Non-Liner Programming

Lecture Notes in Transportation Systems Engineering

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1 Introduction

Optimization is the act of obtaining the best result under given circumstances. It is the process of finding the condtions that give maximum or minimum value of a function.

The general form of the optimization problem will be to find $\bar{x}$ that

$$\begin{aligned} \text{Minimize} \quad & f(\bar{x}) \\ \text{subjected to} \quad & \\ g_j(\bar{x}) &\leq 0 \\ h_j(\bar{x}) &= 0. \end{aligned}$$

where $\bar{x}$ is the decision variable. Figure 1 shows the concept of optimization which is to find the minimum or maximum of a function. Since Minimizing $f(x)$ is same as maximizing $-f(x)$ we shall follow the former notation. The optimum solution is indicated by $\bar{x}^*$

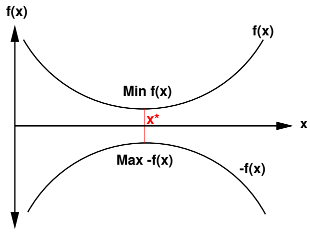

Figure 1: The concept of optimisation

2 Convexity of functions

2.1 Multivariable functions

2.1.1 Example

Determin the convexity of the following function:

$$f(x_1, x_2) = (x_1 - x_2)^2,$$

2.1.2 Problems

Determin the convexity of the following functions:

$$
\begin{aligned}
f(\bar{x}) &= x_1 - 2x_1x_2 + x_2^2 \\
f(\bar{x}) &= x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_2x_3 - x_1x_3 \\
f(x) &= 4x_1^4 + 2x_1^2 - 5x_1
\end{aligned}
$$

3 Classical methods

3.1 Unconstrained single variable

3.1.1 Example

Solve the function and iterpret the results(s):

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5.$$

3.1.2 Problems

Solve the following functions and iterpret the results(s):

$$
\begin{aligned}
f(x) &= (x - 7)^2, \\
f(x) &= x(x - 1.5), \\
f(x) &= -3x^2 + 21.6x + 1.
\end{aligned}
$$

3.2 Unconstrained multi variable

3.2.1 Example

Solve the function and iterpret the results(s):

$$f(\bar{x}) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6.$$

3.2.2 Problems

Solve the following functions and iterpret the results(s):

$$f(\bar{x}) = (x_1 - 10)^2 + (x_2 - 10)^2.$$

3.3 Constrained multi variable with equality constraints

3.3.1 Example

Solve the following constrained program:

$$\begin{aligned} \text{Min} : f(\bar{x}) &= x_1^2 + 2x_2, \\ \text{st} : g(\bar{x}) &= x_1^2 + x_2^2 = 1. \end{aligned}$$

3.3.2 Problems

Solve the following constrained program:

$$\begin{aligned} \text{Min} : f(x,y) &= kx^{-1}y^{-2}, \\ \text{st} : g(x,y) &= x^2 + y^2 = a^2. \end{aligned}$$

Solve the following constrained program:

$$\begin{aligned} \text{Min} : f(x,y,z) &= 3x^2 + 4y^2 + 5z^2, \\ \text{st} : g(x,y,z) &= x + y + z = 10. \end{aligned}$$

3.4 Karush-Kuhn-Tucker conditions

3.4.1 Example

Solve the following constrained program:

$$\begin{aligned} \text{Min} : f(\bar{x}) &= x_1^2 - x_2, \\ \text{st} : x_1 + x_2 &= 6, \\ x_1 &\geq 1, \\ x_1^2 + x_2^2 &\leq 26. \end{aligned}$$

3.4.2 Problems

Solve the following constrained program:

$$\begin{aligned} \text{Max} : f(\bar{x}) &= \log(x_1 + 1) + x_2, \\ \text{st} : 2x_1 + x_2 &\leq 3, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solve the following constrained program:

$$\begin{aligned} \text{Min} : f(x,y,z) &= 3x^2 + 4y^2 + 5z^2, \\ \text{st} : g(x,y,z) &= x + y + z = 10. \end{aligned}$$

4 Numerical methods

4.1 Unconstrained one variable problems

4.1.1 Example

Solve the following unconstrained one dimentional problem:

$$\text{Max} : f(x) = 12x - 3x^4 - 2x^6.$$

4.1.2 Problems

Solve the following unconstrained one dimentional problems

$$\begin{aligned} \text{Min} : f(x) &= x(x-1.5), \quad [0-1]; \\ \text{Min} : f(x) &= 0.65 - \frac{0.75}{1+x^2} - 0.65x\tan^{-1}\left(\frac{1}{x}\right), \\ & \quad [0-3]; \\ \text{Max} : f(x) &= -3x^2 + 21.6x + 1, \quad [0-25]. \end{aligned}$$

4.2 Unconstrained multi variable problems

4.2.1 Example

Solve the following unconstrained multi-variable problem:

$$\text{Max} : f(\bar{x}) = 2x_1x_2 + 2x_2 - x_1^2 - 2x_2^2, \quad \bar{x}^0 = (0,0).$$

4.2.2 Problems

Solve the following unconstrained multi-variable problems:

$$\text{Min} : f(\bar{x}) = 4x_1^2 - 4x_1x_2 + 2x_2^2, \quad \bar{x}^0 = (2,3).$$

4.3 Constrained multi variable problems

4.3.1 Frank-Wolfe Alorithm

It is type of sequential approximation algorithm. This is applicable when the objective function is non liner and the constraints are all liner. It uses a linear approximation of the objetive function and solve by any classical linear programming algorithm.

Given a feasible trail solution ($\bar{x} = \bar{x}^1$), then the linear approximation used for the objective function is the first order Taylor series expansion of $f(\bar{x})$ around $\bar{x} = \bar{x}^1$. Thus

$$
\begin{aligned}
f(\bar{x}) &= f(\bar{x}^1) + \sum_{j=1}^{n} \frac{\partial f(\bar{x}^1)}{\partial x_j}(x_j - x_j^1) && (1)\\
&= f(\bar{x}^1) - \sum_{j=1}^{n} \frac{\partial f(\bar{x}^1)}{\partial x_j} x_j^1 + \sum_{j=1}^{n} \frac{\partial f(\bar{x}^1)}{\partial x_j} x_j && (2)\\
&= f(\bar{x}^1) - \sum_{j=1}^{n} \frac{\partial f(\bar{x}^1)}{\partial x_j} x_j^1 + g(\bar{x}) && (3)
\end{aligned}
$$

Since the first and second term of $f(x)$ is constant, optimzing $f(x)$ is same as optimizing $g(x)$. Therefore, the equivalent linear form of $f(x)$ is $g(x)$, that is,

$$
\begin{aligned}
g(\bar{x}) &= \sum_{j=1}^{n} \frac{\partial f(\bar{x}^1)}{\partial x_j} x_j && (4)\\
&= \sum_{j=1}^{n} c_j\ x_j, \ \text{where } c_j = \left| \frac{\partial f(\bar{x})}{\partial x_j} \right|_{\bar{x}=\bar{x}^1} && (5)
\end{aligned}
$$

Frank-Wolfe Algorithm

1. Set k=0, find initital solution $\bar{x}^k$
2. Set $k = k + 1$ compute $c_j = \left|\frac{\partial f(\bar{x})}{\partial x_j}\right|_{\bar{x}=\bar{x}^{k-1}} \quad \forall j = 1, 2, \ldots, n$
3. Find optimum solution x_{LP}^k for the following LP

$$
\begin{aligned}
\text{Max} : g(\bar{x}) &= c_j\ x_j,\\
\text{s.t. } A\bar{x} &\leq \bar{b},\\
\bar{x} &\geq 0.
\end{aligned}
$$

4. For the variable $t,\ 0 \leq t \leq 1$ set

$$h(t) = f(\bar{x}^k) \quad \text{where} \quad \bar{x}^k = \bar{x}^{k-1} + t\left[\bar{x}_{LP}^k - \bar{x}^{k-1}\right]$$

5. Find t^* that maximizes $h(t)$ in the interval $0 \leq t \leq 1$ and compute the next point by

$$\bar{x}^k = \bar{x}^{k-1} + t^* \left[\bar{x}^k_{LP} - \bar{x}^{k-1}\right] \tag{6}$$

6. If the error $\epsilon \geq |\bar{x}^k - \bar{x}^{k-1}|$ go to step 2, else STOP.

Note that in step 1 use LP to get a feasible initital solution. Also in step 5 use any one dimentional search (either numerical or classical) to maximize the function $h(t)$.

4.3.2 Example

Solve the following program by frank-wolfe alogorithm:

$$\begin{aligned} \text{Max} : f(\bar{x}) &= 5x_1 - x_1^2 + 8x_2 - 2x_2^2, \\ \text{st} : 3x_1 + 2x_2 &\leq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution

- Set $k = 0$, and assume $\bar{x}^0 = (0, 0)$
- compute c_1 and c_2 as follows:

$$\begin{aligned} c_1 &= \left.\frac{\partial f}{\partial x_1}\right|_{x_1=0} = 5 - 2x_1 = 5 \\ c_2 &= \left.\frac{\partial f}{\partial x_2}\right|_{x_2=0} = 8 - 4x_2 = 8 \\ g(x_1, x_2) &= 5x_1 + 8x_2 \end{aligned}$$

- Solve the following LP:

$$\begin{aligned} \text{Max} : g(\bar{x}) &= 5x_1 + 8x_2, \\ \text{s.t.} \quad 3x_1 + 2x_2 &\leq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

- Solving graphically, $\bar{x}^* = (0, 3)$

- Setting $\bar{x}^k = (0,0) + t[(0,3) - (0,0)] = (0,3t)$
- Therefore, $h(t) = f(0,3t) = (5 \times 0) + (0)^2 + 8(3t) - 2(3t)^2 = 24t - 18t^2$
- Solving, $\frac{\partial h(t)}{\partial t} = 24 - 36t = 0, \rightarrow t^* = \frac{24}{36} = \frac{2}{3}$
- Therfore, the new trail solution is $x_k = (0, [3 \times \frac{2}{3}]) = (0,2)$

k	$x^{(k-1)}$	(c_1, c_2)	x^k_{LP}	x(t)	h(t)	t^*	x^k
0	0, 0	5, 8	0, 3	0, 3t	24t-18t^2	2/3	0, 2

4.3.3 Problems

Solve the following program by frank-wolfe alogorithm:

$$\begin{aligned} \text{Min}: f(\bar{x}) &= (x_1 - x_2)^2 + x_2, \\ \text{st}: x_1 + x_2 &\geq 1, \\ x_1 + 2x_2 &\leq 3, \\ x_1, x_2 &\geq 0. \end{aligned}$$

4.3.4 Problems

Solve the following program by frank-wolfe alogorithm:

$$\begin{aligned} \text{Max}: f(\bar{x}) &= 3x_1 + 4x_2 - x_1^3 - x_2^3, \\ \text{st}: x_1 + x_2 &\leq 1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

4.3.5 Problems

Solve the following program by frank-wolfe alogorithm:

$$\begin{aligned} \text{Max}: f(\bar{x}) &= 4x_1 - x_1^4 + 2x_2 - x_2^2, \\ \text{st}: 4x_1 + 2x_2 &\leq 5, \\ x_1, x_2 &\geq 0. \end{aligned}$$

4.3.6 Example

Solve the following program by inner penalty function method:

$$\begin{aligned} \text{Min} : f(\bar{x}) &= \frac{(x_1+1)^3}{3} + x_2, \\ \text{st} : -x_1 + 1 &\leq 0, \\ -x_2 &\leq 0. \end{aligned}$$

4.3.7 Example

Solve the following program by exterior penalty function method:

$$\begin{aligned} \text{Min} : f(\bar{x}) &= \frac{(x_1+1)^3}{3} + x_2, \\ \text{st} : -x_1 + 1 &\leq 0, \\ -x_2 &\leq 0. \end{aligned}$$

Exercises

1. Not Available

References

1. K Deb. *Optimization for engineering design: Algorithms and Examples*. Prentice Hall, India, 1998.
2. F S Hillier and G J Lieberman. *Introduction to operations research*. McGraw Hill, Inc., 2001.
3. Singiresu S Rao. *Engineering optimization: Theory and practice*. New Age International, New Delhi, India, 1996.
4. A Ravindran, D T Philips, and J J Solberg. *Operations research: Principles and practice*. John Wiley and Sons, 1987.
5. H A Taha. *Operations research: An introduction*. Prentice-Hall, India, 1997.

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