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Static solutions for functionally graded simply supported plates

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Abstract In this article mixed semi-analytical and analytical solutions are presented for a rectangular plate made of functionally graded (FG) material. All edges of a plate are under simply supported (diaphragm) end conditions and general stress boundary conditions can be applied on both top and bottom surface of a plate during solution. A mixed semi-analytical model consists in defining a two-point boundary value problem governed by a set of first-order ordinary differential equations in the plate thickness direction. Analytical solutions based on shear-normal deformation theories are also established to show the accuracy, simplicity and effectiveness of mixed semi-analytical model. The FG material is assumed to be exponential in the thickness direction and Poisson's ratio is assumed to be constant.

Keywords FG plates · Mixed semi analytical method · ODE · PDE · BVP

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1 Introduction

A functionally graded (FG) material is a micro-engineered composites where the composite and structures are continuously varied in thickness direction of a structural component so that an optimum distribution of properties can be obtained depending on the functional requirements and are therefore, free from interface weaknesses typically consists in laminated composites and sandwiches. The great feature of this particular material is elimination or reduction of delamination failure associated with the traditional laminated components at their interfaces where a material property varies suddenly. FG material offers many advantages over the monolithic material and traditional laminated composites and sandwiches, and therefore, extensive use of FG material has been seen in the field of aerospace and nuclear technologies. However, directional compositional variation of the constituents within FG materials makes analysis and design more challenging than traditional materials.

Over the years, a number of approaches/models have been developed and presented to analysis of structural components with FG material under transverse/thermal/electric loads. A comprehensive review on FG materials can be found in Tanigawa (1995). Three dimensional (3D) elasticity solutions based on the solution of partial differential equations (PDEs) with appropriate boundary conditions are valuable because they represent a more realistic and closer approximation to the actual behaviour of the structures. Sankar (2001) has presented two dimensional (2D) elasticity solution under plane strain conditions for simply supported FG beams subjected to sinusoidal load by assuming exponential variation of Young's modulus through the thickness of beam. With the same assumption, a mixed semi-analytical solution along with analytical solutions based on shear deformation theories for simply supported FG beams under plane stress condition of elasticity have been presented by Pendhari et al. (2010). Sankar and Tzeng (2002) extended the elasticity solutions for a FG beams subjected to thermal loads.

Kashtalyan (2004) developed exact 3D elasticity solutions for simply supported FG plate subjected to transverse loading by assuming exponential variation of Young's modulus through the thickness of plate. By assuming power-law variation of the volume fractions of the constituents through the thickness of simply supported FG plate, exact solutions are presented by Vel and Batra (2002) for mechanical and thermal loading. Governing PDE for thermo-mechanical deformation is reduced to a set of coupled ordinary differential equations (ODEs) in the thickness coordinate, which is then solved by using the power series method. Anderson (2003) solved sandwich composites with FG core components exactly for circular patch loading. Parametric studies on different degrees of core stiffness at the face sheet interface are also presented.

The shear stiffness and shear correction factors associated with first order shear deformation theories (FOSTs) were calculated by Nguyen et al. (2008) for FG simply supported plates under cylindrical bending. Matsunaga (2009) developed higher order shear deformation theories (HOSTs) for displacements and stresses in FG simply supported plates subjected to thermal and mechanical loads. Khabbaz et al. (2009) used energy concept along with first and HOSTs to evaluate large deformation and through thickness stresses of FG plates. Kang and Li (2009) presented non-linear behaviour of cantilever beam subjected to end force by using large and small deformation theories. Analytical solutions for piezoelectric FG half-spaces under uniform circular surface loading are presented by Han et al. (2006). The effect of different exponential factors of the FG materials on the field response is demonstrated in detail.

Woo and Meguid (2001) developed series solutions for large deflections of FG plates under transverse loading and temperature fields using Von-Karman theory. The material properties of FG materials are assumed to vary according to power-law distribution of the volume fraction of the constituents through the thickness. Further, analytical solution is presented by Woo et al. (2005) for post-buckling analysis of moderately thick FG plates and shells under edge compression loads and a temperature field. Bodaghi and Saidi (2010) presented analytical approach based on a HOST to determine critical buckling loads of thick FG rectangular plates.

Praveen and Reddy (1998) developed finite element (FE) model for static and dynamic analysis of FG ceramic–metal plates with Von-Karman type nonlinearity. Power-law dependence of material properties in the thickness direction of plate is assumed. Reddy and Chin (1998) derived boundary value problem (BVP) by using FOST to study the dynamic thermoelastic response of FG cylinders and plates. The presented formulation accounting for the coupling with 3D heat conduction equation for a FG plate. Further, Reddy (2000) extended same formulation for third order shear deformation plate theory. With the help of classical plate theory (CPT), Shen (2002) and Yang and Shen (2003) studied large deflection and post buckling response of FG plates with temperature dependent material properties. Chakraborty and Gopalakrishnan (2003) have developed a FE model based on the FOST. Ma and Wang (2004) presented a relationship between the third order shear deformation solutions of axisymmetric bending and buckling of FG circular plate with isotropic circular plates based on CPT. GhannadPour and Alinia (2006) studied the large deflection behaviour of FG plate with power-law distribution of the volume fraction of constituents by using CPT. The fundamental equations for rectangular FG plate are obtained using Von-Karman theory and solution is obtained by minimization of the potential energy (PE).

In the present article, mixed semi-analytical model developed by Kant et al. (2008) has been reformulated for 3D stress analysis of simply supported FG plates under transverse loads. The model use of the formation of two-point BVP governed by coupled first-order ODEs along the thickness coordinate of a plate (Kantrovich and Krylov 1958). In addition to this, analytical solutions based on higher order shear-normal deformation theory (HOSNT) are also developed and presented.

2 Formulations

A FG plate (Fig. 1), simply supported on all its four edges is considered. A right-handed orthogonal co-ordinate system (x, y, and z) is chosen such that the plate occupies a domain Ω in the x–y plane and z-axis is normal to the plane. The top surface of a plate is loaded only with transversely distributed load and it can be expressed as,

$$p(x,y) = \sum_{mn} p_{0mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{1}$$

where, m and $n = 1,3,5,\ldots\ldots$ and other surfaces are free from any stresses. The 3D equations of equilibrium are,

$$\begin{aligned} &\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + B_x = 0 \\ &\frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + B_y = 0 \\ &\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + B_z = 0 \end{aligned} \tag{2}$$

where, B_x, B_y and B_z are the body forces per unit volume in x, y and z directions, respectively.

It is assumed that the Poisson's ratio is constant through the thickness and variation of Young's modulus through the plate thickness is given by $E(z) = E_o e^{\lambda z}$ ($\lambda =$ gradation factor). Further, it is assumed here that FG material is isotropic at every point. Therefore, constitutive relations for FG plate can be written as,

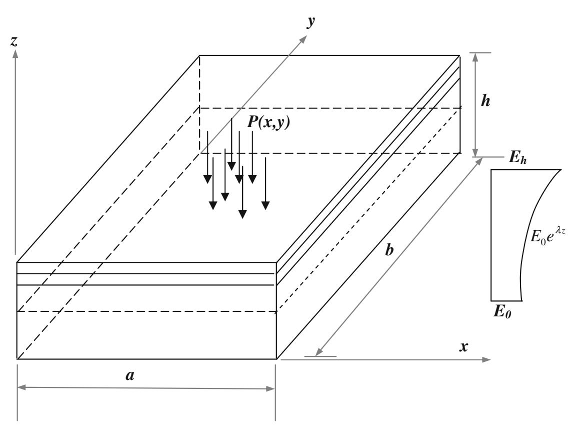

Fig. 1 3D plate domain subjected to transverse loads

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & symmetric & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

Or

$$\underset{\sim}{\sigma} = C_{ij}\, \underset{\sim}{\varepsilon} \tag{3}$$

where,

$$C_{11} = C_{22} = C_{33} = \frac{E_0 e^{\lambda z}(1-\upsilon^2)}{(1-3\upsilon^2-2\upsilon^3)}$$

$$C_{12} = C_{13} = C_{23} = \frac{E_0 e^{\lambda z}(\upsilon+\upsilon^2)}{(1-3\upsilon^2-2\upsilon^3)}$$

$$C_{44} = C_{55} = C_{66} = \frac{E_0 e^{\lambda z}}{2(1+\upsilon)}$$

and $\lambda = -\ln\frac{E_0}{E_h}$ is the Gradation factor, E_o is the Young's modulus at the bottom of the beam, E_h is the Young's modulus at the top of the beam, υ is the Poison's ratio and, general 3D linear strain–displacement relations are,

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ &\varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ &\varepsilon_z = \frac{\partial w}{\partial z} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned} \tag{4}$$

2.1 Mixed semi analytical model

An attempt is made in this section to extend the semi analytical model developed by Kant et al. (2008) for stress analysis of FG plate under transverse loads. The models is based on the formation of two-point BVP governed by a set of first-order ODEs,

$$\frac{\mathrm{d}}{\mathrm{dz}}\mathbf{y}(\mathrm{z}) = \mathbf{A}(\mathrm{z})\mathbf{y}(\mathrm{z}) + \mathbf{p}(\mathrm{z}) \tag{5}$$

in the domain $0<z<h$ with any half of the primary unknowns prescribed at the top and bottom surface of a plate. In Eq. 5, $\mathbf{y}(z)$ is an n-dimensional vector of fundamental variables whose number (n) equals the order of PDE, $\mathbf{A}(z)$ is a $n \times n$ coefficient matrix (which is a function of material properties in thickness direction) and $\mathbf{p}(z)$ is a n-dimensional vector of non-homogenous (loading) terms.

The Eqs. 2–4 have a total of fifteen unknowns $u, v, w, \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}, \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}$ and τ_{yz} in fifteen equations. After a simple algebraic manipulation of the above sets of equations, a set of PDEs involving only six primary dependent variables $u, v, w, \tau_{xz}, \tau_{yz}$ and σ_z are obtained as follows,

$$
\begin{aligned}
&\frac{\partial u}{\partial z}=-\frac{\partial w}{\partial x}+\left(\frac{C_{66}}{C_{55}C_{66}}\right)\tau_{xz}\\
&\frac{\partial v}{\partial z}=-\frac{\partial w}{\partial y}+\left(\frac{C_{55}}{C_{55}C_{66}}\right)\tau_{yz}\\
&\frac{\partial w}{\partial z}=\frac{\sigma_z}{C_{33}}-\frac{1}{C_{33}}\left(C_{31}\frac{\partial u}{\partial x}+C_{32}\frac{\partial v}{\partial y}\right)\\
&\frac{\partial \tau_{xz}}{\partial z}=-\left(C_{11}-\frac{C_{13}C_{31}}{C_{33}}\right)\frac{\partial^2 u}{\partial x^2}\\
&-C_{44}\frac{\partial^2 u}{\partial y^2}-\left(C_{12}+C_{44}-\frac{C_{13}C_{32}}{C_{33}}\right)\frac{\partial^2 v}{\partial x\partial y}-\frac{C_{13}\partial\sigma_z}{C_{33}\,\partial x}-B_x\\
&\frac{\partial \tau_{yz}}{\partial z}=-\left(C_{21}+C_{44}-\frac{C_{23}C_{31}}{C_{33}}\right)\frac{\partial^2 u}{\partial x\partial y}\\
&-C_{44}\frac{\partial^2 u}{\partial x^2}-\left(C_{22}-\frac{C_{23}C_{32}}{C_{33}}\right)\frac{\partial^2 v}{\partial y^2}-\frac{C_{23}\partial\sigma_z}{C_{33}\,\partial y}-B_y\\
&\frac{\partial \sigma_z}{\partial z}=-\frac{\partial \tau_{xz}}{\partial x}-\frac{\partial \tau_{yz}}{\partial y}-B_z
\end{aligned}
\tag{6}
$$

The above PDEs defined by Eq. 6 can be further reduced to a coupled first-order ODEs by using double Fourier trigonometry series expansion for primary displacement variables satisfying the simply support end conditions on all four edges.

$$
\begin{aligned}
u(x,y,z) &= \sum_{mn} u_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
v(x,y,z) &= \sum_{mn} v_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\
w(x,y,z) &= \sum_{mn} w_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned}
\tag{7}
$$

and from the basic relations of theory of elasticity, it can be shown that,

$$
\begin{aligned}
\tau_{xz}(x,y,z) &= \sum_{mn} \tau_{xzmn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\tau_{yz}(x,y,z) &= \sum_{mn} \tau_{yzmn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\
\sigma_z(x,y,z) &= \sum_{mn} \sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned}
\tag{8}
$$

Substituting Eqs. 7–8 and its derivatives into Eq. 6 and noting orthogonality conditions of trigonometric functions, the following ODEs are obtained,

$$
\begin{aligned}
&\frac{du_{mn}(z)}{dz}=-\frac{m\pi}{a}w_{mn}(z)+\frac{1}{C_{55}}\tau_{xzmn}(z)\\
&\frac{dv_{mn}(z)}{dz}=-\frac{n\pi}{b}w_{mn}(z)+\frac{1}{C_{66}}\tau_{yzmn}(z)\\
&\frac{dw_{mn}(z)}{dz}=\frac{C_{31}}{C_{33}}\frac{m\pi}{a}u_{mn}(z)+\frac{C_{32}}{C_{33}}\frac{n\pi}{b}v_{mn}(z)\\
&\quad+\frac{1}{C_{33}}\sigma_{zmn}(z)\\
&\frac{d\tau_{xzmn}(z)}{dz}=\left(C_{11}-\frac{C_{13}C_{31}}{C_{33}}\right)\frac{m^2\pi^2}{a^2}u_{mn}(z)\\
&\quad+C_{44}\frac{n^2\pi^2}{b^2}u_{mn}(z)+\left(C_{12}+C_{44}-\frac{C_{13}C_{32}}{C_{33}}\right)\frac{mn\pi^2}{ab}v_{mn}(z)\\
&\quad-\frac{C_{13}}{C_{33}}\frac{m\pi}{a}\sigma_{zmn}(z)-B_x(x,y,z)\\
&\frac{d\tau_{yzmn}(z)}{dz}=\left(C_{21}+C_{44}-\frac{C_{23}C_{31}}{C_{33}}\right)\frac{mn\pi^2}{ab}u_{mn}(z)\\
&\quad+C_{44}\frac{m^2\pi^2}{a^2}v_{mn}(z)+\left(C_{22}-\frac{C_{23}C_{32}}{C_{33}}\right)\frac{n^2\pi^2}{b^2}v_{mn}(z)\\
&\quad-\frac{C_{23}}{C_{33}}\frac{n\pi}{b}\sigma_{zmn}(z)-B_y(x,y,z)\\
&\frac{d\sigma_{zmn}(z)}{dz}=\frac{m\pi}{a}\tau_{xzmn}(z)+\frac{n\pi}{b}\tau_{yzmn}(z)-B_z(x,y,z)
\end{aligned}
\tag{9}
$$

Further, the secondary variables σ_x, σ_y and τ_{xy} can be expressed as a function of the primary set of variables as follows,

$$
\begin{aligned}
\sigma_x &=\left(C_{11}-\frac{C_{13}C_{31}}{C_{33}}\right)\sum_{mn}u_{mn}(z)\left[\frac{-m\pi}{a}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&\quad+\left(C_{12}-\frac{C_{13}C_{32}}{C_{33}}\right)\sum_{mn}v_{mn}(z)\left[\frac{-n\pi}{b}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&\quad+\frac{C_{13}}{C_{33}}\sum_{mn}\sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\sigma_y &=\left(C_{21}-\frac{C_{23}C_{31}}{C_{33}}\right)\sum_{mn}u_{mn}(z)\left[\frac{-m\pi}{a}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&\quad+\left(C_{22}-\frac{C_{23}C_{32}}{C_{33}}\right)\sum_{mn}v_{mn}(z)\left[\frac{-n\pi}{b}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&\quad+\frac{C_{23}}{C_{33}}\sum_{mn}\sigma_{zmn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\tau_{xy} &=C_{44}\sum_{mn}u_{mn}(z)\left[\frac{n\pi}{b}\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\
&\quad+C_{44}\sum_{mn}v_{mn}(z)\left[\frac{m\pi}{a}\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\end{aligned}
\tag{10}
$$

Table 1 Transformation of a BVP into IVPs

Intg.	Starting edge; $z = 0$						Final edge; $z = h$						Load term
	u	v	w	τ_{xz}	τ_{yz}	σ_z	u	v	w	τ_{xz}	τ_{yz}	σ_z	
1	0 (assumed)	0 (assumed)	0 (assumed)	0 (known)	0 (known)	0 (known)	Y_{11}	Y_{21}	Y_{31}	Y_{41}	Y_{51}	Y_{61}	Include
2	1 (unity)	0 (assumed)	0 (assumed)	0	0	0	Y_{12}	Y_{22}	Y_{32}	Y_{42}	Y_{52}	Y_{62}	Delete
3	0 (assumed)	1 (unity)	0 (assumed)	0	0	0	Y_{13}	Y_{23}	Y_{33}	Y_{43}	Y_{53}	Y_{63}	Delete
4	0 (assumed)	0 (assumed)	1 (unity)	0	0	0	Y_{14}	Y_{24}	Y_{34}	Y_{44}	Y_{54}	Y_{64}	Delete
Final	X_1	X_2	X_3	Known	Known	Known	u_T	v_T	w_T	0	0	$p\ (x,\ y)$	Include

The Eq. 9, defines the governing two-point BVP in ODEs through thickness of the laminate in the domain $0 < z < h$ with stress components known at the top and bottom faces. The basic approach to the numerical integration of the BVP defined in Eq. 9 and the associated boundary conditions *when it contains no boundary layer effects,* is to transform the given BVP into a set of IVPs—one non-homogeneous and *n/2* homogeneous. The solution of BVP defined by Eq. 9 is then obtained by forming a linear combination of one non-homogeneous and *n/2* homogeneous solutions so as to satisfy the boundary conditions at $z = h$ (Kant and Ramesh 1981). This gives rise to a system of *n/2* linear algebraic equations, the solutions of which determines the unknown *n/2* components, X_1, X_2 and X_3 (Table 1) at the starting edge $z = 0$. Then a final numerical integration of Eq. 9 produces the desired results. Numbers of successful and well-tested numerical algorithms are available in literature for solution of IVPs expressed by ODEs. Displacement and stress boundary conditions on all four edges of a 3D FG plate are detailed in Table 2.

2.2 Analytical models based on HOSNT

In order to reduce 3D elasticity problem to a 2D plate problem, the displacement components in all three direction of rectangular Cartension co-ordinate system $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$ at any point in the plate domain are expanded in Taylor's series in terms of thickness co-ordinate. The higher order displacement fields considered here in the formulation are,

$$\begin{aligned} u(x,\ y,\ z) &= u_o(x,\ y) + z\theta_x(x,\ y) + z^2 u_o^*(x,\ y) + z^3\theta_x^*(x,\ y)\\ v(x,\ y,\ z) &= v_o(x,\ y) + z\theta_y(x,\ y) + z^2 v_o^*(x,\ y) + z^3\theta_y^*(x,\ y)\\ w(x,\ y,\ z) &= w_o(x,\ y) + z\theta_z(x,\ y) + z^2 w_o^*(x,\ y) + z^3\theta_z^*(x,\ y) \end{aligned} \tag{11}$$

Table 2 Boundary conditions (BCs)

Edge	BCs on displacement field	BCs on stress field
$x = 0,\ a$	$w = 0$	$\sigma_x = 0$
$x = a/2$	$u = 0$	$\tau_{xz} = 0$
$x = 0,\ b$	$w = 0$	$\sigma_y = 0$
$x = b/2$	$v = 0$	$\tau_{yz} = 0$
$z = 0$	–	$\sigma_z = 0;\ \tau_{xz} = 0;\ \tau_{yz} = 0$
$z = h$	–	$\sigma_z = p_{0mn};\ \tau_{xz} = 0;\ \tau_{yz} = 0$

In the above relations, the terms u, v and w are the displacement at a general point (x, y, z) in the plate domain in the x, y and z directions, respectively. The parameters u_o, v_o are the inplane displacement and w_o is the transverse displacement at a point (x, y) on reference plane. The functions θ_x, θ_y are the rotations of the normal to the reference plane about y and x axes, respectively. The parameters u_0^*, v_0^*, w_o^*, θ_x^*, θ_y^* and θ_z^* are the higher order terms in Taylor's series expansion and they represent higher order transverse cross sectional deformation modes. The terms u_0^* and v_0^* contribute to higher order inplane modes of deformation while the terms θ_x^* and θ_y^* contribute to higher order flexural modes of deformation. The terms w_o^* and θ_z^* defines the non-linear variation of the transverse strain ε_z.

By substitution of the displacement relations given by Eq. 11 into the strain–displacement Eq. 4 of the classical theory of elasticity, the following relations are obtained.

$$\begin{aligned}
\varepsilon_x &= \varepsilon_{xo} + z\chi_x + z^2\varepsilon_{xo}^* + z^3\chi_x^* \\
\varepsilon_y &= \varepsilon_{yo} + z\chi_y + z^2\varepsilon_{yo}^* + z^3\chi_y^* \\
\varepsilon_z &= \varepsilon_{zo} + z\chi_z + z^2\varepsilon_{zo}^* \\
\gamma_{xy} &= \varepsilon_{xyo} + z\chi_{xy} + z^2\varepsilon_{xyo}^* + z^3\chi_{xy}^* \\
\gamma_{xz} &= \phi_x + z\chi_{xz} + z^2\phi_x^* + z^3\chi_{xz}^* \\
\gamma_{yz} &= \phi_y + z\chi_{yz} + z^2\phi_y^* + z^3\chi_{yz}^*
\end{aligned} \tag{12}$$

where,

$$\begin{aligned}
(\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{xyo}) &= \left(\frac{\partial u_o}{\partial x}, \frac{\partial v_o}{\partial y}, \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) \\
\left(\varepsilon_{xo}^*, \varepsilon_{yo}^*, \varepsilon_{xyo}^*\right) &= \left(\frac{\partial u_o^*}{\partial x}, \frac{\partial v_o^*}{\partial y}, \frac{\partial u_o^*}{\partial y} + \frac{\partial v_o^*}{\partial x}\right) \\
(\varepsilon_{zo}) &= (\theta_z) \\
\left(\varepsilon_{zo}^*\right) &= \left(3\theta_z^*\right) \\
(\chi_x, \chi_y, \chi_{xy}) &= \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \\
\left(\chi_x^*, \chi_y^*, \chi_{xy}^*\right) &= \left(\frac{\partial \theta_x^*}{\partial x}, \frac{\partial \theta_y^*}{\partial y}, \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) \\
(\chi_z, \chi_{xz}, \chi_{yz}) &= \left(2w_o^*, 2u_o^* + \frac{\partial \theta_z}{\partial x}, 2v_o^* + \frac{\partial \theta_z}{\partial y}\right) \\
\left(\chi_{xz}^*, \chi_{yz}^*\right) &= \left(\frac{\partial \theta_z^*}{\partial x}, \frac{\partial \theta_z^*}{\partial y}\right) \\
(\phi_x, \phi_y) &= \left(\theta_x + \frac{\partial w_o}{\partial x}, \theta_y + \frac{\partial w_o}{\partial y}\right) \\
\left(\phi_x^*, \phi_y^*\right) &= \left(3\theta_x^* + \frac{\partial w_o^*}{\partial x}, 3\theta_y^* + \frac{\partial w_o^*}{\partial y}\right)
\end{aligned}$$

2.2.1 Stress resultant

It is convenient to integrate the stress distribution through the thickness of the FG plate to replace the usual consideration of stresses by considering stress-resultants due to which the variations with respect to 'z' direction are completely eliminated. Here, the membrane, flexure and shear stress resultants of FG plate are derived as a function of the reference plane stretching, curvature and shear rotation strain terms, respectively.

The total PE $\prod$ of the FG plate with volume V, reference surface A can be written as: $\prod = U - W$

$$\prod = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^t \underset{\sim}{\sigma}\, dV - \int_A \underset{\sim}{\delta}^t \underset{\sim}{p}\, dA \tag{13}$$

where, U is the strain energy stored in the plate, W represents the work-done by externally applied loads and p is the vector of surface load intensities corresponding to the generalized displacement vector δ defined at the reference plane and these can be expressed as,

$$\begin{aligned}
\underset{\sim}{\sigma} &= (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})^t \\
\underset{\sim}{\varepsilon} &= (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})^t \\
\underset{\sim}{\delta} &= (u, v, w)^t \\
\underset{\sim}{p} &= (p_x, p_y, p_z)^t
\end{aligned}$$

The expressions for the strain components (Eq. 12) are substituted in Eq. 13. The following relation results when an explicit integration is carried out through the plate thickness.

$$\prod = \frac{1}{2}\int_A \bar{\varepsilon}^t \bar{\sigma}\, dA - \int_A \bar{\delta}^t p\, dA \tag{14}$$

in which,

$$\begin{aligned}
\bar{\sigma} = (&N_x, M_x, N_x^*, M_x^*, N_y, M_y, N_y^*, M_y^*, N_z, M_z, N_z^*, \\
&N_{xy}, M_{xy}, N_{xy}^*, M_{xy}^*, Q_x, S_x, Q_x^*, S_x^*, Q_y, S_y, Q_y^*, S_y^*)^t \\
\bar{\varepsilon} = (&\varepsilon_{xo}, \chi_x, \varepsilon_{xo}^*, \chi_x^*, \varepsilon_{yo}, \chi_y, \varepsilon_{yo}^*, \chi_y^*, \varepsilon_{zo}, \chi_z, \varepsilon_{zo}^*, \varepsilon_{xyo}, \\
&\chi_{xyo}, \varepsilon_{xyo}^*, \chi_{xyo}^*, \phi_x, \chi_{xz}, \phi_x^*, \chi_{xz}^*, \phi_y, \chi_{yz}, \phi_y^*, \chi_{yz}^*)^t \\
\bar{\delta} = (&u_o, v_o, w_o, \theta_x, \theta_y, \theta_z, u_o^*, v_o^*, w_o^*, \theta_x^*, \theta_y^*, \theta_z^*)^t
\end{aligned}$$

where,

$$\begin{bmatrix} N_x & M_x & N_x^* & M_x^* \\ N_y & M_y & N_y^* & M_y^* \\ N_z & M_z & N_z^* & 0 \end{bmatrix} = \int_0^h \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$= \int_o^h \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ & symmetry & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$\begin{bmatrix} N_{xy} & M_{xy} & N_{xy}^* & M_{xy}^* \end{bmatrix} = \int_o^h \tau_{xy} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$= \int_o^h C_{44}\gamma_{xz} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$\begin{bmatrix} Q_x & S_x & Q_x^* & S_x^* \end{bmatrix} = \int_o^h \tau_{xz} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$= \int_o^h C_{55}\gamma_{xz} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$\begin{bmatrix} Q_y & S_y & Q_y^* & S_y^* \end{bmatrix} = \int_o^h \tau_{yz} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

$$= \int_o^h C_{66}\gamma_{yz} \begin{bmatrix} 1 & z & z^2 & z^3 \end{bmatrix} dz$$

Upon integration, these expressions are rewritten in the matrix form as given below,

$$\begin{Bmatrix} N_x \\ M_x \\ N_x^* \\ M_x^* \\ N_y \\ M_y \\ N_y^* \\ M_y^* \\ N_z \\ M_z \\ N_z^* \end{Bmatrix}_{11x1} = [A]_{11x11} \begin{Bmatrix} \varepsilon_{xo} \\ \chi_x \\ \varepsilon_{xo}^* \\ \chi_x^* \\ \varepsilon_{yo} \\ \chi_y \\ \varepsilon_{yo}^* \\ \chi_y^* \\ \varepsilon_{zo} \\ \chi_z \\ \varepsilon_{zo}^* \end{Bmatrix}_{11x1}$$

$$\begin{Bmatrix} N_{xy} \\ M_{xy} \\ N_{xy}^* \\ M_{xy}^* \end{Bmatrix}_{4x1} = [B]_{4x4} \begin{Bmatrix} \varepsilon_{xyo} \\ \chi_{xyo} \\ \varepsilon_{xyo}^* \\ \chi_{xyo}^* \end{Bmatrix}_{4x1}$$

$$\begin{Bmatrix} Q_x \\ S_x \\ Q_x^* \\ S_x^* \end{Bmatrix}_{4x1} = [D]_{4x4} \begin{Bmatrix} \phi_x \\ \chi_{xz} \\ \phi_x^* \\ \chi_{xz}^* \end{Bmatrix}_{4x1} \quad \text{and} \qquad (15)$$

$$\begin{Bmatrix} Q_y \\ S_y \\ Q_y^* \\ S_y^* \end{Bmatrix}_{4x1} = [E]_{4x4} \begin{Bmatrix} \phi_y \\ \chi_{yz} \\ \phi_y^* \\ \chi_{yz}^* \end{Bmatrix}_{4x1}$$

where $[A]$, $[B]$, $[D]$ and $[E]$ are the matrices of FG plate stiffness whose elements are defined in Appendix. The stress resultants with * as a superscript represent higher order quantities because of the higher order terms in the displacement fields (Eq. 11).

2.2.2 *Equilibrium equations and boundary conditions*

The governing equations of equilibrium for the stress analysis are obtained using the principle of minimum potential energy (PMPE), which states that for equilibrium, the total PE must be stationary. In analytical form it can be written as follows,

$$\delta\left(U - W_s - W_{ex} - W_{ey}\right) = 0$$

The individual terms of the above equation are evaluated as follows

$$\delta U = \int_x \int_y \int_z \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz}\right) dx\, dy\, dz \qquad (16)$$

Substituting the appropriate strain expressions using Eq. 12 and integrating through the thickness to get the stress resultants as defined in Eq. 15 and integrating the resulting expressions by parts transforms the Eq. 16 into the following form

$$\delta U = \oint_x [N_{yx}\delta u_o + N_y\delta v_o + Q_y\delta w_o + M_{yx}\delta\theta_x + M_y\delta\theta_y + S_y\delta\theta_z + N^*_{yx}\delta u^*_o + N^*_y\delta v^*_o + Q^*_y\delta w^*_o + M^*_{yx}\delta\theta^*_x + M^*_y\delta\theta^*_y + S^*_y\delta\theta^*_z]\,dx$$
$$+ \oint_y [N_x\delta u_o + N_{xy}\delta v_o + Q_x\delta w_o + M_x\delta\theta_x + M_{xy}\delta\theta_y + S_x\delta\theta_z + N^*_x\delta u^*_o + N^*_{xy}\delta v^*_o + Q^*_x\delta w^*_o + M^*_x\delta\theta^*_x + M^*_{xy}\delta\theta^*_y + S^*_x\delta\theta^*_z]\,dy$$
$$- \int_x\int_y \left\{ \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}\right)\delta u_o + \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}\right)\delta v_o + \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)\delta w_o + \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x\right)\delta\theta_x + \left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y\right)\delta\theta_y + \left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - Nz\right)\delta\theta_z + \left(\frac{\partial N^*_x}{\partial x} + \frac{\partial N^*_{yx}}{\partial y} - 2S_x\right)\delta u^*_o + \left(\frac{\partial N^*_y}{\partial y} + \frac{\partial N^*_{xy}}{\partial x} - 2S_y\right)\delta v^*_o + \left(\frac{\partial Q^*_x}{\partial x} + \frac{\partial Q^*_y}{\partial y} - 2M_z\right)\delta w^*_o + \left(\frac{\partial M^*_x}{\partial x} + \frac{\partial M^*_{yx}}{\partial y} - 3Q^*_x\right)\delta\theta^*_x + \left(\frac{\partial M^*_y}{\partial y} + \frac{\partial M^*_{xy}}{\partial x} - 3Q^*_y\right)\delta\theta^*_y + \left(\frac{\partial S^*_x}{\partial x} + \frac{\partial S^*_y}{\partial y} - 3N^*_z\right)\delta\theta^*_z \right\} dx\,dy \quad (17)$$

Work done by externally applied load can be calculated by,

$$\delta W_s = \int_x\int_y (p_z^+\delta w^+)\,dx\,dy \quad (18)$$

where, w^+ is the transverse displacement at any point at the top surface of the plate and is given by

$$w^+ = w_o + h^+\theta_z + h^{+2}w^*_o + h^{+3}\theta^*_z$$

Therefore after further simplification for h

$$\delta W_s = \int_x\int_y \{(p_z^+)\delta w_o + (p_z^+ h)\delta\theta_z + (p_z^+ h^2)\delta w^*_o + (p_z^+ h^3)\delta\theta^*_z\}\,dx\,dy \quad (19)$$

The work done by the edge stresses is given by

$$W_{ex} = \frac{1}{2}\int_y\int_z (\overline{\sigma}_x u + \overline{\tau}_{xy} v + \overline{\tau}_{xz} w)\,dy\,dz \quad \text{on an edge } x = \text{constant} \quad (20)$$

$$W_{ey} = \frac{1}{2}\int_x\int_z (\overline{\tau}_{xy} u + \overline{\sigma}_y v + \overline{\tau}_{yz} w)\,dx\,dz \quad \text{on an edge } y = \text{constant} \quad (21)$$

where the bars on the quantities refer to edge values. On integration through the thickness the variation of these expressions take the form

$$\delta W_{ex} = \oint_y (\bar{N}_x\delta u_o + \bar{N}_{xy}\delta v_o + \bar{Q}_x\delta w_o + \bar{M}_x\delta\theta_x + \bar{M}_{xy}\delta\theta_y + \bar{S}_x\delta\theta_z + \bar{N}^*_x\delta u^*_o + \bar{N}^*_{xy}\delta v^*_o + \bar{Q}^*_x\delta w^*_o + \bar{M}^*_x\delta\theta^*_x + \bar{M}^*_{xy}\delta\theta^*_y + \bar{S}^*_x\delta\theta^*_z)\,dy \quad (22)$$

and

$$\delta W_{ey} = \oint_x (\bar{N}_{yx}\delta u_o + \bar{N}_y\delta v_o + \bar{Q}_y\delta w_o + \bar{M}_{yx}\delta\theta_x + \bar{M}_y\delta\theta_y + \bar{S}_y\delta\theta_z + \bar{N}^*_{yx}\delta u^*_o + \bar{N}^*_y\delta v^*_o + \bar{Q}^*_y\delta w^*_o + \bar{M}^*_{yx}\delta\theta^*_x + \bar{M}^*_y\delta\theta^*_y + \bar{S}^*_y\delta\theta^*_z)\,dx \quad (23)$$

The variational Eq. 17 takes the following form when the relevant foregoing expressions are substituted for its individual terms.

$$\int_x\int_y \Bigg[\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}\right)\delta u_o + \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}\right)\delta v_o + \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+\right)\delta w_o + \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x\right)\delta\theta_x + \left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y\right)\delta\theta_y + \left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + h(p_z^+)\right)\delta\theta_z + \left(\frac{\partial N^*_x}{\partial x} + \frac{\partial N^*_{yx}}{\partial y} - 2S_x\right)\delta u^*_o + \left(\frac{\partial N^*_y}{\partial y} + \frac{\partial N^*_{xy}}{\partial x} - 2S_y\right)\delta v^*_o + \left(\frac{\partial Q^*_x}{\partial x} + \frac{\partial Q^*_y}{\partial y} - 2M^*_z + h^2(p_z^+)\right)\delta w^*_o + \left(\frac{\partial M^*_x}{\partial x} + \frac{\partial M^*_{yx}}{\partial y} - 3Q^*_x\right)\delta\theta^*_x + \left(\frac{\partial M^*_y}{\partial y} + \frac{\partial M^*_{xy}}{\partial x} - 3Q^*_y\right)\delta\theta^*_y + \left(\frac{\partial S^*_x}{\partial x} + \frac{\partial S^*_y}{\partial y} - 3N^*_z + h^3(p_z^+)\right)\delta\theta^*_z \Bigg] dx\,dy$$
$$+ \oint_x \Big[(\bar{N}_{yx} - N_{yx})\delta u_o + (\bar{N}_y - N_y)\delta v_o + (\bar{Q}_y - Q_y)\delta w_o + (\bar{M}_{yx} - M_{yx})\delta\theta_x + (\bar{M}_y - M_y)\delta\theta_y + (\bar{S}_y - S_y)\delta\theta_z + (\bar{N}^*_{yx} - N^*_{yx})\delta u^*_o + (\bar{N}^*_y - N^*_y)\delta v^*_o + (\bar{Q}^*_y - Q^*_y)\delta w^*_o + (\bar{M}^*_{yx} - M^*_{yx})\delta\theta^*_x + (\bar{M}^*_y - M^*_y)\delta\theta^*_y + (\bar{S}^*_y - S^*_y)\delta\theta^*_z\Big]dx$$
$$+ \oint_y \Big[(\bar{N}_x - N_x)\delta u_o + (\bar{N}_{xy} - N_{xy})\delta v_o + (\bar{Q}_x - Q_x)\delta w_o + (\bar{M}_x - M_x)\delta\theta_x + (\bar{M}_{xy} - M_{xy})\delta\theta_y + (\bar{S}_x - S_x)\delta\theta_z + (\bar{N}^*_x - N^*_x)\delta u^*_o + (\bar{N}^*_{xy} - N^*_{xy})\delta v^*_o + (\bar{Q}^*_x - Q^*_x)\delta w^*_o + (\bar{M}^*_x - M^*_x)\delta\theta^*_x + (\bar{M}^*_{xy} - M^*_{xy})\delta\theta^*_y + (\bar{S}^*_x - S^*_x)\delta\theta^*_z\Big]dy = 0 \quad (24)$$

The above Eq. 24 will be an identity only if each of the coefficients of the arbitrary variation vanishes. The vanishing of the surface integral defines twelve equilibrium equations, while that of the line integrals defines the consistent natural boundary conditions that are to be used with this theory along the two edges. Setting the individual integral terms in Eq. 24 to zero, the following equations of equilibrium and the consistent boundary conditions are obtained.

$$\begin{aligned}
&\delta u_o : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0\\
&\delta v_o : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0\\
&\delta w_o : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ = 0\\
&\delta \theta_x : \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0\\
&\delta \theta_y : \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0\\
&\delta \theta_z : \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + h(p_z^+) = 0\\
&\delta u_o^* : \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = 0\\
&\delta v_o^* : \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = 0\\
&\delta w_o^* : \frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z + h^2(p_z^+) = 0\\
&\delta \theta_x^* : \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = 0\\
&\delta \theta_y^* : \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = 0\\
&\delta \theta_z^* : \frac{\partial S_x^*}{\partial x} + \frac{\partial S_y^*}{\partial y} - 3N_z^* + h^3(p_z^+) = 0
\end{aligned} \tag{25}$$

The boundary conditions, on the edge x = constant

$$\begin{array}{lclclcl}
u_o = \bar{u}_o & & N_x = \bar{N}_x & & v_o = \bar{v}_o & & N_{xy} = \bar{N}_{xy}\\
w_o = \bar{w}_o & & Q_x = \bar{Q}_x & & \theta_x = \bar{\theta}_x & & M_x = \bar{M}_x\\
\theta_y = \bar{\theta}_y & \text{or} & M_{xy} = \bar{M}_{xy} & \text{and} & \theta_z = \bar{\theta}_z & \text{or} & S_x = \bar{S}_x\\
u_o^* = \bar{u}_o^* & & N_x^* = \bar{N}_x^* & & v_o^* = \bar{v}_o^* & & N_{xy}^* = \bar{N}_{xy}^*\\
w_o^* = \bar{w}_o^* & & Q_x^* = \bar{Q}_x^* & & \theta_x^* = \bar{\theta}_x^* & & M_x^* = \bar{M}_x^*\\
\theta_y^* = \bar{\theta}_y^* & & M_{xy}^* = \bar{M}_{xy}^* & & \theta_z^* = \bar{\theta}_z^* & & S_x^* = \bar{S}_x^*
\end{array} \tag{26}$$

and, on the edge y = constant

$$\begin{array}{lclclcl}
u_o = \bar{u}_o & & N_{xy} = \bar{N}_{xy} & & v_o = \bar{v}_o & & N_y = \bar{N}_y\\
w_o = \bar{w}_o & & Q_y = \bar{Q}_y & & \theta_x = \bar{\theta}_x & & M_{xy} = \bar{M}_{xy}\\
\theta_y = \bar{\theta}_y & \text{or} & M_y = \bar{M}_y & \text{and} & \theta_z = \bar{\theta}_z & \text{or} & S_y = \bar{S}_y\\
u_o^* = \bar{u}_o^* & & N_{xy}^* = \bar{N}_{xy}^* & & v_o^* = \bar{v}_o^* & & N_y^* = \bar{N}_y^*\\
w_o^* = \bar{w}_o^* & & Q_y^* = \bar{Q}_y^* & & \theta_x^* = \bar{\theta}_x^* & & M_{xy}^* = \bar{M}_{xy}^*\\
\theta_y^* = \bar{\theta}_y^* & & M_y^* = \bar{M}_y^* & & \theta_z^* = \bar{\theta}_z^* & & S_y^* = \bar{S}_y^*
\end{array} \tag{27}$$

2.2.3 *Closed-form solution*

Navier's solution technique using the Fourier series is used to obtain closed form solution of the 2D plate problem. All displacements and loads acting on the FG plate are defined in terms of Fourier series. The equilibrium equations are solved for displacement amplitudes by substituting stress resultants in terms of displacements expanded in Fourier series.

For the simply supported boundary conditions, viz., At edges $x = 0$ and $x = a$

$$\begin{aligned}
&v_o = 0; \quad w_o = 0; \quad \theta_y = 0; \quad \theta_z = 0; \quad M_x = 0;\\
&v_o^* = 0; \quad w_o^* = 0; \quad \theta_y^* = 0; \quad \theta_z^* = 0; \quad M_x^* = 0;\\
&N_x = 0; \quad N_x^* = 0.
\end{aligned} \tag{28}$$

At edges $y = 0$ and $y = b$

$$\begin{aligned}
&u_o = 0; \quad w_o = 0; \quad \theta_x = 0; \quad \theta_z = 0; \quad M_y = 0;\\
&u_o^* = 0; \quad w_o^* = 0; \quad \theta_x^* = 0; \quad \theta_z^* = 0; \quad M_y^* = 0;\\
&N_y = 0; \quad N_y^* = 0.
\end{aligned} \tag{29}$$

To satisfy the above boundary conditions the generalized displacement field is expanded in terms of double Fourier series as:

$$
\begin{aligned}
u_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}} \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} & \theta_x &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{xmn} \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
v_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad , & \theta_y &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\
w_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} & \theta_z &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
u_o^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}}^* \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} & \theta_x^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}}^* \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
v_o^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}}^* \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad \text{and} & \theta_y^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}}^* \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\
w_o^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}}^* \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} & \theta_z^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}}^* \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned}
\tag{30}
$$

where $u_{o_n}, \theta_{x_n}, u_{on}^*, \ldots\ldots\ldots$ are called as displacement Fourier amplitudes. Only odd values of m and $n = 1, 3, 5, \ldots$. are taken.

Further, substitution of Eq. 12 into stress resultant relationship (Eq. 15), relation between the stress resultant and the reference plane strain quantities can be obtained.

$$
\begin{Bmatrix} N_x \\ M_x \\ N_x^* \\ M_x^* \\ N_y \\ M_y \\ N_y^* \\ M_y^* \\ N_z \\ M_z \\ N_z^* \end{Bmatrix}_{11x1} = [A]_{11x11} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial u_o^*}{\partial x} \\ \frac{\partial \theta_x^*}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial v_o^*}{\partial y} \\ \frac{\partial \theta_y^*}{\partial y} \\ \theta_z \\ 2w_o^* \\ 3\theta_z^* \end{Bmatrix}_{11x1},
\quad
\begin{Bmatrix} N_{xy} \\ M_{xy} \\ N_{xy}^* \\ M_{xy}^* \end{Bmatrix}_{4x1} = [B]_{4x4} \begin{Bmatrix} \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \\ \frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x} \\ \frac{\partial u_o^*}{\partial y} + \frac{\partial v_o^*}{\partial x} \\ \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x} \end{Bmatrix}_{4x1},
$$

$$
\begin{Bmatrix} Q_x \\ S_x \\ Q_x^* \\ S_x^* \end{Bmatrix}_{4x1} = [D]_{4x4} \begin{Bmatrix} \theta_x + \frac{\partial w_o}{\partial x} \\ 2u_o^* + \frac{\partial \theta_z}{\partial x} \\ 3\theta_x^* + \frac{\partial w_o^*}{\partial x} \\ \frac{\partial \theta_z^*}{\partial x} \end{Bmatrix}_{4x1} \quad \text{and}
\quad
\begin{Bmatrix} Q_y \\ S_y \\ Q_y^* \\ S_y^* \end{Bmatrix}_{4x1} = [E]_{4x4} \begin{Bmatrix} \theta_y + \frac{\partial w_o}{\partial y} \\ 2v_o^* + \frac{\partial \theta_z}{\partial y} \\ 3\theta_y^* + \frac{\partial w_o^*}{\partial y} \\ \frac{\partial \theta_z^*}{\partial y} \end{Bmatrix}_{4x1}
\tag{31}
$$

where,

$$
\left[\frac{\partial u_o}{\partial x}\frac{\partial \theta_x}{\partial x}\frac{\partial u_o^*}{\partial x}\frac{\partial \theta_x^*}{\partial x}\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left\{-\alpha u_{o_{mn}} - \alpha\theta_{x_{mn}} - \alpha u_{o_{mn}}^* - \alpha\theta_{o_{mn}}^*\right\} \sin\alpha x \sin\beta y
$$

$$\left[\frac{\partial v_o}{\partial y}\frac{\partial \theta_y}{\partial y}\frac{\partial v_o^*}{\partial y}\frac{\partial \theta_y^*}{\partial y}\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$$
$$\left\{-\beta u_{o_{mn}} - \beta\theta_{x_{mn}} - \beta u_{o_{mn}}^* - \beta\theta_{o_{mn}}^*\right\}$$
$$\sin\alpha x \sin\beta y$$
$$\left[\theta_z w_o^* \theta_z^*\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\theta_{z_{mn}} w_{o_{mn}}^* \theta_{z_{mn}}^*\right\}\sin\alpha x \sin\beta y$$
$$\left[\frac{\partial u_o}{\partial y}\frac{\partial \theta_x}{\partial y}\frac{\partial u_o^*}{\partial y}\frac{\partial \theta_x^*}{\partial y}\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$$
$$\left\{\beta u_{o_{mn}} \beta\theta_{x_{mn}} \beta u_{o_{mn}}^* \beta\theta_{x_{mn}}^*\right\}\cos\alpha x \sin\beta y$$
$$\left[\frac{\partial v_o}{\partial x}\frac{\partial \theta_y}{\partial x}\frac{\partial v_o^*}{\partial x}\frac{\partial \theta_y^*}{\partial x}\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$$
$$\left\{\alpha v_{o_{mn}} \alpha\theta_{y_{mn}} \alpha v_{o_{mn}}^* \alpha\theta_{y_{mn}}^*\right\}\cos\alpha x \sin\beta y$$
$$\left[\theta_x u_o^* \theta_x^*\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\theta_{x_{mn}} u_{o_{mn}}^* \theta_{x_{mn}}^*\right\}$$
$$\cos\alpha x \sin\beta y$$
$$\left[\frac{\partial w_o}{\partial x}\frac{\partial \theta_z}{\partial x}\frac{\partial w_o^*}{\partial x}\frac{\partial \theta_z^*}{\partial x}\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$$
$$\left\{\alpha w_{o_{mn}} \alpha\theta_{z_{mn}} \alpha w_{o_{mn}}^* \alpha\theta_{z_{mn}}^*\right\}\cos\alpha x \sin\beta y$$
$$\left[\theta_y v_o^* \theta_y^*\right]^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\theta_{y_{mn}} v_{o_{mn}}^* \theta_{y_{mn}}^*\right\}\sin\alpha x \cos\beta y$$
$$\left[\frac{\partial w_o}{\partial y}\frac{\partial \theta_z}{\partial y}\frac{\partial w_o^*}{\partial y}\frac{\partial \theta_z^*}{\partial y}\right]^T$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\beta w_{o_{mn}} \beta\theta_{z_{mn}} \beta w_{o_{mn}}^* \beta\theta_{z_{mn}}^*\right\}\sin\alpha x \cos\beta y$$

in which, $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$

The following steps are taken to obtain the required system of equilibrium equations (Eq. 25) in terms of displacements.

1. Equations 28–31 are substituted in Eq. 25.
2. The twelve equilibrium equations are multiplied with $\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$ and $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ respectively and then integrated between the limits $0 < x < a$ and $0 < y < b$.

After following the above two steps with use of orthogonality conditions for trigonometric functions and collecting the displacement coefficients, one obtains:

$$[X]_{12x12}\begin{Bmatrix} u_{o_{mn}} \\ v_{o_{mn}} \\ w_{o_{mn}} \\ \theta_{x_{mn}} \\ \theta_{y_{mn}} \\ \theta_{z_{mn}} \\ u_{o_{mn}}^* \\ v_{o_{mn}}^* \\ w_{o_{mn}}^* \\ \theta_{x_{mn}}^* \\ \theta_{y_{mn}}^* \\ \theta_{z_{mn}}^* \end{Bmatrix}_{12x1} = \begin{Bmatrix} 0 \\ 0 \\ p_z^+ \\ 0 \\ 0 \\ h(p_z^+) \\ 0 \\ 0 \\ h^2(p_z^+) \\ 0 \\ 0 \\ h^3(p_z^+) \end{Bmatrix}_{12x1} \quad (32)$$

for any fixed value of m and n. The matrix [X] is the coefficient matrix whose elements are listed in Appendix.

The Fourier amplitudes are obtained by solving Eq. 32. The Fourier displacement amplitudes are then used to calculate the generalized displacement components and their derivatives. The values of generalized displacement components and their derivatives are then substituted in Eq. 15 to obtain the values of stress resultants. The same displacement values are also back substituted into the strain–displacement relations (Eq. 4) to obtain the values of strain. The material constitutive relations (Eq. 3) are then used to compute the inplane and transverse stresses using 3D equilibrium equations (Eq. 2).

2.2.4 Evaluation of transverse stresses

The transverse stresses (τ_{xz}, τ_{yz} and σ_z) cannot be accurately estimated by constitutive relations (Eq. 3). Here, the transverse stresses are obtained by integrating the 3D equilibrium equations of elasticity (Eq. 2) for each layer over the plate thickness.

After performing numerical integration along the thickness of a FG plate, first-order and second-order differential equations are obtained.

$$\tau_{xz} = -\int_o^z \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) dz + C_1 \quad (33)$$

$$\tau_{yz} = -\int_o^z \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right) dz + C_2 \quad (34)$$

$$\sigma_z = \int_o^z \left(\int_o^z \left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y}\right) dz\right) dz + zC_3 + C_4 \quad (35)$$

After substitution of Eq. 3 into above Eqs. 33–35, the final expression for the transverse shear and normal stresses are written as,

$$\begin{aligned}\tau_{xz} = \Bigg[&\left(t_1\frac{\partial^2 u_o}{\partial x^2} + t_2\frac{\partial^2 \theta_x}{\partial x^2} + t_3\frac{\partial^2 u_o^*}{\partial x^2} + t_4\frac{\partial^2 \theta_x^*}{\partial x^2}\right)\\ &+ \upsilon\left(t_1\frac{\partial^2 v_o}{\partial x\partial y} + t_2\frac{\partial^2 \theta_y}{\partial x\partial y} + t_3\frac{\partial^2 v_o^*}{\partial x\partial y} + t_4\frac{\partial^2 \theta_y^*}{\partial x\partial y}\right)\\ &+ \upsilon\left(t_1\frac{\partial \theta_z}{\partial x} + 2t_2\frac{\partial w_o^*}{\partial x} + 3t_3\frac{\partial \theta_z^*}{\partial x}\right)\\ &+ \frac{1-\upsilon}{2}\left(t_1\frac{\partial^2 u_o}{\partial y^2} + t_2\frac{\partial^2 \theta_x}{\partial y^2} + t_3\frac{\partial^2 u_o^*}{\partial y^2}\right.\\ &\left. + t_4\frac{\partial^2 \theta_x^*}{\partial y^2} + t_1\frac{\partial^2 v_o}{\partial x\partial y} + t_2\frac{\partial^2 \theta_y}{\partial x\partial y} + t_3\frac{\partial^2 v_o^*}{\partial x\partial y} + t_4\frac{\partial^2 \theta_y^*}{\partial x\partial y}\right)\Bigg] + C_1\end{aligned} \quad (36)$$

$$\begin{aligned}\tau_{yz} = \Bigg[&\upsilon\left(t_1\frac{\partial^2 u_o}{\partial x\partial y} + t_2\frac{\partial^2 \theta_x}{\partial x\partial y} + t_3\frac{\partial^2 u_o^*}{\partial x\partial y} + t_4\frac{\partial^2 \theta_x^*}{\partial x\partial y}\right)\\ &+ \left(t_1\frac{\partial^2 v_o}{\partial y^2} + t_2\frac{\partial^2 \theta_y}{\partial y^2} + t_3\frac{\partial^2 v_o^*}{\partial y^2} + t_4\frac{\partial^2 \theta_x^*}{\partial y^2}\right)\\ &+ \upsilon\left(t_1\frac{\partial \theta_z}{\partial y} + 2t_2\frac{\partial w_o^*}{\partial y} + 3t_3\frac{\partial \theta_x^*}{\partial y}\right)\\ &+ \frac{1-\upsilon}{2}\left(t_1\frac{\partial^2 u_o}{\partial x\partial y} + t_2\frac{\partial^2 \theta_x}{\partial x\partial y} + t_3\frac{\partial^2 u_o^*}{\partial x\partial y}\right.\\ &\left. + t_4\frac{\partial^2 \theta_x^*}{\partial x\partial y} + t_1\frac{\partial^2 v_o}{\partial x^2} + t_2\frac{\partial^2 \theta_y}{\partial x^2} + t_3\frac{\partial^2 u_o^*}{\partial x^2} + t_4\frac{\partial^2 \theta_x^*}{\partial x^2}\right)\Bigg] + C_2\end{aligned} \quad (37)$$

$$\begin{aligned}\sigma_z = \Bigg[&\left(s_1\frac{\partial^3 u_o}{\partial x^3} + s_2\frac{\partial^3 \theta_x}{\partial x^3} + s_3\frac{\partial^3 u_o^*}{\partial x^3} + s_4\frac{\partial^3 \theta_x^*}{\partial x^3}\right)\\ &+ \upsilon\left(s_1\frac{\partial^3 v_o}{\partial x^2\partial y} + s_2\frac{\partial^3 \theta_y}{\partial x^2\partial y} + s_3\frac{\partial^3 v_o^*}{\partial x^2\partial y} + s_4\frac{\partial^3 \theta_y^*}{\partial x^2\partial y}\right)\\ &+ \upsilon\left(s_1\frac{\partial^2 \theta_z}{\partial x^2} + 2s_2\frac{\partial^2 w_o^*}{\partial x^2} + 3s_3\frac{\partial^2 \theta_z^*}{\partial x^2}\right)\\ &+ \upsilon\left(s_1\frac{\partial^3 u_o}{\partial x\partial y^2} + s_2\frac{\partial^3 \theta_x}{\partial x\partial y^2} + s_3\frac{\partial^3 u_o^*}{\partial x\partial y^2} + s_4\frac{\partial^3 \theta_x^*}{\partial x\partial y^2}\right)\\ &+ \left(s_1\frac{\partial^3 v_o}{\partial y^3} + s_2\frac{\partial^3 \theta_y}{\partial y^3} + s_3\frac{\partial^3 v_o^*}{\partial y^3} + s_4\frac{\partial^3 \theta_y^*}{\partial y^3}\right)\\ &+ \upsilon\left(s_1\frac{\partial^2 \theta_z}{\partial y^2} + 2s_2\frac{\partial^2 w_o^*}{\partial y^2} + 3s_3\frac{\partial^2 \theta_z^*}{\partial y^2}\right) + 2\left(\frac{1-\upsilon}{2}\right)\\ &\left(s_1\frac{\partial^3 u_o}{\partial x\partial y^2} + s_2\frac{\partial^3 \theta_x}{\partial x\partial y^2} + s_3\frac{\partial^3 u_o^*}{\partial x\partial y^2} + s_4\frac{\partial^3 \theta_x^*}{\partial x\partial y^2}\right.\\ &\left. + s_1\frac{\partial^3 v_o}{\partial x^2\partial y} + s_2\frac{\partial^3 \theta_y}{\partial x^2\partial y} + s_3\frac{\partial^3 v_o^*}{\partial x^2\partial y} + s_4\frac{\partial^3 \theta_y^*}{\partial x^2\partial y}\right)\Bigg]\\ &+ zC_3 + C_4\end{aligned} \quad (38)$$

where,

$$t_1 = \frac{\bar{E}_z - \bar{E}_o}{\lambda} \quad s_1 = \frac{\bar{E}_z - \bar{E}_o}{\lambda^2} - \frac{z\bar{E}_o}{\lambda}$$

$$t_2 = \frac{z\bar{E}_z - t_1}{\lambda} \quad s_2 = \frac{z\bar{E}_z}{\lambda^2} - \left(\frac{\bar{E}_z - \bar{E}_o}{\lambda^3}\right) - \frac{s_1}{\lambda}$$

$$t_3 = \frac{z^2\bar{E}_z - 2t_2}{\lambda} \quad s_3 = \frac{z^2\bar{E}_z}{\lambda^2} - \frac{2z\bar{E}_z}{\lambda^3} + 2\left(\frac{\bar{E}_z - \bar{E}_o}{\lambda^4}\right) - 2\frac{s_2}{\lambda} \, t_4 = \frac{z^3\bar{E}_z - 3t_3}{\lambda}$$

$$s_4 = \frac{z^3\bar{E}_z}{\lambda^2} - \frac{3z^2\bar{E}_z}{\lambda^3} + \frac{6z\bar{E}_z}{\lambda^4} - 6\left(\frac{\bar{E}_z - \bar{E}_o}{\lambda^5}\right) - 3\frac{s_3}{\lambda}$$

in which, $\bar{E}(z) = \frac{E_o}{1-\upsilon^2}e^{\lambda z}$ and $\bar{E}_o = \frac{E_o}{1-\upsilon^2}$

3 Numerical investigation

A computer code is developed in FORTRAN 90 by incorporating the present mixed semi-analytical and HOSNT formulation for the stress analysis of rectangular FG plates under transverse loads. Numerical

Table 3 Normalized stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xy}}, \overline{\tau_{xz}}, \overline{\tau_{yz}}$) and the transverse displacement ($\bar{w}$) of square FG plate under sinusoidal transverse load

a/h	Source	$\bar{\sigma}_x/\bar{\sigma}_y\left(\frac{a}{2},\frac{b}{2},h\right)$	$\bar{\tau}_{xy}(0,0,-h)$	$\bar{\tau}_{xz}/\bar{\tau}_{yz}(\max)$	$\bar{w}\left(\frac{a}{2},\frac{b}{2},h\right)$
2	Semi-analytical	0.3791	0.0833	0.2361	10.8456
	HOSNT	0.3652	0.0908	0.2341	11.7217
5	Semi-analytical	0.2702	0.0832	0.2393	4.9165
	HOSNT	0.2570	0.0915	0.2389	5.3749
10	Semi-analytical	0.2583	0.0823	0.2404	4.3203
	HOSNT	0.2448	0.0905	0.2403	4.7407
50	Semi-analytical	0.2547	0.0820	0.2407	4.1433
	HOSNT	0.2412	0.0901	0.2407	4.5528

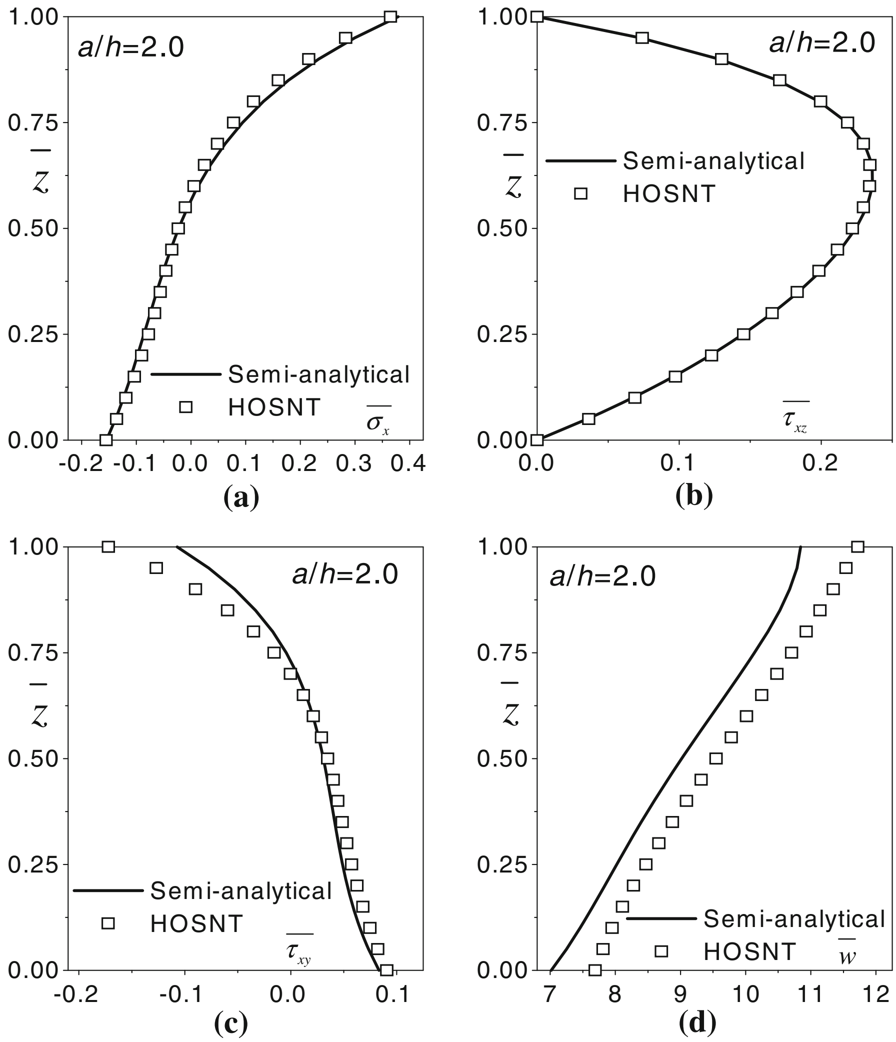

Fig. 2 Through thickness variation of **a** inplane normal stress $\overline{\sigma_x}$, **b** transverse shear stress $\overline{\tau_{xz}}$, **c** inplane shear stress $\overline{\tau_{xy}}$ and **d** transverse displacement ($\bar{w}$) for simply supported square FG plate under sinusoidal load

Table 4 Normalized stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xy}}, \overline{\tau_{xz}}, \overline{\tau_{yz}}$) and the transverse displacement ($\bar{w}$) of rectangular FG plate under sinusoidal transverse load

$^a/_h$	Source	$\bar{\sigma}_x\left(\frac{a}{2},\frac{b}{2},h\right)$	$\bar{\sigma}_y\left(\frac{a}{2},\frac{b}{2},h\right)$	$\bar{\tau}_{xy}(0,0,-h)$	$\bar{\tau}_{xz}(\max)$	$\bar{\tau}_{yz}(\max)$	$\bar{w}\left(\frac{a}{2},\frac{b}{2},h\right)$
2	Semi-analytical	0.6545	0.3872	0.1080	0.3791	0.1896	20.0471
	HOSNT	0.6536	0.3461	0.1184	0.3767	0.1884	21.7327
5	Semi-analytical	0.5536	0.2916	0.1060	0.3837	0.1919	11.8021
	HOSNT	0.5463	0.2562	0.1165	0.3833	0.1917	12.9252
10	Semi-analytical	0.5424	0.2796	0.1053	0.3848	0.1924	10.8808
	HOSNT	0.5341	0.2447	0.1157	0.3847	0.1923	11.9460
50	Semi-analytical	0.5391	0.2759	0.1050	0.3851	0.1926	10.5999
	HOSNT	0.5304	0.2412	0.1154	0.3851	0.1926	11.6477

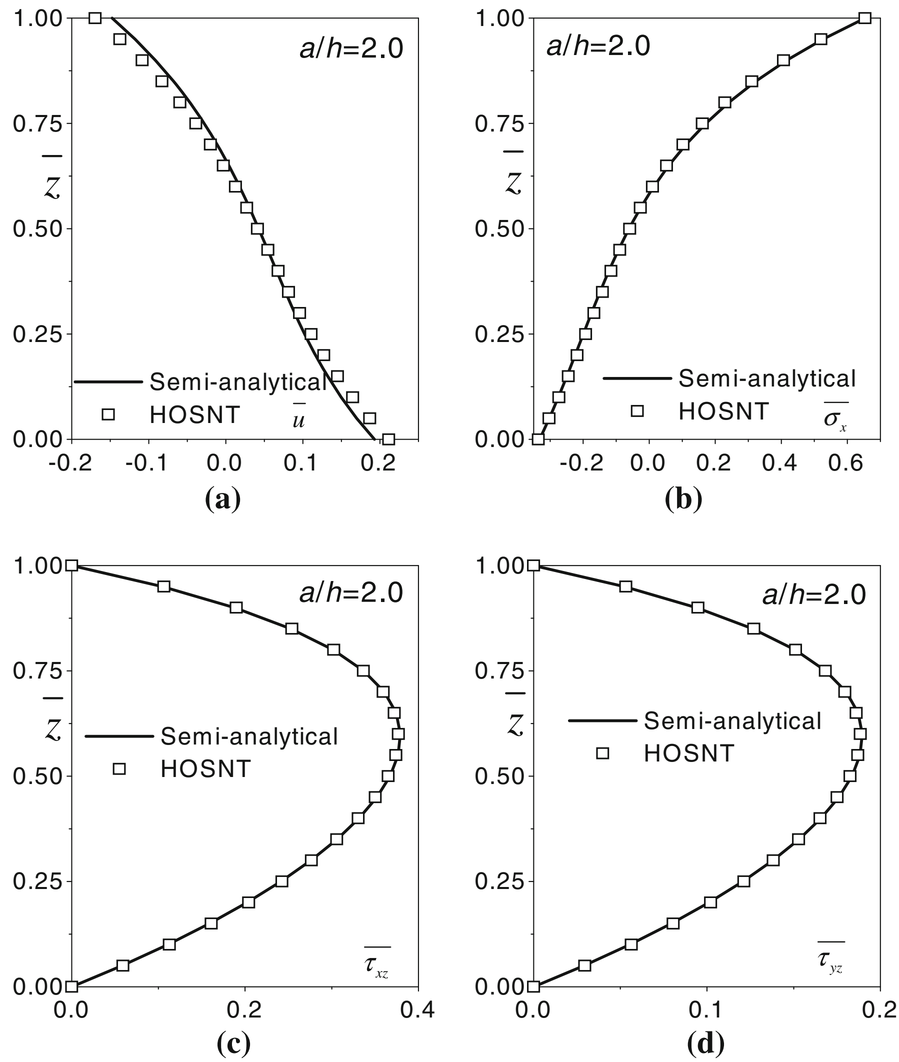

Fig. 3 Through thickness variation of **a** inplane displacement $\bar{u}$, **b** inplane normal stress $\overline{\sigma_x}$, **c** transverse shear stress $\overline{\tau_{xz}}$ and **d** transverse shear stress $\overline{\tau_{yz}}$ for simply supported rectangular FG plate under sinusoidal load

investigations on various simply supported rectangular FG plates have been performed to establish the accuracy of the simplified model presented in the preceding sections of the paper. Comparison is demonstrated between present semi-analytical and analytical solutions based on shear-normal deformation theories (HOSNT).

Following normalizations are used here in all numerical examples for the uniform comparison of the results.

$$s = \frac{a \text{ or } b}{h} \quad \overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xy}} = \frac{\sigma_x, \sigma_y, \tau_{xy}}{p_0 s^2}$$
$$\overline{w} = \frac{100 E_h h^3 w}{p_0 a^4} \quad \overline{\tau_{xz}}, \overline{\tau_{yz}} = \frac{\tau_{xz}, \tau_{yz}}{p_0 s} \tag{39}$$
$$u, v = \frac{E_h(u, v)}{p_0 h s^3} \quad \sigma_z = \frac{\sigma_z}{p_0}$$

in which a bar over the variable defines its normalized value. The following sets of material properties are used:

Material set 1 (Praveen and Reddy 1998; Reddy 2000)

$E_o = 70$ GPa (Aluminum)
$E_h = 151$ GPa (Zirconia)

$\upsilon = 0.3$

Material set 2 (Sankar 2001)

$E_o = 1$ GPa
$E_h/E_o = 5$, 10, 20 and 40

$\upsilon = 0.3$

Illustrative examples considered in the present work are discussed next in detail.

Table 5 Normalized stresses ($\overline{\sigma_x}, \overline{\sigma_y}, \overline{\tau_{xy}}, \overline{\tau_{xz}}, \overline{\tau_{yz}}$) and the transverse displacement ($\bar{w}$) of square FG plate under uniformly distributed transverse load for different gradation factors

a/h	E_h/E_0	$\bar{\sigma}_x/\bar{\sigma}_y\left(\frac{a}{2},\frac{b}{2},h\right)$		$\bar{\tau}_{xy}(0,0,-h)$	
		Semi-analytical	HOSNT	Semi-analytical	HOSNT
2	5	0.6341	0.5903	0.0993	0.1230
	10	0.7931	0.7355	0.0727	0.0899
	20	0.9868	0.9119	0.0523	0.0647
	40	1.2186	1.1233	0.0369	0.0457
5	5	0.5091	0.4803	0.1102	0.1232
	10	0.6365	0.6000	0.0839	0.0938
	20	0.7938	0.7476	0.0629	0.0702
	40	0.9863	0.9282	0.0463	0.0517
10	5	0.4921	0.4654	0.1114	0.1229
	10	0.6150	0.5815	0.0857	0.0945
	20	0.7670	0.7250	0.0649	0.0715
	40	0.9536	0.9012	0.0482	0.0532
a/h	E_h/E_0	$\bar{\tau}_{xz}/\bar{\tau}_{yz}(\max)$		$\bar{w}\left(\frac{a}{2},\frac{b}{2},h\right)$	
		Semi-analytical	HOSNT	Semi-analytical	HOSNT
2	5	0.5280	0.6479	23.1297	24.7710
	10	0.5778	0.6881	31.9892	33.9580
	20	0.6367	0.7541	43.6258	46.1202
	40	0.6998	0.8129	58.7232	61.9529
5	5	0.5061	0.5095	11.8598	12.9185
	10	0.5366	0.5509	16.9658	18.4424
	20	0.5763	0.6026	24.1295	26.1991
	40	0.6142	0.6353	34.0076	36.9109
10	5	0.5092	0.5179	10.5360	11.5496
	10	0.5331	0.5476	15.1162	16.5607
	20	0.5616	0.5739	21.6259	23.6848
	40	0.6043	0.6235	30.7213	33.6431

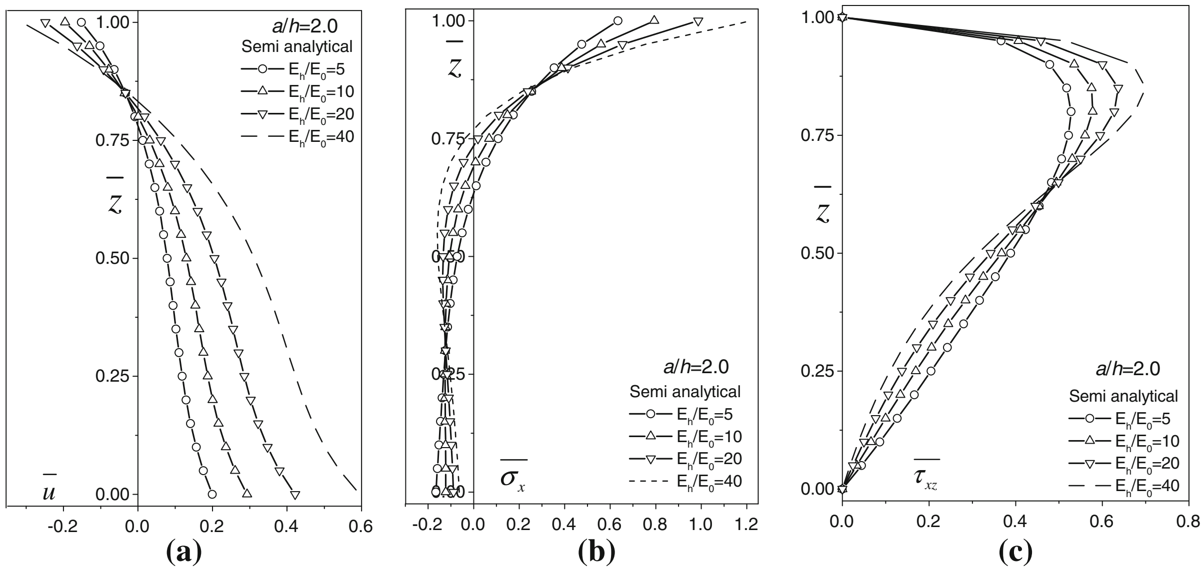

Fig. 4 Through thickness variation of **a** inplane displacement $\bar{u}$, **b** inplane normal stress $\overline{\sigma_x}$ and **c** transverse shear stress $\overline{\tau_{xz}}$ for simply supported square FG plate under uniformly distributed load for different gradation factors

3.1 Example 1

A square FG plate with simply supported end condition on all four edges and subjected to bidirectional sinusoidal load has been considered to show the effectiveness of mixed semi-analytical model over the simplified plate models. Material set 1 is used. The normalized inplane normal stress ($\overline{\sigma_x}$, $\overline{\sigma_y}$), inplane shear stress ($\overline{\tau_{xy}}$), transverse shear stress ($\overline{\tau_{xz}}$) and transverse displacement ($\overline{w}$) for different aspect ratios are presented in Table 3. Moreover, through thickness variations of inplane normal stress ($\overline{\sigma_x}$), inplane and transverse shear stresses ($\overline{\tau_{xy}}$ and $\overline{\tau_{xz}}$) as well as transverse displacement ($\overline{w}$) for an aspect ratio of 2 are depicted in Fig. 2. The analytical solutions (HOSNT) are used for comparison.

3.2 Example 2

A simply supported rectangular FG plate under bidirectional sinusoidal load has been considered here. Material set 1 is used. The normalized inplane normal stresses ($\overline{\sigma_x}$, $\overline{\sigma_y}$), inplane and transverse shear stress ($\overline{\tau_{xy}}$, $\overline{\tau_{xz}}$, $\overline{\tau_{yz}}$) and transverse displacement ($\overline{w}$) for different aspect ratios are detailed in Table 4. Through thickness variations of inplane displacement ($\overline{u}$), inplane normal stresses ($\overline{\sigma_x}$), transverse shear stresses ($\overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$) for an aspect ratio of 2 are shown in Fig. 3. The HOSNT solutions are used for comparison.

3.3 Example 3

A simply supported square FG plate under uniformly distributed load with various gradation factor (λ) is considered in this example to study the effect of gradation factor and to show the capability of presented models to handle the different kind of loading. Material set 2 is used. The normalized inplane normal stress ($\overline{\sigma_x}$, $\overline{\sigma_y}$), inplane and transverse shear stress ($\overline{\tau_{xy}}$, $\overline{\tau_{xz}}$, $\overline{\tau_{yz}}$) and transverse displacement ($\overline{w}$) for different aspect ratios and for different gradation factors ($\lambda = 5$, 10, 20 and 40) are detailed in Table 5. The HOSNT solutions are used for comparison. Through thickness variations of inplane displacement ($\overline{u}$), inplane normal stress ($\overline{\sigma_x}$) and transverse shear stresses ($\overline{\tau_{xz}}$) for an aspect ratio of 2 are shown in Fig. 4.

4 Concluding remarks

A simple mixed semi-analytical model developed by Kant et al. (2008) is extended here for 3D stress analysis of simply supported FG plate. A two-point BVP governed by a set of linear first-order ODEs is formed by assuming all primary variables in the form of trigonometry functions along the inplane direction. The solutions ensure the fundamental relationship between stress, strain and displacement fields within the elastic continuum. No any simplifying assumptions are made through the thickness of a FG plate. Analytical solutions based on shear-normal deformation theory (HOSNT) are also developed and presented for comparison and to show the effectiveness, simplicity and accuracy of a newly developed mixed semi-analytical model over the other simplified plate models. The main feature of mixed semi-analytical model is that the governing equation system is not transformed into an algebraic equation system, thus the intrinsic behaviour of the physical system is retained to a greater degree of accuracy.

Appendix

The coefficients of matrix [**A**] are,

$$A_{1,1}=\frac{\bar{E}_h-\bar{E}_o}{\lambda}\quad A_{1,2}=\frac{h\bar{E}_h-A_{1,1}}{\lambda}$$
$$A_{1,3}=\frac{h^2\bar{E}_h-2A_{1,2}}{\lambda}\quad A_{1,4}=\frac{h^3\bar{E}_h-3A_{1,3}}{\lambda}$$
$$A_{1,5}=\upsilon A_{1,1}\quad A_{1,6}=\upsilon A_{1,2}\quad A_{1,7}=\upsilon A_{1,3}$$
$$A_{1,8}=\upsilon A_{1,4}\quad A_{1,9}=A_{1,5}$$
$$A_{1,10}=A_{1,6}\quad A_{1,11}=A_{1,7}$$

$$A_{2,2}=A_{1,3}\quad A_{2,3}=A_{1,4}\quad A_{2,4}=\frac{h^4\bar{E}_h-4A_{2,3}}{\lambda}$$
$$A_{2,5}=\upsilon A_{1,2}\quad A_{2,6}=\upsilon A_{2,2}\ A_{2,7}=\upsilon A_{2,3}$$
$$A_{2,8}=\upsilon A_{2,4}\quad A_{2,9}=A_{2,5}$$
$$A_{2,10}=A_{2,6}\quad A_{2,11}=A_{2,7}$$

$$A_{3,3}=A_{2,4}\quad A_{3,4}=\frac{h^5\bar{E}_h-5A_{3,3}}{\lambda}\quad A_{3,5}=\upsilon A_{1,3}$$
$$A_{3,6}=\upsilon A_{2,3}\quad A_{3,7}=\upsilon A_{3,3}\ A_{3,8}=\upsilon A_{3,4}\quad A_{3,9}=A_{3,5}$$
$$A_{3,10}=A_{3,6}\quad A_{3,11}=A_{3,7}$$

$$A_{4,4}=\frac{h^6\bar{E}_h-6A_{4,3}}{\lambda}\quad A_{4,5}=\upsilon A_{1,4}\quad A_{4,6}=\upsilon A_{2,4}$$
$$A_{4,7}=\upsilon A_{3,4}\quad A_{4,8}=\upsilon A_{4,4}\ A_{4,9}=A_{4,5}$$
$$A_{4,10}=A_{4,6}\quad A_{4,11}=A_{4,7}$$

$$A_{5,5}=A_{1,1}\quad A_{5,6}=A_{1,2}\quad A_{5,7}=A_{1,3}\quad A_{5,8}=A_{1,4}\quad A_{5,9}=A_{1,5}\quad A_{5,10}=A_{1,6}\quad A_{5,11}=A_{1,7}$$

$$A_{6,6}=A_{2,2}\quad A_{6,7}=A_{2,3}\quad A_{6,8}=A_{2,4}\quad A_{6,9}=A_{2,5}\quad A_{6,10}=A_{2,6}\quad A_{6,11}=A_{2,7}$$

$$A_{7,7}=A_{3,3}\quad A_{7,8}=A_{3,4}\quad A_{7,9}=A_{3,5}$$
$$A_{7,10}=A_{3,6}\quad A_{7,11}=A_{3,7}$$

$$A_{8,8}=A_{4,4}\quad A_{8,9}=A_{4,5}\quad A_{8,10}=A_{4,6}$$
$$A_{8,11}=A_{4,7}$$

$$A_{9,9}=A_{1,1}\quad A_{9,10}=A_{1,2}\quad A_{9,11}=A_{1,3}$$

$$A_{10,10}=A_{2,2}\quad A_{10,11}=A_{2,3}$$

$$A_{11,11}=A_{3,3}$$

where, $\bar{E}_h=\left(\frac{E_o}{1-\upsilon^2}\right)e^{\lambda}$ and $\bar{E}_o=\frac{E_o}{(1-\upsilon^2)}$ and $A_{i,j}=A_{j,i}$, (i, j = 1 to 11)

The coefficients of matrix [**B**] are,

$$B_{1,1}=\frac{1-\upsilon}{2}A_{1,1}\quad B_{1,2}=\frac{1-\upsilon}{2}A_{1,2}$$
$$B_{1,3}=\frac{1-\upsilon}{2}A_{1,3}\quad B_{1,4}=\frac{1-\upsilon}{2}A_{1,4}$$
$$B_{2,2}=\frac{1-\upsilon}{2}A_{2,2}\quad B_{2,3}=\frac{1-\upsilon}{2}A_{2,3}\quad B_{2,4}=\frac{1-\upsilon}{2}A_{2,4}$$
$$B_{3,3}=\frac{1-\upsilon}{2}A_{3,3}\quad B_{3,4}=\frac{1-\upsilon}{2}A_{3,4}$$
$$B_{4,4}=\frac{1-\upsilon}{2}A_{4,4}$$

and $B_{i,j}=B_{j,i}$ (i, j = 1 to 4)

[D] and [E] matrix are same as [B] matrix.

The coefficients of vector {**I**} are

$$I_1=\frac{\rho_h-\rho_o}{\lambda_1}\quad I_2=\frac{h\,\rho_h-I_1}{\lambda_1}\quad I_3=\frac{h^2\rho_h-2I_2}{\lambda_1}$$
$$I_4=\frac{h^3\rho_h-3I_3}{\lambda_1}\quad I_5=\frac{h^4\rho_h-4I_4}{\lambda_1}\quad I_6=\frac{h^5\rho_h-5I_5}{\lambda_1}$$
$$I_7=\frac{h^6\rho_h-6I_6}{\lambda_1}$$

The coefficients of matrix [**X**] are

$$X_{1,1}=A_{1,1}\alpha^2+B_{1,1}\beta^2$$
$$X_{1,2}=A_{1,5}\alpha\beta+B_{1,1}\alpha\beta\quad X_{1,3}=0$$
$$X_{1,4}=A_{1,2}\alpha^2+B_{1,2}\beta^2$$
$$X_{1,5}=A_{1,6}\alpha\beta+B_{1,2}\alpha\beta$$
$$X_{1,6}=-A_{1,9}\alpha$$
$$X_{1,7}=A_{1,3}\alpha^2+B_{1,3}\beta^2$$
$$X_{1,8}=A_{1,7}\alpha\beta+B_{1,3}\alpha\beta$$
$$X_{1,9}=-2A_{1,10}\alpha$$

$X_{1,10} = A_{1,4}\alpha^2 + B_{1,4}\beta^2$
$X_{1,11} = A_{1,8}\alpha\beta + B_{1,4}\alpha\beta$
$X_{1,12} = -3A_{1,11}\alpha$

$X_{2,2} = A_{1,5}\beta^2 + B_{1,1}\alpha^2$
$X_{2,3} = 0$
$X_{2,4} = A_{5,2}\alpha\beta + B_{1,2}\alpha\beta$
$X_{2,5} = A_{5,6}\beta^2 + B_{1,2}\alpha^2$
$X_{2,6} = -A_{5,9}\beta$
$X_{2,7} = A_{5,3}\alpha\beta + B_{1,3}\alpha\beta$
$X_{2,8} = A_{5,7}\beta^2 + B_{1,3}\alpha^2$
$X_{2,9} = -2A_{5,10}\beta$
$X_{2,10} = A_{5,4}\alpha\beta + B_{1,4}\alpha\beta$
$X_{2,11} = A_{5,8}\beta^2 + B_{1,4}\alpha^2$
$X_{2,12} = -3A_{5,11}\beta$

$X_{3,3} = D_{1,1}\alpha^2 + E_{1,1}\beta^2$
$X_{3,4} = D_{1,1}\alpha$
$X_{3,5} = E_{1,1}\beta$
$X_{3,6} = D_{1,2}\alpha^2 + E_{1,2}\beta^2$
$X_{3,7} = 2D_{1,2}\alpha$
$X_{3,8} = 2E_{1,2}\beta$
$X_{3,9} = D_{1,3}\alpha^2 + E_{1,3}\beta^2$
$X_{3,10} = 3D_{1,3}\alpha$
$X_{3,11} = 3E_{1,3}\beta$
$X_{3,12} = D_{1,4}\alpha^2 + E_{1,4}\beta^2$

$X_{4,4} = A_{2,2}\alpha^2 + B_{2,2}\beta^2 + D_{1,1}$
$X_{4,5} = A_{2,6}\alpha\beta + B_{2,2}\alpha\beta$
$X_{4,6} = -A_{2,9}\alpha + D_{1,2}\alpha$
$X_{4,7} = A_{2,3}\alpha^2 + B_{2,3}\beta^2 + 2D_{1,2}$
$X_{4,8} = A_{2,7}\alpha\beta + B_{2,3}\alpha\beta$
$X_{4,9} = -2A_{2,10}\alpha + D_{1,3}\alpha$
$X_{4,10} = A_{2,4}\alpha^2 + B_{2,4}\beta^2 + 3D_{1,3}$
$X_{4,11} = A_{2,8}\alpha\beta + B_{2,4}\alpha\beta$
$X_{4,12} = -3A_{2,11}\alpha + D_{1,4}\alpha$

$X_{5,5} = A_{6,6}\beta^2 + B_{2,2}\alpha^2 + E_{1,1}$
$X_{5,6} = -A_{6,9}\beta + E_{1,2}\beta$
$X_{5,7} = A_{6,3}\alpha\beta + B_{2,3}\alpha\beta$
$X_{5,8} = A_{6,7}\beta^2 + B_{2,3}\alpha^2 + 2E_{1,2}$
$X_{5,9} = -2A_{6,10}\beta + E_{1,3}\beta$
$X_{5,10} = A_{6,4}\alpha\beta + B_{2,4}\alpha\beta$
$X_{5,11} = A_{6,8}\beta^2 + B_{2,4}\alpha^2 + 3E_{1,3}$
$X_{5,12} = -3A_{6,11}\beta + E_{1,4}\beta$

$X_{6,6} = D_{2,2}\alpha^2 + E_{2,2}\beta^2 + A_{9,9}$
$X_{6,7} = -A_{9,3}\alpha + 2D_{2,2}\beta$
$X_{6,8} = -A_{9,7}\beta + 2E_{2,2}\beta$
$X_{6,9} = D_{2,3}\alpha^2 + E_{2,3}\beta^2 + 2A_{9,10}$
$X_{6,10} = -A_{9,4}\alpha + 3D_{2,3}\alpha$
$X_{6,11} = -A_{9,8}\beta + 3E_{2,3}\beta$
$X_{6,12} = D_{2,4}\alpha^2 + E_{2,4}\beta^2 + 3A_{9,11}$

$X_{7,7} = A_{3,3}\alpha^2 + B_{3,3}\beta^2 + 4D_{2,2}$
$X_{7,8} = A_{3,7}\alpha\beta + B_{3,3}\alpha\beta$
$X_{7,9} = -2A_{3,10}\alpha + 2D_{2,3}\alpha$
$X_{7,10} = A_{3,4}\alpha^2 + B_{3,4}\beta^2 + 6D_{2,4}$
$X_{7,11} = A_{3,8}\alpha\beta + B_{3,4}\alpha\beta$

$X_{8,8} = A_{7,7}\beta^2 + B_{3,3}\alpha^2 + 4E_{2,2}$
$X_{8,9} = -2A_{7,10}\beta + 2E_{2,3}\beta$
$X_{8,10} = A_{7,4}\alpha\beta + B_{3,4}\alpha\beta$
$X_{8,11} = A_{7,8}\beta^2 + B_{3,4}\alpha^2 + 6E_{2,3}$
$X_{8,12} = -3A_{7,11}\beta + 2E_{2,4}\beta$

$X_{9,9} = D_{3,3}\alpha^2 + E_{3,3}\beta^2 + 4A_{10,10}$
$X_{9,10} = -2A_{10,4}\alpha + 3D_{3,3}\alpha$
$X_{9,11} = -2A_{10,8}\beta + 3E_{3,3}\beta$
$X_{9,12} = D_{3,4}\alpha^2 + E_{3,4}\beta^2 + 6A_{10,11}$

$X_{10,10} = A_{4,4}\alpha^2 + B_{4,4}\beta^2 + 9D_{3,3}$
$X_{10,11} = A_{4,8}\alpha\beta + B_{4,4}\alpha\beta$
$A_{10,12} = -3A_{4,11}\alpha + 3D_{3,4}\alpha$

$X_{11,11} = A_{8,8}\beta^2 + B_{4,4}\alpha^2 + 9E_{3,3}$
$X_{11,12} = -3A_{8,11}\beta + 3E_{3,4}\beta$

$X_{12,12} = D_{4,4}\alpha^2 + E_{4,4}\beta^2 + 9A_{11,11}$

in which, $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$ and $X_{\mathrm{i,j}} = X_{\mathrm{j,i}}$, (i, j = 1 to 12)

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