes membrane, flexure, and the transverse shear effects, and $\mathbf{M}^e$ is the element mass matrix, which are given by

$$\mathbf{K}^e = \int_A \mathbf{B}^t \mathbf{D} \mathbf{B} \, \mathrm{d}A \quad \text{and} \quad \mathbf{M}^e = \int_A \mathbf{N}^t \mathbf{m} \mathbf{N} \, \mathrm{d}A \tag{16}$$

The element stiffness matrix and mass matrix can be obtained by using the standard relation

$$\mathbf{K}_{ij}^e = \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j |\mathbf{J}| \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{17}$$

and

$$\mathbf{M}_{ij}^e = \int_{-1}^{1} \int_{-1}^{1} \mathbf{N}_i^{\mathrm{T}} \mathbf{m} \mathbf{N}_j |\mathbf{J}| \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{18}$$

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where $|\mathbf{J}|$ is the determinant of the standard Jacobian matrix, $\mathbf{D}$ is the rigidity matrix, and $\mathbf{m}$ is the inertia matrix, which is defined in Appendix B.

SKEW BOUNDARY TRANSFORMATION

For skew plates supported on two adjacent edges, the edges of the boundary elements may not be parallel to the global axes (x, y, z). To specify boundary conditions at skew edges, it becomes necessary to use edge displacements u_o^l, v_o^l, w_o^l, etc. in local coordinates (x^l, y^l, z^l) (Figure 1). It is thus required to transform the element matrices corresponding to global axes to local edge axes with respect to which the boundary conditions can be conveniently specified. The relation between the global and local degrees of freedom of a node can be obtained through the simple transformation rules [30,31] and the same can be expressed as:

$$\boldsymbol{d}_i = \mathbf{L}_g \boldsymbol{d}_i^l \tag{19}$$

in which $\boldsymbol{d}_i$ and $\boldsymbol{d}_i^l$ are the generalized displacement vectors in the global and local edge coordinate system, respectively of node i and they are defined as:

$$\begin{aligned} \boldsymbol{d}_i &= \left\{ u_o, v_o, w_o, \theta_x, \theta_y, u_o^*, v_o^*, \theta_x^*, \theta_y^* \right\}^{\mathrm{T}} \\ \boldsymbol{d}_i^l &= \left\{ u_o^l, v_o^l, w_o^l, \theta_x^l, \theta_y^l, u_o^{*l}, v_o^{*l}, \theta_x^{*l}, \theta_y^{*l} \right\}^{\mathrm{T}} \end{aligned} \tag{20}$$

The node transformation matrix for a node i, on the skew boundary is

$$\mathbf{L}_g = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s & c \end{bmatrix} \tag{21}$$

in which $c = \cos(\psi)$ and $s = \sin(\psi)$, where ψ is the skew angle of the plate. It may be noted that for nodes, which are not lying on skew edges, the node transformation matrix consists of all elements being zero except the

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principal diagonal elements, which are equal to unity. Thus for the complete element, the element transformation matrix is written as:

$$\mathbf{T}_e = \begin{bmatrix} \mathbf{L}_\mathrm{g} & 0 & 0 & \cdots \\ 0 & \mathbf{L}_\mathrm{g} & 0 & \\ 0 & 0 & \mathbf{L}_\mathrm{g} & \\ \vdots & & & \end{bmatrix} \tag{22}$$

in which the number of $\mathbf{L}_\mathrm{g}$ matrices are equal to the number of nodes in the element. For those elements whose nodes are on the skew edges, the element matrices are transformed to the local axes using the element transformation matrix, $\mathbf{T}_e$. After the transformation of element matrices, Equation (15) is expressed as:

$$\int_{t_1}^{t_2} \sum_{e=1}^{NE} \delta \boldsymbol{d}_e^t [\mathbf{K}^l \boldsymbol{d}_e + \mathbf{M}^l \ddot{\boldsymbol{d}}_e]\, \mathrm{d}t = 0 \tag{23}$$

where

$$\mathbf{K}^l = \mathbf{T}_e^\mathrm{T} \mathbf{K}^e \mathbf{T}_e, \quad \text{and} \quad \mathbf{M}^l = \mathbf{T}_e^\mathrm{T} \mathbf{M}^e \mathbf{T}_e \tag{24}$$

This relation is valid for every virtual displacement in an arbitrary time interval t_1 and t_2. We thus have

$$\sum_{e=1}^{NE} \mathbf{K}^l \boldsymbol{d}_e + \sum_{e=1}^{NE} \mathbf{M}^l \ddot{\boldsymbol{d}}_e = 0 \tag{25}$$

The global discrete equation for free vibration in matrix form can be written as:

$$\mathbf{K}\boldsymbol{d} + \mathbf{M}\ddot{\boldsymbol{d}} = 0 \tag{26}$$

where **K** and **M** are the assembled stiffness and mass matrices, respectively for the structure, $\boldsymbol{d}$ is the nodal displacement vector and $\ddot{\boldsymbol{d}}$ is the second derivative of the displacements of the structure with respect to time.

To find natural modes and frequencies, we assume that the field variables can be expressed as:

$$\boldsymbol{d} = \bar{\boldsymbol{d}}_\mathrm{a} e^{i\omega t} \tag{27}$$

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where $\bar{\boldsymbol{d}}_{\mathrm{a}}$ is the vector of unknown amplitudes at time $t = 0$ at the nodes, and ω is the circular natural frequency of the system. Substituting Equation (27) into Equation (26), we get

$$(\mathbf{K} - \omega^2\mathbf{M})\bar{\boldsymbol{d}}_{\mathrm{a}} = 0 \tag{28}$$

The subspace iteration method [32] is used here to obtain the numerical solution of the eigenvalue problem.

NUMERICAL RESULTS AND DISCUSSION

A computer program has been developed, based on the foregoing finite element model to solve a number of numerical examples on free vibration of skew composite laminates and sandwiches. A 6×6 skew mesh of sixteen-noded quadrilateral Lagrangian elements is used in the computations. This scheme is arrived at on the basis of a convergence study in which the fundamental natural frequency converges monotonically from a higher value. The full integration scheme (4×4) is used. A parallel computer code was also developed based on the Reissner–Mindlin's FOST to compare its results with those of HOST. A shear correction factor of 5/6 is used in the former.

A clamped isotropic thin skew plate with a skew angle of 45° is considered for convergence study for 8-, 9-, 12-, and 16-node elements. To check for shear-locking nature of the element, a very thin geometry with $a/h = 1000$ is taken. Selective (SI) and full (FI) integration schemes are used. Both FOST and HOST models have predicted the same results. Graphical results normalized with respect to Durvasula [10] are presented in Figure 2 for different mesh patterns. Shear locking effect is clearly seen in the serendipity elements, while this effect is very small in Lagrangian elements. Between the two Lagrangian elements, the 16-noded element is less susceptible to shear locking. A 6×6 mesh of 16-node elements has given a slightly improved result when compared to a 9×9 mesh of 9-node elements.

The accuracy of the present finite element formulations is evaluated first for isotropic and composite skew plates with the available literature results. Subsequently, some new results are presented for laminated sandwich skew plates. The notations used in the tables presented in this study for boundary conditions SSSS and CCCC represent all edges simply supported and all edges clamped, respectively.

Isotropic Skew Plates

The nondimensional natural frequencies of isotropic thin skew plates with a side-to-thickness ratio of 1000 are determined for both simply supported

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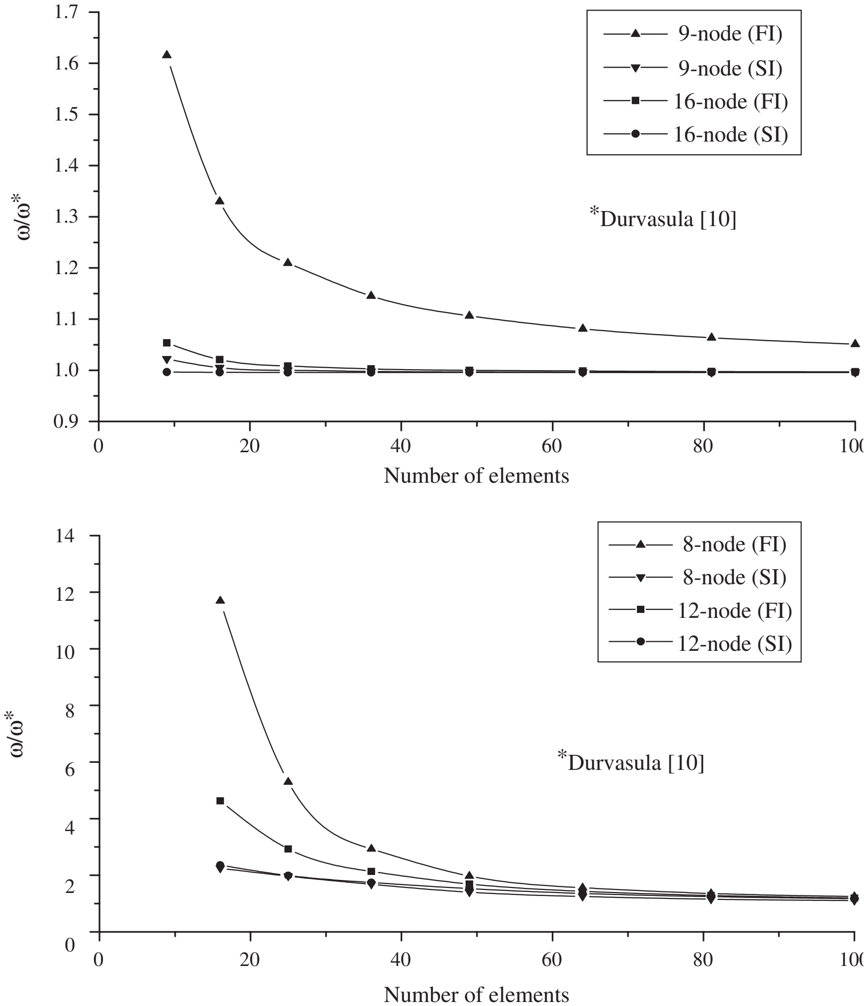

Figure 2. *Convergence of nondimensional fundamental frequency with mesh refinement for clamped isotropic skew plate ($\psi = 45^\circ$).*

and clamped boundary conditions. The frequencies are evaluated in nondimensional form, expressed as $\lambda = \omega b^2/\pi^2(\rho h/D)^2$, where $D = Eh^3/[12(1-\nu^2)]$ and E is the Young's modulus and the Poisson's ratio ν is taken to be 0.3. The results are presented in Table 1 along with the other literature results for four different values of $\psi = 0$, 15, 30, and 45°. It is to be noted that the solutions of the present FOST and HOST match well with the literature results.

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Table 1. Nondimensional fundamental frequencies for isotropic skew plates.

Boundary conditions	Source	Skew angle 0°	15°	30°	45°
SSSS	Durvasula [10]	–	2.110	2.520	3.530
	Liew and Lam [33]	2.000	2.110	2.540	3.540
	Reddy and Palaninathan [21]	2.000	2.120	2.530	3.510
	FOST	2.000	2.116	2.539	3.629
	HOST	2.000	2.115	2.524	3.629
CCCC	Durvasula [10]	–	–	4.675	6.680
	Bardell [11]	3.646	3.869	4.670	6.651
	Reddy and Palaninathan [21]	3.648	3.872	4.672	6.663
	FOST	3.648	3.872	4.680	6.699
	HOST	3.648	3.872	4.680	6.699

Cross-ply Skew Laminates

Cross-ply skew laminates with symmetric (90°/0°/90°/0°/90°) and antisymmetric (0°/90°/0°/90°) layups are considered in this section. Results are presented in the nondimensional form as $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$, showing the effects of skew angle and boundary conditions on the natural frequencies. The orthotropic material properties of each layer are $E_1/E_2 = 40$, $E_2 = E_3$, $G_{12} = 0.6E_2$, $G_{13} = G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. Simply supported and clamped skew laminates having $a/h = 10$ are analyzed.

The solutions of present formulations along with those obtained by Wang [23] using the RRM with the FOST is given in Tables 2–5 for symmetric and antisymmetric, simply supported and clamped skew plates. It is to be noted that the results of FOST and RRM are in very good agreement for all the layups considered. But the results of these two models are higher in comparison to that of HOST. The difference may be due to the assumption of an arbitrary value of 5/6 for shear correction factor, which depends on several factors, such as number of layers, lamination angle, etc.

Angle-ply Skew Laminates

The nondimensional fundamental frequencies for antisymmetric laminated composite (45°/−45°/45°/−45°), simply supported and clamped skew plates are presented in Tables 6 and 7. The orthotropic material properties for the antisymmetric angle ply laminates considered are $E_1/E_2 = 40$, $E_2 = E_3$, $G_{12} = 0.6E_2$, $G_{13} = G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. Results of FOST are identical to the solutions of RRM based on the

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Table 2. Nondimensional frequency parameter $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$ for simply supported symmetric cross-ply skew composite laminates $(90^\circ/0^\circ/90^\circ/0^\circ/90^\circ)$ with $a/h = 10$.

Skew angle (°)	Source	Modes							
		1	2	3	4	5	6	7	8
0	Wang [23]	1.5699	3.0371	3.7324	4.5664	5.1469	6.0343	6.1820	6.5894
	FOST	1.5699	3.0372	3.7325	4.5664	5.1475	6.0347	6.1825	6.5898
	HOST	1.5703	2.8917	3.8041	4.5314	4.7273	5.8888	6.2125	6.6768
15	FOST	1.6874	3.1413	3.9600	4.6073	5.4606	6.0880	6.3125	7.0046
	HOST	1.6877	3.0458	4.0264	4.4818	5.1533	5.8277	6.4910	6.9075
30	Wang [23]	2.0844	3.5127	4.6997	4.8855	6.2494	6.3118	7.2533	7.6518
	FOST	2.0884	3.5147	4.7033	4.8864	6.2514	6.3149	7.2570	7.6523
	HOST	2.0840	3.4023	4.7176	4.7674	5.9852	6.1177	7.1966	7.5472
45	Wang [23]	2.8825	4.2823	5.5868	6.1808	6.9022	7.8390	8.2114	9.3057
	FOST	2.8932	4.2852	5.5886	6.1874	6.9035	7.8366	8.1971	9.2610
	HOST	2.8925	4.1906	5.4149	6.2868	6.6302	7.7433	7.8398	8.9606

Table 3. Nondimensional frequency parameter $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$ for clamped symmetric cross-ply skew composite laminates $(90^\circ/0^\circ/90^\circ/0^\circ/90^\circ)$ with $a/h = 10$.

Skew angle (°)	Source	Modes							
		1	2	3	4	5	6	7	8
0	Wang [23]	2.3820	3.7383	4.1122	5.0366	5.6127	6.2393	6.5591	6.8899
	FOST	2.3820	3.7383	4.1122	5.0366	5.6133	6.2397	6.5595	6.8901
	HOST	2.3687	3.5399	4.2134	4.9852	5.1895	6.2801	6.4666	6.9999
15	FOST	2.4750	3.7872	4.3418	5.0599	5.8762	6.4405	6.5178	7.2850
	HOST	2.4663	3.6255	4.4247	4.9395	5.5603	6.2086	6.7439	7.2133
30	Wang [23]	2.7921	4.0566	5.0220	5.2906	6.5698	6.6042	7.4485	7.8880
	FOST	2.7922	4.0568	5.0222	5.2909	6.5702	6.6047	7.4487	7.8848
	HOST	2.8001	3.9557	5.1079	5.1675	6.3624	6.4236	7.5067	7.7769
45	Wang [23]	3.4738	4.7393	5.9554	6.3750	7.1961	8.0031	8.4468	9.4414
	FOST	3.4739	4.7396	5.9553	6.3745	7.1920	7.9978	8.4213	9.3946
	HOST	3.5215	4.7129	5.8789	6.5290	7.0275	7.9963	8.1687	9.2793

FOST given by Wang [23]. Results of HOST are slightly on the lower side. It is observed that the frequencies of angle-ply laminates $(45^\circ/-45^\circ/45^\circ/-45^\circ)$ for skew angle 45° are identical with cross-ply laminates $(0^\circ/90^\circ/0^\circ/90^\circ)$.

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Table 4. Nondimensional frequency parameter $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$ for simply supported antisymmetric cross-ply skew composite laminates (0°/90°/0°/90°) with $a/h = 10$.

Skew angle (°)	Source	Modes 1	2	3	4	5	6	7	8
0	Wang [23]	1.5119	2.4656	2.4656	3.3904	3.3904	4.5679	4.9312	4.9312
	FOST	1.5119	2.4656	2.4656	3.3905	3.3905	4.5679	4.9313	4.9313
	HOST	1.4829	2.4656	2.4656	3.2522	3.2522	4.3794	4.9313	4.9313
15	FOST	1.6081	2.5364	3.0298	3.3557	3.7179	4.6504	5.2595	5.7893
	HOST	1.5741	2.5351	3.0270	3.2185	3.5637	4.4505	5.2572	5.5165
30	Wang [23]	1.9410	2.9063	3.6124	4.1159	4.4583	4.9554	5.9702	6.3831
	FOST	1.9439	2.9389	3.6131	4.1543	4.4618	4.9564	6.0137	6.3846
	HOST	1.8871	2.9372	3.4489	4.1374	4.2763	4.7172	6.0079	6.0585
45	Wang [23]	2.6652	3.2716	4.2757	5.6255	5.8915	6.3345	6.5997	7.0085
	FOST	2.6752	3.3131	4.2772	5.6276	5.9050	6.5823	7.6600	7.0110
	HOST	2.5609	3.3126	4.0617	5.3338	5.6047	6.5630	6.6501	6.6571

Table 5. Nondimensional frequency parameter $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$ for clamped antisymmetric cross-ply skew composite laminates (0°/90°/0°/90°) with $a/h = 10$.

Skew angle (°)	Source	Modes 1	2	3	4	5	6	7	8
0	Wang [23]	2.3947	3.9532	3.9532	5.0665	5.9680	5.9757	6.7679	6.7679
	FOST	2.3947	3.9533	3.9533	5.0666	5.9685	5.9762	6.7684	6.7684
	HOST	2.2990	3.7880	3.7880	4.8610	5.7129	5.7203	6.4927	6.4927
15	FOST	2.4803	3.9145	4.2554	5.1261	6.1358	6.2917	6.5617	7.2854
	HOST	2.3809	3.7516	4.0785	4.9155	5.8768	6.0260	6.2926	6.9854
30	Wang [23]	2.7796	4.1564	4.9237	5.3983	6.7204	6.7240	7.2729	8.0467
	FOST	2.7798	4.1566	4.9240	5.3989	6.7216	6.7248	7.2738	8.0459
	HOST	2.6666	3.9851	4.7227	5.1752	6.4445	6.4510	6.9755	7.7218
45	Wang [23]	3.4430	4.8219	6.0850	6.2414	7.3720	8.0239	8.6233	9.3320
	FOST	3.4434	4.8223	6.0858	6.2421	7.3717	8.0226	8.6118	9.3262
	HOST	3.3015	4.6290	5.8423	6.0039	7.0792	7.7269	8.2726	8.9874

Honeycomb Skew Sandwich Plates

The natural frequencies of a 1828.8 mm long by 1219.2 mm wide simply supported skew sandwich plate are presented in Table 8. The plate

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Table 6. Nondimensional frequency parameter $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$ for simply supported antisymmetric angle-ply skew composite laminates ($45^\circ/-45^\circ/45^\circ/-45^\circ$) with $a/h = 10$.

Skew angle (°)	Source	Modes 1	2	3	4	5	6	7	8
0	Wang [23]	1.9171	3.4869	3.5290	3.5290	5.1351	5.4942	5.5251	6.7981
	FOST	1.9171	3.4869	3.5391	3.5291	5.1353	5.4948	5.5258	6.7986
	HOST	1.7974	3.3351	3.3351	4.8042	5.2277	5.2750	6.3931	6.3931
15	FOST	1.9366	3.4206	3.8755	4.4321	4.9802	5.7004	6.0045	6.4449
	HOST	1.8313	3.2490	3.6724	4.3957	4.7109	5.4241	5.7043	6.0814
30	Wang [23]	2.1158	3.6138	4.3015	4.5763	5.0336	5.8271	6.3885	6.4506
	FOST	2.1196	3.6146	4.3068	4.5801	5.0350	5.8435	6.3899	6.4601
	HOST	2.0270	3.4431	4.2361	4.4079	4.7739	5.7845	6.0911	6.1203
45	Wang [23]	2.6652	3.2716	4.2757	5.6255	5.8915	6.3345	6.5997	7.0085
	FOST	2.6752	3.3131	4.2772	5.6276	5.9050	6.5825	6.6599	7.0110
	HOST	2.5609	3.3125	4.0617	5.3338	5.6047	6.5632	6.6502	6.6570

Table 7. Nondimensional frequency parameter $\lambda = \omega b^2/\pi^2 h(\rho/E_2)^{1/2}$ for clamped antisymmetric angle-ply skew composite laminates ($45^\circ/-45^\circ/45^\circ/-45^\circ$) with $a/h = 10$.

Skew angle (°)	Source	Modes 1	2	3	4	5	6	7	8
0	Wang [23]	2.2964	3.8919	3.8919	5.3042	5.7449	5.8196	6.9647	6.9647
	FOST	2.2965	3.8921	3.8921	5.3045	5.7455	5.8202	6.9654	6.9654
	HOST	2.2119	3.7339	3.7339	5.0618	5.5129	5.5810	6.6473	6.6473
15	FOST	2.4007	3.8555	4.2172	5.2635	5.9487	6.2267	6.7141	7.3632
	HOST	2.3099	3.6997	4.0438	5.0322	5.7087	5.9648	6.4131	7.0431
30	Wang [23]	2.7416	4.1219	4.9126	5.4395	6.6183	6.7842	7.2753	8.0427
	FOST	2.7418	4.1221	4.9130	5.4402	6.6192	6.7854	7.2763	8.0421
	HOST	2.6325	3.9549	4.7125	5.2107	6.3577	6.4954	6.9760	7.7176
45	Wang [23]	3.4430	4.8219	6.0850	6.2414	7.3720	8.0239	8.6233	9.3320
	FOST	3.4434	4.8223	6.0858	6.2421	7.3717	8.0226	8.6118	9.3262
	HOST	3.3015	4.6290	5.8423	6.0039	7.0792	7.7269	8.2726	8.9874

has two identical aluminum faceplates, 0.4064 mm thick, and an aluminum honeycomb core, 6.35 mm thick. The physical properties for the faceplates are

$$E = 68.948\,\text{GPa}, \quad G = 25.924\,\text{GPa}, \quad \nu = 0.33, \quad \rho = 2768.0\,\text{kg/m}^3$$

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Table 8. Natural frequencies (Hz) of simply supported honeycomb skew sandwich plates.

Skew angle (°)	Source	Modes 1	2	3	4	5	6
0	FOST	23.5279	45.2496	72.3990	81.5919	94.0832	130.3219
	HOST	23.4514	44.8867	71.7909	80.2416	92.8687	127.6216
	Raville and Veng [34]	23.0000	44.0000	71.0000	80.0000	91.0000	126.0000
15	FOST	24.9315	47.0681	77.5111	82.2836	101.8814	125.6111
	HOST	24.8438	46.6782	76.8076	80.9709	100.3713	122.5909
30	FOST	30.0623	53.7200	88.5274	95.8735	125.9725	130.6636
	HOST	29.9153	53.2053	87.0499	94.7506	123.5169	127.2716
45	FOST	43.1677	70.4135	107.7857	140.0649	156.0967	180.1823
	HOST	42.7936	69.4561	105.4600	136.9906	151.2371	174.2296

and for the core

$$G_{23} = 0.05171\,\text{GPa}, \quad G_{13} = 0.13445\,\text{GPa}, \quad \rho = 121.83\,\text{kg/m}^3$$

To be consistent with the simple representation of core behavior of [34], it is assumed that the core in-plane direct and shear stiffnesses are zero. The results of the present theories are in good agreement with those of the analytical results of Raville and Veng [34] for rectangular plates. Natural frequencies for skew sandwich plates are presented for skew angles 15, 30, and 45°.

Cross-ply Skew Sandwich Laminates

The variation of fundamental frequency with respect to the lamination scheme of simply supported cross-ply skew sandwich plate with antisymmetric and symmetric cross-ply faceplates is given in Table 9. The simply supported sandwich plates of length 1270 mm and skew side 635 mm wide are considered. The core thickness is 25.4 mm and each faceplate is 3.175 mm thick and is made of two equal-thickness plies. The core is defined as orthotropic and is assumed to carry through-thickness shearing action only, with shear moduli and density values as considered by Yuan and Dawe [35];

$$G_{13} = 0.11721\,\text{GPa}, \quad G_{23} = 0.24132\,\text{GPa} \quad \text{and} \quad \rho = 2351.2\,\text{kg/m}^3$$

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Table 9. Natural frequencies (Hz) of simply supported cross-ply skew sandwich plates.

Lamination scheme	Source	Skew angle 0°	15°	30°	45°
0°/90°/core/0°/90°	FOST	166.3086	177.6942	217.7630	310.6456
	HOST	152.2992	161.7182	194.3770	267.3398
	Yuan and Dawe [35]	152.58	–	–	–
0°/90°/core/90°/0°	FOST	159.8275	170.7568	209.3430	299.3778
	HOST	146.5089	155.5495	186.9801	257.5617
	Yuan and Dawe [35]	145.99	–	–	–
90°/0°/core/0°/90°	FOST	172.7237	184.5342	225.9660	321.4230
	HOST	158.0954	167.8775	201.7029	276.9311
	Yuan and Dawe [35]	159.30	–	–	–

The faceplates are considered to be classical thin plates, which are made of graphite–epoxy material, with ply properties defined as:

$$E_1 = 206.84\,\text{GPa}, \quad E_3 = 5.1711\,\text{GPa}, \quad G_{12} = 5.1711\,\text{GPa}, \quad \nu_{12} = 0.25,$$
$$\text{and} \quad \rho = 1603.1\,\text{kg/m}^3.$$

The fundamental frequencies obtained by the present HOST are in very good agreement with the spline finite strip method solutions given by Yuan and Dawe [35] for the skew angle 0° for various lamination schemes. The present first-order theory predicts much higher frequencies. Results for skew angles 15, 30, and 45° are presented as benchmark results for further investigations.

CONCLUSIONS

A simple C^o isoparametric finite element formulation based on the HOST is presented for the free vibration analysis of fiber-reinforced composite and sandwich laminates. By the use of transformation matrices for the nodes lying on the skew edges, the general finite element formulation in orthogonal coordinates is extended to the analysis of skew plates. The accuracy of the present formulation is evaluated by obtaining the solutions to a wide range of problems and comparing them with the available results in the literature along with the solutions obtained using FOST. Numerical results are presented for isotropic, anisotropic, and sandwich plates with skew geometries.

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The parametric effects of width-to-thickness ratio, variation in skew angle, fiber, orientation and boundary conditions upon the frequencies and mode shapes are studied. The influence of skew angle on natural frequencies is more pronounced as the skew angle increases. In thick laminates, the transverse shear deformation effect is significant and this effect increases with increasing skew angle. It is observed that in case of composite laminates, the difference in prediction of results by FOST and HOST are small. However, for sandwich laminates, in comparison to HOST, FOST overestimates the natural frequency with significant margin and the margin increases as the skew angle increases. It is believed that the present results on skew sandwich laminates will serve as benchmark solutions for other researchers to validate their numerical techniques.

APPENDIX A

Rigidity Matrices

Assuming $H_i = (z^i_{L+1} - z^i_L)/i$, where i takes an integer value from one to seven, the elements of the submatrices of the rigidity matrix can be readily obtained in the following forms:

$$\mathbf{D}_\mathrm{m} = \sum_{L=1}^{NL} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{13}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_3 \\ & Q_{22}H_1 & Q_{23}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_3 \\ & & Q_{33}H_1 & Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_3 \\ & & & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 \\ & \text{Symm.} & & & Q_{22}H_5 & Q_{23}H_5 \\ & & & & & Q_{33}H_5 \end{bmatrix}$$

$$\mathbf{D}_\mathrm{s} = \sum_{L=1}^{NL} \begin{bmatrix} Q_{44}H_1 & Q_{45}H_1 & Q_{44}H_3 & Q_{45}H_3 & Q_{44}H_2 & Q_{45}H_2 \\ & Q_{55}H_1 & Q_{45}H_3 & Q_{55}H_3 & Q_{45}H_2 & Q_{55}H_2 \\ & & Q_{44}H_5 & Q_{45}H_5 & Q_{44}H_4 & Q_{45}H_4 \\ & & & Q_{55}H_5 & Q_{45}H_4 & Q_{55}H_4 \\ & \text{Symm.} & & & Q_{44}H_3 & Q_{45}H_3 \\ & & & & & Q_{55}H_3 \end{bmatrix}$$

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The elements of the $\mathbf{D}_c$ and $\mathbf{D}_b$ matrices are obtained by replacing (H_1, H_3, and H_5) by (H_2, H_4, and H_6) and (H_3, H_5, and H_7), respectively in the $\mathbf{D}_m$ matrix.

APPENDIX B

Inertia Matrix

The inertia matrix **m** for the present higher-order theory is given by:

$$\mathbf{m} = \begin{bmatrix} I_1 & 0 & 0 & 0 & I_2 & I_3 & 0 & 0 & I_4 \\ 0 & I_1 & 0 & -I_2 & 0 & 0 & I_3 & -I_4 & 0 \\ 0 & 0 & I_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & I_3 & 0 & 0 & -I_4 & I_5 & 0 \\ I_2 & 0 & 0 & 0 & I_3 & I_4 & 0 & 0 & I_5 \\ I_3 & 0 & 0 & 0 & I_4 & I_5 & 0 & 0 & I_6 \\ 0 & I_3 & 0 & -I_4 & 0 & 0 & I_5 & -I_6 & 0 \\ 0 & -I_4 & 0 & I_5 & 0 & 0 & -I_6 & I_7 & 0 \\ I_4 & 0 & 0 & 0 & I_5 & I_6 & 0 & 0 & I_7 \end{bmatrix}$$

The parameters I_1, I_2, and (I_5, I_7) are linear inertia, rotary inertia, and higher-order inertia terms, respectively. The parameters I_2, I_4, and I_6 are the coupling inertia terms. They are defined as follows:

$$(I_1, I_2, I_3, I_4, I_5, I_6, I_7) = \sum_{L=1}^{NL} \int_{Z_L}^{Z_{L+1}} (1, z, z^2, z^3, z^4, z^5, z^6)\rho_L \, dz$$

where ρ_L is the material density of the Lth layer.

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