UTILITY THEORY

- Each alternative has attractiveness or utility associated with it
- Decision maker is assumed to chose that alternative which yields the highest utility
- Utilities are expressed as sum of measured attractiveness and a random term
- Measured attractiveness is a function of the attributes of the alternative as well as the decision maker’s characteristics

UTILITY THEORY CONTINUED

$$U_{ji} = V_{ji} + \varepsilon_{ji}$$

$$V_{ji} = \beta^{/} Z_{ji} \qquad Z_{ji} = (X_{ji}, S_i)$$

Where,

U_{ji} =utility of alternative *j* for individual *i*

V_{ji} = measured attractiveness of alternative *j* for individual *i*

ε_{ji} = random part

Z_{ji} = column vector of characteristics of the individual *i* and attributes of the alternative *j*

β = column vector of parameters

UTILITY THEORY CONTINUED

The alternative *j* is chosen by *i* when

$$U_{ji} > U_{li} \quad for\ all\ l \neq j$$

The probability P_{ji} for the j^{th} alternative to be chosen is

$$P_{ji} = \Pr[V_{ji} + \varepsilon_{ji} > V_{li} + \varepsilon_{li}] \quad for\ all\ l \neq j$$

$$= \Pr[(\varepsilon_{li} - \varepsilon_{ji}) < (V_{ji} - V_{li})]$$

UTILITY THEORY CONTINUED

If εs are independently and identically *Weibull* distributed i.e.,

$$\Pr(\varepsilon_j < \varepsilon) = \exp\left(-e^{-(\varepsilon+\alpha_j)}\right) \text{ then } P_{ji} = \frac{e^{V_{ji}}}{\sum_{\text{all } l} e^{V_{li}}}$$

i.e., $$P_{ji} = \frac{e^{\beta' z_{ji}}}{\sum_{l=1}^{J} e^{\beta' z_{li}}}$$

TYPICAL UTILITY FUNCTIONS

V_{Car}= - 0.023* *TIME* -0.021* *COST* +0.003* *INCOME* -0.001

V_{Bus}= -0.023* *TIME* -0.021* *COST*-0.001* *INCOME*

V_{Train}= -0.023* *TIME* -0.021* *COST*+0.003

TIME and *COST* are generic variables

INCOME is alternative specific variable

VARIABLES CAN ENTER IN Three WAYS

- Generic Variable

  -Variable that appears in the utility functions of all alternatives in a generic sense and has same coefficient estimate for all the alternatives

- Alternative Specific Variable

  -Variable that appears only in the utility function of those alternatives to which it is specific and has different coefficient estimate for each of the alternatives

- Alternative Specific Constant

  -Takes care of unexplained effects

MAXIMUM LIKELIHOOD ESTIMATION

Likelihood function:-

$$L = f(x_1)f(x_2)\ldots f(x_n)$$

$f(x)$ = probability function of random variable X *with a single parameter* θ

$x_1, x_2, \ldots x_n$ = sample of n independent values of X

If L is a differentible function of θ*, a necessary condition for L to have maximum is*

$$\frac{\partial L}{\partial \theta} = 0$$

LIKELIHOOD FUNCTION FOR LOGIT MODEL

$$L = \prod_{i=1}^{N} P_{1i}^{\ f_{1i}} P_{2i}^{\ f_{2i}} \ldots P_{Ji}^{\ f_{Ji}}$$

Where, f_{ji} is the observed frequency for individual *i* to choose alternative *j*

J = number of alternatives

N = number of individuals

The log-likelihood *function is*

$$\ln L = \sum_{i=1}^{n} \sum_{j=1}^{J_i} f_{ji} \ln P_{ji} = \sum_{i} \sum_{j} f_{ji} \ln \frac{e^{\beta' z_{ji}}}{\sum_{l} e^{\beta' z_{li}}}$$

$$= \sum_{i} \sum_{j} f_{ji} \beta' z_{ji} - \sum_{i} \sum_{j} f_{ji} \ln\left(\sum_{l} e^{\beta' z_{li}} \right)$$

The log-likelihood *function is maximised by differentiation*

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i} \sum_{j} f_{ji} z_{ji} - \sum_{i} \sum_{j} f_{ji} \left(\sum_{l} e^{\beta' z_{li}} \right)^{-1} \sum_{l} e^{\beta' z_{li}} z_{li}$$

$$= \sum_{i} \sum_{j} f_{ji} (z_{ji} - \bar{z}_i) = 0 \qquad \text{(A)}$$

Where,

$$\bar{z}_i = \left(\sum_l e^{\beta' z_{li}}\right)^{-1} \sum_j e^{\beta' z_{ji}} z_{ji} = \sum_{j=1}^{J_i} P_{ji} z_{ji}$$

Using the fact that, $\sum_j P_{ji}(z_{ji} - \bar{z}_i) = 0$

The first order condition can be rewritten as

$$\sum_{i=1}^{n} \sum_{j=1}^{J_i} (f_{ji} - P_{ji})(z_{ji} - \bar{z}_i) = 0$$

To apply Newton-Raphson method to maximise ln *L* we find the matrix of second partial derivatives by differentiating (A) with respect to the row vector β'. Accordingly,

$$\frac{\partial^2 \ln L}{\partial\beta\, \partial\beta'} = -\sum_i \sum_j P_{ji} z_{ji} (z_{ji} - \bar{z}_i)' = -\sum_i \sum_j P_{ji} (z_{ji} - \bar{z}_i)(z_{ji} - \bar{z}_i)'$$

Then, a typical iteration of Newton-Raphson algorithm would be,

$$\beta^1 = \beta^0 + \left[\sum_i \sum_j P_{ji}^0 (z_{ji} - \bar{z}_i^0)(z_{ji} - \bar{z}_i^0)'\right]^{-1} \sum_i \sum_j (f_{ji} - P_{ji}^0)(z_{ji} - \bar{z}_i^0)$$

GOODNESS-OF-FIT STATISTICS

- Logical sign
- Significance of variable based on *t*-test

  t-statistic = value of parameter/standard error of estimate

- ρ^2- statistic

$$\rho^2 = 1 - \frac{\ln(\hat{\beta})}{\ln(\bar{\beta})}$$

Likelihood Ratio Test

- Two alternative models, with one being a subset of the other can be compared using likelihood ratio.
- Likelihood ratio is the ratio between the values of likelihood at convergence.
- The fundamental property of the likelihood ratio is that when the log of the likelihood ratio is multiplied by 2, it is distributed as χ^2 with degrees of freedom equal to the number of additional parameters in the more highly specified model
- The likelihood ratio $\chi^2 = 2\left[\ln L(\beta) - \ln L(\beta_1)\right]$ is distributed asymptotically as $\chi2$ (k-k_1) where, k is the number of parameters in β and k_1 is the number of parameters in β_1

Variable Selection Process

Sign	Significance	Decision	
		Policy	**Other**
Correct sign	Significant	Include	Include
	Not significant	Include	May reject
Wrong sign	Significant	Big problem	Reject
	Not significant	Problem	Reject

DEMAND ELASTICITIES

Direct Choice Elasticity

$$E_j^i(j,k) = \frac{\Delta P_{ji}}{P_{ji}} \Big/ \frac{\Delta Z_{ji}^k}{Z_{ji}^k} = \frac{\partial P_{ji}}{\partial Z_{ji}^k}\frac{Z_{ji}^k}{P_{ji}} = \beta^k Z_{ji}^k (1 - P_{ji})$$

Cross Elasticity of Choice

$$E_j^i(l,k) = -\beta^k Z_{li}^k P_{li}$$

Aggregate Choice Elasticities

SUBJECTIVE VALUE OF TIME

Utility functions are normally of the following form

$$V_j = ... + \alpha_t \, TIME + \alpha_c COST$$

The rate at which time would be traded for cost to leave the utility function unchanged is

$$\frac{\dfrac{\partial V_j}{\partial \, TIME}}{\dfrac{\partial V_j}{\partial \, COST}} = \frac{\alpha_t}{\alpha_c}$$

Thus the value of one unit of travel time is equal to α_t/α_c cost units

Prediction Success Table

- Prediction success table is a cross classification between observed choices and predicted choices

Observed Choices	Predicted Choices			Row Totals	Observed Share
	TW	Car	PT		
TW	25	1	4	30	30%
CAR	2	7	1	10	10%
PT	4	1	55	60	60%
Column Totals	31	9	60	100	100%
Predicted Share	31%	9%	60%		
% Correctly Predicted	83%	70%	92%		

Overall prediction success rate = 87%

INDEPENDENCE OF IRRELEVANT ALTERNATIVES

Luce's Axiom (IIA Property)

If a set of alternative choices exists, then the relative probability of choice among any two alternative is unaffected by the removal (or addition) of any set of other alternatives.

OTHER ISSUES

- Taste Variation

  - Segmentation
  - Random Coefficients

- Aggregation

- Transferability