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Elasticity solution for cross-ply composite and sandwich laminates

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Abstract

A novel semi-analytical model is presented here for accurate estimation of stresses and displacements in composite and sandwich laminates. Displacements and corresponding transverse stresses are considered as primary variables of interest. Formulation is based on solution of a two-point boundary value problem (BVP) governed by a set of linear first-order ordinary differential equations (ODEs) through the thickness of a laminate. These first-order ODEs are numerically integrated by using fourth-order Runge–Kutta–Gill routine. Present model is free from any simplifying assumptions and also satisfies the continuity requirements of displacements and interlaminar transverse stresses at the laminae interfaces. Solutions for a wide range of composite and sandwich laminates are obtained to validate the present formulation. Results obtained through this technique are seen to compare well with the available three dimensional (3D) elasticity and other two dimensional (2D) analytical and 2D/3D finite element (FE) solutions. Few new benchmark solutions are also presented for future reference.

Keywords: Semi-analytical method; Laminate; Composites; Sandwich; Runge–Kutta–Gill algorithm; Numerical integration

1. Introduction

Fiber reinforced polymer composites (FRPCs) laminates posses ideal engineering properties like high stiffness/strength along the fiber direction, light weight, etc. and therefore these materials are used in many engineering applications. Simple theories applicable to homogenous material like steel cannot be directly employed to analyze layered composites because of mismatch of properties of different layers. Furthermore, delamination is one of the major failure modes of laminated composites that involve separation of layers of composite laminate due to the transverse/interlaminar stresses. Thus, a laminate theory which predicts these stresses accurately is an essential prerequisite for understanding of the failure behavior.

The behavior of composite and sandwich laminates can be characterized by a complex 3D state of stress. In many instances, these laminated structural elements are moderately thick in relation to their span dimensions. As a result, a refined analysis is required by incorporating transverse shear deformation in analytical 2D models. The classical laminate plate theory (CLPT) based on the Kirchhoff [1] hypothesis ignores effects of transverse shear deformation, normal stress/strain and non-linear inplane normal strain distribution through the thickness. On the other hand, the first-order shear deformation theory (FOST) based on Reissner [2] and Mindlin [3] considers effects of the transverse shear deformation by assuming it to be constant through the thickness. Thus a fictitious shear coefficient is introduced to correct the strain energy due to the shear deformation. In order to remove the limitations of FOST, higher-order shear deformation theories (HOSTs) involving higher order terms in the Taylor's expansion of displacements in the thickness coordinate were developed. In these theories, an additional dependent unknown was introduced [4–11] with each additional power of the thickness coordinate. All these theories are referred to as equivalent single layer (ESL) theories. The ESL theories have been reported to predict the overall response like gross deflections, buckling modes, inplane stresses, etc. reasonably well. However, these fail to capture the transverse stresses accurately.

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Nomenclature

1, 2, 3 local coordinate system of lamina (principal material directions)
x, y, z reference coordinate system of laminate
a, b, h length, width and thickness of laminate
E_1, E_2, E_3 Young's moduli of lamina in principal material directions
G_{12}, G_{13}, G_{23} Shear moduli of lamina in three orthogonal planes
u, v, w displacement components along reference directions x, y and z, respectively, at a point
σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}, components of stress at a point
ε_x, ε_y, ε_z, γ_{xy}, γ_{xz}, γ_{yz} components of strain at a point
n order of partial differential equation

Elasticity solution of layered laminates [12,13] indicates that the interlaminar continuity of the interlaminar transverse normal and shear stresses as well as the layerwise continuous displacement field through the thickness of a laminate components is an essential requirement for their accurate estimation. Thus, a layerwise analysis is necessary for composite and sandwich laminates. Various displacement based layerwise theories have been proposed by Reddy [14], Soldatos [15], Wu and Kuo [16], Wu and Hsu [17] and others. However, only continuity of displacement field through the thickness of a laminate could be satisfied in the displacement based layerwise models and continuity of the transverse normal and shear stresses at the layer interfaces could not be enforced. To overcome this lacks, another group of researchers including Spilker [18], Wu and Lin [19], Shin and Chen [20], Ramtekkar et al. [21,22] have worked on development of layerwise mixed/hybrid FE models with displacement and the transverse stresses as primary variables. Such models satisfy requirements of continuity of displacements and transverse stresses through the thickness of composite and sandwich laminates.

An attempt is made here to look at the governing exact 3D partial differential equations (PDEs) of laminate. Taking a cue from Kantorovich and Krylov [23] for the dimension reduction through an assumption of global solution functions in all but one independent coordinate, mathematical model as a two point BVP governed by a set of linear coupled first order ODEs,

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{y}(z) = \mathbf{A}(z)\mathbf{y}(z) + \mathbf{p}(z) \tag{1}$$

in the interval $-h/2 \leqslant z \leqslant h/2$ with any half of the dependent variables prescribed at the edges $z = \pm h/2$ is formulated. Clearly, mixed and/or non-homogenous boundary conditions are easily admitted in the formulation. Here, $\mathbf{y}(z)$ is an n-dimensional vector of dependent variables whose number (n) equals the order of PDE, $\mathbf{A}(z)$ is a $n \times n$ coefficient matrix and $\mathbf{p}(z)$ is an n-dimensional vector of non-homogenous (loading) terms.

2. Theoretical formulation

A laminate composed of a number of isotropic/orthotropic, linear elastic laminae of uniform thickness with plan dimension axb and thickness 'h' is considered (Fig. 1). The angle between the fiber direction and reference axis 'x' is measured in anticlockwise direction as shown in Fig. 1. Simply (diaphragm) supported end conditions on all four edges of laminates are considered (Table 1). The top surface of laminate is loaded with transversely distributed load. The intensity of transverse loading is expressed in the form of a double Fourier series as

$$p(x,y) = \sum_m^{\infty}\sum_n^{\infty} p_{0mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{2}$$

where p_{0mn} is the peak intensity of distributed load.

With m and n assumed to be odd in Eq. (2), the loading is symmetric about the center of the plate.

2.1. Constitute relations

Each lamina in the laminate has been considered to be in a 3D state of stress so that the constitutive relation for a typical ith lamina with reference to the principal material coordinate axes (1, 2 and 3) can be written as

$$\begin{aligned}
(\varepsilon_1)^i &= \left(\frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{31}}{E_3}\sigma_3\right)^i,\\
(\varepsilon_2)^i &= \left(-\frac{\nu_{12}}{E_1}\sigma_1 + \frac{1}{E_2}\sigma_2 - \frac{\nu_{32}}{E_3}\sigma_3\right)^i,\\
(\varepsilon_3)^i &= \left(-\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 + \frac{1}{E_3}\sigma_3\right)^i,\\
(\gamma_{12})^i &= \left(\frac{\tau_{12}}{G_{12}}\right)^i; \quad (\gamma_{13})^i = \left(\frac{\tau_{13}}{G_{13}}\right)^i \quad \text{and} \quad (\gamma_{23})^i = \left(\frac{\tau_{23}}{G_{23}}\right)^i.
\end{aligned} \tag{3}$$

These can be written as

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{Bmatrix}^i = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & \text{Sym.} & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}^i \begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{Bmatrix}^i, \tag{4}$$

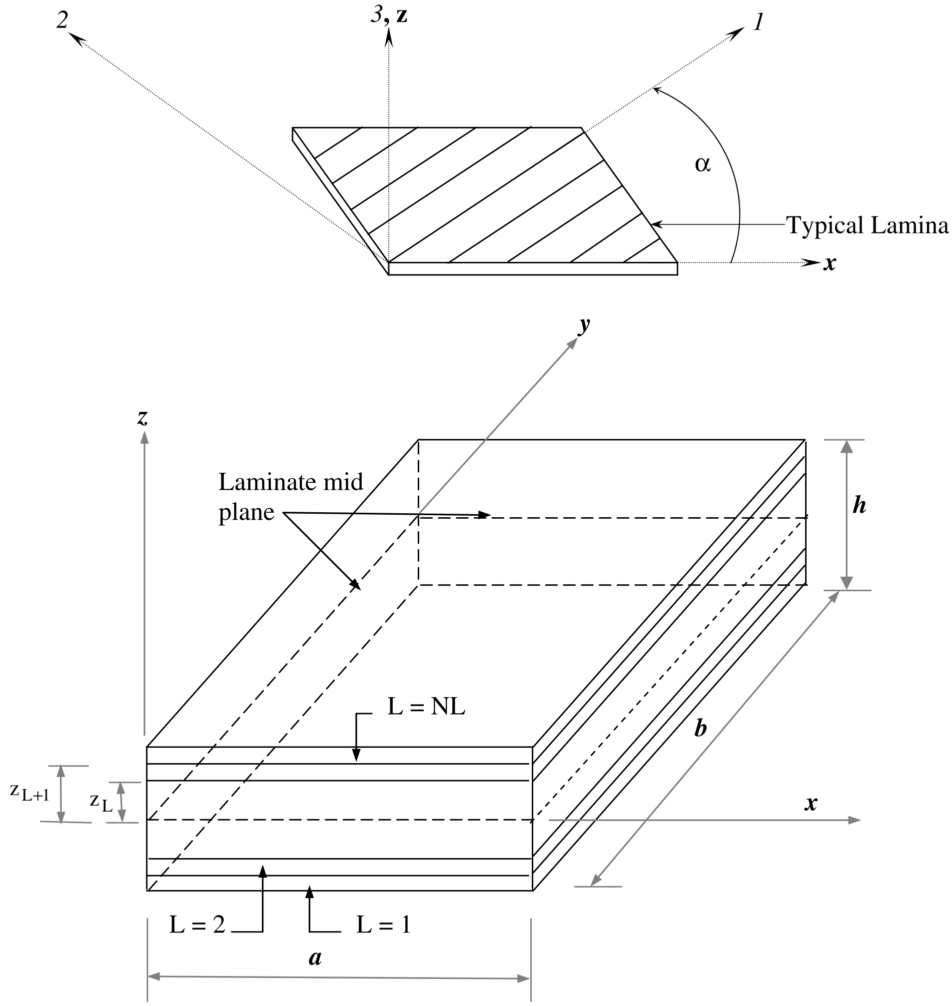

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes and fiber orientation.

Table 1
Boundary conditions (BCs)

	BC imposed on displacement field	BC imposed on stress field
Face $x = 0$	$v = w = 0$	$\sigma_x = 0$ (true)
Face $x = a/2$	$u = 0$	$\tau_{xz} = 0$
Face $y = 0$	$u = w = 0$	$\sigma_y = 0$ (true)
Face $y = b/2$	$v = 0$	$\tau_{yz} = 0$
Top face $z = h/2$	–	$\tau_{xz} = \tau_{yz} = 0$ and $\sigma_z = p_0(x, y)$
Bottom face $z = -h/2$	–	$\tau_{xz} = \tau_{yz} = \sigma_z = 0$

where $\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{13}, \tau_{23}$ are stresses and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{13}, \gamma_{23}$ are linear strain components with reference to the lamina coordinates 1, 2, and 3. C'_{mn}s $(m, n = 1, \ldots, 6)$ are elasticity constants of the ith lamina with reference to the fiber axes (1, 2, 3) defined in Appendix A. Stress–strain relations for the ith lamina in laminate coordinates (x, y, z) can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ & & Q_{33} & Q_{34} & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ & \text{Sym.} & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{5}$$

where $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are stresses and $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ are strain components with respect to laminate axes (x, y, z) and Q'_{mn}s $(m, n = 1, \ldots, 6)$ are the transformed elasticity constants of the ith lamina with reference to the laminate axes. Elements of matrix $[Q]$ are defined in Appendix B.

2.2. Strain–displacement relationship

General 3D linear strain–displacement relations can be written as

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$
$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}. \tag{6}$$

2.3. Equations of equilibrium

The 3D differential equations of equilibrium are

$$\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+B_x=0,\\
&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+B_y=0,\\
&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+B_z=0.
\end{aligned}\tag{7}$$

Here, B_x, B_y and B_z are components of body force in x, y and z directions, respectively.

2.4. Partial differential equations

Eqs. (5)–(7) have a total of 15 unknowns, 6 stresses$(\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz})$, 6 strains $(\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{xz},\gamma_{yz})$ and 3 displacements (u,v,w) in 15 equations. After simple algebraic manipulations, a system of PDEs involving only six fundamental dependent variables $u,v,w,\tau_{xz},\tau_{yz}$ and σ_z called 'primary variables' are obtained as follows:

$$\begin{aligned}
\frac{\partial u}{\partial z}&=\frac{1}{(Q_{55}Q_{66}-Q_{56}Q_{65})}[-Q_{65}\tau_{yz}+Q_{66}\tau_{xz}]-\frac{\partial w}{\partial x},\\
\frac{\partial v}{\partial z}&=\frac{1}{(Q_{55}Q_{66}-Q_{56}Q_{65})}[Q_{55}\tau_{yz}-Q_{56}\tau_{xz}]-\frac{\partial w}{\partial y},\\
\frac{\partial w}{\partial z}&=\frac{1}{Q_{33}}\left[\sigma_z-Q_{31}\frac{\partial u}{\partial x}-Q_{34}\frac{\partial u}{\partial y}-Q_{32}\frac{\partial v}{\partial y}-Q_{34}\frac{\partial v}{\partial x}\right],\\
\frac{\partial\tau_{xz}}{\partial z}&=\left(-Q_{11}+\frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\partial^2u}{\partial x^2}\\
&\quad+\left(-Q_{41}-Q_{14}+\frac{Q_{13}Q_{34}}{Q_{33}}+\frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial^2u}{\partial x\partial y}\\
&\quad+\left(-Q_{44}+\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2u}{\partial y^2}+\left(-Q_{14}+\frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\partial^2v}{\partial x^2}\\
&\quad+\left(-Q_{12}-Q_{44}+\frac{Q_{13}Q_{32}}{Q_{33}}+\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2v}{\partial x\partial y}\\
&\quad+\left(-Q_{42}+\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial^2v}{\partial y^2}-\left(\frac{Q_{13}}{Q_{33}}\right)\frac{\partial\sigma_z}{\partial x}-\left(\frac{Q_{43}}{Q_{33}}\right)\frac{\partial\sigma_z}{\partial y}-B_x,\\
\frac{\partial\tau_{yz}}{\partial z}&=\left(-Q_{41}+\frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial^2u}{\partial x^2}\\
&\quad+\left(-Q_{21}-Q_{44}+\frac{Q_{23}Q_{31}}{Q_{33}}+\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2u}{\partial x\partial y}\\
&\quad+\left(-Q_{24}+\frac{Q_{23}Q_{34}}{Q_{33}}\right)\frac{\partial^2u}{\partial y^2}+\left(-Q_{44}+\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2v}{\partial x^2}\\
&\quad+\left(-Q_{24}-Q_{42}+\frac{Q_{23}Q_{34}}{Q_{33}}+\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial^2v}{\partial x\partial y}\\
&\quad+\left(-Q_{22}+\frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\partial^2v}{\partial y^2}-\left(\frac{Q_{43}}{Q_{33}}\right)\frac{\partial\sigma_z}{\partial x}-\left(\frac{Q_{23}}{Q_{33}}\right)\frac{\partial\sigma_z}{\partial y}-B_y\\
\frac{\partial\sigma_z}{\partial z}&=-\frac{\partial\tau_{xz}}{\partial x}-\frac{\partial\tau_{yz}}{\partial y}-B_z.
\end{aligned}\tag{8}$$

2.5. Inplane variation of primary variables

The above PDEs defined by Eq. (8) can be reduced to a coupled first-order ODEs by using a double Fourier trigonometric series for primary variables satisfying completely the simple (diaphragm) end conditions at all four edges, $x=0$, a and $y=0$, b, as follows:

$$\begin{aligned}
u(x,y,z)&=\sum_{mn}u_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\
v(x,y,z)&=\sum_{mn}v_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},\\
w(x,y,z)&=\sum_{mn}w_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\
\tau_{xz}(x,y,z)&=\sum_{mn}\tau_{xzmn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\
\tau_{yz}(x,y,z)&=\sum_{mn}\tau_{yzmn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},\\
\sigma_z(x,y,z)&=\sum_{mn}\sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned}\tag{9}$$

in the above both m, n are 1, 3, 5, 7, ...

2.6. Linear first-order ordinary differential equations (ODEs)

Substituting Eq. (9) into Eq. (8), a set of linear coupled first-order ODEs involving only primary variables $u(z),v(z),w(z),\tau_{xz}(z),\tau_{yz}(z)$ and $\sigma_z(z)$ are obtained as

$$\begin{aligned}
\frac{\mathrm{d}u_{mn}(z)}{\mathrm{d}z}&=-w_{mn}(z)\frac{m\pi}{a}+\left(\frac{Q_{66}}{Q_{55}Q_{66}-Q_{56}Q_{65}}\right)\tau_{xzmn}(z),\\
\frac{\mathrm{d}v_{mn}(z)}{\mathrm{d}z}&=-w_{mn}(z)\frac{n\pi}{b}+\left(\frac{Q_{55}}{Q_{55}Q_{66}-Q_{56}Q_{65}}\right)\tau_{yzmn}(z),\\
\frac{\mathrm{d}w_{mn}(z)}{\mathrm{d}z}&=\frac{Q_{31}}{Q_{33}}u_{mn}(z)\frac{m\pi}{a}+\frac{Q_{32}}{Q_{33}}v_{mn}(z)\frac{n\pi}{b}+\frac{1}{Q_{33}}\sigma_{zmn}(z),\\
\frac{\mathrm{d}\tau_{xzmn}(z)}{\mathrm{d}z}&=\left\{\left[Q_{11}-\left(\frac{Q_{13}Q_{31}}{Q_{33}}\right)\right]\frac{m^2\pi^2}{a^2}\right.\\
&\quad\left.+\left[Q_{44}-\left(\frac{Q_{43}Q_{34}}{Q_{33}}\right)\right]\frac{n^2\pi^2}{b^2}\right\}u_{mn}(z)\\
&\quad+\left[Q_{12}-\left(\frac{Q_{13}Q_{32}}{Q_{33}}\right)-\left(\frac{Q_{43}Q_{34}}{Q_{33}}\right)+Q_{44}\right]\frac{mn\pi^2}{ab}v_{mn}(z)\\
&\quad-\left(\frac{Q_{13}}{Q_{33}}\frac{m\pi}{a}\right)\sigma_{zmn}(z)-B_x(x,y,z),\\
\frac{\mathrm{d}\tau_{yzmn}(z)}{\mathrm{d}z}&=\left[Q_{21}-\left(\frac{Q_{31}Q_{23}}{Q_{33}}\right)-\left(\frac{Q_{43}Q_{34}}{Q_{33}}\right)+Q_{44}\right]\frac{mn\pi^2}{ab}u_{mn}(z)\\
&\quad+\left\{\left[Q_{22}-\left(\frac{Q_{23}Q_{32}}{Q_{33}}\right)\right]\frac{n^2\pi^2}{b^2}\right.\\
&\quad\left.+\left[Q_{44}-\left(\frac{Q_{43}Q_{34}}{Q_{33}}\right)\right]\frac{m^2\pi^2}{a^2}\right\}v_{mn}(z)-\left(\frac{Q_{23}}{Q_{33}}\frac{n\pi}{b}\right)\sigma_{zmn}(z)\\
&\quad-B_y(x,y,z),\\
\frac{\mathrm{d}\sigma_{zmn}(z)}{\mathrm{d}z}&=\left(\frac{m\pi}{a}\right)\tau_{xzmn}(z)+\left(\frac{n\pi}{b}\right)\tau_{yzmn}(z)-B_z(x,y,z).
\end{aligned}\tag{10}$$

Eq. (10) defines the governing two-point BVP in ODEs through thickness of the laminate in the domain $-h/2<z<h/2$ with stress components known at the top and bottom faces. The basic approach to the numerical

integration of the BVP defined in Eq. (10) and the associated boundary conditions *when it contains no boundary layer effects* is to transform the given BVP into a set of initial value problems (IVPs) – one non-homogeneous and $n/2$ homogeneous. The solution of BVP defined by Eq. (10) is then obtained by forming a linear combination of one non-homogeneous and $n/2$ homogeneous solutions so as to satisfy the boundary conditions at $z = +h/2$ [24]. This gives rise to a system of $n/2$ linear algebraic equations, the solutions of which determines the unknown $n/2$ components, X_1, X_2 and X_3 (Table 2) at the starting edge $z = -h/2$. Then a final numerical integration of Eq. (10) produces the desired results. Availability of efficient, accurate and robust ODE numerical integrators for IVPs helps in computing reliable values of the primary variables through the thickness. Change in material properties are incorporated by changing coefficients of material matrix appropriately for each lamina.

2.7. Secondary relations

Secondary variables, σ_x, σ_y and τ_{xy} can be expressed in terms of primary variables with the help of constitutive and strain–displacement relation as

$$\begin{aligned}\sigma_x = {} & \left(\frac{Q_{13}Q_{31}}{Q_{33}} - Q_{11}\right)\sum_{mn} u_{mn}(z)\left(\frac{m\pi}{a}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ & + \left(\frac{Q_{13}Q_{32}}{Q_{33}} - Q_{12}\right)\sum_{mn} v_{mn}(z)\left(\frac{n\pi}{b}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ & + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right)\sum_{mn} u_{mn}(z)\left(\frac{n\pi}{b}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ & + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right)\sum_{mn} v_{mn}(z)\left(\frac{m\pi}{a}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ & + \left(\frac{Q_{13}}{Q_{33}}\right)\sum_{mn} \sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b} \end{aligned} \tag{11}$$

$$\begin{aligned}\sigma_y = {} & \left(\frac{Q_{23}Q_{31}}{Q_{33}} - Q_{11}\right)\sum_{mn} u_{mn}(z)\left(\frac{m\pi}{a}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ & + \left(\frac{Q_{23}Q_{32}}{Q_{33}} - Q_{22}\right)\sum_{mn} v_{mn}(z)\left(\frac{n\pi}{b}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ & + \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}}\right)\sum_{mn} u_{mn}(z)\left(\frac{n\pi}{b}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ & + \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}}\right)\sum_{mn} v_{mn}(z)\left(\frac{m\pi}{a}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ & + \left(\frac{Q_{23}}{Q_{33}}\right)\sum_{mn} \sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b} \end{aligned} \tag{12}$$

$$\begin{aligned}\tau_{xy} = {} & \left(\frac{Q_{43}Q_{31}}{Q_{33}} - Q_{11}\right)\sum_{mn} u_{mn}(z)\left(\frac{m\pi}{a}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ & + \left(\frac{Q_{43}Q_{32}}{Q_{33}} - Q_{42}\right)\sum_{mn} v_{mn}(z)\left(\frac{n\pi}{b}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ & + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)\sum_{mn} u_{mn}(z)\left(\frac{n\pi}{b}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ & + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)\sum_{mn} v_{mn}(z)\left(\frac{m\pi}{a}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ & + \left(\frac{Q_{43}}{Q_{33}}\right)\sum_{mn} \sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b}. \end{aligned} \tag{13}$$

3. Numerical results and discussion

A computer code is developed by incorporating the present approach in FORTRAN-90 for the analysis of composite and sandwich laminates. Numerical investigations on various examples have been performed for validation of the present semi-analytical formulation. The 3D elasticity solution given by Pagano [13] and various other analytical and FE solutions available in the literature have been used for proper comparison of the obtained results. Material properties used here have been tabulated in Table 3.

Following normalizations have been used in all numerical examples considered here for the comparison of the results excluding Example 1.

$$s = \frac{a}{h}; \quad \bar{u} = \frac{E_2 u}{h p_0 s^3}; \quad \bar{w} = \frac{100 E_2 h^3 w}{p_0 a^4}; \overline{\sigma_z} = \frac{\sigma_z}{p_0},$$
$$(\overline{\sigma_x}; \overline{\sigma_y}; \overline{\tau_{xy}}) = \frac{1}{p_0 s^2}(\sigma_x; \sigma_y; \tau_{xy}); \quad (\overline{\tau_{xz}}; \overline{\tau_{yz}}) = \frac{1}{p_0 s}(\tau_{xz}; \tau_{yz}) \tag{14}$$

in which bar over the variable defines its normalized value.

Table 3
Material properties

Examples	Source	Property		
2 and 4	Pagano [13]	$E_1 = 172.4$ GPa	$\nu_{12} = 0.25$	$G_{12} = 3.45$ GPa
		$E_2 = 6.89$ GPa	$\nu_{13} = 0.25$	$G_{13} = 3.45$ GPa
		$E_3 = 6.89$ GPa	$\nu_{23} = 0.25$	$G_{23} = 1.378$ GPa
3	Pagano [13]	*Face sheet*		
		$E_1 = 172.4$ GPa	$\nu_{12} = 0.25$	$G_{12} = 3.45$ GPa
		$E_2 = 6.89$ GPa	$\nu_{13} = 0.25$	$G_{13} = 3.45$ GPa
		$E_3 = 6.89$ GPa	$\nu_{23} = 0.25$	$G_{23} = 1.378$ GPa
		Core sheet		
		$E_1 = 0.276$ GPa	$\nu_{12} = 0.25$	$G_{12} = 0.1104$ GPa
		$E_2 = 0.276$ GPa	$\nu_{31} = 0.25$	$G_{13} = 0.414$ GPa
		$E_3 = 3.450$ GPa	$\nu_{32} = 0.25$	$G_{23} = 0.414$ GPa

Table 2
Transformation of a BVP into IVPs

Intg.	Starting edge; $z = -h/2$						Final edge; $z = h/2$						Load term
	u	v	w	τ_{xz}	τ_{yz}	σ_z	u	v	w	τ_{xz}	τ_{yz}	σ_z	
1	0 (assumed)	0 (assumed)	0 (assumed)	0 (known)	0 (known)	0 (known)	Y_{11}	Y_{21}	Y_{31}	Y_{41}	Y_{51}	Y_{61}	Include
2	1 (unity)	0 (assumed)	0 (assumed)	0	0	0	Y_{12}	Y_{22}	Y_{32}	Y_{42}	Y_{52}	Y_{62}	Delete
3	0 (assumed)	1 (unity)	0 (assumed)	0	0	0	Y_{13}	Y_{23}	Y_{33}	Y_{43}	Y_{53}	Y_{63}	Delete
4	0 (assumed)	0 (assumed)	1 (unity)	0	0	0	Y_{14}	Y_{24}	Y_{34}	Y_{44}	Y_{54}	Y_{64}	Delete
Final	X_1	X_2	X_3	Known	Known	Known	u_T	v_T	w_T	0	0	$p(x,y)$	Include

Table 4
Transverse displacement ($\bar{w}$) and shear stress ($\overline{\tau_{xz}}$) in homogenous isotropic plates under an uniform distributed load

υ	$\frac{a}{b}$	$\frac{a}{h}$	$\bar{w}(a/2, b/2, 0)$		$\overline{\tau_{xz}}(0, b/2, 0)$	
			Present analysis	Elasticity solution[a]	Present analysis	Elasticity solution[a]
0.3	0.2	20	8770.7200	8769.6000	14.7100	14.7960
		10	558.1750	558.0600	7.3130	7.3120
		7.14	148.3900	148.5800	5.1830	5.1700
	0.5	20	6855.0700	6855.0000	13.6150	13.6990
		10	437.5200	437.5200	6.7650	6.7920
		7.14	116.7300	116.9400	4.7920	4.7990
	1.0	20	2761.3100	2761.3000	9.8000	9.8330
		10	178.4460	178.4500	4.8500	4.8810
		7.14	48.4460	48.4010	3.4220	3.4340
	2.0	20	437.5200	437.5200	5.1900	5.3600
		10	29.6040	29.6040	2.5520	2.5890
		7.14	8.4400	8.45180	1.7850	1.7970
0.2	1.0	10	203.1500	203.1500	4.8460	4.8810
0.4	1.0	10	153.7600	153.7500	4.8460	4.8810

[a] Srinivas and Rao [12].

Table 5
Maximum stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xy}}, \overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$) and the transverse displacement ($\bar{w}$) of symmetric cross-ply ($0^0/90^0/0^0$) square laminated plates under bi-directional transverse sinusoidal load

s	Source	$\overline{\sigma_x}(\frac{a}{2}, \frac{b}{2}; \pm\frac{h}{2})$		$\overline{\sigma_y}(\frac{a}{2}, \frac{b}{2}; \pm\frac{h}{6})$		$\overline{\tau_{xy}}(0, 0, \pm\frac{h}{2})$		$\overline{\tau_{xz}}(0, \frac{b}{2}, 0)$		$\overline{\tau_{yz}}(\frac{a}{2}, 0, 0)$		$\bar{w}(\frac{a}{2}, \frac{b}{2}, 0)$
2	Present analysis	1.4360	−0.9370	0.6690	−0.7420	−0.0859	0.0702	0.1640	0.3090 (.33)	0.2590	0.2600 (.03)	5.0950
	Elasticity solution[a]	1.4360	−0.9380	0.6690	−0.7420	−0.0850	0.0700	0.1640	0.3090 (.33)	0.2590	0.2600 (.03)	–
	Mixed FE analysis[b]	–	−0.9760	–	–	–	0.0900	–	0.3350 (.33)	–	–	–
	Mixed FE analysis[c]	1.4600	−0.9540	0.6790	−0.7540	−0.0870	0.0710	0.1660	0.3110 (.33)	0.2600	0.2610 (.03)	5.1100
	HOST[d]	1.0910	–	0.6330	–	–	0.0803	–	–	–	–	5.2150
4	Present analysis	0.8010	−0.7550	0.5340	−0.5560	−0.0510	0.0505	0.2560	0.2820 (.27)	0.2170		2.0060
	Elasticity solution[a]	0.8010	−0.7550	0.5340	−0.5560	−0.0510 0.0500	0.2560	0.2820 (.27)	0.2170		–	
	Mixed FE analysis[b]	–	−0.7850	–	–	–	0.0540	–	0.3090 (.27)	–		–
	Mixed FE analysis[c]	0.8080	−0.7600	0.5380	−0.5600	−0.0510 0.0500	0.2570	0.2830 (.27)	0.2210		2.0070	
	HOST[d]	0.7670	–	0.5079 –	–	0.0500	–	–	–		1.9260	
10	Present analysis	±0.5900		0.2845	−0.2880	∓0.0290		0.3570		0.1230		0.7530
	Elasticity solution[a]	±0.5900		0.2850	−0.2880	∓0.0290		0.3570		0.1230		–
	Mixed FE analysis[b]	0.6100		–	–	0.0300		0.3820		–		–
	Mixed FE analysis[c]	±0.5940		0.2860	−0.2890	∓0.0290		0.3580		0.1240		0.8560
	HOST[d]	0.5850		0.2712 –	0.0281		–		–		0.7176	
20	Present analysis	±0.5520		±0.2100		∓0.0234		0.3850		0.0940		0.5164
	Elasticity solution[a]	±0.5520		±0.2100		∓0.0234		0.3850		0.0940		–
	Mixed FE analysis[c]	±0.5550		±0.2100		∓0.0230		0.3880		0.1010		0.5170
	HOST[d]	0.5507		0.2050		0.0231		–		–		0.5058

'–' Indicates results are not available.
Number within '( )' indicates position in the thickness dimension where stress is maximum.
[a] Pagano [13].
[b] Wu and Kuo [16].
[c] Ramtekkar et al. [21].
[d] Kant and Swaminathan [11].

A convergence study on number of steps required for numerical integration in the thickness direction of the laminate is performed first for all examples. It is observed in all examples that 20–30 steps are enough for converged solution. Details of the convergence studies are not presented here for the sake of brevity. Illustrative examples considered in the present work are discussed next.

Example 1. A homogenous isotropic plate with simple support end conditions (Table 1) on all four edges and subjected to an uniformly distributed load has been considered to show the ability of present formulation to handle different loadings. The convergence study on number of harmonics required to define the uniformly distributed load is performed along with the convergence study on number of steps for numerical integration and displacements as well as stresses were observed to converge after 17 harmonics. Normalized Young's modulus of elasticity for isotropic plate has been considered $E = 1$ GPa and Poisson's ratios are varied from 0.2 to 0.4 in steps of 0.1. The normalized transverse displacement ($\bar{w}$) and the transverse shear stress ($\overline{\tau_{xz}}$) obtained by the proposed approach for isotropic plate have been compared in Table 4 with elasticity solution given by Srinivas and Rao [12] for various a/b ratios as well as Poisson's ratios. The transverse displacement and the shear stress have been normalized by using relations $\bar{w} = wG/(hp_0)$ and $\overline{\tau_{xz}} = \tau_{xz}/p_0$, respectively for a consistent comparison with the data available in the literature. It is observed from Table 4 that the present approach predicts highly accurate values of the transverse deflection and the transverse shear stress for thin as well as thick isotropic plate.

Example 2. Various three-layered symmetric cross-ply ($0^0/90^0/0^0$) square laminates with aspect ratios, $s = 2$, 4, 10 and 20 and simple supports end condition on all four edges (Table 1) subjected to bidirectional sinusoidal load on its top surface are considered next for validation. Material properties are presented in Table 3. Results of the normalized maximum inplane normal stresses ($\overline{\sigma_x}$ and $\overline{\sigma_y}$), the inplane shear stress ($\overline{\tau_{xy}}$), the transverse shear stresses ($\overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$) and the transverse displacement ($\bar{w}$) are pre-

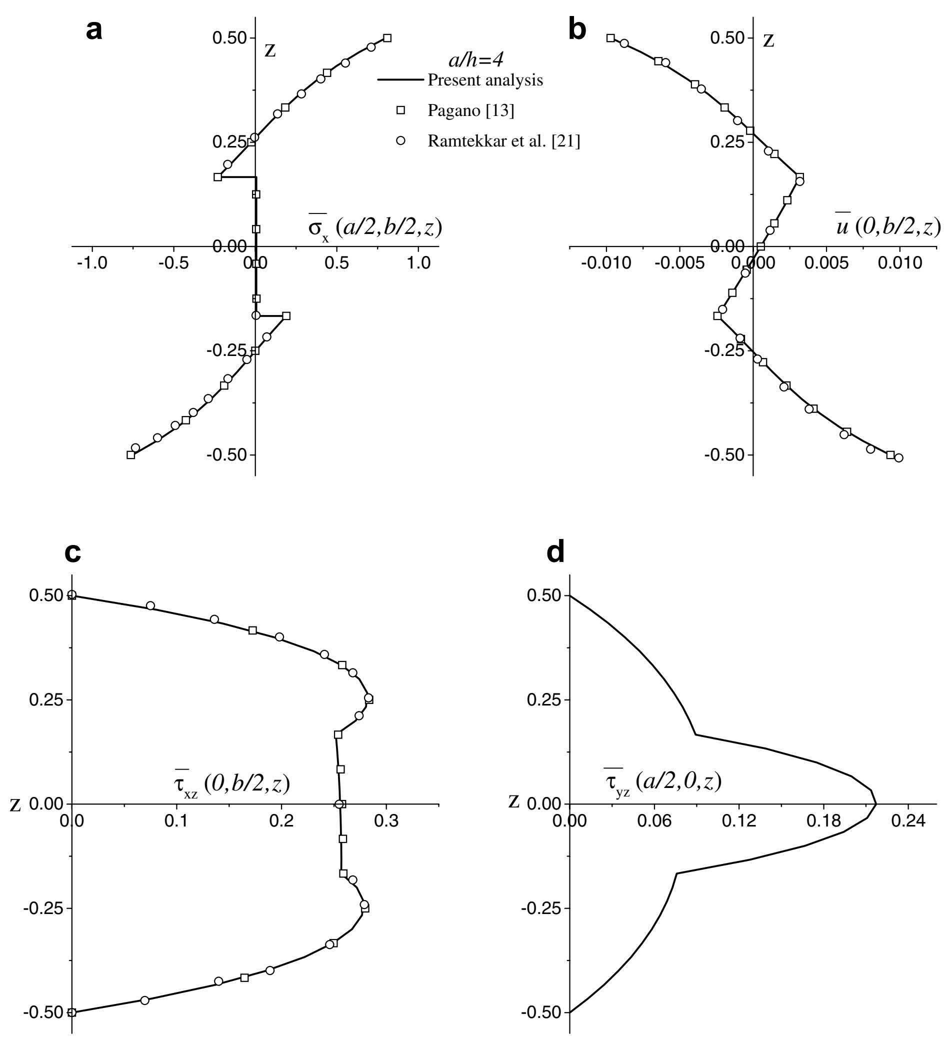

Fig. 2. Variation of normalized (a) inplane normal stress $\overline{\sigma_x}$ (b) inplane displacement $\bar{u}$ (c) transverse shear stress $\overline{\tau_{xz}}$ (d) transverse shear stress $\overline{\tau_{yz}}$ through thickness of a $0^0/90^0/0^0$ symmetric laminate subjected to bidirectional sinusoidal load.

sented in Table 5. Moreover, the variations of the inplane normal stress ($\overline{\sigma_x}$), the inplane displacement ($\bar{u}$) and transverse shear stresses ($\overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$) through the laminate thickness are shown in Fig. 2 for an aspect ratio of 4. Results have been compared with the 3D elasticity solution given by Pagano [13] and also with the HOST of Kant and Swaminathan [11] and the mixed FE solution by Wu and Kuo [16], Ramtekkar et al. [21]. This comparison clearly indicates that the present results are very close to the elasticity solution compared to those obtained by others and thus proves the superiority of the present formulation.

Example 3. A symmetric square sandwich laminate (0^0/core/0^0) with simple support end conditions on all four edges (Table 1) and subjected to bi-directional sinusoidal load on its top surface has been considered here. Material properties of the face sheets and core materials have been presented in Table 3. Thickness of the face sheets is one tenth of the total thickness of the sandwich plate. Results for aspect ratios, s=2, 4, 10 and 20, have been compared in Table 6 with elasticity solution given by Pagano [13] as well as analytical and FE solutions presented by others. Present results are seen to be closest to the elasticity solution. Through thickness variations of all the normalized stress components ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\sigma_z}, \overline{\tau_{xy}}, \overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$) and inplane and transverse displacements ($\bar{u}, \bar{w}$) for an aspect ratio of 4 have been presented in Figs. 3 and 4. Excellent agreement of the results with the elasticity solution suggests that problem with sudden change in material properties can be analyzed accurately by using the present approach.

Example 4. A two layered unsymmetric cross-ply ($0^0/90^0$) square laminate with equal thicknesses under bi-directional transverse load on its top surface is considered in this example with simple support boundary conditions (Table 1). Exact solution of this example is not available in the literature. Material properties are presented in Table 3. The normalized maximum stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xy}}, \overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$) have been presented in Table 7. Results with HOST presented by Kant and Swaminathan [11] and 3D analytical given by Vel and Batra [25] have been used for general comparison purpose. Fig. 5 shows the through thickness variations of inplane normal stress ($\overline{\sigma_x}$), inplane displacement ($\bar{u}$), transverse shear stress ($\overline{\tau_{xz}}$) and transverse displacement ($\bar{w}$) for an aspect ratio of 4. It is noted that, only for a thin laminate, results show good agreement with HOST.

Table 6
Maximum stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xz}}, \overline{\tau_{yz}}$ and $\overline{T_{xy}}$) of symmetric (0^0/core/0^0) square sandwich plates under bi-directional transverse sinusoidal load

s	Source	$\overline{\sigma_x}(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2})$		$\overline{\sigma_x}(\frac{a}{2},\frac{b}{2};\pm 0.4h)$		$\overline{\sigma_y}(\frac{a}{2},\frac{b}{2};\pm\frac{h}{6})$		$\overline{\tau_{xz}}(0,\frac{b}{2},0)$		$\overline{\tau_{yz}}(\frac{a}{2},0,0)$		$\overline{\tau_{xy}}(0,0,\pm\frac{h}{2})$	
2	Present analysis	3.2780	−2.6520	−2.2200	1.6680	0.4520	−0.3920	0.1850	0.3200 (.44)	0.1390	0.1400 (.08)	−0.2400	0.2340
	Elasticity analysis[a]	3.2780	−2.6530	−2.2200	1.6680	0.4520	−0.3920	0.1850	0.3200 (.44)	0.1390	0.1400 (.08)	−0.2400	0.2340
	Mixed FEM[d]	3.3250	−2.6840	−2.2320	1.6710	0.4560	−0.3960	0.1860	0.3230 (.44)	0.1420	0.1420 (.08)	−0.2430	0.2360
4	Present analysis	1.5560	−1.5120	−0.2330	0.1960	0.2590	−0.2590	0.2390		0.1070		−0.1440	0.1480
	Elasticity analysis[a]	1.5560	−1.5120	−0.2330	0.1960	0.2590	−0.2530	0.2390		0.1070		−0.1440	–
	FEM-HOST[b]	1.5230	–	−0.0120	–	0.2410	–	0.2750		–		−0.1420	–
	Mixed FEM[c]	1.5480 –	–	0.2410	–	0.2490		–		−0.1340		–	
	Mixed FEM[d]	1.5700	−1.5240	−0.2320	0.1940	0.2600	−0.2550	0.2370		0.1040		–	–
10	Present analysis	1.1530	−1.1520	0.6280	−0.6290	0.1100	−0.1100	0.3000		0.0527		−0.0707	0.0720
	Elasticity analysis[a]	1.1530	−1.1520	0.6280	−0.6290	0.1100	−0.1100	0.3000		0.0530		−0.071	0.0720
	FEM-HOST[b]	1.1660	–	0.6880	–	0.1050	–	0.3400		–		−0.0690	–
	Mixed FEM[c]	1.2100	–	0.6890	–	0.1110	–	0.3240		–		−0.0710	–
	Mixed FEM[d]	1.1590	−1.1580	−0.6330	0.6290	0.1110	−0.1100	0.3030		0.0550		−0.0710	0.0720
20	Present analysis	±1.1100		±0.8100		±0.0700		0.3170		0.0360		∓0.0510	
	Elasticity analysis[a]	±1.1100		±0.8100		±0.0700		0.3170		0.0360		∓0.0510	
	Mixed FEM[c]	1.1730		0.8610		0.0720		0.3530		–		0.0520	
	Mixed FEM[d]	±1.1150		±0.8150		±0.070		0.3170		0.0360		∓0.0510	

'–' Indicates results are not available.
Number within '( )' indicates position in the thickness dimension where stress is maximum.
[a] Pagano [13].
[b] Pandya and Kant [9].
[c] Wu and Kuo [16].
[d] Ramtekkar et al. [21].

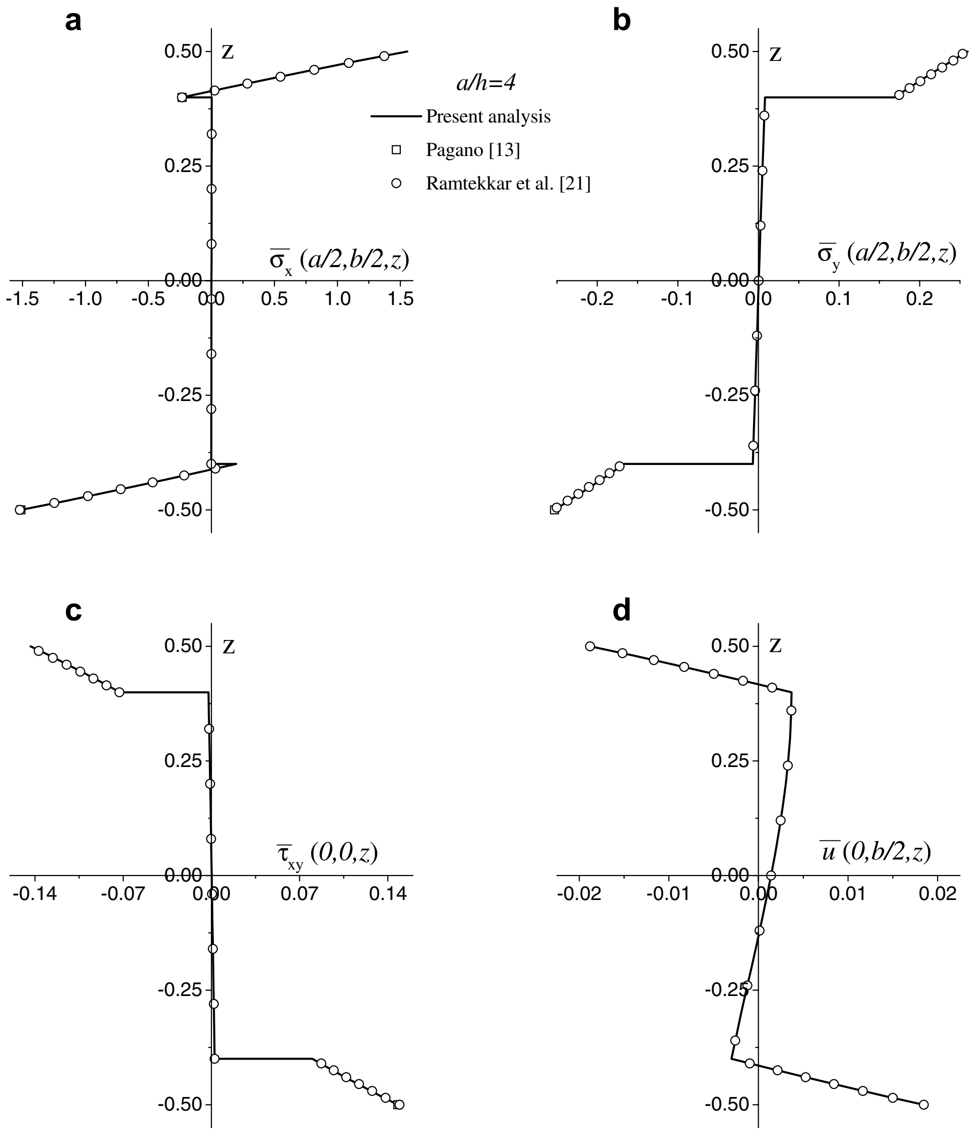

Fig. 3. Variation of normalized (a) inplane normal stress $\overline{\sigma_x}$ (b) inplane normal stress $\overline{\sigma_y}$ (c) inplane shear stress $\overline{\tau_{xy}}$ (d) inplane displacement $\bar{u}$ through thickness of a 0^0/core/0^0 symmetric sandwich subjected to bidirectional sinusoidal load.

4. Concluding remarks

A novel, semi-analytical methodology with mixed variables (displacements and transverse stresses) for static analysis of composite and sandwich laminates under transversely distributed load has been presented in this paper. A two point BVP governed by a set of linear coupled first-order ODEs is formed by assuming all primary variables in the form of trigonometric functions along the inplane directions. The solution ensures the fundamental elasticity relationship between stress, strain and displacement fields within the elastic continuum and implicitly maintains the continuity of displacements and transverse stresses at the laminae interfaces. It is shown through numerical investigation that results obtained by present approach are highly accurate. Since loading term is expanded in the form of a Fourier series, any system of loading can be handled with this formulation. Another important feature of this approach is that both the displacements and the stresses are computed simultaneously with the same degree of accuracy.

Appendix A. Coefficients of [C] matrix

$$C_{11} = \frac{E_1(1-\nu_{23}\nu_{32})}{\Delta}; \quad C_{12} = \frac{E_1(\nu_{21}+\nu_{31}\nu_{23})}{\Delta};$$

$$C_{13} = \frac{E_1(\nu_{31}+\nu_{21}\nu_{32})}{\Delta}; \quad C_{22} = \frac{E_2(1-\nu_{13}\nu_{31})}{\Delta};$$

$$C_{23} = \frac{E_2(\nu_{32}+\nu_{12}\nu_{31})}{\Delta}; \quad C_{33} = \frac{E_3(1-\nu_{12}\nu_{21})}{\Delta};$$

$$C_{44} = G_{12}; \quad C_{55} = G_{13}; \quad C_{66} = G_{23},$$

where

$$\Delta = (1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{12}\nu_{23}\nu_{31}).$$

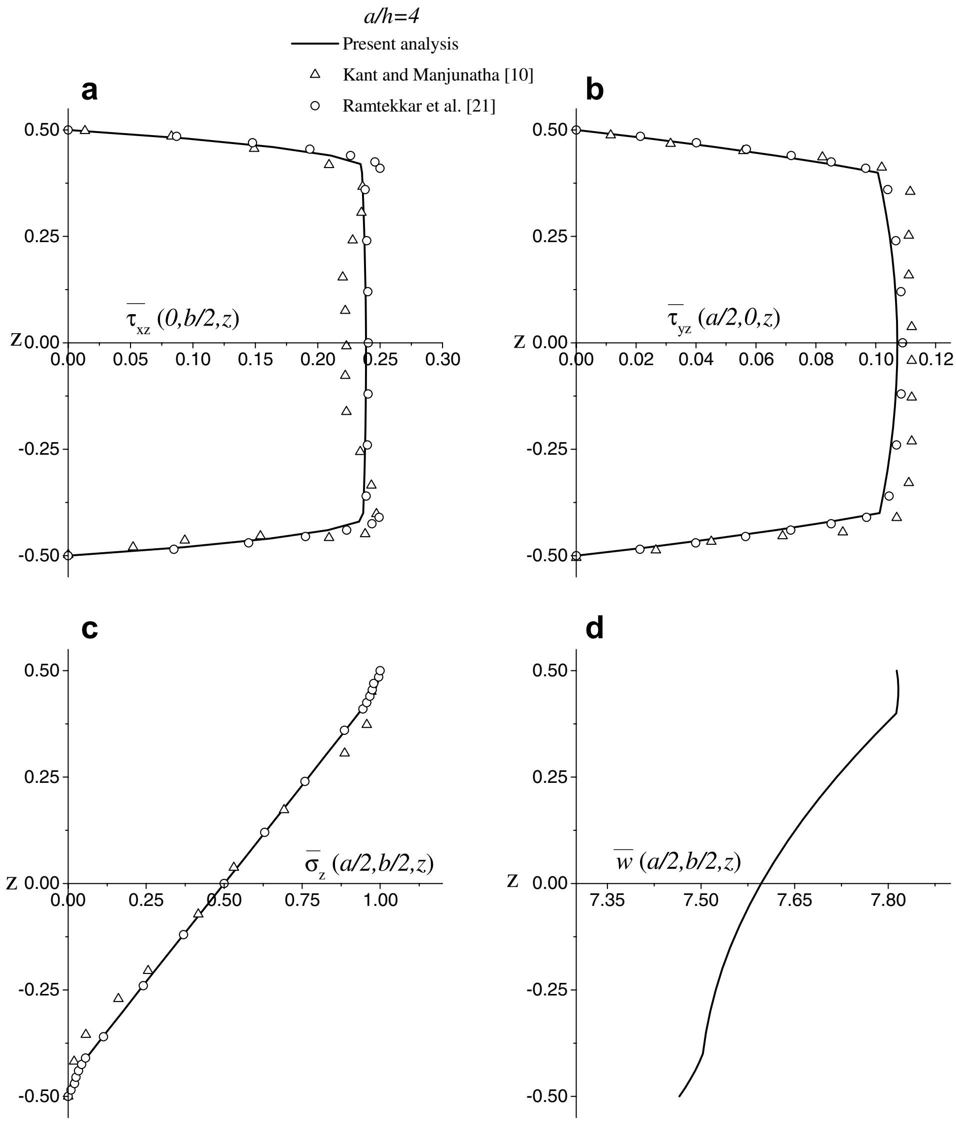

Fig. 4. Variation of normalized (a) transverse shear stress $\overline{\tau_{zx}}$ (b) transverse shear stress $\overline{\tau_{yz}}$ (c) transverse normal stress $\overline{\sigma_z}$ (d) transverse displacement $\bar{w}$ through thickness of a 0^0/core/0^0 symmetric sandwich subjected to bidirectional sinusoidal load.

Table 7
Maximum stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xz}}, \overline{\tau_{yz}}$ and $\overline{T_{xy}}$) of symmetric cross-ply ($0^0/90^0$) square laminated plates under bi-directional transverse sinusoidal load

s	Source	$\overline{\sigma_x}(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2})$		$\overline{\sigma_y}(\frac{a}{2},\frac{b}{2};\pm\frac{h}{6})$		$\overline{\tau_{xz}}(0,\frac{b}{2},0)$		$\overline{\tau_{yz}}(\frac{a}{2},0,0)$		$\overline{\tau_{xy}}(0,0,\pm\frac{h}{2})$	
2	Present analysis	0.1830	−0.8880	1.4040	−0.1110	0.1490	0.2570 (−.187)	0.0668	0.0673 (.28)	−0.0791	0.0681
	HOST[a]	–	−0.8270	1.1946	–	–	–	–	–	−0.0729	–
5	Present analysis	0.1010	−0.7670	0.7900	−0.0920	0.1316	0.3220 (−.218)	0.1211	0.3240 (.28)	−0.0566	0.0570
	HOST[a]	–	−0.7510	0.7720	–	–	–	–	–	−0.0557	–
	3D Analytical[b]	–	−0.7671	0.7894 –	–	−0.1211	–	–	–		
10	Present analysis	0.0890	−0.7300	0.7310	−0.0865	0.1250	0.3330 (−.218)	0.1220	0.3320 (.28)	−0.0536	0.0537
	HOST[a]	–	−0.7270	0.7270	–	–	–	–	–	−0.0533	–
	3D Analytical[b]	—	−0.7304	0.7309	–	–	–	0.1220	–	–	–
20	Present analysis	0.0854	−0.7200	0.7200	−0.0849	0.1230	0.3350 (−.218)	0.1220	0.3340 (.28)	−0.0528	0.0528
	HOST[a]	–	−0.7190	0.7190	–	–	–	–	–	−0.0527	–

'–' Indicates results are not available.
Number within '( )' indicates position in the thickness dimension where stress is maximum.
[a] Kant and Swaminathan [11].
[b] Vel and Batra [25].

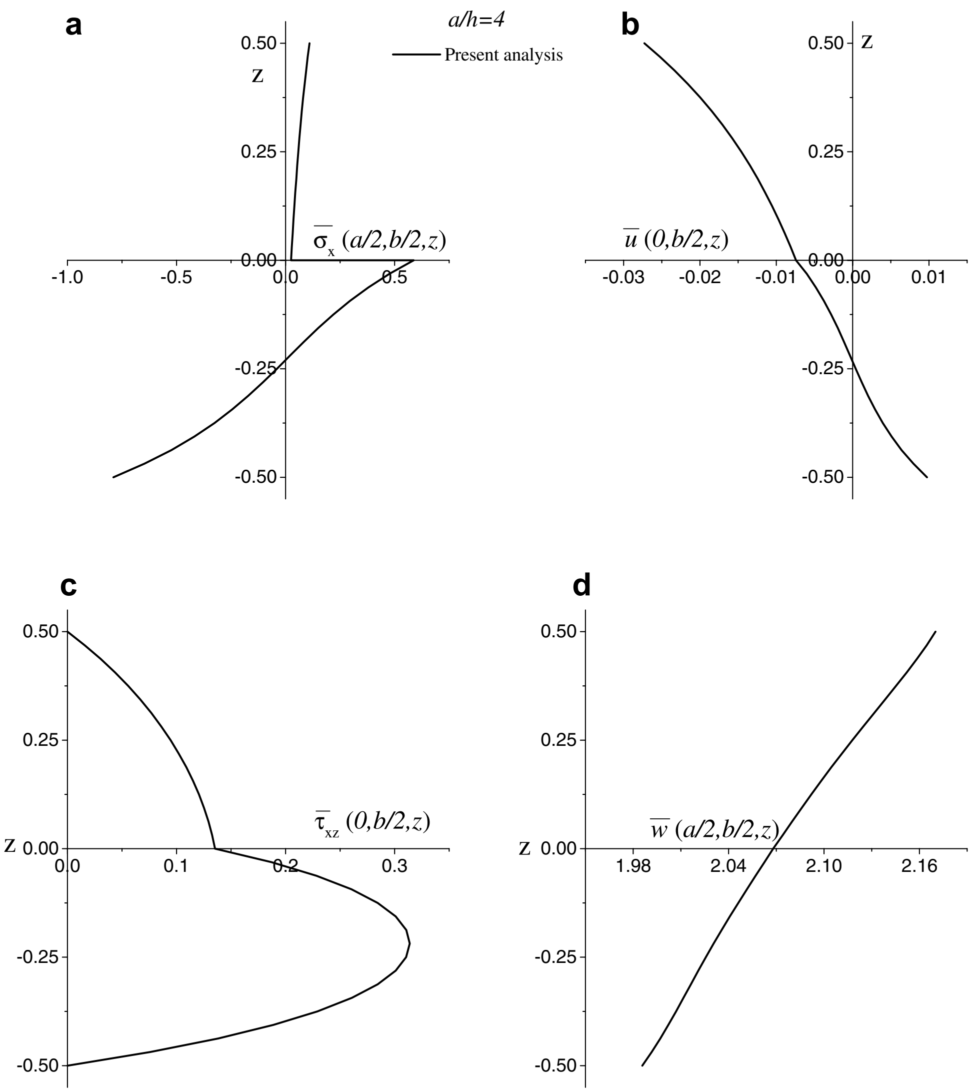

Fig. 5. Variation of normalized (a) inplane normal stress $\overline{\sigma_x}$ (b) inplane displacement $\bar{u}$ (c) transverse shear stress $\overline{\tau_{xz}}$ (d) transverse displacement $\bar{w}$ through thickness of a $0^0/90^0$ unsymmetric laminate subjected to bidirectional sinusoidal load.

Appendix B. Coefficients of [Q] matrix

$$Q_{11} = C_{11}c^4 + 2(C_{12} + 2C_{44})c^2s^2 + C_{22}s^4,$$
$$Q_{12} = C_{12}(c^4 + s^4) + (C_{11} + C_{22} - 4C_{44})c^2s^2,$$
$$Q_{13} = C_{13}c^2 + C_{23}s^2,$$
$$Q_{14} = (C_{11} - C_{12} - 2C_{44})c^3s + (C_{12} - C_{22} + 2C_{44})cs^3,$$
$$Q_{22} = C_{22}c^4 + 2(C_{12} + 2C_{44})c^2s^2 + C_{11}s^4,$$
$$Q_{23} = C_{23}c^2 + C_{13}s^2,$$
$$Q_{24} = (C_{12} - C_{22} + 2C_{44})c^3s + (C_{11} - C_{12} - 2C_{44})cs^3,$$
$$Q_{33} = C_{33},$$
$$Q_{34} = (C_{31} - C_{32})cs,$$
$$Q_{44} = (C_{11} - 2C_{12} + C_{22} - 2C_{44})c^2s^2 + C_{44}(c^4 + s^4),$$
$$Q_{55} = C_{55}c^2 + C_{66}s^2,$$
$$Q_{56} = (C_{55} - C_{66})cs,$$
$$Q_{66} = C_{55}s^2 + C_{66}c^2.$$

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