Nonlinear Dyn (2007) 47:389–403
DOI 10.1007/s11071-006-9038-8

ORIGINAL ARTICLE

Nonlinear transient dynamic response of damped plates using a higher order shear deformation theory

Suraj Narendra Khante · Vijay Rode · Tarun Kant

Received: 8 April 2005 / Accepted: 3 March 2006 / Published online: 16 September 2006

Abstract Damped transient dynamic elasto-plastic analysis of plate is investigated. A finite element model based on a C^0 higher order shear deformation theory has been developed. Nine noded Lagrangian elements with five degrees of freedom per node are used. Selective Gauss integration is used to evaluate energy terms so as to avoid shear locking and spurious mechanisms. Von Mises and Tresca yield criteria are incorporated along with associated flow rules. Explicit central difference time stepping scheme is employed to integrate temporal equations. The mass matrix is diagonalized by using the efficient proportional mass lumping scheme. A program is developed for damped transient dynamic finite element analysis of elasto-plastic plate. Several numerical examples are studied to unfold different facets of damping of elasto-plastic plates.

Keywords Damping · Dynamics · Elasto-plastic · Lumped mass · Plate · Transient

S. N. Khante (✉) · V. Rode
CE-AMD, Shri G. S. Institute of Technology and Science, Indore-452 003, India
e-mail: snkhante@yahoo.com

T. Kant
Department of Civil Engineering, Indian Institute of Technology, Bombay, Powai, Mumbai-400 076, India

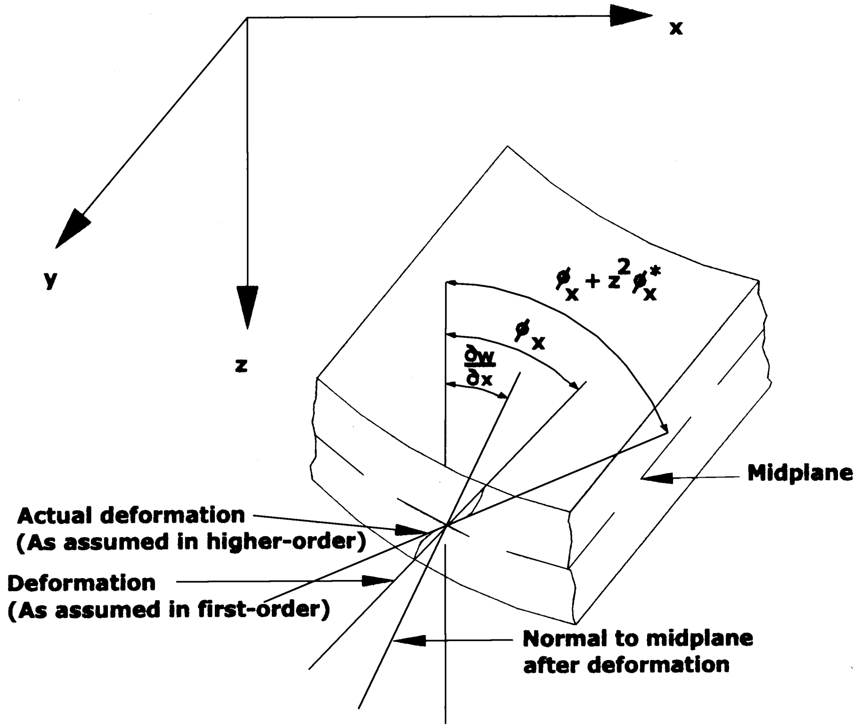
1. Introduction

There are many important sophisticated structures such as nuclear reactors, pressure vessels, submarine hulls, rocket casings, aircrafts, etc., in which a flat plate forms an integral structural component. The high diversity severity of demands as well as operating conditions to which these components are exposed results in dynamic non-linear elasto-plastic behavior of the plates.

Another area where dynamic plastic response of plate is of utmost significance is crashworthiness protection of aircraft, automobiles, buses, ships, and trains. To minimize the damages, efforts should be directed towards reducing displacement response of plate. At this stage it becomes pertinent to explore methodologies, which will help this cause. Damping of elasto-plastic plate subjected to transient catastrophic load is one such aspect that needs an attention. Surprisingly such an important field of research has not received due consideration by the researching fraternity, which is evident from the scarcity of literature available on the topic.

The dynamic non-linear elasto-plastic bending response of plate is certainly dependent on the modeling of plates. In the classical theory of thin plates, there is no provision for transverse normal and transverse shear strains. However, the transverse shear and the normal stresses are obtained using the equilibrium equations. Obviously this involves violation of the constitutive law. Reissner [1] and Mindlin [2] suggested first-order

Fig. 1 Shear deformation of plate as per different theories



shear deformation theories. They aim at improving the theory by incorporating the transverse shear strains. The main drawback of these theories is the consideration of constant shear strain along the thickness, which in turn compels for an approximation in the form of a shear correction coefficient. It can be concluded that the major disadvantage of the first order theories is that although they account for the transverse shear deformations, they can not correctly represent its through thickness distribution as shown in Fig. 1. To overcome these limitations and to make the theory more generic higher order shear deformation theories were proposed [3, 4]. In this the displacement field is chosen in a suitable form and the energy principle is used to formulate the governing equations. Mostly the dynamic material non-linear analyses are dealt with first order theory [5–8]. In some geometric non-linear dynamic analysis of plates, higher order shear deformation theories are used [9, 10]. In higher order shear deformation theory (HOST) the modification of displacement fields enables to include higher order terms. Thus for thick plates the prominent effect of warping is taken care of. However, the HOST formulation requires more number of degrees of freedom per node, i.e. more number of boundary conditions. The boundary conditions, related to higher order terms, cannot be assigned physical meaning. The present formulation includes only two additional degrees of freedom to conserve computational efficiency along with improved accuracy. The geometric nonlinear damped circular plate analysis has been dealt in [23]. The material nonlinear dynamic analysis of the plate with special emphasis on damping is not investigated till date to the best of authors' knowledge and this paper aims at bridging this gap.

In this paper transient elasto-plastic C^0 finite element formulation using a higher order shear deformation theory for nine noded Lagrangian elements is presented. Tresca [11] and von Mises [12] yield criteria along with associated flow rules are employed. The response of the plate in linear and non-linear range is obtained along with effect of varying degree of damping. Lumped mass scheme is used for explicit time integration. Reduced integration is selectively applied to evaluate the shear energy terms.

2. Higher order shear deformation theory

The displacement fields, using Taylor's series, can be expressed as (Kant [3]),

$$u(x, y, z, t) = z\theta_x(x, y, t) + z^3\theta_x^*(x, y, t)$$
$$v(x, y, z, t) = z\theta_y(x, y, t) + z^3\theta_y^*(x, y, t)$$

$$w(x, y, z, t) = w_0(x, y, t) \tag{1}$$

The parameters u and v are in-plane displacements and w is transverse displacement. θ_x and θ_y are rotations about y and x axes respectively at time t. The terms θ_x^* and θ_y^* denote higher order transverse deformation modes. The linear relationship between displacements and strains can be obtained by using definitions of strains from the theory of elasticity.

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = z\chi_x + z^3\chi_x^* \\
\varepsilon_y &= \frac{\partial v}{\partial y} = z\chi_y + z^3\chi_y^* \\
\varepsilon_z &= 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = z\chi_{xy} + z^3\chi_{xy}^* \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \phi_y + z^3\phi_y^* \\
\gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x + z^2\phi_x^*
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
(\chi_x, \chi_y, \chi_{xy}) &= \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \\
(\chi_x^*, \chi_y^*, \chi_{xy}^*) &= \left(\frac{\partial \theta_x^*}{\partial x}, \frac{\partial \theta_y^*}{\partial y}, \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) \\
(\phi_x, \phi_y, \phi_x^*, \phi_y^*) &= \left(\theta_x + \frac{\partial w_0}{\partial x}, \theta_y + \frac{\partial w_0}{\partial y}, 3\theta_x^*, 3\theta_y^*\right)
\end{aligned} \tag{3}$$

The total energy of a system can be given by

$$L = T - \Pi \tag{4}$$

in which T is the kinetic energy. The total potential energy Π of the plate with volume V, surface area A and at time t can be written as,

$$\Pi = U - W \tag{5}$$

Here U is strain Energy of the plate, W is work done by the external forces

Thus,

$$L = \frac{1}{2}\int_v \dot{u}^T \rho \dot{u}\, dv - \left[\frac{1}{2}\int_v \varepsilon^T \boldsymbol{\sigma} dv - \int_v u^T \mathbf{p} dv\right] \tag{6}$$

where $\mathbf{p}$ is the vector of force intensities corresponding to generalized displacement vector $\mathbf{u}$ and

$$\begin{aligned}
\boldsymbol{\varepsilon}^T &= (\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}) \text{ and} \\
\boldsymbol{\sigma}^T &= (\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz})
\end{aligned} \tag{7}$$

Π in the Equation (5) may be expressed in terms of resultants as,

$$\Pi = \frac{1}{2}\int_A \bar{\varepsilon}^T \bar{\boldsymbol{\sigma}}\, dA - \int_A \bar{u}^T \mathbf{P} dA \tag{8}$$

$$\begin{aligned}
\bar{u} &= \{w, \theta_x, \theta_y, \theta_x^*, \theta_y^*\}^T \\
\dot{\bar{u}} &= \{\dot{w}, \dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_x^*, \dot{\theta}_y^*\}^T
\end{aligned}$$

where

$$\begin{aligned}
\bar{\varepsilon} &= \{\chi_x, \chi_y, \chi_{xy}, \chi_x^*, \chi_y^*, \chi_{xy}^*, \\
&\quad \varphi_x, \varphi_y, \varphi_x^*, \varphi_y^*\}^T \\
\bar{\boldsymbol{\sigma}} &= \{M_x, M_y, M_{xy}, M_x^*, M_y^*, \\
&\quad M_{xy}^*, Q_x, Q_y, Q_x^*, Q_y^*\}^T
\end{aligned} \tag{9}$$

The components of the stress-resultant vector $\bar{\boldsymbol{\sigma}}$ for the plate are:

$$\begin{aligned}
(M_x, M_y, M_{xy}) &= \int_{-h/2}^{+h/2} (\sigma_x, \sigma_y, \tau_{xy}) z dz \\
(M_x^*, M_y^*, M_{xy}^*) &= \int_{-h/2}^{+h/2} (\sigma_x, \sigma_y, \tau_{xy}) z^3 dz \\
(Q_x, Q_y) &= \int_{-h/2}^{+h/2} (\tau_{xz}, \tau_{yz}) dz \\
(Q_x^*, Q_y^*) &= \int_{-h/2}^{+h/2} (\tau_{xz}, \tau_{yz}) z^2 dz
\end{aligned} \tag{10}$$

By integrating stress components through the plate thickness stress resultants can be obtained. The constitutive relations in terms of stress-resultants and other details are available elsewhere [27].

3. Elasto-plastic incremental stress strain relationship

For elasto-plastic problems the evaluation of $\boldsymbol{\sigma}$ requires the generation of an elasto-plastic matrix to relate the increments of stress and strain, which results in relationship of the form as below.

$$d\boldsymbol{\sigma}^f = \mathbf{D}_{\mathrm{ep}}^f d\boldsymbol{\varepsilon}_0^f \tag{11}$$

Where $d\boldsymbol{\sigma}^f$ and $d\boldsymbol{\varepsilon}_0^f$ are the flexural stress and strain increments respectively. The elasto-plastic matrix $\mathbf{D}_{\mathrm{ep}}^f$ is expressed as [13],

$$\mathbf{D}_{\mathrm{ep}}^f = \mathbf{D}^f - \mathbf{D}^f \left[\frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}^f}\right]\left[\frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}^f}\right]^T \mathbf{D}^f \left\{ H + \left[\frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}^f}\right]^T \mathbf{D}^f \left[\frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}^f}\right] \right\}^{-1} \tag{12}$$

in which $\mathbf{F}$ is the yield function being employed and H is known as strain-hardening parameter. The matrix $\mathbf{D}_{\mathrm{ep}}^f$ replaces the usual elastic rigidity matrix $\mathbf{D}^f$ after onset of yielding at a point, and is symmetric and positive definite. The incremental stresses defined by (11) are then accumulated to give the total elastic-plastic stresses. The reader is referred to ref. [13] for details.

Thus, when the stress at a Gauss point satisfies the yield condition, the sub matrix $\mathbf{D}_{\mathrm{ep}}^f$, replaces the sub matrix $\mathbf{D}^f$ of the complete stress-strain relationship as given by following relation.

$$\begin{Bmatrix} d\hat{\boldsymbol{\sigma}}_f \\ d\hat{\boldsymbol{\sigma}}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_{ep}^f & 0 \\ 0 & \mathbf{D}^s \end{bmatrix} \begin{Bmatrix} d\hat{\boldsymbol{\varepsilon}}_f \\ d\hat{\boldsymbol{\varepsilon}}_s \end{Bmatrix}$$
$$d\hat{\boldsymbol{\sigma}} = \mathbf{D}_{\mathrm{ep}} d\hat{\boldsymbol{\varepsilon}} \tag{13}$$

4. Damping

Damping in structures is not viscous; rather, it is due to mechanisms such as hysteresis in the material and slip in connections. These mechanisms are not well understood. Very limited information is available on damping in linear solid mechanics problems and there is even less data available for damping in nonlinear situations. Moreover, they are awkward to incorporate into the equations of structural dynamics, or they make the equations computationally difficult. Therefore, the actual damping mechanism is usually approximated by viscous damping. Comparisons of theory and experiments show that this approach is sufficiently accurate in most cases [24].

For plate as stated above the determination of the damping matrix $\mathbf{C}$ is, in practice, difficult because of the lack of knowledge of the viscous term μ. The most usual approximation for $\mathbf{C}$ is so-called Rayleigh damping, given by a linear combination of mass and stiffness matrices.

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \tag{14}$$

where α and β are to be determined experimentally. It can be shown that, in low frequency dominant system, the term $\beta\mathbf{K}$ can be neglected. More so ever, in present analysis, matrix $\mathbf{K}$ is not formed and hence effects of $\beta\mathbf{K}$ are not studied. By approximating $\beta = 0$ in the central difference method,

$$\mathbf{C} = \alpha \mathbf{M} \tag{15}$$

Another type of damping scheme available is 'Adaptive Damping' [14, 15]. In this, damping factor is constantly updated on the basis of the information gained during current iteration, in contrast to a constant value of α throughout the analysis, which appears to be more realistic. On the other hand, for geometric nonlinearity, adaptive damping is shown to be complete failure in certain situations [15]. In light of above information and scarcity of pinpointed literature availability on the topic, only kinetic damping has been applied and investigated in this paper.

The various plates considered in this investigation responded in different fashion to different degree of damping for applied dynamic loading. This varied response is due to frequency dependence of viscous damping as shown in Fig. 2. The first part of the curve (low frequency) is mass damping predominant as is evident. When the plate pertains to this zone, it is very sensitive to mass proportional damping i.e. absolute damping. At this stage it is felt necessary to devise one non-dimensional parameter, which will ascertain to which zone the plate belongs. The authors found that

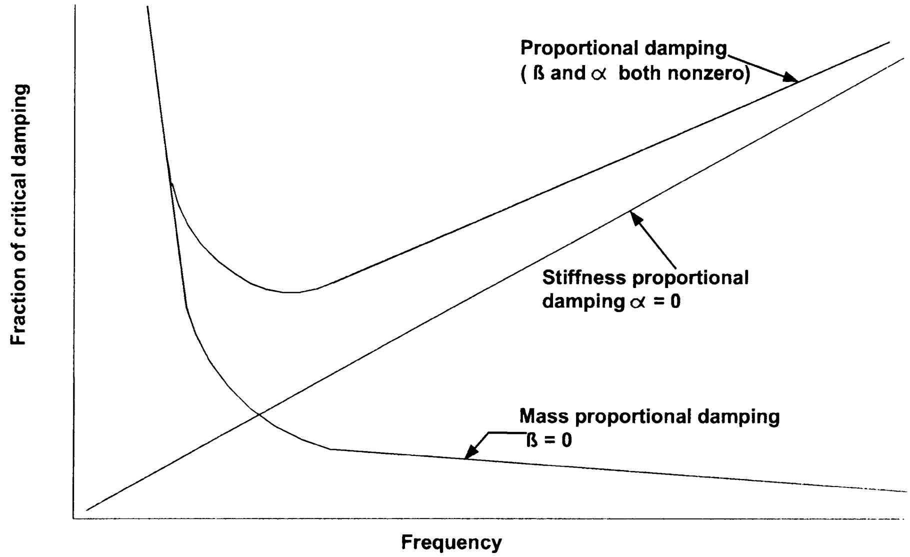


Fig. 2 Proportional damping as a function of frequency and contribution of mass and stiffness damping to overall damping

non-dimensional parameter (NDP) presented below predicts the dependence of response very efficiently.

$$\mathrm{NDP} = \sqrt{(D/\rho h)}/a^2 \tag{16}$$

where D is flexural rigidity of plate, ρ is mass density, h is plate thickness and a is length of side of square plate.

The mass proportional damping sensitivity of plate and its dependence on NDP is discussed in detail later in section 'Numerical Experimentation'.

5. The time stepping scheme

It is well known that considerable computational efforts are needed in nonlinear transient analysis of structures. In the present work, the very popular and easily implemented explicit time integration scheme is being employed for time stepping. During each time step, relatively little computational effort is required, since no stiffness and mass matrices of the complete element assemblage need to be formed, the solution can essentially be carried out on the element level and relatively little high speed storage is required. Using this scheme, system of very large order can be solved effectively. Unfortunately the method is conditionally stable and very small time steps are needed. Therefore the computational advantages of the central difference scheme are counterbalanced by the very small size time step necessary when some stiff and/or small elements are present.

5.1. Central difference approximation

Upon finite element discretization of the dynamic equilibrium equations, a nonlinear system in motion can be written in matrix form as

$$\mathbf{M}\ddot{\mathbf{d}}_n + \mathbf{C}\dot{\mathbf{d}}_n + \mathbf{p}_n = \mathbf{f}_n \tag{17}$$

where **M, C** and $\mathbf{p}_n$ are the global mass, damping and internal resisting nodal forces matrices respectively; **f** is the external load vector; and $\ddot{\mathbf{d}}$ and $\dot{\mathbf{d}}$ are the acceleration and velocity vectors of the finite element assemblage respectively. Instead of satisfying this equation at any time t, in the central difference scheme, it is satisfied only at discrete time intervals Δt apart. This yields

$$\mathbf{d}_{n+1} = \left(\mathbf{M} + \frac{\Delta t}{2}\mathbf{C}\right)^{-1}\left[(\Delta t)^2(-\mathbf{p}_n + \mathbf{f}_n) + 2\mathbf{M}\mathbf{d}_n - \left(\mathbf{M} - \frac{\Delta t}{2}\mathbf{C}\right)\mathbf{d}_{n-1}\right] \tag{18}$$

It should be noted that the displacements at time station $t_n + \Delta t$ are given explicitly in terms of the displacements at time stations t_n and $t_n - \Delta t$.

If the mass matrix **M** and damping matrix **C** are diagonal then the solution of Equation (21) become trivial as follows,

$$d_{n+1}^{(i)} = \left(m_{ii} + \frac{\Delta t}{2}c_{ii}\right)^{-1}\left[(\Delta t)^2\left(-p_n^i + f_n^i\right)\right.$$

$$+2m_{ii}d_n^{(i)} - \left(m_{ii} - \frac{\Delta t}{2}c_{ii}\right)d_{n-1}^{(i)}\Bigg] \tag{19}$$

Where superscript (i) denotes ith component of the corresponding displacement vectors and m_{ii} and c_{ii} are the diagonal terms of the mass and damping matrices. From Equation (19) it is obvious that for each displacement degree of freedom at time $t_n - \Delta t$, there is a separate equation involving information at the degree of freedom at times t_n and $t_n - \Delta t$. No matrix factorization or sophisticated equation solving is therefore necessary.

5.2. Starting algorithm

The governing equilibrium equation at time station $t_n + \Delta t$ in the central difference method involves information at two previous time stations t_n and $t_n - \Delta t$. A starting algorithm is therefore necessary. From initial conditions, the values $\mathbf{d}_{(0-\Delta t)}$ may be obtained as,

$$\mathbf{d}_{(0-\Delta t)} = 2\Delta t\dot{\mathbf{d}}_{(0)} + \mathbf{d}_{(0+\Delta t)} \tag{20}$$

and hence

$$d_1^{(i)} = \frac{(\Delta t)^2}{2m_{ii}}\left(-p_0^{(i)} + f_0^{(i)}\right) + d_0^{(i)} + \Delta t\left(1 - \frac{c_{ii}\Delta t}{2m_{ii}}\right)\dot{d}_0^{(i)} \tag{21}$$

6. Special mass matrix diagonalization scheme

The inertia force vector requires the evaluation of the mass matrix $\mathbf{M}$. This consistent mass matrix is not diagonal and it must be therefore be diagonalized in some way if it is to be useful in the explicit marching scheme. The use of lumped mass matrix need not necessarily incorporate approximation; rather in some cases it is proved to improve accuracy of the solution. For the quadratic isoparametric elements used here, several alternatives were investigated by Hinton et al. [16]. The most efficient scheme found to date could be summarized as follows:

(i) Only the diagonal coefficients of the consistent mass matrix are computed.

$$M = \int_A \mathbf{N}^T \bar{\mathbf{m}} \mathbf{N}\, dA \tag{22a}$$

where

$$\bar{\mathbf{m}} = \begin{bmatrix} I_1 & & & & 0 \\ & I_2 & & & \\ & & I_2 & & \\ & & & I_3 & \\ 0 & & & & I_3 \end{bmatrix} \tag{22b}$$

in which I_1, I_2 and I_3 are normal inertia, rotary inertia and higher-order inertia terms respectively. These are given by,

$$(I_1, I_2, I_3) = \sum_{L=1}^{NN} \int \left(1, z^2, z^6\right)\rho\, dz \tag{22c}$$

and ρ is the material density and NN are number of nodes in an element.

(ii) The total mass of the element is computed,

$$\mathbf{M}_t = \int_{\text{vol}} \rho\; d(\text{vol.}) \tag{23}$$

(iii) The diagonal coefficients M_{ii} associated with translation (but not rotation) degrees of freedom, are summed such that

$$\mathbf{SUM} = \sum \mathbf{M}_{ii} \tag{24}$$

(iv) All the diagonal coefficients of the consistent mass matrix are scaled in the following manner.

$$\mathbf{M}_{ii}^d = \mathbf{M}_{ii}\frac{\mathbf{M}_t}{\mathbf{SUM}} \tag{25}$$

7. Numerical experimentation

The superiority of the present higher order model is already established for linear and non-linear dynamic analysis of plates [9, 10, 17, 18, 22, 25, 26]. This paper is an attempt to highlight damped dynamic response of the elasto-plastic plates. Plates are subjected to transverse dynamic loading, which will incorporate linear as well as elasto-plastic behavior. Isoparametric nine noded lagrangian elements are used to discretize

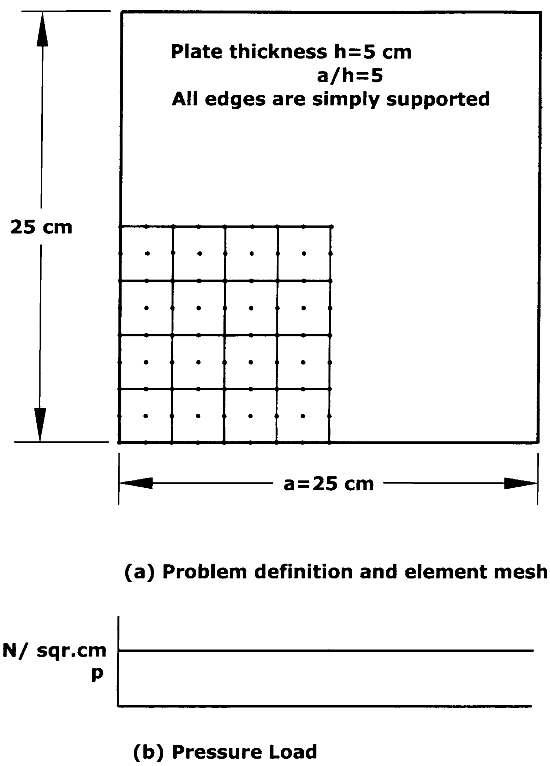


Fig. 3 Plate discretization and geometry of plate for linear dynamic analysis

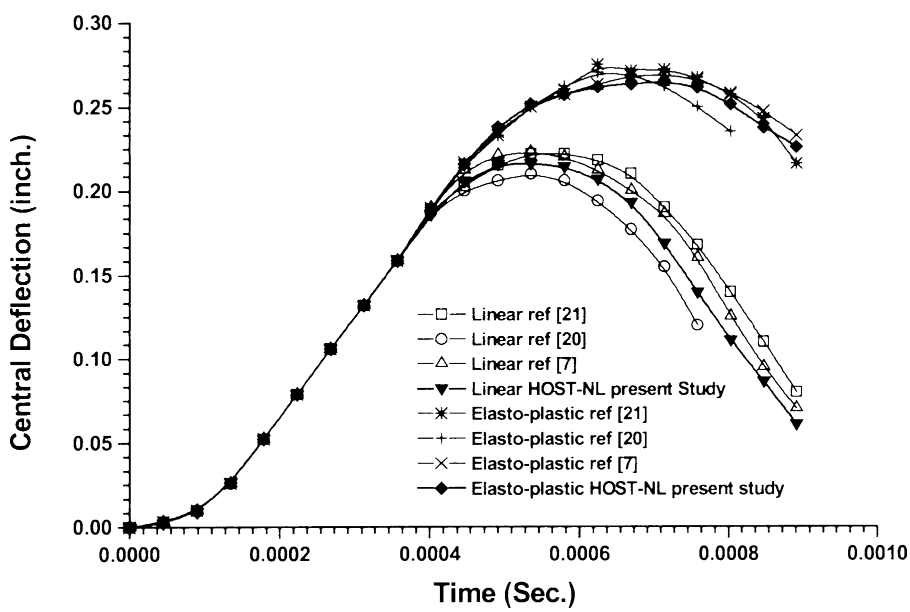


Fig. 4 Comparison of central deflection of simply supported elastic and elasto-plastic plate using different theories

the plate. Programs are developed for analysis of plates using higher order shear deformation theory (HOST-NL) as well as for first order shear deformation theory (FOST-NL) for comparison. In this paper specifically effects of viscous damping are studied on elasto-plastic dynamic response of plates with different boundary conditions. Since the loading considered is symmetrical in nature for all the problems considered

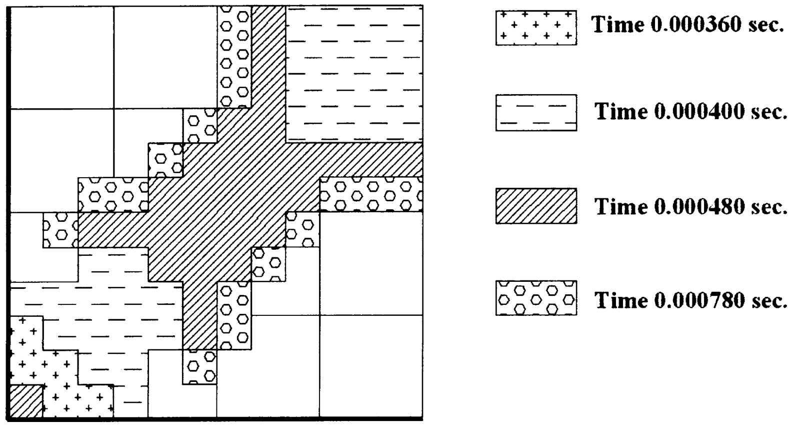


Fig. 5 Progressive yielding of Gauss points for different damping parameters $\alpha = 0.0$ to 0.50 at different time for quarter plate

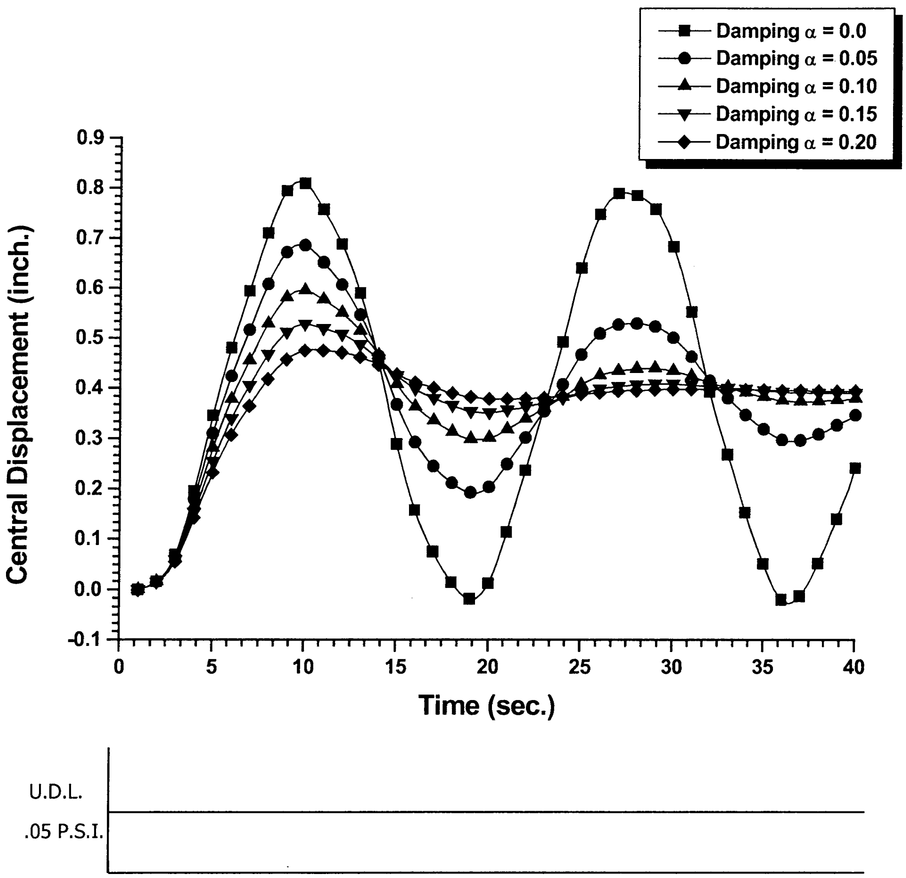


Fig. 6 Effect of damping on simply supported square elastic plate using present HOST-NL formulation and the loading pulse applied

in this study, only quarter of the plate is considered for analysis. The mesh size is arrived at after a convergence study conducted on plates.

Example 1. Simply supported square elastic plate subjected to a uniform step load.

A simply supported plate shown in Fig. 3 is solved for linear response. A uniform pressure $q = 10\,\mathrm{N/cm^2}$ is applied in the z-direction as shown. The plate properties are as follows. $E = 2.1 \times 10^6\ \mathrm{N/mm^2}$, $\nu = 0.25$, $\rho = 8 \times 10^{-6}\ \mathrm{N\ sec^2/cm^4}$. Quarter of plate is discretized

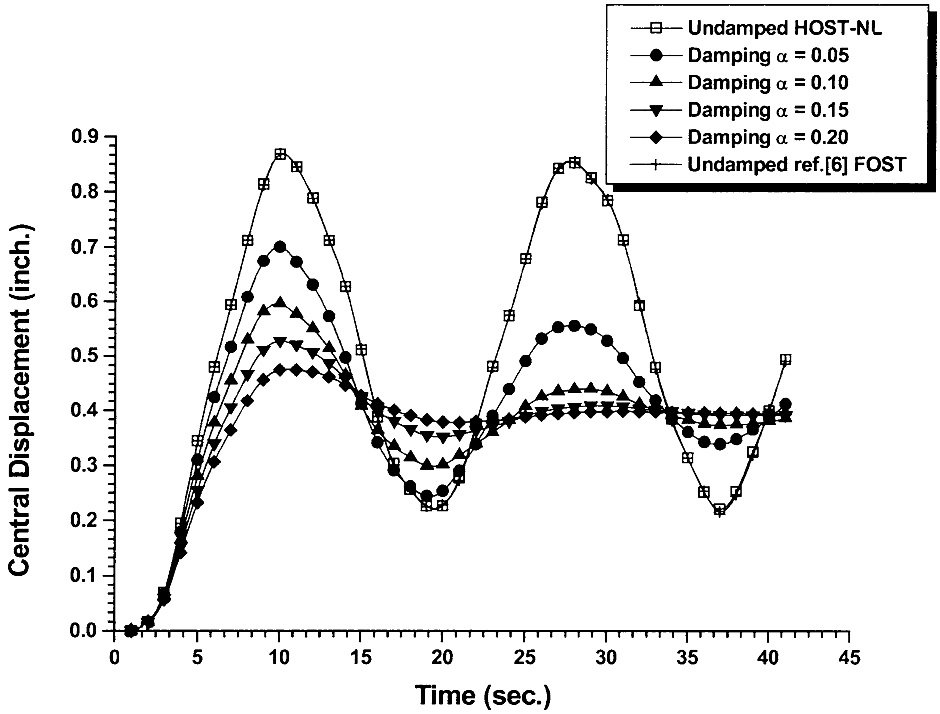


Fig. 7 Damped (HOST-NL) and undamped responses of simply supported square elasto-plastic plate with UDL 0.05 lb/sq. inch

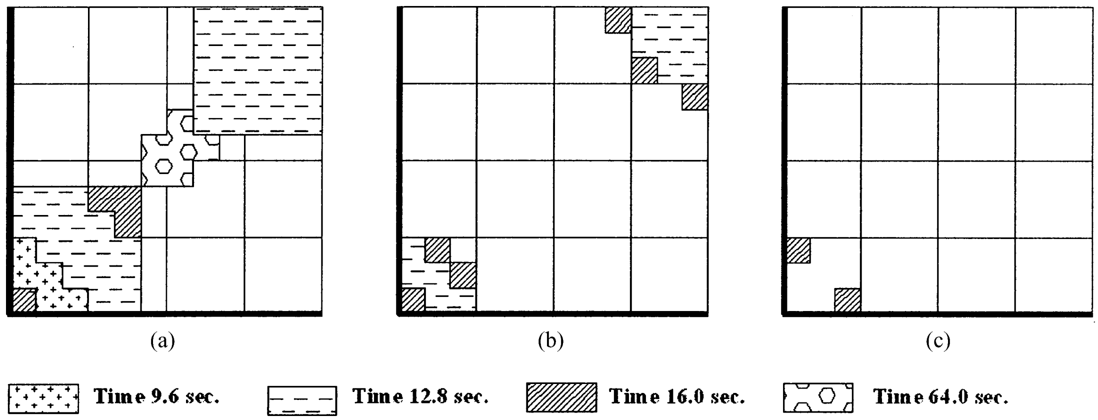


Fig. 8 Progressive yielding of Gauss points for different damping parameters at different time for quarter simply supported plate. (a) Undamped plate (b) Plate with $\alpha = 0.5$ (c) Plate with $\alpha = 0.10$

by 16 elements (4×4 mesh). The results of the linear response of HOST-NL are shown in Table 1 and are compared with other Finite Element solution [19], along with different degree of damping in terms of α. The plate is moderately thick plate. It is evident from Table 1 that the FOST solution [19] is under predicting the displacements. As mentioned previously, the non-dimensional parameter devised and introduced, as per (16) so as to ascertain the vulnerability of plate to absolute damping i.e. mass proportional damping for the present plate has a value of 6110.101. It is clear from Table 1 that there are little changes in displacements with increase in α because of high value of NDP. The plate is rated as extremely insensitive to viscous damping in Table 3.

Example 2. Simply supported square plate subjected to a uniform step load.

Table 1 Central deflection in cm for undamped and damped response of simply-supported elastic plate

			Present HOST-NL				
Time (μs)	Mindlin Theory [33]	FOST-NL	Undamped $\alpha = 0.0$	Damped $\alpha = 0.05$	Damped $\alpha = 0.10$	Damped $\alpha = 0.15$	Damped $\alpha = 0.20$
20	3.9900×10^{-5}	5.0168×10^{-5}	5.0300×10^{-5}	5.0300×10^{-5}	5.0300×10^{-5}	5.0300×10^{-5}	5.0300×10^{-5}
40	1.8550×10^{-4}	2.1852×10^{-4}	2.1870×10^{-4}	2.1870×10^{-4}	2.1870×10^{-4}	2.1870×10^{-4}	2.1870×10^{-4}
60	5.3390×10^{-4}	5.8715×10^{-4}	5.9316×10^{-4}	5.9316×10^{-4}	5.9316×10^{-4}	5.9316×10^{-4}	5.9316×10^{-4}
80	9.2490×10^{-4}	9.6969×10^{-4}	9.7144×10^{-4}	9.7144×10^{-4}	9.7144×10^{-4}	9.7144×10^{-4}	9.7143×10^{-4}
100	1.2278×10^{-3}	1.2717×10^{-3}	1.2691×10^{-3}	1.2691×10^{-3}	1.2691×10^{-3}	1.2691×10^{-3}	1.2691×10^{-3}
120	1.4591×10^{-3}	1.4904×10^{-3}	1.4953×10^{-3}	1.4953×10^{-3}	1.4953×10^{-3}	1.4953×10^{-3}	1.4953×10^{-3}
140	1.6537×10^{-3}	1.6768×10^{-3}	1.6819×10^{-3}	1.6819×10^{-3}	1.6819×10^{-3}	1.6819×10^{-3}	1.6819×10^{-3}
160	1.6667×10^{-3}	1.6624×10^{-3}	1.6563×10^{-3}	1.6563×10^{-3}	1.6563×10^{-3}	1.6563×10^{-3}	1.6563×10^{-3}
180	1.4604×10^{-3}	1.4573×10^{-3}	1.4429×10^{-3}	1.4429×10^{-3}	1.4429×10^{-3}	1.4429×10^{-3}	1.4429×10^{-3}
200	1.1728×10^{-3}	1.1353×10^{-3}	1.1331×10^{-3}	1.1331×10^{-3}	1.1331×10^{-3}	1.1331×10^{-3}	1.1331×10^{-3}
220	8.6690×10^{-4}	8.4738×10^{-4}	8.4444×10^{-4}	8.4444×10^{-4}	8.4444×10^{-4}	8.4444×10^{-4}	8.4444×10^{-4}
240	5.4100×10^{-4}	5.0252×10^{-4}	4.8393×10^{-4}	4.8393×10^{-4}	4.8393×10^{-4}	4.8394×10^{-4}	4.8394×10^{-4}
260	1.7110×10^{-4}	1.6128×10^{-4}	1.4904×10^{-4}	1.4904×10^{-4}	1.4905×10^{-4}	1.4905×10^{-4}	1.4906×10^{-4}
280	-4.1100×10^{-6}	-2.6418×10^{-5}	-2.0506×10^{-5}	-2.0500×10^{-5}	-2.0490×10^{-5}	-2.0490×10^{-5}	-2.0480×10^{-5}
300	9.0000×10^{-5}	4.4036×10^{-6}	1.3500×10^{-5}	1.3500×10^{-5}	1.3510×10^{-5}	1.3510×10^{-5}	1.3520×10^{-5}
320	1.0450×10^{-4}	1.2273×10^{-4}	1.2104×10^{-4}	1.2105×10^{-4}	1.2105×10^{-4}	1.2106×10^{-4}	1.2106×10^{-4}
340	2.9580×10^{-4}	3.3330×10^{-4}	3.3272×10^{-4}	3.3272×10^{-4}	3.3272×10^{-4}	3.3273×10^{-4}	3.3273×10^{-4}
360	6.2950×10^{-4}	6.5210×10^{-4}	6.8463×10^{-4}	6.8463×10^{-4}	6.8463×10^{-4}	6.8463×10^{-4}	6.8463×10^{-4}

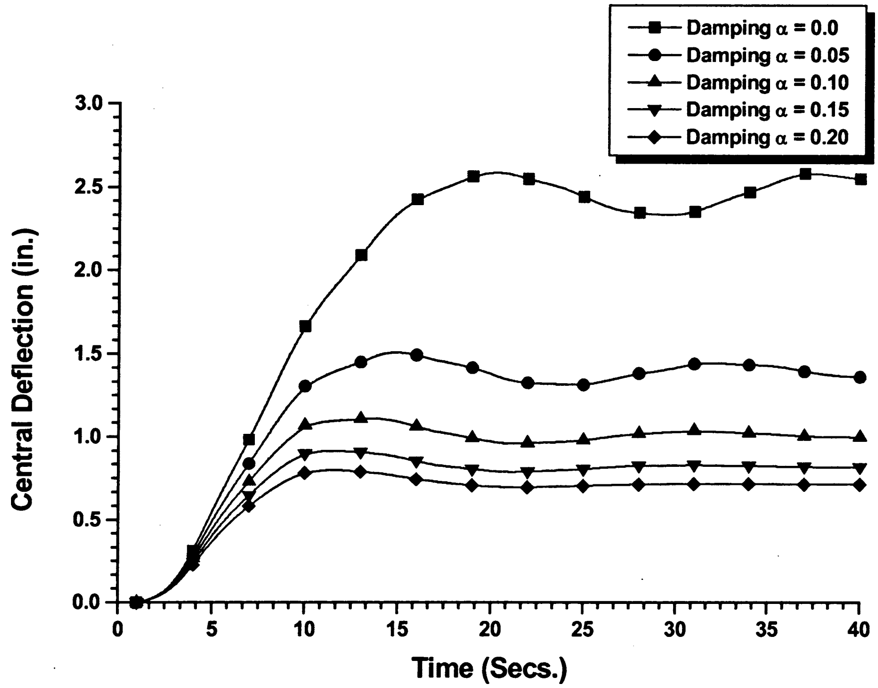


Fig. 9 Effect of damping on responses of simply supported square elasto-plastic plate with higher UDL 0.08 lb/sq. inch using HOST-NL

This problem is one of the standard problems. All sides of plate are simply supported. Both elastic and elasto-plastic materials are considered. Pressure $q = 300$ psi (2.07 MPa) is applied in the z-direction. The plate properties are as follows. $E = 1 \times 10^7$ lb/in^2 (69.0 $\times 10^3$ MPa), $\nu = 0.3$, $\rho = 0.259 \times 10^{-6}$ lb sec^2/in^4 (2768 kg/m^3) and $\sigma_y = 30$ ksi (207 MPa). The present formulation (HOST-NL) results are compared with those obtained with analytical solution of Liu and Lin [20], first order solution [7] and classical plate theory solution [21] for elastic and elasto-plastic materials respectively. Time histories for elastic and elasto-plastic of the deflection of the center point without damping are plotted in Fig. 4. It is clear that even for thin plate, the HOST-NL results are matching well with the various thin plate solutions. Thus the present formulation is effective in linear as well as nonlinear zone for thin plates. The effect of damping is presented in Table 2. The value of non-dimensional parameter in this case is 148.65. As in previous case the changes in response with change in α are very minute due to high value of non-dimensional damping parameter. The same observation is emphasized in Fig. 5, where it is clearly evident that the yielded positions of Gauss point are unchanged, irrespective of value of damping parameter α (which is varying from 0.05 to 0.50).

Example 3. Linear and non-linear response of a simply supported square plate and non-linear response of clamped square plate under a uniformly distributed impulsive loading.

A square plate is subjected to a uniformly distributed impulsive load. The plate size is $32'' \times 32'' \times 4''$ (812.8 mm $\times$ 812.8 mm $\times$ 101.6 mm) along with following properties. $q = 0.05$ lb/in^2 (3.45×10^{-4} N/mm^2), $E = 100$ lb/in^2(0.6897 N/mm^2), $\nu = 0.3$, $\rho = 1.0$ lb-sec^2/in^4 (1.0687×10^7 kg/m^3), $\Delta T = 0.08$ sec. and yield moment/unit length = 2.0 lb in/in (8.9 N mm/mm). Elastic response of the plate to different degrees of damping is obtained for this plate as shown in Fig. 6 using HOST-NL. Elasto-plastic small deflection response for von Mises yield criteria for HOST-NL for simply supported case is compared with reference [6] first order undamped solution in Fig. 7. The effect of damping is presented using HOST-NL. Figure 8 depicts effects of damping on yielding of Gauss points, area-wise as well as along time dimension. The effects as can be seen in Fig. 8 are very prominent. The increase in damping delays the inception of plasticity in the plate and also curbs the area of yielded Gauss points substantially. In order to investigate the response of the same plate to higher load the plate is subjected to impulse of $q = 0.08$ lb / in^2 (5.52×10^{-4} N/mm^2) and the effect of

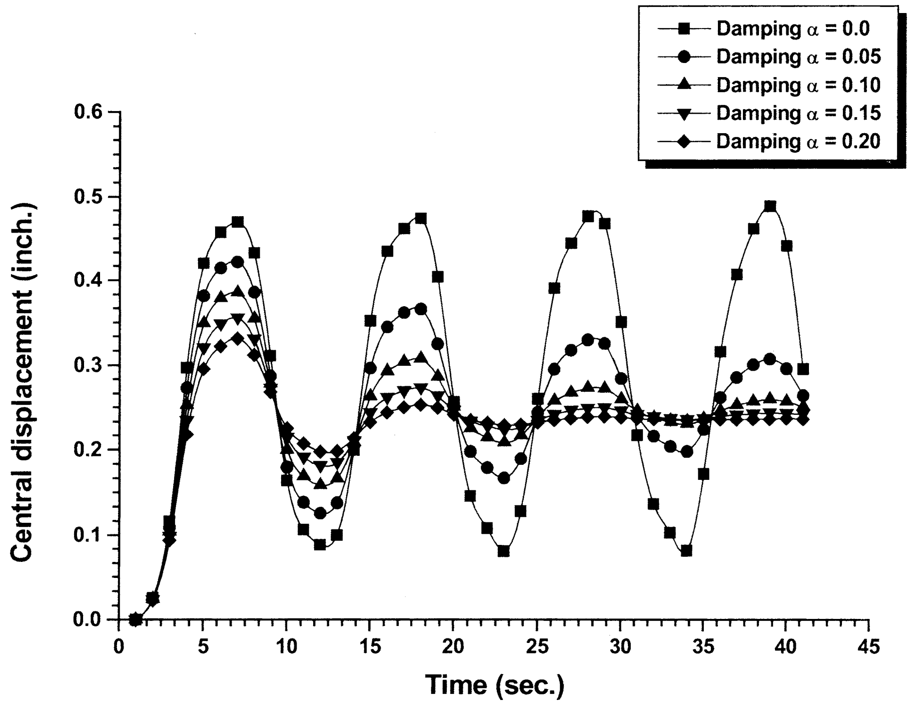


Fig. 10 Effect of damping on responses of clamped square elasto-plastic plate with UDL 0.08 lb/sq. inch, using HOST-NL

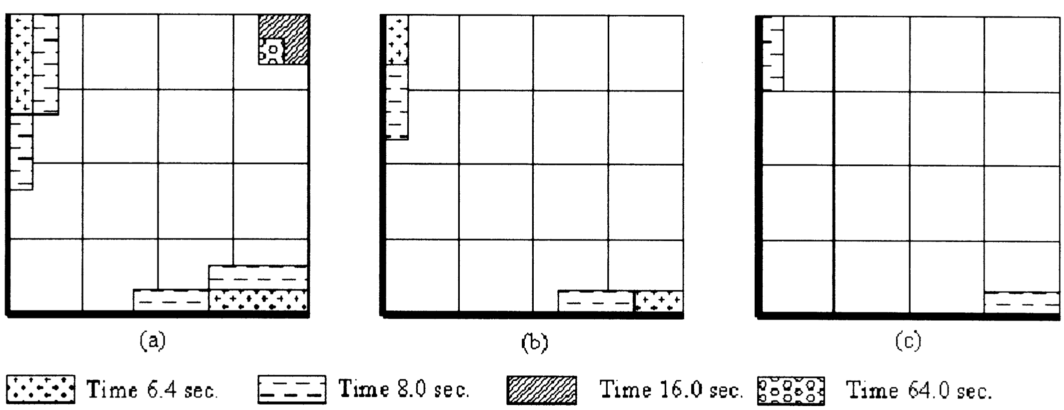


Fig. 11 Progressive yielding of gauss points for different damping parameters at different time for quarter fixed plate, (a) Undamped plate (b) Plate with $\alpha = 0.10$ (c) Plate with $\alpha = 0.20$

different values of α is noted. The response of simply supported plate with higher load is shown in Fig. 9. To study the effect of boundary conditions, the same plate with clamped boundary condition and q = 0.08 lb / in^2(5.52×10^{-4} N/mm^2) was subjected to different degree of damping as shown in Fig. 10. The central deflection in case of simply supported plate is many times more in comparison to clamped plate under same condition of loading, material and geometry as expected. The effective frequency of vibration is more in clamped plates as compared to simply supported plate. The Gauss point's yielding dependence on damping parameter is shown in Fig. 11. In all these cases the non-dimensional parameter has very low value of 0.047283. As a consequence the plate is sensitive to applied damping. The response varies to a great deal with changes in α. The sensitivity of plate is independent of boundary conditions.

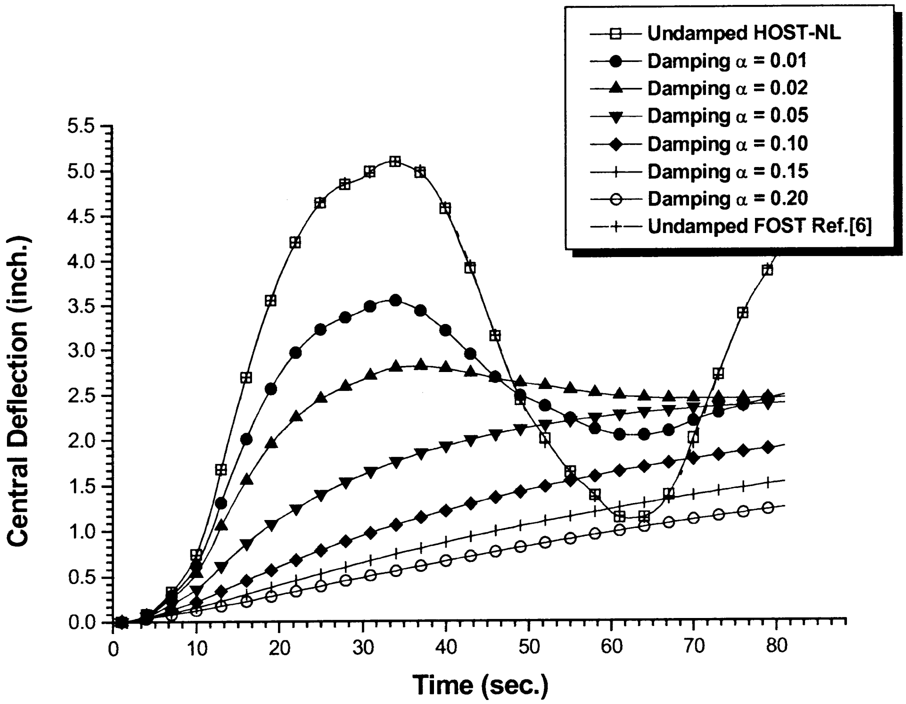


Fig. 12 Damped (HOST-NL) and undamped central deflection of circular elasto-plastic clamped plate

Example 4. Non-linear Response of a clamped circular plate under a uniformly distributed step load.

A clamped circular plate is modeled using 20 elements in symmetric quarter and is subjected to a uniformly distributed step load. The plate diameter is 200 inches (5080 mm) and thickness is 20 inches (508 mm) along with following properties. $q = 0.1$ lb/in^2 (6.897×10^{-4} N/mm^2), $E = 100$ lb/in^2 (0.6897 N/mm^2), $\nu = 0.3$, $\rho = 10$ lb sec^2/in^4 (1.0687×10^8 kg/m^3), $\Delta T = 0.6$ sec. and yield moment/ unit length = 150 lb in /in (667.47 N mm/mm). The elasto-plastic small deflection response for von Mises yield criterion for HOST-NL solution is compared with reference [6] (a first order solution) along with effect of damping in Fig. 12. The non-dimensional parameter has still lower value of 0.030261. As a consequence the plate is extremely sensitive to applied damping. The response variation with changes in α is very conspicuously evident from Fig. 12.

8. Conclusions

A formulation using higher-order shear deformation theory (HOST-NL) is presented for damped elasto-plastic dynamic bending analysis of plates. It is well acknowledged and established that the C^0 HOST formulation has certain edge over first-order shear deformation theory owing to its more realistic assumption of transverse shear deformation over the thickness. Moreover this formulation does not use approximation in the form of shear correction coefficient. Main aim of this paper was to study the response of the plate under different degrees of damping measured by α (defined previously) and to justify the varied behavior. The sensitivity of response of the plate to absolute damping i.e. mass proportional damping is seen to be dependent on non-dimensional parameter (NDP). For the larger NDP values the plate is observed to be insensitive to damping considered. As is summarized in Table 3, the sensitivity of plate is described qualitatively in relation to NDP and in terms of minimum value of damping coefficient α, which introduced appreciable change in central deflection response of different plates. For the sensitive plates, it is found that with increase in damping coefficient α the central displacement decreases without affecting effective period of vibration of plate as is true in case of elastic plates. In contrast to damped behavior of elastic plate, where the central displacement response approaches the equilibrium or

Table 2 Central deflection in inches for undamped and damped response of simply-supported elastic and elasto-plastic plate using HOST-NL

Time (Sec.)	Undamped $\alpha = 0.0$	Damped $\alpha = 0.05$	Damped $\alpha = 0.10$	Damped $\alpha = 0.15$	Damped $\alpha = 0.20$	Damped $\alpha = 0.50$
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.000054	0.003281	0.003281	0.003281	0.003281	0.003281	0.003281
0.000108	0.015097	0.015097	0.015097	0.015097	0.015097	0.015097
0.000162	0.042315	0.042315	0.042315	0.042315	0.042315	0.042314
0.000216	0.077072	0.077071	0.077071	0.077071	0.077071	0.077069
0.000270	0.108202	0.108201	0.108201	0.108200	0.108200	0.108197
0.000324	0.139499	0.139498	0.139497	0.139496	0.139496	0.139491
0.000378	0.172115	0.172114	0.172113	0.172112	0.172111	0.172105
0.000432	0.207728	0.207726	0.207725	0.207724	0.207722	0.207713
0.000486	0.236146	0.236144	0.236142	0.236140	0.236138	0.236126
0.000540	0.252293	0.252290	0.252288	0.252286	0.252283	0.252269
0.000594	0.258815	0.258813	0.258810	0.258807	0.258804	0.258787
0.000648	0.262523	0.262520	0.262517	0.262513	0.262510	0.262491
0.000702	0.264495	0.264491	0.264488	0.264484	0.264480	0.264458
0.000756	0.261711	0.261706	0.261702	0.261698	0.261693	0.261667
0.000810	0.249037	0.249032	0.249027	0.249022	0.249018	0.248989
0.000864	0.232750	0.232745	0.232740	0.232735	0.232730	0.232702
0.000918	0.218802	0.218798	0.218793	0.218788	0.218783	0.218754
0.000972	0.204767	0.204762	0.204757	0.204753	0.204748	0.204718
0.001026	0.188168	0.188163	0.188159	0.188154	0.188149	0.188121
0.001080	0.172070	0.172066	0.172061	0.172057	0.172053	0.172027
0.001134	0.164490	0.164486	0.164482	0.164479	0.164475	0.164452
0.001188	0.162823	0.162819	0.162816	0.162812	0.162809	0.162788
0.001242	0.162799	0.162796	0.162793	0.162790	0.162787	0.162768
0.001296	0.165516	0.165514	0.165511	0.165508	0.165505	0.165489
0.001350	0.176204	0.176201	0.176199	0.176196	0.176194	0.176179
0.001404	0.193402	0.193399	0.193397	0.193394	0.193392	0.193376
0.001458	0.208531	0.208529	0.208526	0.208523	0.208520	0.208504
0.001512	0.222428	0.222425	0.222422	0.222419	0.222416	0.222398
0.001566	0.237431	0.237427	0.237424	0.237421	0.237418	0.237398
0.001620	0.253725	0.253721	0.253717	0.253713	0.253709	0.253686
0.001674	0.263247	0.263243	0.263238	0.263234	0.263229	0.263202
0.001728	0.264246	0.264241	0.264236	0.264231	0.264226	0.264197
0.001782	0.263687	0.263681	0.263676	0.263671	0.263666	0.263634

Table 3 Effect of NDP on damped response of plate

Sr. No.	NDP Value	Minimum α to which plate is sensitive	Qualitative response of a plate to α
1.	6110.101	–	Extremely insensitive
2.	148.650	–	Insensitive
3.	0.047238	0.05	Sensitive
4.	0.030261	0.01	Very Sensitive

steady state position of oscillations, the elasto-plastic response is characterized by rigid body shift on lower side. This effect is more pronounced in case of plate with higher load and very low value of NDP, which will eventually incorporate more elasto-plasticity. One important conclusion can be drawn from the above discussion that in case of elastic plates damping only reduces the period for which the plate shall continue to vibrate while the equilibrium position or permanent deformation is independent of damping. On the other hand for elasto-plastic plate the permanent deformation is function of damping and is inversely proportional to damping. For plates with lower NDP, damping introduces varying degree of elasto-plasticity. The more is value of α the lesser number of Gauss points yield while the plate vibrates. Further it can be stated that damping delays inception of plasticity for sensitive plates. The sensitivity to damping is independent of boundary conditions. No unexpected qualitative response was observed for elastic and elasto-plastic plates with change in boundary conditions.

References

1. Reissner, E.: The effects of transverse shear deformation on the bending of elastic plates. ASME J. Appl. Mech. **12**, A69–A77 (1945)
2. Mindlin, R.D.: Influence of rotatory inertia and shear and flexural motions on isotropic elastic plates. ASME J. Appl. Mech. **18**(1), 31–38 (1951)
3. Kant, T.: Numerical analysis of thick plates. Comput. Methods Appl. Mech. Eng. **31**, 1–18 (1982)
4. Lo, K.H., Christensen, R.M., Wu, E.M.: A higher order theory of plate deformation: Part 1. Homogeneous plates. ASME J. Appl. Mech. **99**, 663–668 (1977)
5. Hinton, E., Owen, D.R.J., Shantaram, D.: Dynamic transient linear and nonlinear behavior of thick and thin plates. In: J. R. Whiteman (ed.) Mathematics of Finite Elements and Applications, MAFELAP II. Academic Press, London, pp. 423–438 (1977)
6. Owen, D.R.J., Hinton, E., Shantaram, D.: Nonlinear dynamic transient analysis of plates using parabolic isoparametric elements. In: Finite Element Methods in Engineering. The University of Adelaide 44/1–16 (1976)
7. Paul, D.K., Huq, M.M., Hinton, E.: Nonlinear static and transient dynamic analysis of Mindlin plates. In: First International Conference on Numerical Methods for Nonlinear Problems, University College of Swansea, pp. 221–235 (1980)
8. Shantaram, D., Owen, D.R.J., Zeinkiewicz, O.C.: Dynamic transient behavior of two- and three-dimensional structures including plasticity, large deformation effects and fluid interaction. Earthquake Eng. Struct. Dyn. **4**, 561–578 (1976)
9. Kant, T., Mallikarjuna: Nonlinear dynamics of laminated plates with a higher order theory and C^0 finite elements. Int. J. Non-Linear Mech. **26**(3/4), 335–343 (1991)
10. Kant, T. Kommineni, J.R.: Large amplitude free vibration analysis of cross-ply composite and sandwich laminates with a refined theory and C^0 finite elements. Comput. Struct. **50**, 123–134 (1994)
11. Tresca, H.: Surl'ecoulement des corps solides soumis a de fortes pression. C. R. Acad. Sci. Paris **59**, 754 (1864)
12. von Mises. R.: Mechanicsanik der festen korper in plastisch deformablen zustant. Nachr. Ges. Wiss. Gottingen 582 (1930)
13. Owen, D.R.J., Hinton, E., Finite Elements in Plasticity: Theory and Practice, 1st ed., Pineridge Press, Swansea (1980)
14. Underwood, P.: Dynamic relaxation, In: T. Belytschko and TJR. Hughes (eds.) Computational Methods for Transient Analysis. Elsevier Science Publisher, Amsterdam, pp. 245–265 (1983)
15. Pica, A. and Hinton, E.: Further developments in transient and pseudo-transient analysis of Mindlin plates. Int. J. Num. Methods Eng. **17**, 1749–1761 (1981)
16. Hinton, E., Rock, T., Zienkiewicz, O.C.: A note on mass lumping and related processes in the finite element method. Earthquake Eng. Struct. Dyn. **4**, 245–249 (1976)
17. Khante, S.N., Rode, V.R.: Elasto-plastic dynamic analysis of plates using higher order shear deformation theory. In: P.K., K., Gupta, Manoj (eds.) Structural Engineering and Mechanics BITS Pilani, pp. 224–234 (2004)
18. Khante, S.N., Rode, V.R.: Effect of tangent modulus on transient non-linear response of plates using higher order shear deformation theory. In: Sudarsana Rao (ed.) Recent Treads in Structural Engineering. JNTU College of Engineering, Anantpur, pp. 222–231 (2004)
19. Reddy, J.N.: Dynamic (transient) analysis of layered anisotropic composite material plates. Int. J. Num. Methods Eng. **19**, 237–255 (1983)
20. Liu, S.C., Lin, T.H.: Elastic-plastic dynamic analysis of structures using known elastic solutions. Earthquake Eng. Struct. Dyn. **7**, 147–159 (1979)
21. Bathe, K.J., Bolourchi, S.A.: Geometric and material nonlinear plate and shell elements. Comput. Struct. **11**, 23–48 (1980)
22. Kant, T., Ravichandran, R.V., Pandya, B.N. Mallikarjuna: Finite element transient dynamic analysis of isotropic and fiber reinforced composite plates using higher order theory. Composite Struct. **19**, 319–342 (1988)
23. Chandrasekharappa, G., Srirangarajan, H.R.: Nonlinear dynamic damped response of an orthotropic circular plate. Comput. Struct. **33**(5), 1163–1165 (1989)
24. Cook, R.D., Malkus, D.S., Plesha, M.E.: Concepts and applications of finite element analysis. Wiley, NY (2000)
25. Kant, T., Swaminathan, K.: Analytical solutions for free vibration of laminated composite and sandwich plates based on a higher-order refined theory. Composite Struct. **53**, 73–85 (2001)
26. Kant, T., Swaminathan, K.: Analytical solutions for the static analysis of laminated composite and sandwich plates based on a higher order refined theory. Composite Struct. **56**, 329–344 (2002)
27. Khante, S.N., Rode, V.R.: Non-linear dynamic bending analysis of plates using a higher order shear deformation theory. Nonlinear Dyn. **43**, 257–275 (2006)