Stress Transformation

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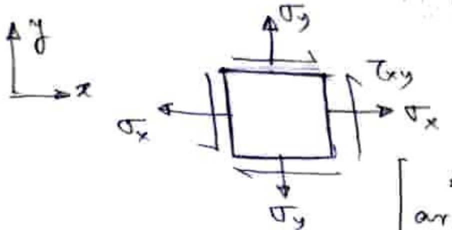
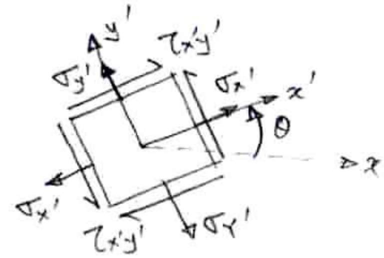
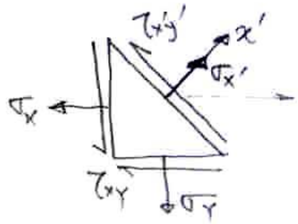


[All quantities are positive in both stress blocks]



General relationship



Take Equilibrium (Derived in class)

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos 2\theta \quad \text{--- (1)}$$

$$= \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{--- (2)}$$

$$\tau_{x'y'} = \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) + (\sigma_y-\sigma_x)\sin\theta\cos\theta \quad \text{--- (3)}$$

$$= -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \quad \text{--- (4)}$$

Note: $\sigma_{y'} = \sigma_{x'}(\theta+90^\circ) = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$

- Determination of max^m/min^m normal stress (i.e principal stresses) and max^m/min^m shearing stress as well as their orientations is simply a maximization/minimization problem.

For example, let us calculate the principal stresses (σ_1 & σ_2)

We set $\dfrac{d\sigma_{x'}}{d\theta} = 0 \Rightarrow \tan 2\theta = \dfrac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)} = \tan 2\theta_p$ (P denotes principal stress planes)

This will give us two values θ_p differ by 90°, for $0 < \theta < 180^\circ$. These are the two mutually perpendicular principal directions, along which principal stresses (σ_1, σ_2) act.

In order to obtain σ_1 & σ_2, we then substitute θ_p in (2) as follows:

$\sin 2\theta_p = \pm \dfrac{\tau_{xy}}{R}$ & $\cos 2\theta_p = \pm \dfrac{\left(\frac{\sigma_x-\sigma_y}{2}\right)}{R}$ where $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Two value of $\sigma_{x'}$ $\Rightarrow \sigma_{1,2} = \dfrac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Note: $\boxed{\underset{\max^m}{\sigma_1} > \underset{\min^m}{\sigma_2}}$ Always true!

Method 1 (DIRECT APPROACH)

Steps: θ is always measure from $+x$ axis

Principal stresses & orientation

$$\frac{d\sigma_{x'}}{d\theta} = 0 \Rightarrow \tan 2\theta_p = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$$

obtain two values of θ_p differ by 90°

Take one value of θ_p, plug it in the expression for $\sigma_{x'}$, we must obtain any one of the following, defined by the principal stresses (σ_1, σ_2),

$$\sigma_{1,2} = \sigma_{ave} \pm R \quad , \begin{cases} \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \\ R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \end{cases}$$

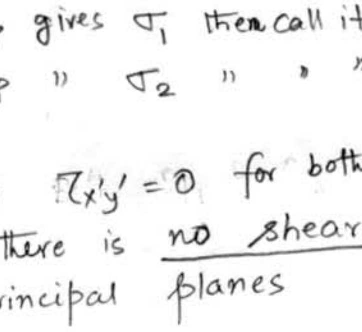
If θ_p gives σ_1 then call it θ_{p_1} or θ_p'
" θ_p " σ_2 " " " θ_{p_2} or θ_p''

Note: $\tau_{x'y'} = 0$ for both θ_{p_1} & θ_{p_2}
Thus, there is no shear stress on principal planes

Show the principal stresses in a properly oriented block

Say, for a given values of $\sigma_x, \sigma_y, \tau_{xy}$ you obtained

$\theta_{p_1} = -25^\circ$, $\sigma_1 = 100$ MPa
$\theta_{p_2} = 65^\circ$, $\sigma_2 = -200$ MPa



Max^m / Min^m shearing stress & orientation

① $$\frac{d\tau_{x'y'}}{d\theta} = 0 \Rightarrow \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

obtain two values of θ_s differ by 90°, Observe that $\theta_s = \theta_p \pm 45^\circ$

② Take any one value of θ_s, plug it in the expression for $\tau_{x'y'}$, we must obtain any one of the following, defined by max^m / min^m shearing stresses (τ_1, τ_2)

$$\left.\begin{aligned} \tau_1 &= +\tau_{max} \\ \tau_2 &= -\tau_{max} \end{aligned}\right\} \tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

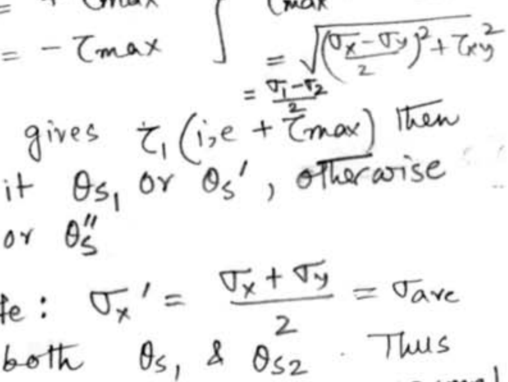
If θ_s gives τ_1 (i.e. $+\tau_{max}$) then call it θ_{s_1} or θ_s', otherwise θ_{s2} or θ_s''

③ Note: $\sigma_x' = \frac{\sigma_x + \sigma_y}{2} = \sigma_{ave}$ for both θ_{s_1} & θ_{s2}. Thus there is an average normal stress in τ_{max} (+ & −) planes

④ Show the τ_{max} planes

Say, $\theta_{s_1} = -70^\circ$, $\tau_1 = +100$ MPa
$\theta_{s_2} = +20^\circ$ $\tau_2 = -100$ MPa
$\sigma_{ave} = 50$ MPa

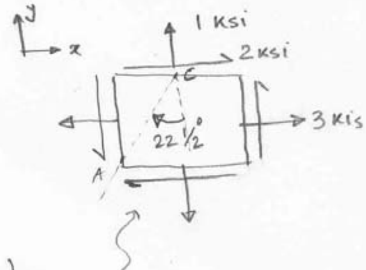
Note: τ_1 & τ_2 differ by sign here, since you always look at x' plane because θ is measured from x to x'

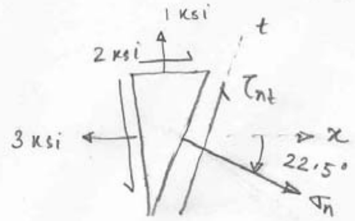


Replace n by x′ and t by y′ !

Example

(a) Find σ_n and τ_{nt} for $\theta = -22.5°$





Note

$\sigma_x = 3$

$\sigma_y = 1$

$\tau_{xy} = 2$

In the element They are positive according to the sign convention

Use,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_n(-22.5°) = \frac{3+1}{2} + \frac{3-1}{2}\cos(-45°) + 2\sin(-45°)$$

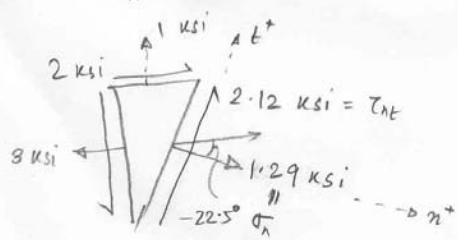
$$= 2 + 1(0.707) - 2(0.707) = \underline{\underline{1.29 \text{ ksi}}}$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\left(\frac{3-1}{2}\right)\sin(-45°) + 2\cos(-45°)$$

$$= 1(0.707) + 2(0.707) = \underline{\underline{2.12 \text{ ksi}}}$$

Both values are positive, so they act in the positive n and t directions, as shown below



(b) Find principal stresses (σ_1 & σ_2) and show on stress block w.r.t x-axis

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$= \frac{3+1}{2} \pm \sqrt{\left(\frac{3-1}{2}\right)^2+2^2} = 2 \pm \sqrt{5}$$

$$\underline{\underline{\sigma_1 = 4.25 \text{ ksi}}} \quad \text{and} \quad \underline{\underline{\sigma_2 = -0.24 \text{ ksi}}}$$

Orientation. $\tan 2\theta_p = \dfrac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)} = \dfrac{2}{\left(\frac{3-1}{2}\right)} = 2$

$$\Rightarrow 2\theta_p = 63.43^\circ \text{ and } 243.43^\circ$$

Therefore $\theta_p = 31.7^\circ$ and 121.7° (both positive, measured CC from x-axis)

(*) Which angle is θ_{p1} (gives σ_1 plane)?
& " " is θ_{p2} (gives σ_2 plane)?

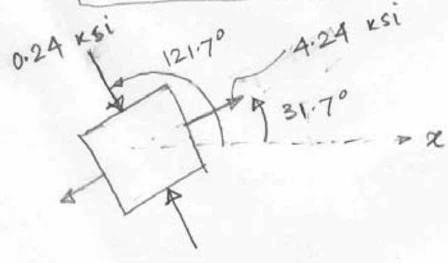
Try 31.7° in $\sigma_n(31.7^\circ) = \frac{3+1}{2} + \frac{3-1}{2}\cos(63.43^\circ) + 2\sin(63.43^\circ)$

$$= \underline{\underline{4.24}} = \sigma_1 \;!$$

↑ we obtained this already using direct forumula above

Remark – This is a good check also

Hence, $\boxed{\begin{aligned}\theta_{p1} &= 31.7^\circ \\ \theta_{p2} &= 121.7^\circ\end{aligned}}$



(c) Find τ_{max} and show its plane w.r.t x axis

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \pm\sqrt{\left(\frac{3-1}{2}\right)^2 + 2^2} = \pm\sqrt{5} = \pm 2.24 \text{ ksi}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}} = -0.5$$

$$2\theta_s = \tan^{-1}(-0.5) = -26.56^\circ \text{ and}$$
$$-26.56^\circ + 180^\circ = 153.4^\circ$$

Therefore $\theta_s = -13.28^\circ$ (measured clockwise) and 76.7° (measured cc)

[observe they differ by 45° from θ_p !]

Which angle is θ_{s1} (gives $+\tau_{max}$ plane) ?

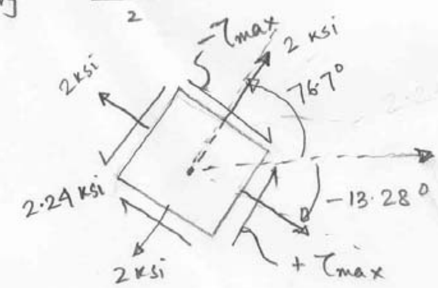
& " " is θ_{s2} (gives $-\tau_{max}$ plane) ?

Try 76.7° in $\tau_{nt}(76.7^\circ) = -\left(\frac{3-1}{2}\right)\sin(153.4^\circ)$
$+ 2\cos(153.4^\circ)$
$= -2.24 = -\tau_{max}$ (great !)

Hence $\theta_{s1} = -13.28^\circ$

$\theta_{s2} = 76.7^\circ$

Note, $\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = 2$ ksi (presents in max^m shear stress planes)



{ Note for 76.7° τ_{max} is negative, means must act along $(-t)$ directi.