

A General Partial Discretization Methodology for Interlaminar Stress Computation in Composite Laminates

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Abstract: A two-point boundary value problem (BVP) is formed in the present work governed by a set of first-order coupled ordinary differential equations (ODEs) in terms of displacements and the transverse stresses through the thickness of laminate (in domain $-h/2 < z < h/2$) by introducing partial discretization methodology only in the plan area of the three dimensional (3D) laminate. The primary dependent variables in the ODEs are those which occur naturally on a plane $z=a$ constant. An effective numerical integration (NI) technique is utilized for tackling the two-point BVP in an efficient manner. Numerical studies on cross-ply and angle-ply composite plates are performed and presented, involving both validation and solution of new problems.

Keyword: composite laminates, partial finite element, boundary value problem, initial value problem, numerical integration method

1 Introduction

Use of fiber reinforced polymer composite (FRPC) materials has grown rapidly in the last three decades, especially in weight sensitive structural members, owing to their high strength-to-weight and stiffness-to-weight ratios. Therefore, analyses of laminated composite plates have been of significant concern in many advanced engineering structures. Furthermore, study concerning failure due to delamination is of considerable

importance. This involves separation of composite laminae along the interfaces due to high transverse/interlaminar stresses. Thus an analytical-numerical methodology which can predict these stresses accurately becomes necessary for understanding failure mechanism due to delamination.

A 3D elasticity solution of laminated composite beams, plates and shells is extremely complex. Pagano (1969, 1970), Srinivas and Rao (1970) and Srinivas et al. (1970) have given flexure, vibration and buckling response of simply supported plates by solving analytically the governing BVP defined by 3D partial differential equation (PDE) system. Their solutions have been used in the over the last three decades, as benchmark solution by researchers especially involved in developing general numerical techniques. However, 3D elasticity solutions are available only for simple geometry and boundary conditions and hence solutions lack generality. Vel and Batra (1999), for example, presented 3D analytical solutions for different boundary conditions by a generalization of Eshelby-Stroh formalism. Solution was obtained in form of infinite series with infinite number of unknowns. In such an approach, accuracy of solution depends on number of terms considered in the series and more number of terms in the series generally leads to increased computational efforts.

Family of conventional equivalent single layer (ESL) theories includes classical laminated plate theory (CLPT) (Jones 2003), first-order shear deformation theory (FOST) (Reissner 1945, Mindlin 1951) and higher-order shear deformation theories (HOSTs) (Lo et al. 1977, Kant 1982, Reddy 1984, Kant and Manjunatha 1994, Kant and Swaminathan 2002). The first two ESL theories have generally produced satisfactory results

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for global analysis of thin laminated composites (e.g. deflections and flexural stresses). However, CLPT and FOST based analytical or FE models generally yield poor results for thick or moderately thick laminated composites because they cannot precisely model deformed kinematic configuration of laminated composites. HOST models have been advantageously employed for such situation. However, the transverse interlaminar stresses are most commonly estimated using a post-processing technique by integration 3D equilibrium equations of elasticity along the laminate thickness and there is serious limitations in the ESL theories. The estimates are not only inaccurate but the methods are unreliable and the whole methodology lacks robustness. Therefore, accurate and reliable evaluation/computation of the transverse stresses have not been addressed adequately (Kant and Swaminathan 2002) in ESL theories.

A number of displacement based layerwise theories have also been proposed by Reddy (1987), Soldatos (1992), Wu and Kuo (1993), Wu and Hsu (1993) and others. However, only continuity of displacement field through thickness of laminates could be satisfied in such displacement based layerwise models and continuity of the transverse stresses at the laminae interfaces could not be enforced. Elasticity solutions of layered components (Pagano 1969, 1970) indicate that the interlaminar continuity of the transverse normal and shear fields as well as layerwise continuous displacement fields through the thickness of laminated components are essential requirements for their accurate analysis. Thus, layerwise analysis with mixed (displacements and transverse stresses) primary variables is often required for laminated composite structures. In order to improve the displacement based layerwise theories, a group of researchers including Spilker (1984), Shin and Chen (1992), Wu and Lin (1993), Ramtekkar et al. (2002, 2003), Cazzani et al. (2005) have worked on development of layerwise mixed FE models with displacements and the transverse stresses as primary variables of interest. Such models satisfy continuity requirements of displacements and the transverse stresses

at laminae interfaces.

The various simple analytical/FE models developed in past based on the assumed variations over the global/element domain in all considered directions in space. An attempt is made here to present a general method starting from the fundamentals, that is, the exact 3D partial differential equation (PDE) system of a laminated composite. A mixed partial finite element model is developed for elastostatic of composite plates in such a way that it results in solution of a two-point BVP governed by a system of first-order ODEs,

$$\frac{d}{dz}\mathbf{y}(z) = \mathbf{A}(z)\mathbf{y}(z) + \mathbf{p}(z) \quad (1)$$

in the interval $-h/2 \leq z \leq h/2$ with any half of the dependent variables prescribed at the edges $z = \pm h/2$. The solution vector $\mathbf{y}(z)$ consists of a set of primary variables whose number equals the order of PDE system times the number of discrete FE mesh nodes. Availability of efficient, accurate and above all proven robust ODE numerical integrators for initial value problems (IVPs) helps in obtaining the set of primary variables at all nodal points through the thickness. Ingenuity lies here in transforming the BVP into a set of IVPs (Kant and Ramesh 1981). Once the fundamental set is known, the auxiliary set of dependent variables over the entire nodal set can be simply computed by substitution of values of the fundamental set of variables on the right hand side of algebraic expressions node-by-node. Accuracy of the model is verified against the results available in literature for laminated composite plates. New results have also been presented for the clamped supported composite plates using standard material properties available in the literature.

2 Formulation

A laminate composed of a number of isotropic/orthotropic, linear elastic laminae of uniform thicknesses with plan dimension $a \times b$ and thickness ' h ' is considered (Fig. 1). The angle between the fiber direction and reference axis ' x ' is measured in anticlockwise direction as shown in Fig. 1. Transversely distributed load is applied on the top surface of laminate.

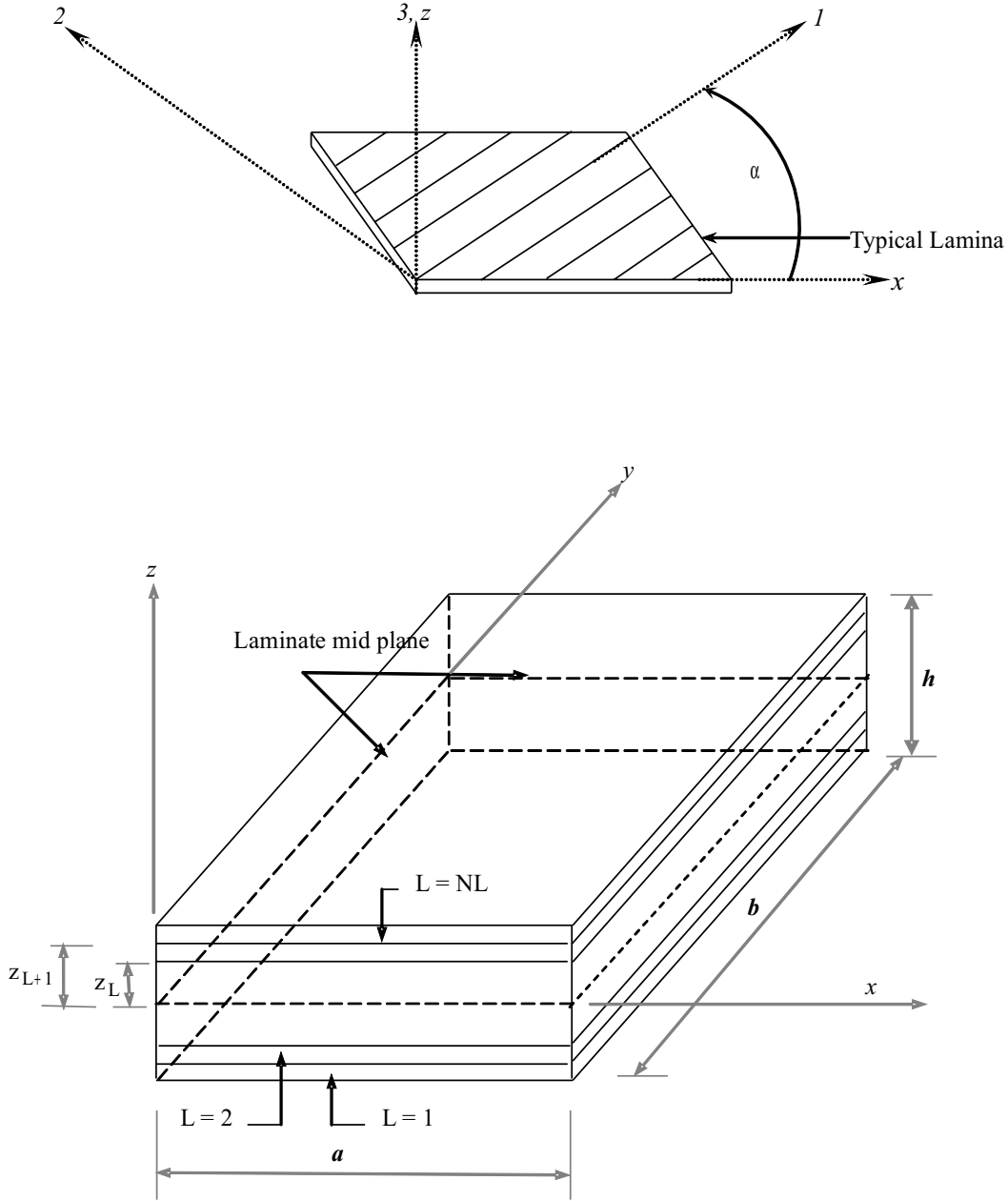


Figure 1: Laminate geometry with positive set of lamina/laminate reference axes and fibre orientation

2.1 Constitutive relations

Each lamina in the laminate has been considered to be in a 3D state of stress so that the constitutive relation for a typical orthotropic i^{th} lamina with reference to the principal material coordinate axes (1, 2 and 3) can be written as

$$(\varepsilon_1)^i = \left(\frac{1}{E_1} \sigma_1 - \frac{\nu_{21}}{E_2} \sigma_2 - \frac{\nu_{31}}{E_3} \sigma_3 \right)^i$$

$$\begin{aligned} (\varepsilon_2)^i &= \left(-\frac{\nu_{12}}{E_1} \sigma_1 + \frac{1}{E_2} \sigma_2 - \frac{\nu_{32}}{E_3} \sigma_3 \right)^i \\ (\varepsilon_3)^i &= \left(-\frac{\nu_{13}}{E_1} \sigma_1 - \frac{\nu_{23}}{E_2} \sigma_2 + \frac{1}{E_3} \sigma_3 \right)^i \\ (\gamma_{12})^i &= \left(\frac{\tau_{12}}{G_{12}} \right)^i ; \quad (\gamma_{13})^i = \left(\frac{\tau_{13}}{G_{13}} \right)^i \\ \text{and } (\gamma_{23})^i &= \left(\frac{\tau_{23}}{G_{23}} \right)^i \end{aligned} \quad (2)$$

These can be also written as,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}^i = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ \text{Sym.} & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}^i \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix}^i \quad (3)$$

where $\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{13}, \tau_{23}$ are stresses and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{13}, \gamma_{23}$ are linear strain components with reference to the lamina coordinates 1, 2, and 3. C_{mn} 's ($m, n = 1, \dots, 6$) are the elastic constants of the i^{th} lamina.

The stress-strain relations for the i^{th} lamina with reference to global laminate axes (x, y, z) can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ & & Q_{33} & Q_{34} & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ \text{Sym.} & & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (4)$$

where $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are stresses and $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ are strains components with respect to laminate axes (x, y, z) and Q_{mn} 's ($m, n = 1, \dots, 6$) are the transformed elasticity constants of the i^{th} lamina with reference to the laminate axes. Elements of matrices $[C]$ and $[Q]$ can be found in any standard text (Jones 2003).

2.2 Strain-displacement relations

General 3D linear strain-displacement relations are

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} & \varepsilon_y &= \frac{\partial v}{\partial y} & \varepsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned} \quad (5)$$

2.3 Equilibrium equations

The 3D differential equations of equilibrium are,

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + B_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + B_y &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z &= 0 \end{aligned} \quad (6)$$

Here, B_x, B_y and B_z are components of body force per unit volume in x, y and z directions, respectively.

2.4 Partial differential equations

Eqs. (4)-(6) have a total of fifteen unknowns; six stresses ($\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$), six strains ($\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$) and three displacements (u, v, w) in fifteen equations. The transverse stresses and the displacements (Fig. 2) are continuous through laminae interfaces for perfect bonded laminate (Pagano 1969, 1970). These conditions are naturally enforced in the present formulation. After a simple algebraic manipulation, PDEs in terms of only six particular dependent variables $u, v, w, \tau_{xz}, \tau_{yz}$ and σ_z are obtained as follows

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})} [-Q_{65}\tau_{yz} + Q_{66}\tau_{xz}] - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} &= \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})} [Q_{55}\tau_{yz} - Q_{56}\tau_{xz}] - \frac{\partial w}{\partial y} \end{aligned}$$

$$\begin{aligned}
 \frac{\partial w}{\partial z} &= \\
 \frac{1}{Q_{33}} &\left[\sigma_z - Q_{31} \frac{\partial u}{\partial x} - Q_{34} \frac{\partial u}{\partial y} - Q_{32} \frac{\partial v}{\partial y} - Q_{34} \frac{\partial v}{\partial x} \right] \\
 \frac{\partial \tau_{xz}}{\partial z} &= \left(-Q_{11} + \frac{Q_{13}Q_{31}}{Q_{33}} \right) \frac{\partial^2 u}{\partial x^2} \\
 &+ \left(-Q_{41} - Q_{14} + \frac{Q_{13}Q_{34}}{Q_{33}} + \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{\partial^2 u}{\partial x \partial y} \\
 &+ \left(-Q_{44} + \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 u}{\partial y^2} \\
 &+ \left(-Q_{14} + \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{\partial^2 v}{\partial x^2} \\
 &+ \left(-Q_{12} - Q_{44} + \frac{Q_{13}Q_{32}}{Q_{33}} + \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 v}{\partial x \partial y} \\
 &+ \left(-Q_{42} + \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{\partial^2 v}{\partial y^2} - \left(\frac{Q_{13}}{Q_{33}} \right) \frac{\partial \sigma_z}{\partial x} \\
 &- \left(\frac{Q_{43}}{Q_{33}} \right) \frac{\partial \sigma_z}{\partial y} - B_x \\
 \frac{\partial \tau_{yz}}{\partial z} &= \left(-Q_{41} + \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{\partial^2 u}{\partial x^2} \\
 &+ \left(-Q_{21} - Q_{44} + \frac{Q_{23}Q_{31}}{Q_{33}} + \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 u}{\partial x \partial y} \\
 &+ \left(-Q_{24} + \frac{Q_{23}Q_{34}}{Q_{33}} \right) \frac{\partial^2 u}{\partial y^2} \\
 &+ \left(-Q_{44} + \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 v}{\partial x^2} \\
 &+ \left(-Q_{24} - Q_{42} + \frac{Q_{23}Q_{34}}{Q_{33}} + \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{\partial^2 v}{\partial x \partial y} \\
 &+ \left(-Q_{22} + \frac{Q_{23}Q_{32}}{Q_{33}} \right) \frac{\partial^2 v}{\partial y^2} - \left(\frac{Q_{43}}{Q_{33}} \right) \frac{\partial \sigma_z}{\partial x} \\
 &- \left(\frac{Q_{23}}{Q_{33}} \right) \frac{\partial \sigma_z}{\partial y} - B_y \\
 \frac{\partial \sigma_z}{\partial z} &= -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - B_z
 \end{aligned} \tag{7}$$

This primary set of dependent variables is naturally defined at a plane $z = a$ constant. The secondary variables σ_x , σ_y and τ_{xy} can simply be ex-

pressed as a function of the primary set of variables as

$$\begin{aligned}
 \sigma_x &= \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) \frac{\partial u}{\partial x} \\
 &+ \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
 &+ \left(Q_{12} - \frac{Q_{13}Q_{32}}{Q_{33}} \right) \frac{\partial v}{\partial y} + \frac{Q_{13}}{Q_{33}} \sigma_z \\
 \sigma_y &= \left(Q_{21} - \frac{Q_{23}Q_{31}}{Q_{33}} \right) \frac{\partial u}{\partial x} \\
 &+ \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}} \right) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
 &+ \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \frac{\partial v}{\partial y} + \frac{Q_{23}}{Q_{33}} \sigma_z \\
 \tau_{xy} &= \left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{\partial u}{\partial x} \\
 &+ \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
 &+ \left(Q_{22} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{\partial v}{\partial y} + \frac{Q_{43}}{Q_{33}} \sigma_z
 \end{aligned} \tag{8}$$

The primary set of variables u , v , w , τ_{xz} , τ_{yz} and σ_z is a function of independent coordinates x , y and z . It is proposed here to perform FE (partial) discretization only in the $x-y$ plane such that the discrete dependent vector $y(z)$ will be a function of the independent coordinate z and a system of coupled discrete first order ODEs connecting all FE nodes results. This new formulation is described below with reference to a four-noded bi-linear element in $x-y$ plane with mixed set of primary variables as nodal degrees of freedom which are function of z (Fig. 3).

2.5 Kinematics

The approximate variation of displacement field over the element domain in the $x-y$ plane can be written as

$$\begin{aligned}
 u &\simeq \hat{u}(x, y, z) = \sum_{i=1}^4 N_i(x, y) u_i(z) \\
 v &\simeq \hat{v}(x, y, z) = \sum_{i=1}^4 N_i(x, y) v_i(z) \\
 w &\simeq \hat{w}(x, y, z) = \sum_{i=1}^4 N_i(x, y) w_i(z)
 \end{aligned} \tag{9}$$

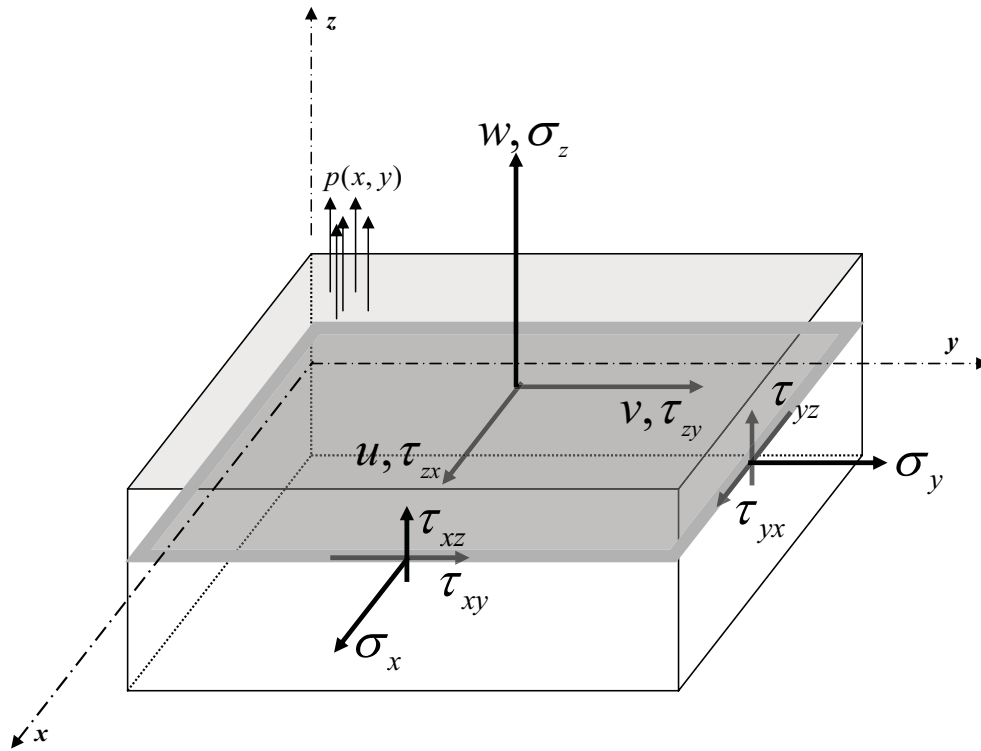


Figure 2: 3D domain subjected to the transverse loading

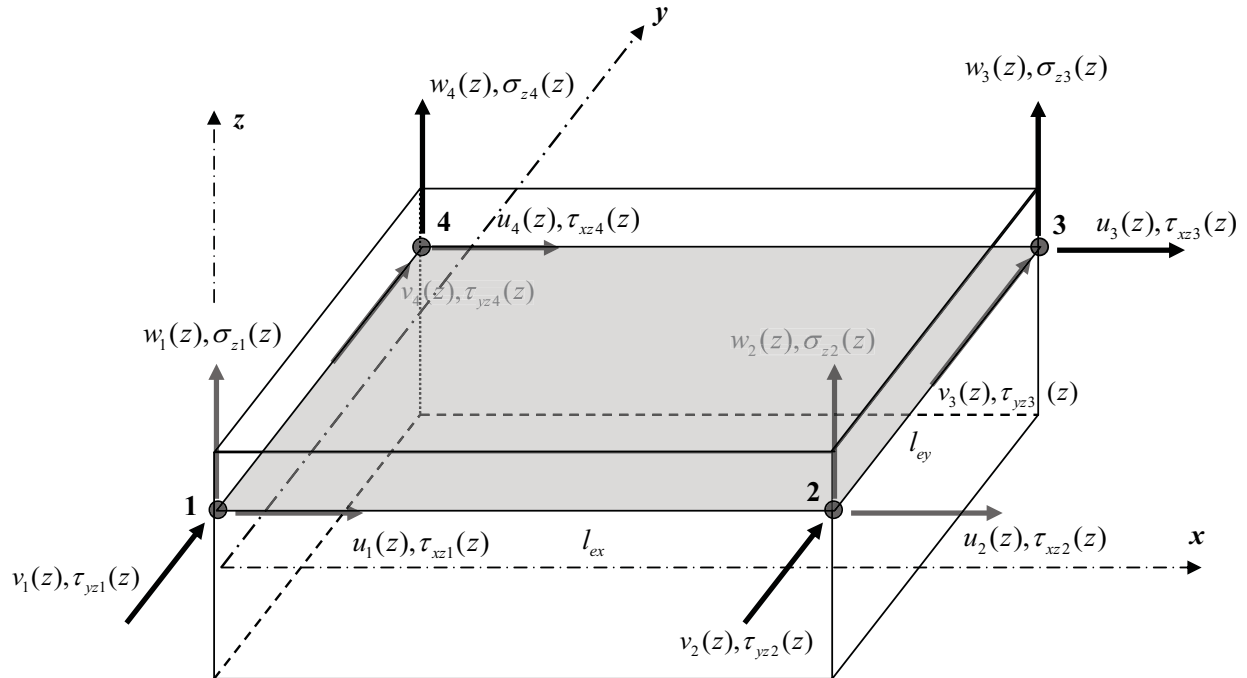


Figure 3: Bi-linear plate element with dependent variables

Further, it can be shown from the basic 3D elasticity relations that

$$\begin{aligned}\tau_{xz} &\simeq \hat{\tau}_{xz}(x, y, z) = \sum_{i=1}^4 N_i(x, y) \tau_{xzi}(z) \\ \tau_{yz} &\simeq \hat{\tau}_{yz}(x, y, z) = \sum_{i=1}^4 N_i(x, y) \tau_{yzi}(z) \\ \sigma_z &\simeq \hat{\sigma}_z(x, y, z) = \sum_{i=1}^4 N_i(x, y) \sigma_{zi}(z)\end{aligned}\quad (10)$$

Here,

$$\begin{aligned}N_1(x, y) &= 1 - \frac{x}{l_{ex}} - \frac{y}{l_{ey}} - \frac{xy}{l_{ex}l_{ey}} \\ N_2(x, y) &= \frac{x}{l_{ex}} - \frac{xy}{l_{ex}l_{ey}} \\ N_3(x, y) &= \frac{xy}{l_{ex}l_{ey}} \\ N_4(x, y) &= \frac{y}{l_{ey}} - \frac{xy}{l_{ex}l_{ey}}\end{aligned}$$

and l_{ex} , l_{ey} are the length and width of element in x and y directions, respectively.

The domain residuals are obtained by substituting Eqs. (9) and (10) into Eq. (7) as

$$\begin{aligned}\frac{\partial \hat{u}(x, y, z)}{\partial z} + \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})} [Q_{65} \hat{\tau}_{yz}(x, y, z) \\ - Q_{66} \hat{\tau}_{xz}(x, y, z)] + \frac{\partial \hat{w}(x, y, z)}{\partial x} \\ = R_{1D}(x, y)\end{aligned}\quad (11)$$

$$\begin{aligned}\frac{\partial \hat{v}(x, y, z)}{\partial z} + \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})} [-Q_{55} \hat{\tau}_{yz}(x, y, z) \\ + Q_{56} \hat{\tau}_{xz}(x, y, z)] + \frac{\partial \hat{w}(x, y, z)}{\partial y} \\ = R_{2D}(x, y)\end{aligned}\quad (12)$$

$$\begin{aligned}\frac{\partial \hat{w}(x, y, z)}{\partial z} - \frac{1}{Q_{33}} \left[\hat{\sigma}_z(x, y, z) - Q_{31} \frac{\partial \hat{u}(x, y, z)}{\partial x} \right. \\ \left. - Q_{34} \frac{\partial \hat{u}(x, y, z)}{\partial y} - Q_{32} \frac{\partial \hat{v}(x, y, z)}{\partial y} - Q_{34} \frac{\partial \hat{v}(x, y, z)}{\partial x} \right] \\ = R_{3D}(x, y)\end{aligned}$$

$$\begin{aligned}\frac{\partial \hat{\tau}_{xz}(x, y, z)}{\partial z} + \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x^2} \\ + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial y^2} \\ + \left(Q_{41} + Q_{14} - \frac{Q_{43}Q_{31}}{Q_{33}} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x \partial y} \\ + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x^2} \\ + \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial y^2} \\ + \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x \partial y} \\ + \frac{Q_{13}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial x} + \frac{Q_{43}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial y} + \hat{B}_x(x, y, z) \\ = R_{4D}(x, y)\end{aligned}\quad (14)$$

$$\begin{aligned}\frac{\partial \hat{\tau}_{yz}(x, y, z)}{\partial z} + \left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x^2} \\ + \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial y^2} \\ + \left(Q_{21} + Q_{44} - \frac{Q_{23}Q_{31}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x \partial y} \\ + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x^2} \\ + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial y^2} \\ + \left(Q_{24} + Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} - \frac{Q_{23}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x \partial y} \\ + \frac{Q_{43}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial x} + \frac{Q_{23}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial y} + \hat{B}_y(x, y, z) \\ = R_{5D}(x, y)\end{aligned}\quad (15)$$

$$\begin{aligned}\frac{\partial \hat{\sigma}_z(x, y, z)}{\partial z} + \frac{\partial \hat{\tau}_{xz}(x, y, z)}{\partial x} + \frac{\partial \hat{\tau}_{yz}(x, y, z)}{\partial y} \\ + \hat{B}_z(x, y, z) \\ = R_{6D}(x, y)\end{aligned}\quad (16)$$

Further, with the help of Eqs. (11-16) the strong Bubnov-Galerkin weighted residual statements (Zienkiewicz 1991) can be written as

$$\iint_A N_i(x, y) \left(\frac{\partial \hat{u}(x, y, z)}{\partial z} + \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})} \cdot [Q_{65} \hat{\tau}_{yz}(x, y, z) - Q_{66} \hat{\tau}_{xz}(x, y, z)] + \frac{\partial \hat{w}(x, y, z)}{\partial x} \right) dA = 0 \quad (17)$$

$$\iint_A N_i(x, y) \left(\frac{\partial \hat{v}(x, y, z)}{\partial z} + \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})} \cdot [-Q_{55} \hat{\tau}_{yz}(x, y, z) + Q_{56} \hat{\tau}_{xz}(x, y, z)] + \frac{\partial \hat{w}(x, y, z)}{\partial y} \right) dA = 0 \quad (18)$$

$$\int_A N_i(x, y) \left\{ \frac{\partial \hat{w}(x, y, z)}{\partial z} - \frac{1}{Q_{33}} [\hat{\sigma}_z(x, y, z) - Q_{31} \frac{\partial \hat{u}(x, y, z)}{\partial x} - Q_{34} \frac{\partial \hat{u}(x, y, z)}{\partial y} - Q_{32} \frac{\partial \hat{v}(x, y, z)}{\partial y} - Q_{34} \frac{\partial \hat{v}(x, y, z)}{\partial x}] \right\} dA = 0 \quad (19)$$

$$\int_A N_i(x, y) \left\{ \frac{\partial \hat{\tau}_{xz}(x, y, z)}{\partial z} + \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x^2} + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial y^2} + \left(Q_{41} + Q_{14} - \frac{Q_{43}Q_{31}}{Q_{33}} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x \partial y} + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x^2} + \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial y^2} + \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x \partial y} + \frac{Q_{13}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial x} + \frac{Q_{43}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial y} + \hat{B}_x(x, y, z) \right\} dA = 0 \quad (20)$$

$$\iint_A N_i(x, y) \left\{ \frac{\partial \hat{\tau}_{yz}(x, y, z)}{\partial z} + \left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x^2} + \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial y^2} + \left(Q_{21} + Q_{44} - \frac{Q_{23}Q_{31}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{u}(x, y, z)}{\partial x \partial y} + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x^2} + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial y^2} + \left(Q_{24} + Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} - \frac{Q_{23}Q_{34}}{Q_{33}} \right) \frac{\partial^2 \hat{v}(x, y, z)}{\partial x \partial y} + \frac{Q_{43}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial x} + \frac{Q_{23}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial y} + \hat{B}_y(x, y, z) \right\} dA = 0 \quad (21)$$

$$\int_A N_i(x, y) \left(\frac{\partial \hat{\sigma}_z(x, y, z)}{\partial z} + \frac{\partial \hat{\tau}_{xz}(x, y, z)}{\partial x} + \frac{\partial \hat{\tau}_{yz}(x, y, z)}{\partial y} + \hat{B}_z(x, y, z) \right) dA = 0 \quad (22)$$

Eqs. (20) and (21) contain second order derivatives of \hat{u} and \hat{v} . These equations can be replaced by their weak forms with the help of integration by parts as

$$\iint_A N_i(x, y) \left[\frac{\partial \hat{\tau}_{xz}(x, y, z)}{\partial z} + \frac{Q_{13}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial x} + \frac{Q_{43}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x, y, z)}{\partial y} \right] dA - \iint_A \frac{dN_i(x, y)}{dy} \left[\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{d\hat{u}(x, y, z)}{dy} + \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{d\hat{v}(x, y, z)}{dy} \right] dA$$

$$\begin{aligned}
 & - \iint_A \frac{dN_i(x,y)}{dx} \left[\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dx} \right. \\
 & + \left(Q_{41} + Q_{14} - \frac{Q_{43}Q_{31}}{Q_{33}} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dy} \\
 & + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dx} + \left(Q_{12} + Q_{44} \right. \\
 & \quad \left. - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dy} \Big] dA \\
 & + \oint_s N_i(x,y) \left[\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dx} n_x \right. \\
 & + \left(Q_{41} + Q_{14} - \frac{Q_{43}Q_{31}}{Q_{33}} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \\
 & \quad \left. \cdot \frac{d\hat{u}(x,y,z)}{dy} n_y \right] ds \\
 & + \oint_s N_i(x,y) \left[\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dy} n_y \right. \\
 & + \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 & \quad \left. \cdot \frac{d\hat{v}(x,y,z)}{dy} n_x \right] ds \\
 & + \oint_s N_i(x,y) \left[\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dx} n_x \right. \\
 & \quad \left. + \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dy} n_y \right] ds \\
 & + \iint_A N_i(x,y) \hat{B}_x(x,y,z) dA = 0 \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 & \iint_A N_i(x,y) \left[\frac{\partial \hat{\tau}_{yz}(x,y,z)}{\partial z} + \frac{Q_{23}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x,y,z)}{\partial y} \right. \\
 & \quad \left. + \frac{Q_{43}}{Q_{33}} \frac{\partial \hat{\sigma}_z(x,y,z)}{\partial x} \right] dA \\
 & - \iint_A \frac{dN_i(x,y)}{dx} \left[\left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dx} \right. \\
 & \quad \left. + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dx} \right] dA
 \end{aligned}$$

$$\begin{aligned}
 & - \iint_A \frac{dN_i(x,y)}{dy} \left[\left(Q_{24} - \frac{Q_{34}Q_{23}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dy} \right. \\
 & + \left(Q_{21} + Q_{44} - \frac{Q_{23}Q_{31}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dx} \\
 & + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dy} + \left(Q_{24} + Q_{42} \right. \\
 & \quad \left. - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dx} \Big] dA \\
 & + \oint_s N_i(x,y) \left[\left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dx} n_x \right. \\
 & + \left(Q_{21} + Q_{44} - \frac{Q_{23}Q_{31}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 & \quad \left. \cdot \frac{d\hat{u}(x,y,z)}{dx} n_y \right] ds \\
 & + \oint_s N_i(x,y) \left[\left(Q_{24} - \frac{Q_{34}Q_{23}}{Q_{33}} \right) \frac{d\hat{u}(x,y,z)}{dy} n_y \right. \\
 & + \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 & \quad \left. \cdot \frac{d\hat{v}(x,y,z)}{dx} n_y \right] ds \\
 & + \oint_s N_i(x,y) \left[\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dx} n_x \right. \\
 & \quad \left. + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \frac{d\hat{v}(x,y,z)}{dy} n_y \right] ds \\
 & + \iint_A N_i(x,y) \hat{B}_y(x,y,z) dA = 0 \tag{24}
 \end{aligned}$$

On substitution of discrete relations described by Eqs. (9) and (10), following twenty-four coupled linear first-order ODEs are obtained.

$$\begin{aligned}
 & \begin{bmatrix} [\mathbf{A}_{01}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{03}^e] & [\mathbf{A}_{02}^e] \\ [\mathbf{A}_{02}^e] & [\mathbf{A}_{01}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{03}^e] \\ [\mathbf{A}_{03}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{01}^e] & [\mathbf{A}_{02}^e] \\ [\mathbf{A}_{02}^e] & [\mathbf{A}_{03}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{01}^e] \end{bmatrix} \frac{d}{dz} \begin{Bmatrix} \mathbf{y}_1^e(z) \\ \mathbf{y}_2^e(z) \\ \mathbf{y}_3^e(z) \\ \mathbf{y}_4^e(z) \end{Bmatrix} = \\
 & \begin{bmatrix} [\mathbf{B}_{01}^e] & [\mathbf{B}_{02}^e] & [\mathbf{B}_{03}^e] & [\mathbf{B}_{04}^e] \\ [\mathbf{B}_{05}^e] & [\mathbf{B}_{06}^e] & [\mathbf{B}_{07}^e] & [\mathbf{B}_{08}^e] \\ [\mathbf{B}_{09}^e] & [\mathbf{B}_{10}^e] & [\mathbf{B}_{11}^e] & [\mathbf{B}_{12}^e] \\ [\mathbf{B}_{13}^e] & [\mathbf{B}_{14}^e] & [\mathbf{B}_{15}^e] & [\mathbf{B}_{16}^e] \end{bmatrix} \begin{Bmatrix} \mathbf{y}_1^e(z) \\ \mathbf{y}_2^e(z) \\ \mathbf{y}_3^e(z) \\ \mathbf{y}_4^e(z) \end{Bmatrix} + \begin{Bmatrix} \mathbf{p}_1^e \\ \mathbf{p}_2^e \\ \mathbf{p}_3^e \\ \mathbf{p}_4^e \end{Bmatrix} \tag{25}
 \end{aligned}$$

where vectors $\mathbf{y}_i^e(z)$ and $\mathbf{p}_i^e(x, y, z)$ are

$$\mathbf{y}_i^e(z) = [u_i^e(z), v_i^e(z), w_i^e(z), \tau_{xzi}^e(z), \tau_{yzi}^e(z), \sigma_{zi}^e(z)]^t$$

$$\mathbf{p}_i^e(x, y, z) = [0, 0, 0, p_{i4}^e, p_{i5}^e, p_{i6}^e]^t, \quad i = 1, 2, 3, 4$$

Coefficients of individual submatrices and vectors are defined in Appendix I.

Eq. (25) can be written in a compact form as

$$\mathbf{C}^e(x, y) \frac{d}{dz} \mathbf{y}^e(z) = \mathbf{D}^e(x, y, z) \mathbf{y}^e(z) + \mathbf{p}^e(x, y, z) \quad (26)$$

When the total plan area is discretized with n finite elements in $x - y$ plane (Fig. 4), the semi-discrete system of equations for the entire domain can be shown to be,

$$\begin{aligned} \sum_{k=1}^n \mathbf{C}^e(x, y) \frac{d}{dz} \mathbf{y}^e(z) \\ = \sum_{k=1}^n \mathbf{D}^e(x, y, z) \mathbf{y}^e(z) + \sum_{k=1}^n \mathbf{p}^e(x, y, z) \end{aligned}$$

or

$$\mathbf{C}(x, y) \frac{d}{dz} \mathbf{y}(z) = \mathbf{D}(x, y, z) \mathbf{y}(z) + \mathbf{p}(x, y, z) \quad (27)$$

Multiplication of Eq. (27) by $[\mathbf{C}(x, y)]^{-1}$ on both sides results in,

$$\frac{d}{dz} \mathbf{y}(z) = \mathbf{K}(x, y, z) \mathbf{y}(z) + \mathbf{f}(x, y, z) \quad (28)$$

where $\mathbf{K}(x, y, z) = [\mathbf{C}(x, y)]^{-1} \mathbf{D}(x, y, z)$ and $\mathbf{f}(x, y, z) = [\mathbf{C}(x, y)]^{-1} \mathbf{p}(x, y, z)$

Eq. (28) define the governing equations of a two-point BVP in terms of ODEs in the domain $-h/2 < z < h/2$. $\mathbf{y}(z)$ is an m -dimensional ($m = \text{nos of nodes} \times 6$) vector of dependent variables, $\mathbf{K}(x, y, z)$ is an $m \times m$ coefficient matrix (which is a function of element geometry in $x - y$ plane and material properties variation in x, y and z directions) and $\mathbf{f}(x, y, z)$ is an m -dimensional vector of non-homogeneous (loading) terms. Any $m/2$ elements of $\mathbf{y}(z)$ are prescribed at the two ends, $z = -h/2$ and $h/2$ as boundary conditions. It is clearly seen that mixed and/or non-homogeneous boundary conditions can be easily admitted in this formulation.

The basic approach to the numerical integration of the BVP defined by Eq. (28) is to transform the given BVP into a set of IVPs – one particular (nonhomogeneous) and $m/2$ complimentary (homogeneous). Numbers of successful and well tested algorithms are available in literature for numerical solution of IVPs expressed by ODEs. Solution of the original BVP defined by Eq. (28) is obtained by forming a linear combination of one nonhomogeneous and $m/2$ homogeneous solutions so as to satisfy the boundary conditions at $z = h/2$. This gives rise to a system of $m/2$ linear algebraic equations, the solution of which determines the unknown $m/2$ components of the vector of initial values $\mathbf{y}(z)$. Then a final numerical integration of Eq. (28) with completely known initial vector of dependent variables $\mathbf{y}(z)$ produces the desired results. It is intended here to extend the applicability of this procedure, which was previously documented by Kant and Ramesh (1981) and applied in original form by Kant and Setlur (1973), Ramesh et al. (1974), Kant (1981, 1982) and Kant and Hinton (1983) for a class of two dimensions (2D) BVPs of plates and shells.

3 Numerical investigations

A four-noded bi-linear element with mixed (displacements/stresses) degrees of freedom is employed in the present numerical investigations. A computer code was developed in FORTRAN 90 by incorporating the foregoing partial FE formulation for the static analyses of laminated composite plates. Accuracy of the proposed formulation for layered composites is established by comparison of the present numerical results with 3D elasticity/analytical, 2D closed-form analytical and 2D/3D FE solutions. The layer elastic coefficients of a unidirectional graphite/epoxy composite have been consistently considered in all examples (Pagano 1969, 1970).

$$\begin{aligned} \mathbf{E}_L &= 25 \times 10^6 \text{ psi (172.4 GPa)}; \\ E_2 &= 10^6 \text{ psi (6.89 GPa)} \\ \mathbf{G}_{LT} &= 0.5 \times 10^6 \text{ psi (3.450 GPa)}; \\ \mathbf{G}_{TT} &= 0.2 \times 10^6 \text{ psi (1.378 GPa)} \\ \nu_{LT} &= \nu_{TT} = 0.25 \end{aligned} \quad (29)$$

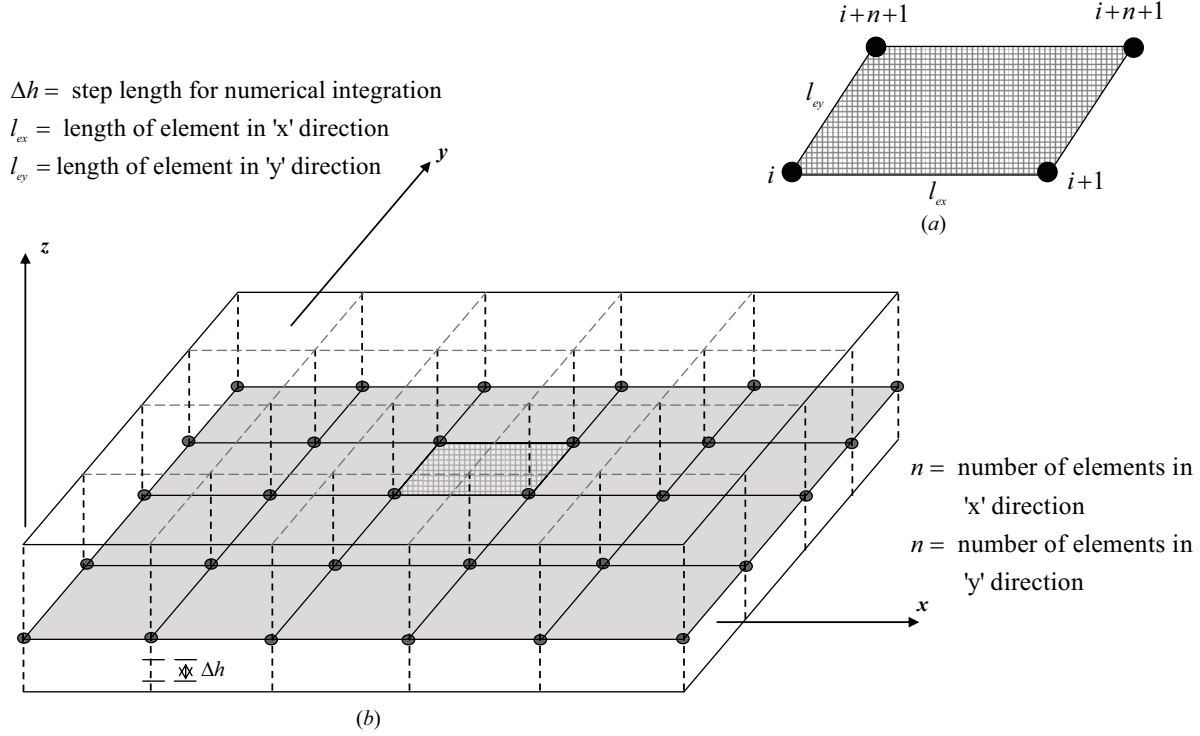


Figure 4: Bi-linear plate elements (concept of partial discretization) [a] plate discretization [b] typical bi-linear element

Table 1: Boundary conditions (BCs)

Description	Edge	BCs on displacement field	BCs on stress field
Nodes along simple support	$x=0$ and a	$v = w = 0$	$\sigma_x = 0$ (true)
	$y=0$ and b	$u = w = 0$	$\sigma_y = 0$ (true)
Nodes along clamped Support	$x=0$ and a	$u = v = w = 0$	-
	$y=0$ and b	$u = v = w = 0$	-
For all nodes	$z=h/2$	-	$\tau_{xz} = \tau_{yz} = 0$ and $\sigma_z = p_0(x, y)$
	$z=-h/2$	-	$\tau_{xz} = \tau_{yz} = \sigma_z = 0$

'-' indicates no BCs

where subscript L and T refer to the fiber direction and the transverse direction perpendicular to the fiber direction.

Support conditions considered have been tabulated in Tab. 1. All laminates are subjected to the bi-directional sinusoidal transverse load on their top surface. Intensity of sinusoidal loading can be expressed as

$$p(x, y) = p_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (30)$$

where p_0 is the peak intensity of distributed load.

Non-dimensionalized displacements and stresses reported in all tables and figures are defined by

$$\begin{aligned}
 s &= \frac{a}{h}; \quad \bar{u} = \frac{E_2 u}{h p_0 s^3}; \quad \bar{w} = \frac{100 E_2 h^3 w}{p_0 a^4}; \quad \bar{\sigma}_z = \frac{\sigma_z}{p_0} \\
 (\bar{\sigma}_x; \bar{\sigma}_y; \bar{\tau}_{xy}) &= \frac{1}{p_0 s^2} (\sigma_x; \sigma_y; \tau_{xy}); \\
 (\bar{\tau}_{xz}; \bar{\tau}_{yz}) &= \frac{1}{p_0 s} (\tau_{xz}; \tau_{yz})
 \end{aligned} \quad (31)$$

for proper comparison of results in which bar over the variable defines its normalized value.

Table 2: Normalized stresses ($\overline{\sigma}_x, \overline{\sigma}_y, \overline{\tau}_{xz}, \overline{\tau}_{yz}$ and $\overline{\tau}_{xy}$) and mid-plane transverse displacement (\overline{w}) of symmetric ($0^\circ/90^\circ/0^\circ$) square composite plates

s	Source	$\overline{\sigma}_x(\frac{a}{2}, \frac{b}{2}; \pm \frac{a}{2})$	$\overline{\sigma}_y(\frac{a}{2}, \frac{b}{2}; \pm \frac{a}{2})$	$\overline{\tau}_{xz}(0, 0, \pm \frac{a}{2})$	$\overline{\tau}_{yz}(\frac{a}{2}, 0, 0)$	$\overline{\tau}_{xy}(0, 0, \pm \frac{a}{2})$	$\overline{\tau}_{xz}(0, \frac{b}{2}, 0)$	$\overline{\tau}_{yz}(\frac{a}{2}, 0, 0)$	$\overline{w}(\frac{a}{2}, \frac{b}{2}, 0)$
4	Present Partial FEM	0.7556	-0.7128	0.5057	-0.5250	-0.0464	0.0458	0.2583	0.2231
	¹ Elasticity analysis	0.8010	-0.7550	0.5340	-0.5560	-0.0510	0.0500	0.2560	0.2170
	² Mixed FEM	-	-0.7850	-	-	-	0.0540	-	-
	³ Mixed FEM	0.8080	-0.7600	0.5380	-0.5600	-0.0510	0.0500	0.2570	0.2210
	⁴ HOST	0.7670	-	0.5079	-	-	0.0500	-	-
	*Semi-analytical	0.8010	-0.7550	0.5340	-0.5560	-0.0510	0.0505	0.2560	0.2170
5	Present Partial FEM	0.6770	-0.6575	0.4384	-0.4512	0.0400	0.0400	0.2886	0.1955
	⁵ 3D Analytical	0.7180	-0.6987	0.4639	-0.4784	-	-	-	0.1911
10	Present Partial FEM	± 0.5750	0.2780	-0.2815	∓ 0.0268	∓ 0.0268	0.3550	0.1200	0.7471
	¹ Elasticity analysis	± 0.5900	0.2850	-0.2880	∓ 0.0290	∓ 0.0290	0.3570	0.1230	-
	² Mixed FEM	± 0.6100	-	-	0.0300	0.0300	0.3820	-	-
	³ Mixed FEM	± 0.5840	0.2860	-0.2890	∓ 0.0290	∓ 0.0290	0.3580	0.1240	0.8560
	⁴ HOST	0.5850	0.2712	-	0.0281	0.0281	-	-	0.7176
	⁵ 3D Analytical	0.5906	-0.5898	0.2845	-0.2882	-	-	0.1228	0.7530
	*Semi-analytical	± 0.5900	0.2845	-0.2880	∓ 0.0290	∓ 0.0290	0.3570	0.1230	0.7530
20	Present Partial FEM	± 0.5410	± 0.2070	± 0.2070	∓ 0.0221	∓ 0.0221	0.3780	0.0900	0.5164
	¹ Elasticity analysis	± 0.5520	± 0.2100	± 0.2100	∓ 0.0234	∓ 0.0234	0.3850	0.0940	-
	³ Mixed FEM	± 0.5550	± 0.2100	± 0.2100	∓ 0.0230	∓ 0.0230	0.3880	0.1010	0.5170
	⁴ HOST	0.5507	0.2050	0.2050	0.0231	0.0231	-	-	0.5058
	*Semi-analytical	± 0.5520	± 0.2100	± 0.2100	∓ 0.0234	∓ 0.0234	0.3850	0.0940	0.5164
50	Partial FEM	± 0.5245	± 0.1810	± 0.1810	∓ 0.0206	∓ 0.0206	0.3840	0.0700	0.4385
	*Semi-analytical	± 0.5410	± 0.1846	± 0.1846	∓ 0.0216	∓ 0.0216	0.3934	0.0842	0.4451

¹ Pagano (1970) ² Wu and Kuo (1993), ³ Ramtekkar et al. (2002) ⁴ Kant and Swaminathan (2002) ⁵ Vel and Batra (1999)

‘-’ indicates results are not available

*Kant et al. –Communicated to Composite Structures (under review)

A 10×10 full mesh of the four-noded bi-linear elements in $x - y$ plane has been used in computation. Mesh density has been selected here on the basis of a convergence study in which the mid-plane transverse displacement and the transverse shear stress converges monotonically. Fig. 5 shows the converged results with respect to number of elements for the transverse shear stress and displacement. Illustrative numerical examples considered in the present work are discussed next.

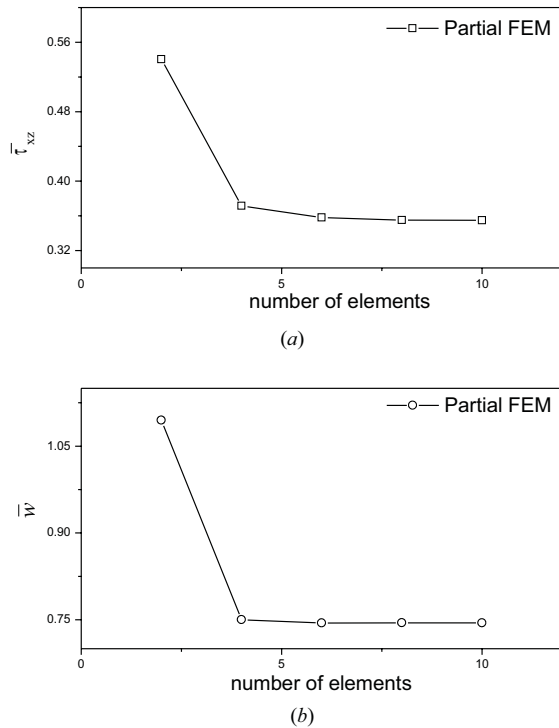


Figure 5: Converged (a) maximum transverse shear stress (τ_{xz}); and (b) mid-plane transverse displacement (w) with number of elements for $0^0/90^0/0^0$ composite plate for aspect ratio of 10

Example 1

A symmetric, square, three layered ($0^0/90^0/0^0$) composite plate, supported on simple supports (Tab. 1) on all the four edges has been considered to establish accuracy of stress predictions through the thickness by the present method. Numerical results of normalized inplane normal

stresses (σ_x, σ_y), inplane shear stress (τ_{xy}), transverse shear stresses (τ_{xz}, τ_{yz}) and transverse displacement (w) for aspect ratios, $s = 4, 10, 20$ and 50 are tabulated under Tab. 2 and compared with the 3D elasticity solution given by Pagano (1970), 3D analytical solutions presented by Vel and Batra (1999) as well as 2D/3D FE solutions presented by Wu and Kuo (1993), Ramtekkar et al. (2002) and Kant and Swaminathan (2002). Moreover, through thickness variations of inplane normal stress (σ_x), transverse displacement (w), inplane displacement (u) and transverse shear stress (τ_{xz}) for an aspect ratio of 4 have been shown in Fig. 6. Comparison clearly shows that the present results are in good agreement with the 3D elasticity solutions and thus proves the superiority of the present formulation.

Example 2

A square, asymmetric ($-15^0/15^0$) angle-ply composite plate with simple support end conditions on all four edges (Tab. 1) has been considered next for numerical investigation. Normalized inplane normal stresses (σ_x, σ_y), inplane shear stress (τ_{xy}), transverse shear stresses (τ_{xz}, τ_{yz}) and transverse displacement (w) for aspect ratios, 4, 10, 20 and 50 are given in Tab. 3. Through thickness variations of normalized inplane normal stress (σ_x), transverse displacement (w), inplane shear stress (τ_{xy}) and transverse shear stress (τ_{yz}) have been shown graphically in Fig. 7 for an aspect ratio of 4. Exact solutions presented by Savoia and Reddy (1992) have been used for proper comparison. It is observed that there is excellent agreement between the present and that of Savoia and Reddy (1992) results and thus shows the ability of the present approach to handle the problem for different fiber angle.

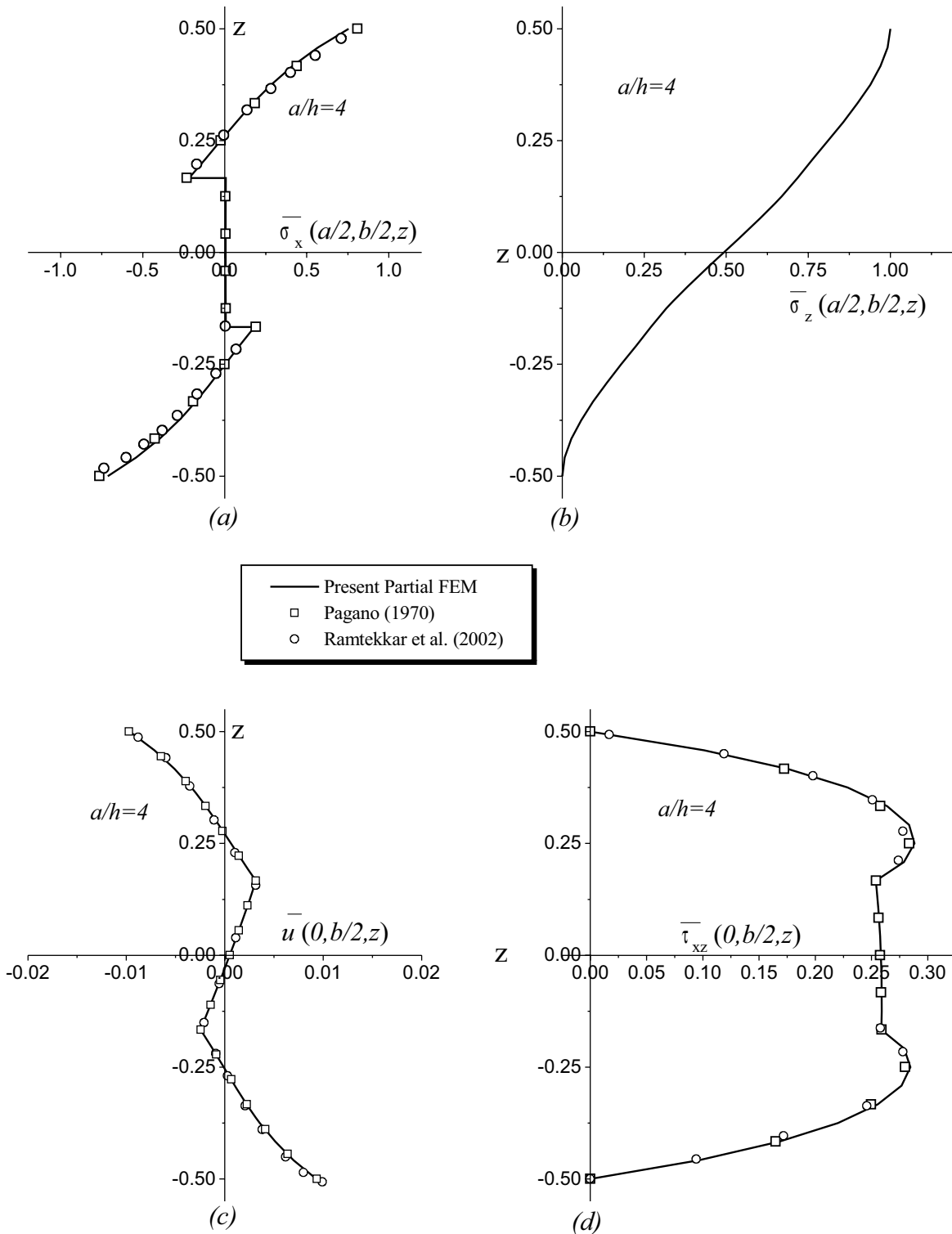


Figure 6: Variation of the normalized (a) inplane normal stress $\bar{\sigma}_x$; (b) transverse normal stress $\bar{\sigma}_z$; (c) inplane displacement \bar{u} ; and (d) transverse shear stress $\bar{\tau}_{xz}$ through thickness of a simply supported symmetric composite $0^\circ/90^\circ/0^\circ$ plate

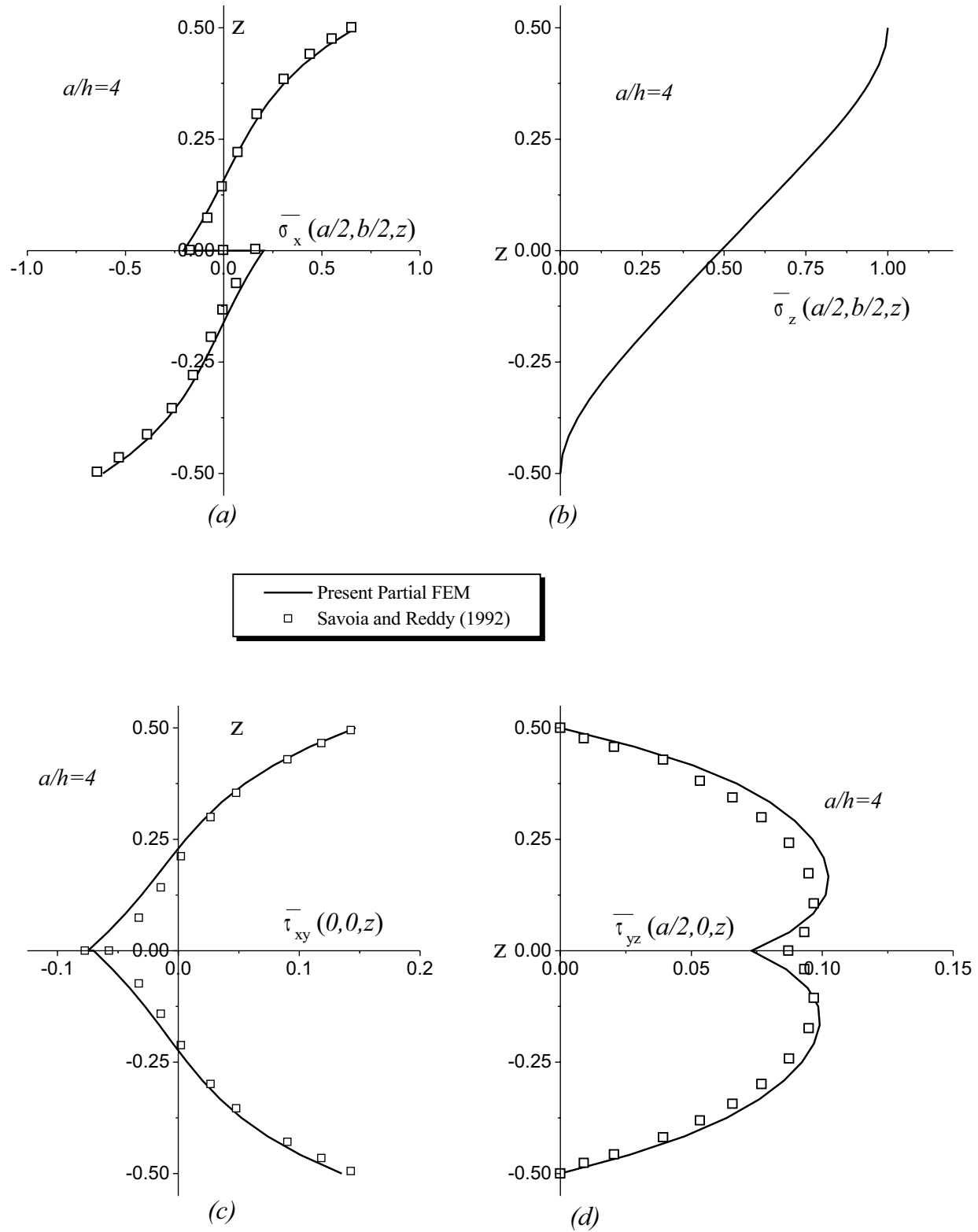


Figure 7: Variation of the normalized (a) inplane normal stress $\bar{\sigma}_x$; (b) transverse normal stress $\bar{\sigma}_z$; (c) inplane shear stress $\bar{\tau}_{xy}$; and (d) transverse shear stress $\bar{\tau}_{yz}$ through thickness of a simply supported unsymmetric angle-ply composite ($-15^\circ/15^\circ$) plate

Table 3: Normalized stresses ($\overline{\sigma}_x, \overline{\sigma}_y, \overline{\tau}_{xz}, \overline{\tau}_{yz}$ and $\overline{\tau}_{xy}$) and mid-plane transverse displacement (\overline{w}) of unsymmetric ($-15^\circ/15^\circ$) square angle-ply composite plates

s	Source	$\overline{\sigma}_x(\frac{a}{2}, \frac{b}{2}, \pm \frac{h}{2})$	$\overline{\sigma}_y(\frac{a}{2}, \frac{b}{2}, \pm \frac{h}{2})$	$\overline{\tau}_{xz}(0, 0, \pm \frac{h}{2})$	$\overline{\tau}_{yz}(\frac{a}{2}, 0, \max)$	$\overline{\tau}_{xy}(0, 0, \pm \frac{h}{2})$	$\overline{\tau}_{xz}(0, \frac{b}{2}, \max)$	$\overline{\tau}_{yz}(\frac{a}{2}, 0, \max)$	$\overline{w}(\frac{a}{2}, \frac{b}{2}, 0)$
4	Present Partial FEM	0.6739	-0.6146	0.1172	-0.1039	0.1466	0.1352	0.3652	0.1044
	⁶ Elasticity analysis	0.6545	-	0.1135	-	0.1489	-	0.3145	0.0960
10	Present Partial FEM	0.5701	-0.5659	0.0815	-0.1518	0.1238	0.1234	0.3849	0.0806
	⁶ Elasticity analysis	0.5633	-	0.0806	-	0.1244	-	0.3421	0.0814
20	Present Partial FEM	0.5572	-0.5564	0.0749	-0.0746	0.1217	0.1216	0.3909	0.0544
50	Present Partial FEM	± 0.5545	± 0.0730	± 0.1215	± 0.0730	± 0.1215	± 0.0730	± 0.3815	± 0.0172

⁶ Savoia and Reddy (1992) ‘-’ indicates results are not available

Table 4: Normalized stresses ($\overline{\sigma}_x, \overline{\sigma}_y, \overline{\tau}_{xz}, \overline{\tau}_{yz}$ and $\overline{\tau}_{xy}$) and mid-plane transverse displacement (\overline{w}) of unsymmetric ($0^\circ/90^\circ$) square composite plates

s	Source	$\overline{\sigma}_x(\frac{a}{2}, \frac{b}{2}, \pm \frac{h}{2})$	$\overline{\sigma}_y(\frac{a}{2}, \frac{b}{2}, \pm \frac{h}{2})$	$\overline{\tau}_{xz}(0, 0, \pm \frac{h}{2})$	$\overline{\tau}_{yz}(0, \frac{b}{2}, 0)$	$\overline{\tau}_{xy}(0, 0, \pm \frac{h}{2})$	$\overline{\tau}_{xz}(0, \frac{b}{2}, 0)$	$\overline{\tau}_{yz}(\frac{a}{2}, 0, 0)$	$\overline{w}(\frac{a}{2}, \frac{b}{2}, 0)$
5	Present Partial FEM	0.0949	-0.7179	0.7399	-0.0858	-0.0507	0.0510	0.1307	0.1203
	⁴ HOST	-	-0.7510	0.7720	-	-	-	-	1.6800
	⁵ 3D Analytical	-	-0.7671	0.7894	-	-	-	0.1211	1.7246
10	Present Partial FEM	0.0858	-0.7060	0.7065	-0.0836	-0.0488	0.0500	0.1176	0.1146
	⁴ HOST	-	-0.7270	0.7270	-	-	-	-	1.2190
	⁵ 3D Analytical	-	-0.7304	0.7309	-	-	-	0.1219	1.2270
	*Semi-analytical	0.0890	-0.7300	0.7310	-0.0865	-0.0536	0.0537	0.1250	1.2270
20	Present Partial FEM	0.0820	-0.6900	0.7000	-0.0811	-0.0483	0.0483	0.1104	0.1096
	⁴ HOST	-	-0.7190	0.7190	-	-	-	-	1.1020
	*Semi-analytical	0.0854	-0.7200	0.7200	-0.0849	-0.0528	0.0528	0.1250	1.1050
50	Present Partial FEM	0.0802	-0.6800	0.6800	-0.0802	-0.0482	0.0482	0.1077	0.1076
	*Semi-analytical	0.0845	-0.7164	0.7163	-0.0844	-0.0525	0.0525	0.1222	1.0702

⁴Kant and Swaminathan (2002) ⁵ Vel and Batra (1999) ‘-’ indicates results are not available

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Table 5: Normalized stresses ($\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_z$ and $\bar{\tau}_{yz}$) and mid-plane transverse displacement (\bar{w}) of unsymmetric ($0^\circ/90^\circ$) and symmetric ($0^\circ/90^\circ/0^\circ$) square composite plates simply supported on two opposite edges ($y = 0, b$) and clamped on two opposite edges ($x = 0, a$)

Lamination scheme	s	Source	$\bar{\sigma}_x(\frac{a}{2}, \frac{b}{2}, -\frac{h}{2})$	$\bar{\sigma}_y(\frac{a}{2}, \frac{b}{2}, \frac{h}{2})$	$\bar{\sigma}_z(\frac{a}{2}, \frac{b}{2}, 0)$	$\bar{\tau}_{yz}(\frac{a}{2}, 0, 0)$	$\bar{w}(\frac{a}{2}, \frac{b}{2}, 0)$
$0^\circ/90^\circ$ unsymmetric composite plate	5	Present Partial FEM	-0.4205 (-9.179)	0.5360 (-6.343)	0.5703 (-1.502)	0.0843 (-3.657)	1.2085 (-0.698)
	10	⁵ 3D Analytical Present Partial FEM	-0.4630 -0.4229	0.5723 0.3794	0.5790 0.6302	0.0875 0.0731	1.217 0.6538
$53D$ Analytical			(-9.112)	(-2.418)	(-1.531)	(2.524)	(0.740)
			-0.4653	0.3888	0.6400	0.0713	0.6490

⁴Kant and Swaminathan (2002) ⁵ Vel and Batra (1999) '-' indicates results are not available

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Example 3

A two layered asymmetric cross-ply ($0^0/90^0$) square laminate with equal thickness and simply support boundary condition (Tab. 1) is considered here. Exact solution of this example is not available in literature. Normalized maximum stresses ($\overline{\sigma}_x, \overline{\sigma}_y, \overline{\tau}_{xy}, \overline{\tau}_{xz}, \overline{\tau}_{yz}$) and the mid-plane transverse displacement are presented in Tab. 4. Results with 3D analytical solution given by Vel and Batra (1999) as well as HOST solutions presented by Kant and Swaminathan (2002) have been used for general comparison and it is found that obtained results are match well with them. Fig. 8 shows through thickness variations of the inplane normal stress ($\overline{\sigma}_x$), transverse displacement (\overline{w}), inplane displacement (\overline{u}) and transverse shear stress ($\overline{\tau}_{xz}$) for aspect ratio 5 and will be helpful in future for comparison.

Example 4

Cross-ply asymmetric ($0^0/90^0$) laminated composite plates with two opposite edges clamped at $x=0, a$ and other two opposite edges simply supported at $y=0, b$ are considered to show the ability of the present formulation to handle general boundary conditions. Numerical results for aspect ratios, $s=5$ and 10 are documented in Tab. 5 and compared with 3D analytical solutions (Ven and Batra 1999). Percentage difference are calculated by Eq. (32) and reported in parentheses.

$$\% \text{ difference} = \frac{\text{Present Partial FEM} - \text{3D Analytical}}{\text{3D Analytical}} \times 100 \quad (32)$$

Comparison showed well match with 3D analytical solutions and established the generality of the present development.

Parametric investigation and general discussion

All previously considered Examples 1, 2 and 3 are reanalyzed with clamped end conditions for parametric investigations. Laminations scheme, material properties and geometrical details are kept same as in Examples 1, 2 and 3. Boundary conditions are specified in Tab. 1. Numerical results for the normalized inplane normal stress ($\overline{\sigma}_x$),

transverse shear stress ($\overline{\tau}_{xz}$) and transverse displacement (\overline{w}) for different aspect ratios are documented in Tab. 6. Through thickness variation of normalized transverse shear stress ($\overline{\tau}_{xz}$) at the end, $x=0$ is presented in Fig. 9. These results should serve as benchmark solutions for future reference. High values of the transverse shear stresses are observed at the top and bottom face of laminate at the end for clamped boundary conditions (Fig. 9). Non-linear variations of the transverse displacement (\overline{w}) along the thickness of laminates can be observed from Fig. 10, for very small aspect ratio which highlights importance of straining of the normal to the mid-surface for thick plates. On the other hand, transverse displacements (\overline{w}) are found to be constant through the thickness for thin plates.

In the present work, simple linear variations for all primary variables (degrees of freedom) are used for the partial FE development. Weak weighted residual approach helps to use such simple linear approximation and hence, complexity in the development of FE model has been greatly reduced. Numerical studies show that the present partial FE results are match well as expected with available solutions. However, use of higher-order approximations certainly improves the accuracy and further refinement is required in this direction.

4 Concluding remarks

A new partial discretization formulation involving two dimensional (2D) FEs only in $x - y$ plane is presented and thus formulation is free from any assumption along the thickness of laminate. The foregoing methodology gives rise to solution of a two-point BVP through the thickness of laminated plate. The primary requirement of continuity of both displacements and the transverse stresses are implicitly satisfied in the formulation and the solution maintains the fundamental elasticity relations between components of stress, strain and displacement within the elastic domain. Accuracy of the present development is demonstrated by comparing the solutions with the available 3D solutions. Numerical investigations are also presented to show the generality of the formulation to handle different boundary conditions. Main ad-

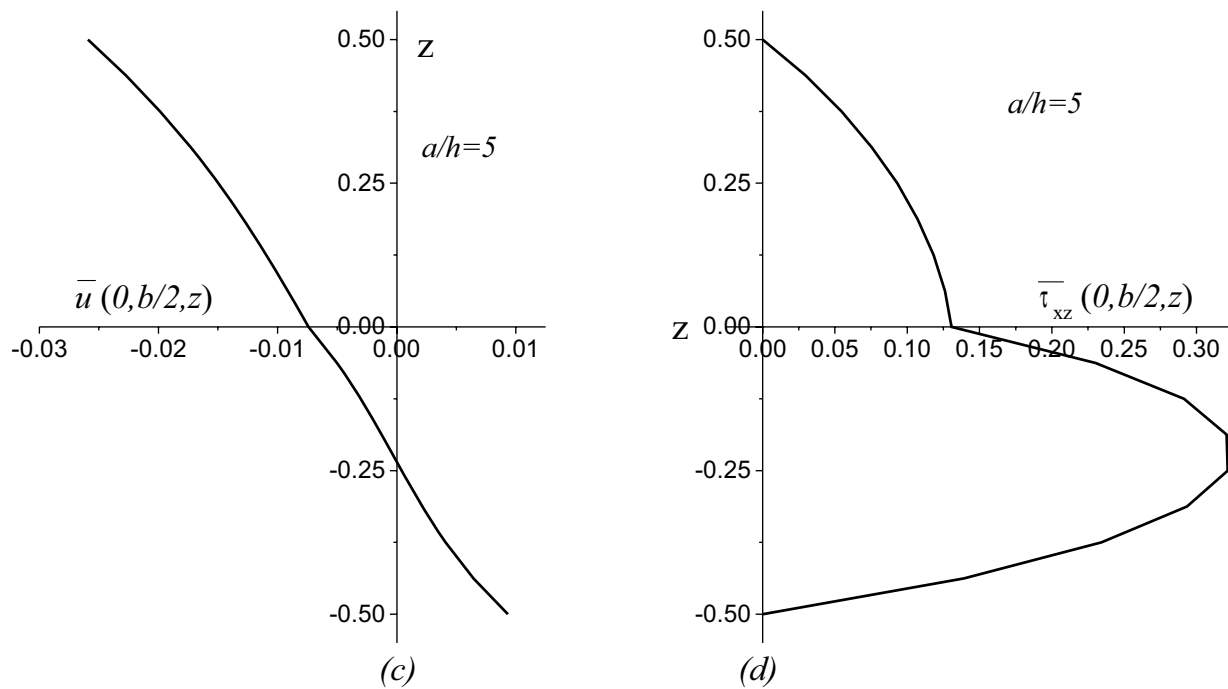
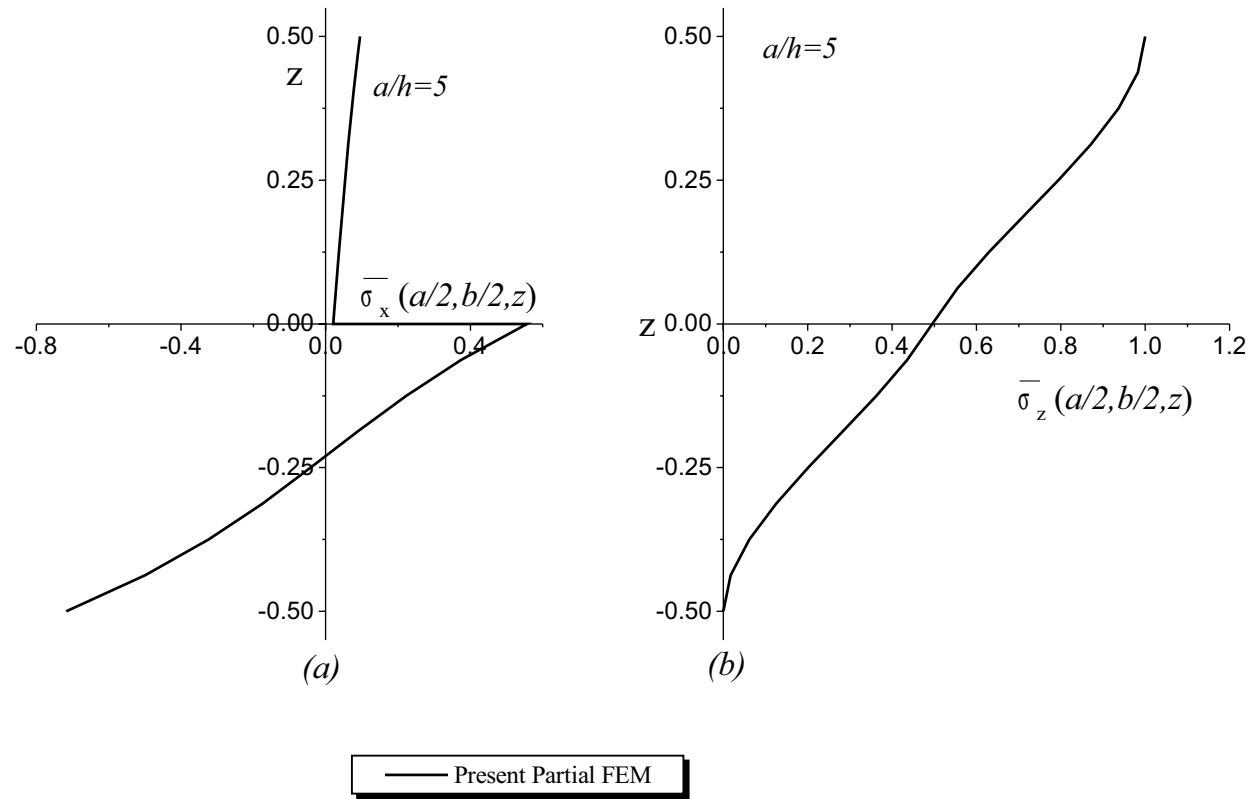


Figure 8: Variation of the normalized (a) inplane normal stress $\bar{\sigma}_x$; (b) transverse normal stress $\bar{\sigma}_z$; (c) inplane displacement \bar{u} ; and (d) transverse shear stress $\bar{\tau}_{xz}$ through thickness of a simply supported unsymmetric composite ($0^\circ/90^\circ$) plate

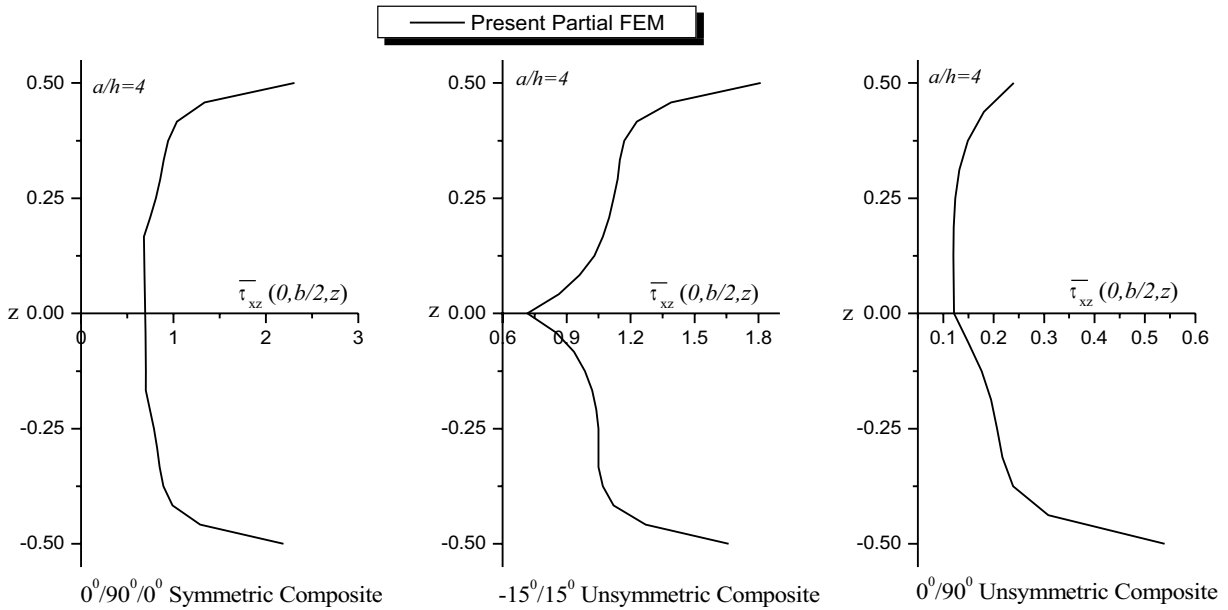


Figure 9: Through thickness variation of the normalized transverse shear stress $\bar{\tau}_{xz}$ of clamped supported

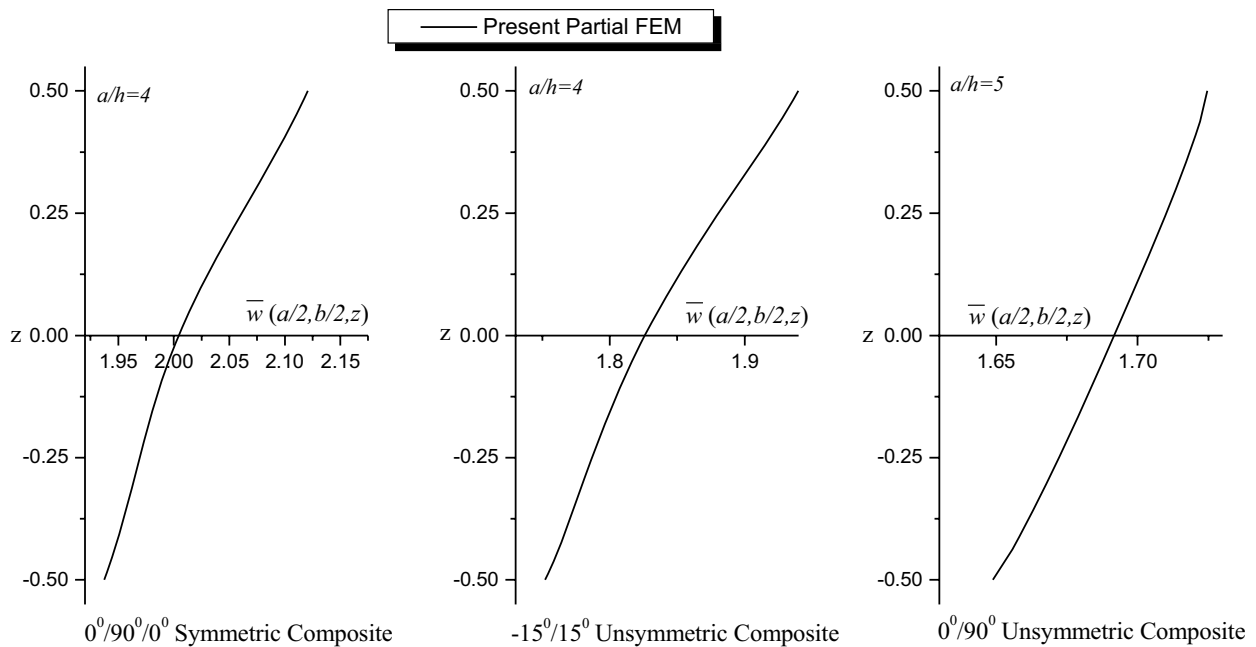


Figure 10: Through thickness variation of the normalized transverse displacement \bar{w} of simply supported composite plates

vantage of the present methodology lies in the fact that both displacements and stresses are evaluated simultaneously at a FE node with the same degree of accuracy through the numerical integration process. Post-processing module, which is

required in other analytical models for calculation of transverse stresses from inplane stresses is altogether eliminated.

It is to be emphasized here very clearly that the semi-discrete form of Eq. (27) is unique as long

Table 6: Normalized inplane normal stress ($\bar{\sigma}_x$), transverse shear stresses ($\bar{\tau}_{xz}$) and transverse displacement (\bar{w}) of symmetric and unsymmetric clamped supported square sandwich plate

0 ⁰ /90 ⁰ /0 ⁰ symmetric sandwich plate						
s	Source	$\bar{\sigma}_x(\frac{a}{2}, \frac{b}{2}; \pm \frac{h}{2})$		$\bar{\tau}_{xz}(0, \frac{b}{2}, \pm \frac{h}{6})$		$\bar{w}(\frac{a}{2}, \frac{b}{2}, 0)$
4	Partial FEM	0.3627	-0.3273	0.5766	0.5465	1.3647
10	Partial FEM	0.2400	-0.2381	0.8714	0.8775	0.3850
20	Partial FEM	0.2075	-0.2071	0.1997	1.2033	0.1692
50	Partial FEM	±0.1867		1.3055	1.3062	0.0947

-15 ⁰ /15 ⁰ unsymmetric sandwich plate						
s	Source	$\bar{\sigma}_x(\frac{a}{2}, \frac{b}{2}; \pm \frac{h}{2})$		$\bar{\tau}_{xz}(0, \frac{b}{2}, \pm \frac{h}{2})$		$\bar{w}(\frac{a}{2}, \frac{b}{2}, 0)$
4	Partial FEM	0.2046	-0.2005	0.4531	0.4144	1.2230
10	Partial FEM	0.2066	-0.2038	0.6867	0.6856	0.3277
20	Partial FEM	0.2157	-0.2152	1.2059	1.0269	0.2114
50	Partial FEM	±0.2160		1.2547	1.2550	0.1592

0 ⁰ /90 ⁰ symmetric sandwich plate						
s	Source	$\bar{\sigma}_x(\frac{a}{2}, \frac{b}{2}; \pm \frac{h}{2})$		$\bar{\tau}_{xz}(0, \frac{b}{2}, \pm \frac{h}{2})$		$\bar{w}(\frac{a}{2}, \frac{b}{2}, 0)$
4	Partial FEM	0.0629	-0.3347	0.2933	0.5391	1.3061
10	Partial FEM	0.0409	-0.2715	0.4357	0.7766	0.4694
20	Partial FEM	0.0354	-0.2636	0.6452	0.8798	0.2989
50	Partial FEM	0.0318	-0.2611	0.7648	0.8884	0.2324

as all but one independent coordinates (z here) are discretized in a BVP. From this viewpoint also the proposed methodology can be considered as a novel and standard (modular) one. Indeed, this method has its unique appeal for accurate estimation of transverse interlaminar stresses.

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Appendix A: Appendix I
Non zero coefficients of vector, $p_i^e(x, y, z)$

$$\begin{aligned}
 p_{pi4}^e = & - \left[\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{du(x, y, z)}{dx} n_x ds \right. \\
 & + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{du(x, y, z)}{dy} n_y ds \\
 & + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{dv(x, y, z)}{dx} n_x ds \\
 & + \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{dv(x, y, z)}{dy} n_y ds \\
 & + \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \\
 & \quad \cdot \oint_s N_i(x, y) \frac{du(x, y, z)}{dy} n_x ds \\
 & + \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 & \quad \cdot \oint_s N_i(x, y) \frac{dv(x, y, z)}{dy} n_x ds \\
 & \left. + \iint_A N_i(x, y) B_x(x, y, z) dA \right]
 \end{aligned}$$

$$\begin{aligned}
 p_{pi5}^e = & - \left[\left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{du(x, y, z)}{dx} n_x ds \right. \\
 & + \left(Q_{24} - \frac{Q_{34}Q_{23}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{du(x, y, z)}{dy} n_y ds \\
 & + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{dv(x, y, z)}{dx} n_x ds \\
 & + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \oint_s N_i(x, y) \frac{dv(x, y, z)}{dy} n_y ds \\
 & + \left(Q_{12} + Q_{44} - \frac{Q_{23}Q_{31}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 & \quad \cdot \oint_s N_i(x, y) \frac{du(x, y, z)}{dx} n_y ds \\
 & + \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 & \quad \cdot \oint_s N_i(x, y) \frac{dv(x, y, z)}{dx} n_y ds \\
 & \left. + \iint_A N_i(x, y) B_y(x, y, z) dA \right]
 \end{aligned}$$

$$p_{pi6}^e = - \iint_A N_i(x, y) B_z(x, y, z) dA$$

Coefficients of diagonal submatrices $[A_s^e]$ (for $s=1-3$)

$$A_{01}^e = 2A_{02}^e = 4A_{03}^e \quad A_{03}^e = a_{01}^e I \quad \text{where, } a_{01}^e = \frac{l_{ex} l_{ey}}{36} \text{ and } \mathbf{I} \text{ is } 6 \times 6 \text{ identity matrix}$$

Coefficients of submatrices $[B_j^e]$ (for $j=1-16$)

$$B_{01}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & 2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ k_{06}^e & k_{07}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{09}^e & k_{10}^e & 0 & 0 & 0 & k_{06}^e \\ k_{10}^e & k_{11}^e & 0 & 0 & 0 & k_{07}^e \\ 0 & 0 & 0 & 2k_{01}^e & 2k_{02}^e & 0 \end{bmatrix}$$

$$B_{02}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{12}^e & k_{13}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{14}^e & k_{15}^e & 0 & 0 & 0 & k_{12}^e \\ k_{16}^e & k_{17}^e & 0 & 0 & 0 & k_{13}^e \\ 0 & 0 & 0 & -2k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$B_{03}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ k_{18}^e & k_{19}^e & 0 & 0 & 0 & k_{08}^e \\ k_{20}^e & k_{21}^e & 0 & 0 & 0 & k_{18}^e \\ k_{21}^e & k_{22}^e & 0 & 0 & 0 & k_{19}^e \\ 0 & 0 & 0 & -k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$B_{04}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{23}^e & k_{24}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{25}^e & k_{26}^e & 0 & 0 & 0 & k_{23}^e \\ k_{27}^e & k_{28}^e & 0 & 0 & 0 & k_{24}^e \\ 0 & 0 & 0 & k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

$$B_{05}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{29}^e & k_{30}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{31}^e & k_{32}^e & 0 & 0 & 0 & k_{29}^e \\ k_{33}^e & k_{34}^e & 0 & 0 & 0 & k_{30}^e \\ 0 & 0 & 0 & 2k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$B_{06}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & 2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ k_{35}^e & k_{36}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{37}^e & k_{38}^e & 0 & 0 & 0 & k_{35}^e \\ k_{38}^e & k_{39}^e & 0 & 0 & 0 & k_{36}^e \\ 0 & 0 & 0 & -2k_{01}^e & 2k_{02}^e & 0 \end{bmatrix}$$

$$B_{07}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{40}^e & k_{41}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{42}^e & k_{43}^e & 0 & 0 & 0 & k_{40}^e \\ k_{44}^e & k_{45}^e & 0 & 0 & 0 & k_{41}^e \\ 0 & 0 & 0 & -k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

$$B_{14}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ -k_{46}^e & -k_{47}^e & 0 & 0 & 0 & k_{08}^e \\ k_{48}^e & k_{49}^e & 0 & 0 & 0 & -k_{46}^e \\ k_{49}^e & k_{50}^e & 0 & 0 & 0 & -k_{47}^e \\ 0 & 0 & 0 & -k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$B_{08}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ k_{46}^e & k_{47}^e & 0 & 0 & 0 & k_{08}^e \\ k_{48}^e & k_{49}^e & 0 & 0 & 0 & k_{46}^e \\ k_{49}^e & k_{50}^e & 0 & 0 & 0 & k_{47}^e \\ 0 & 0 & 0 & k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$B_{15}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ -k_{29}^e & -k_{30}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{31}^e & k_{32}^e & 0 & 0 & 0 & -k_{29}^e \\ k_{33}^e & k_{34}^e & 0 & 0 & 0 & -k_{30}^e \\ 0 & 0 & 0 & -2k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$B_{09}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ -k_{18}^e & -k_{19}^e & 0 & 0 & 0 & k_{08}^e \\ k_{20}^e & k_{21}^e & 0 & 0 & 0 & -k_{18}^e \\ k_{21}^e & k_{22}^e & 0 & 0 & 0 & -k_{19}^e \\ 0 & 0 & 0 & k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$B_{16}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ -k_{35}^e & -k_{36}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{37}^e & k_{38}^e & 0 & 0 & 0 & -k_{35}^e \\ k_{38}^e & k_{39}^e & 0 & 0 & 0 & -k_{36}^e \\ 0 & 0 & 0 & 2k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

where,

$$B_{10}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & 2k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ -k_{23}^e & -k_{24}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{25}^e & k_{26}^e & 0 & 0 & 0 & -k_{23}^e \\ k_{27}^e & k_{28}^e & 0 & 0 & 0 & -k_{24}^e \\ 0 & 0 & 0 & -k_{01}^e & 2k_{02}^e & 0 \end{bmatrix}$$

$$k_{01}^e = \frac{l_{ey}}{12}$$

$$k_{02}^e = \frac{l_{ex}}{12}$$

$$k_{03}^e = \frac{l_{ex}l_{ey}}{36} \left(\frac{Q_{66}}{Q_{55}Q_{66} - Q_{56}Q_{65}} \right)$$

$$B_{11}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ -k_{06}^e & -k_{07}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{09}^e & k_{10}^e & 0 & 0 & 0 & -k_{06}^e \\ k_{10}^e & k_{11}^e & 0 & 0 & 0 & -k_{07}^e \\ 0 & 0 & 0 & -2k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

$$k_{04}^e = \frac{l_{ex}l_{ey}}{36} \left(\frac{Q_{65}}{Q_{55}Q_{66} - Q_{56}Q_{65}} \right)$$

$$k_{05}^e = \frac{l_{ex}l_{ey}}{36} \left(\frac{Q_{55}}{Q_{55}Q_{66} - Q_{56}Q_{65}} \right)$$

$$k_{06}^e = \frac{l_{ey}}{6} \frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{6} \frac{Q_{34}}{Q_{33}}$$

$$B_{12}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ -k_{12}^e & -k_{13}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{14}^e & k_{15}^e & 0 & 0 & 0 & -k_{12}^e \\ k_{16}^e & k_{17}^e & 0 & 0 & 0 & -k_{13}^e \\ 0 & 0 & 0 & 2k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$k_{07}^e = \frac{l_{ex}}{6} \frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{6} \frac{Q_{34}}{Q_{33}}$$

$$k_{08}^e = \frac{l_{ex}l_{ey}}{36} \left(\frac{1}{Q_{33}} \right)$$

$$B_{13}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & 2k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ -k_{40}^e & -k_{41}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{42}^e & k_{43}^e & 0 & 0 & 0 & -k_{40}^e \\ k_{44}^e & k_{45}^e & 0 & 0 & 0 & -k_{41}^e \\ 0 & 0 & 0 & k_{01}^e & 2k_{02}^e & 0 \end{bmatrix}$$

$$k_{09}^e = \frac{l_{ey}}{3l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) + \frac{l_{ex}}{3l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right)$$

$$\begin{aligned}
 k_{10}^e &= \frac{l_{ey}}{3l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &+ \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \\
 &\quad + \frac{l_{ex}}{3l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 k_{11}^e &= \frac{l_{ey}}{3l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &+ \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{3l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \\
 k_{12}^e &= -\frac{l_{ey}}{6} \frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{12} \frac{Q_{34}}{Q_{33}} \\
 k_{13}^e &= \frac{l_{ex}}{12} \frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{6} \frac{Q_{34}}{Q_{33}} \\
 k_{14}^e &= -\frac{l_{ey}}{3l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 &+ \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) - \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &\quad + \frac{l_{ex}}{6l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 k_{15}^e &= -\frac{l_{ey}}{3l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &+ \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \\
 &\quad + \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 k_{16}^e &= -\frac{l_{ey}}{3l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &+ \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \\
 &\quad + \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 k_{17}^e &= -\frac{l_{ey}}{3l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &+ \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) + \frac{l_{ex}}{12} \frac{Q_{34}}{Q_{33}} \\
 k_{18}^e &= -\frac{l_{ey}}{12} \frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{12} \frac{Q_{34}}{Q_{33}} \\
 k_{19}^e &= -\frac{l_{ex}}{12} \frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{12} \frac{Q_{34}}{Q_{33}} \\
 k_{20}^e &= -\frac{l_{ey}}{6l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \\
 &\quad - \frac{l_{ex}}{6l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 k_{21}^e &= -\frac{l_{ey}}{6l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) \\
 &\quad - \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 k_{22}^e &= -\frac{l_{ey}}{6l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) \\
 &\quad - \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right) \\
 k_{23}^e &= \frac{l_{ey}}{12} \frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{6} \frac{Q_{34}}{Q_{33}} \\
 k_{24}^e &= -\frac{l_{ex}}{6} \frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{12} \frac{Q_{34}}{Q_{33}} \\
 k_{25}^e &= \frac{l_{ey}}{6l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) \\
 &\quad - \frac{l_{ex}}{3l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right)
 \end{aligned}$$

$$k_{26}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{27}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{28}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right)$$

$$k_{29}^e = \frac{l_{ey}}{6} \frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{12} \frac{Q_{34}}{Q_{33}}$$

$$k_{30}^e = \frac{l_{ex}}{12} \frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{6} \frac{Q_{34}}{Q_{33}}$$

$$k_{31}^e = -\frac{l_{ey}}{3l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right)$$

$$k_{32}^e = -\frac{l_{ey}}{3l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{33}^e = -\frac{l_{ey}}{3l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right)$$

$$+ \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{34}^e = -\frac{l_{ey}}{3l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right)$$

$$k_{35}^e = -\frac{l_{ey}}{6} \frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{6} \frac{Q_{34}}{Q_{33}}$$

$$k_{36}^e = \frac{l_{ex}}{6} \frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{6} \frac{Q_{34}}{Q_{33}}$$

$$k_{37}^e = \frac{l_{ey}}{3l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) + \frac{l_{ex}}{3l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right)$$

$$k_{38}^e = \frac{l_{ey}}{3l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{l_{ex}}{3l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{39}^e = \frac{l_{ey}}{3l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) + \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right)$$

$$k_{40}^e = -\frac{l_{ey}}{12} \frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{6} \frac{Q_{34}}{Q_{33}}$$

$$k_{41}^e = -\frac{l_{ex}}{6} \frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{12} \frac{Q_{34}}{Q_{33}}$$

$$k_{42}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{l_{ex}}{6l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{43}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) - \frac{l_{ex}}{6l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right)$$

$$k_{44}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} \right)$$

$$k_{45}^e = \frac{l_{ey}}{6l_{ex}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right) - \frac{1}{4} \left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}} \right) - \frac{l_{ex}}{3l_{ey}} \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}} \right)$$

$$k_{46}^e = \frac{l_{ey}}{12} \frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{12} \frac{Q_{34}}{Q_{33}}$$

$$k_{47}^e = -\frac{l_{ex}}{12} \frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{12} \frac{Q_{34}}{Q_{33}}$$

$$k_{48}^e = -\frac{l_{ey}}{6l_{ex}} \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}} \right) + \frac{1}{4} \left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}} \right) - \frac{l_{ex}}{6l_{ey}} \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}} \right)$$

$$k_{49}^e = -\frac{l_{ey}}{6l_{ex}} \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}} \right)$$

