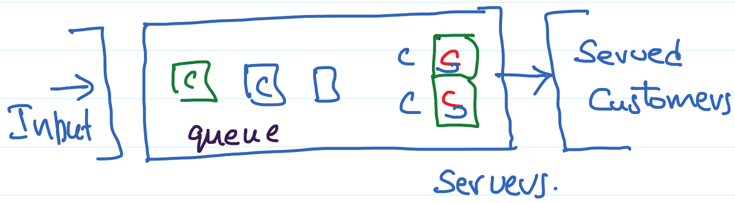
11 - Queue Analysis and Tolling System

11 January 2022 08:53

E.g. Toll Booth
Signalized I/S
Parking bay.
Ticketing Counter

1. Basic Components



2. Customer Arrival.

→ Poisson Process
↳ arrival of one Customer is not affected by other.

3. Input Source

→ Finite ← Difficult
→ Infinite ← Easy.
> The arrival rate is not affected by the state of the system.

4. Queue Capacity

→ Finite/Limited ← Difficult
→ Infinite ← Easy

5. Queue Discipline.

→ First In First Out (FIFO) ↳ common.
→ First In, Last Out
→ Priority
→ Random.

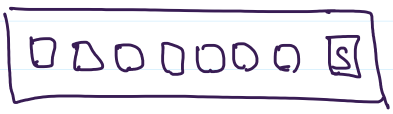
6. Server Mechanism.

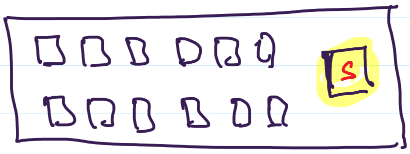
→ Simple
→ Complex

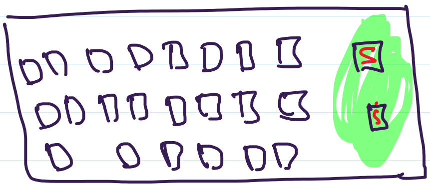
7. State.

→ steady state ← state of the sys. not affected by the initial condition. (Constant) time-independent → easy
→ transient. ↳ Difficult. state affected by the initial condition.

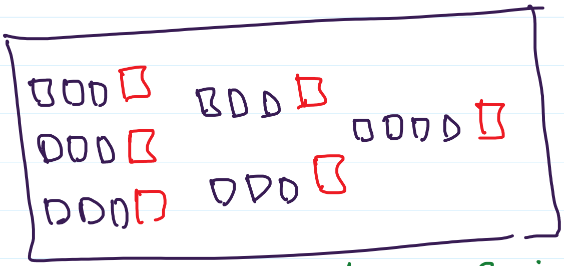
8. Examples.







Toll Both.



Air port.

Gate   Bording Pass   Security Check.

Jan 13, 202

Kendall Classification.

A / B / S - C / P / q

- **A**: Arrival Process (Distributio)
- **B**: Service Process (Dist..)
- **S**: No. of Servers 1, 2, ...
- **C**: Capacity of the System — K → Finite, ∞ → Infinit
- **P**: Population Source
- **q**: Queue Discipline — FIFO, FILO, ⋮

→ Markovian Process

> the Conditional pb: of future state depends only on the present state (and NOT on the past..) (loss of memory...)

→ Degenerate Di

→ Erlang D

→ General D

Simplest Queuing System   M/M/1 - ∞/∞/FIFO

Assumptions

1. Arrival → Markovian Process (Poisson D)
2. Service → Markovian Process (Exp. D)
3. One Server.
4. Queue Sy: Cap : Infinite
5. Population Source : Infinite.
6. Queue discipline: FIFO.

7. Steady State Condition.

a. Arrival Rate is Constant

b. Service Rate is Constant.

c. $\frac{\text{Arrival Rate}}{\text{Service Rate}} < 1 \quad (\rho < 1)$

utilization factor.

d. there is No balking

$M/M/S - \infty/\infty/FIFO$

multiple server.

Operations Research

Hillier & Liberman.

Notations

$\lambda \rightarrow$ Arrival Rate

$\mu \rightarrow$ Service Rate

$\rho \rightarrow$ Utilization factor.

$s \rightarrow$ no. of Server.

$P_0 \Rightarrow$ pb. of zero customers in the system.

$P_n \Rightarrow$ pb. of n Customers in the sys.

$L \rightarrow$ Exp. no. of Customers in the system.

$L_q \rightarrow$ " " " queue.

$W \rightarrow$ Expected waiting time in the system.

$W_q \rightarrow$ " " " " queue

Little's Formulae

$$L = \lambda W$$

$$L_q = \lambda W_q.$$

$$W = W_q + \frac{1}{\mu}$$

$M/M/1 \ \infty/\infty/FIFO$

M/M/1 ∞/∞/FIFO

$$p_0 = (1 - \rho) \qquad \rho = \frac{\lambda}{\mu}.$$

$$p_n = (1-\rho)\cdot \rho^n \qquad n \to \text{no. of customers.}$$

$$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$W = W_q + \frac{1}{\mu} = \frac{1}{\mu - \lambda}$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{L_q}{\lambda}$$

M/M/S – ∞/∞/FIFO

$$\rho = \frac{\lambda}{s\mu} \qquad s \to \text{no. of server.}$$

$$p_0 = \frac{1}{\sum_{n\geq 0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{\quad} \frac{\quad}{1 - \frac{\lambda}{s \cdot \mu}}}$$

$$p_n = \begin{cases} \frac{\lambda^n}{n!} \times p_0 & \text{if } 0 \leq n \leq s \\ \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} \times p_0 & n \geq s \end{cases}$$

$$L_q = \frac{p_0 \, (\lambda/\mu)^s \cdot \rho}{s! \, (1-\rho)^2}$$

$$W_q = L_q/\lambda$$

$$W = W_q + \frac{1}{\mu}.$$

$$L = L_q + \lambda/\mu.$$

Numerical Example

Given (1) Vehicle arrival rate: one every 30 mts.

(2) Service rate: 20 mts per vehicle.

Arrival Rate $\lambda = 2$ veh/hour.

Service Rate $\mu = 3$ veh/hour.

Utilization Factor $\rho = \frac{2}{3} < 1 \Rightarrow$ steady state.

Option I

c c c c c $\boxed{S}$

Option II

c c c c c c $\begin{matrix} c \\ c \end{matrix}$

Option III

c c c c c c $\boxed{S}$

c c c c c $\boxed{S}$

Jan 17.

M/M/1

steady state possible.

$\rho = \frac{2}{3} \quad p_0 = 1-\rho = 1/3$

$$p_n = (1-\rho)\,\rho^n = \frac{1}{3} \cdot (2/3)^n = \frac{2^n}{3^{n+1}}$$

$L = \rho \ldots = \frac{2/3}{\ldots} = 2$ veh.

$$L = \frac{\rho}{1-\rho} = \frac{2/3}{1-2/3} = 2 \text{ veh.}$$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{2\times 2}{3(3-2)} = 4/3 = 1\tfrac{1}{3} \text{ veh.}$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{2}{3(3-2)} = 2/3 \text{ hours}$$

$$W = \frac{1}{\mu-\lambda} = \frac{1}{3-2} = 1 \text{ hour.}$$

M/M/2.

$$\rho = \frac{2}{3\times 2} = 1/3 \qquad p_0 = \frac{1}{\frac{(2/3)^0}{0!} + \frac{(2/3)^1}{1!} + \frac{(2/3)^2}{2!} \times \frac{1}{1-\frac{2}{2\times 3}}} = 1/2$$

$$p_1 = \ldots\ldots = 1/3$$

$$p_2 = \qquad = 1/2$$

$$p_3 = \ldots \quad = 1/27$$

$$L_q = \frac{1/2 \cdot (2/3)^2 \times 1/3}{2!\ (1-1/3)^2} =$$

$$W_q = \frac{L_q}{\lambda} = \frac{1/12}{\lambda} = 1/24$$

$$L = L_q + \frac{\lambda}{\mu} = \frac{1}{12} + 2/3 = 3/4$$

$$W = W_q + \frac{1}{\mu} = \frac{1}{24} + \frac{1}{3} = 3/8$$

M/M/1. → 2 Nos.

$\lambda = 1$ veh/hr $\quad \mu = 3 \quad \rho = 1/3$

$p_0 = 1 - 1/3 = 2/3$

$= 2/9$

$= 2/27$

$L = \dfrac{\lambda}{\mu - \lambda} = 1/2 \qquad W_q = \dfrac{\lambda}{\mu(\mu-\lambda)} = \quad = 1/6$

$L_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = 1/6 \qquad W = \dfrac{1}{\mu - \lambda} = \quad = 1/2$

	M/M/1	M/M/2	2 * M/M/1
ρ	2/3	1/3	1/3
p_0	1/3	1/2	2/3
p_1	2/9	1/3	2/9
L	2	3/4	1/2
L_q	4/3	1/12	1/6
W	60 mts	22.5 mts	30 mts
W_q	40 mts	2.5 mts.	10 mt.