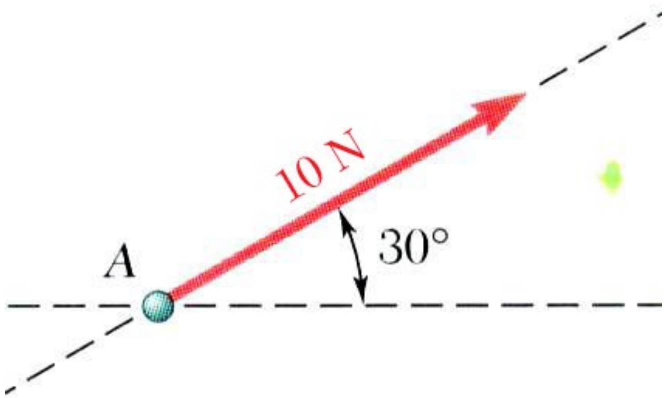
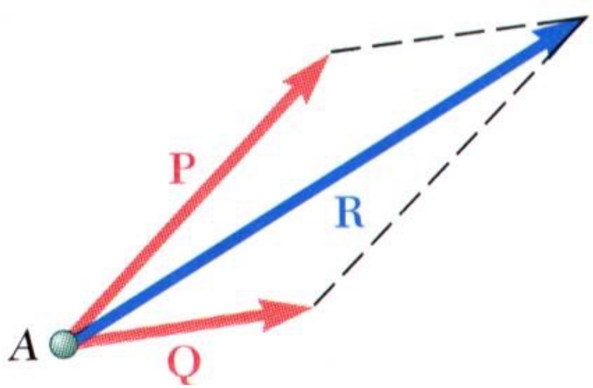
Engineering Mechanics

Force resultants, Torques, Scalar Products, Equivalent Force systems

Resultant of Two Forces

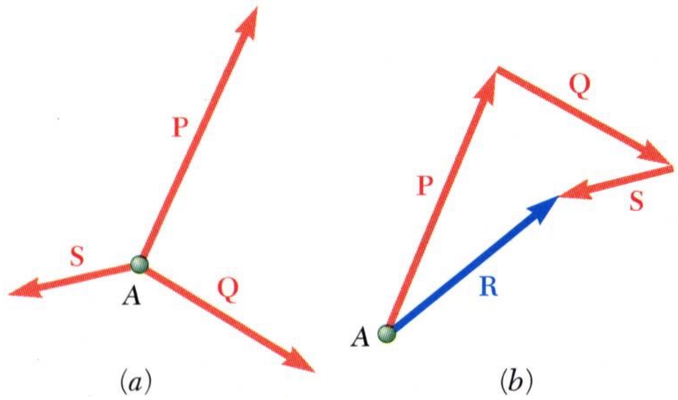


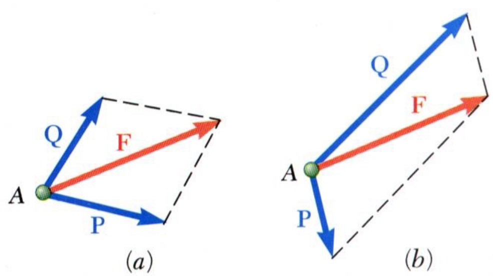
- force: action of one body on another; characterized by its *point of application*, *magnitude*, *line of action*, and *sense*.

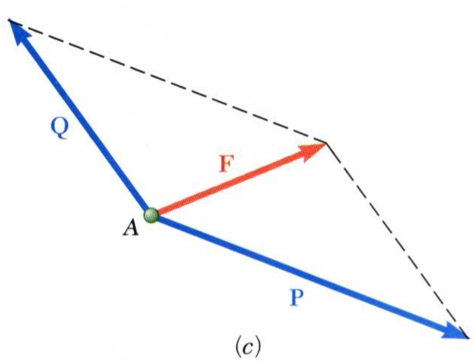


- Experimental evidence shows that the combined effect of two forces may be represented by a single *resultant* force.
- The resultant is equivalent to the diagonal of a parallelogram which contains the two forces in adjacent legs.
- Force is a *vector* quantity.

Resultant of Several Concurrent Forces





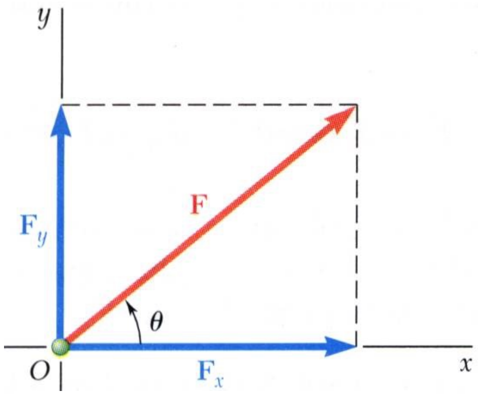


- *Concurrent forces*: set of forces which all pass through the same point.

  A set of concurrent forces applied to a particle may be replaced by a single resultant force which is the vector sum of the applied forces.

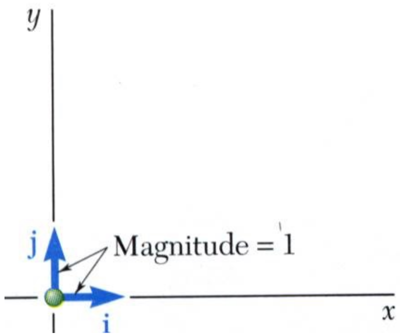
- *Vector force components*: two or more force vectors which, together, have the same effect as a single force vector.

Rectangular Components of a Force: Unit Vectors

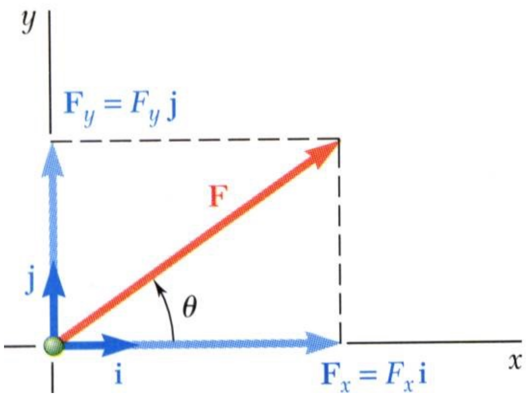


- May resolve a force vector into perpendicular components so that the resulting parallelogram is a rectangle. $\vec{F}_x$ and $\vec{F}_y$ are referred to as *rectangular vector components* and

$$\vec{F} = \vec{F}_x + \vec{F}_y$$



- Define perpendicular *unit vectors* $\vec{i}$ and $\vec{j}$ which are parallel to the x and y axes.

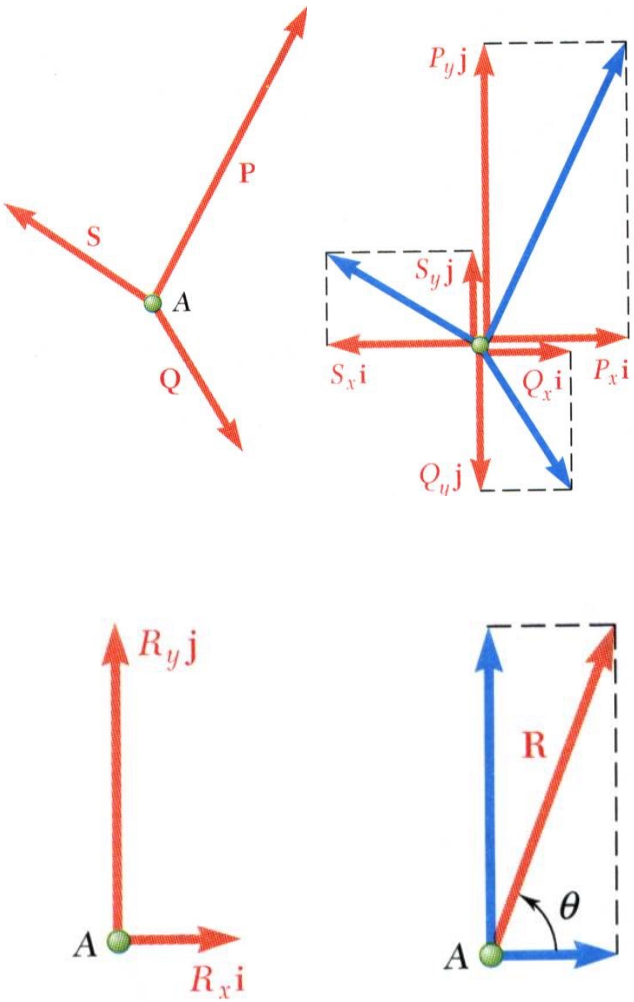


- Vector components may be expressed as products of the unit vectors with the scalar magnitudes of the vector components.

$$\vec{F} = F_x\vec{i} + F_y\vec{j}$$

F_x and F_y are referred to as the *scalar components* of $\vec{F}$

Addition of Forces by Summing Components



- Wish to find the resultant of 3 or more concurrent forces,

$$\vec{R} = \vec{P} + \vec{Q} + \vec{S}$$

- Resolve each force into rectangular components

$$\begin{aligned} R_x\vec{i} + R_y\vec{j} &= P_x\vec{i} + P_y\vec{j} + Q_x\vec{i} + Q_y\vec{j} + S_x\vec{i} + S_y\vec{j} \\ &= \left(P_x + Q_x + S_x\right)\vec{i} + \left(P_y + Q_y + S_y\right)\vec{j} \end{aligned}$$

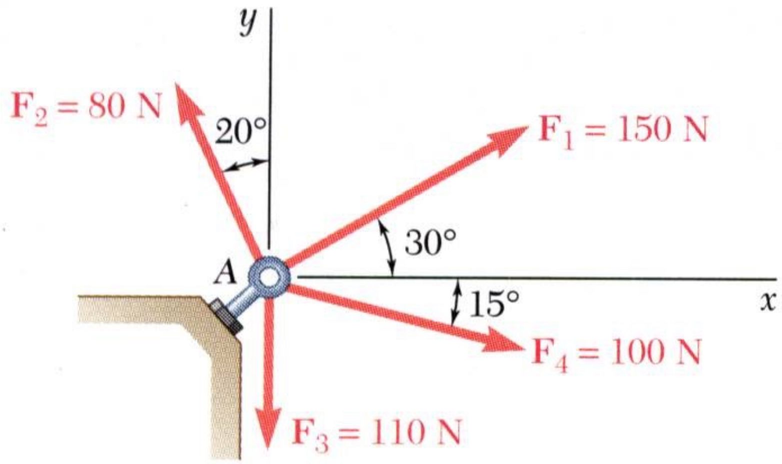
- The scalar components of the resultant are equal to the sum of the corresponding scalar components of the given forces.

$$\begin{aligned} R_x &= P_x + Q_x + S_x \\ &= \sum F_x \end{aligned} \qquad \begin{aligned} R_y &= P_y + Q_y + S_y \\ &= \sum F_y \end{aligned}$$

- To find the resultant magnitude and direction,

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$

Sample Problem 2.3

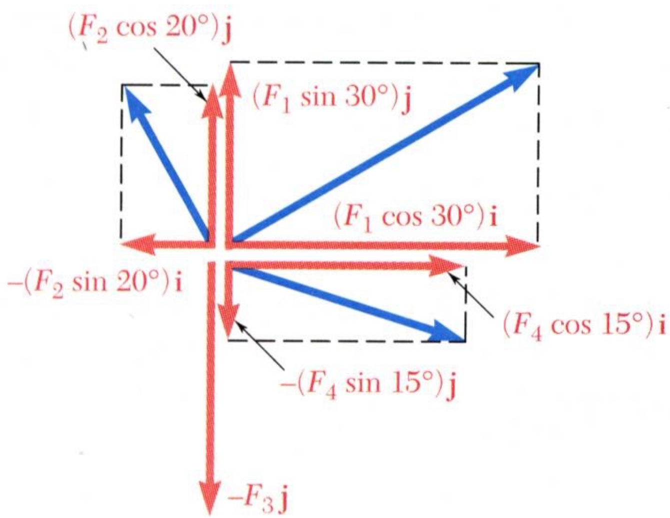


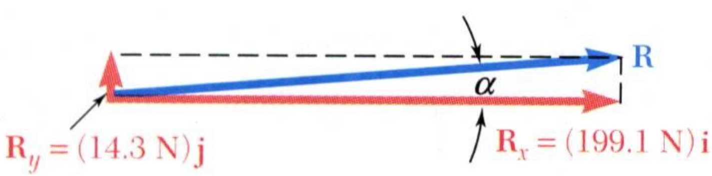
Four forces act on bolt *A* as shown. Determine the resultant of the force on the bolt.

SOLUTION:

- Resolve each force into rectangular components.
- Determine the components of the resultant by adding the corresponding force components.
- Calculate the magnitude and direction of the resultant.

Sample Problem 2.3





SOLUTION:

- Resolve each force into rectangular components.

force	*mag*	$x-comp$	$y-comp$
$\vec{F}_1$	150	$+129.9$	$+75.0$
$\vec{F}_2$	80	-27.4	$+75.2$
$\vec{F}_3$	110	0	-110.0
$\vec{F}_4$	100	$+96.6$	-25.9
		$R_x = +199.1$	$R_y = +14.3$

- Determine the components of the resultant by adding the corresponding force components.
- Calculate the magnitude and direction.

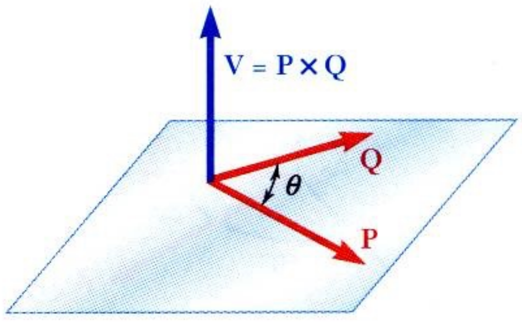
$$R = \sqrt{199.1^2 + 14.3^2} \qquad \boxed{R = 199.6\text{N}}$$

$$\tan\alpha = \frac{14.3\,\text{N}}{199.1\,\text{N}} \qquad \boxed{\alpha = 4.1^\circ}$$

Vector Product of Two Vectors

- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
  1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
  2. Magnitude of $\boldsymbol{V}$ is, $V = PQ\sin\theta$
  3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.



(*a*)

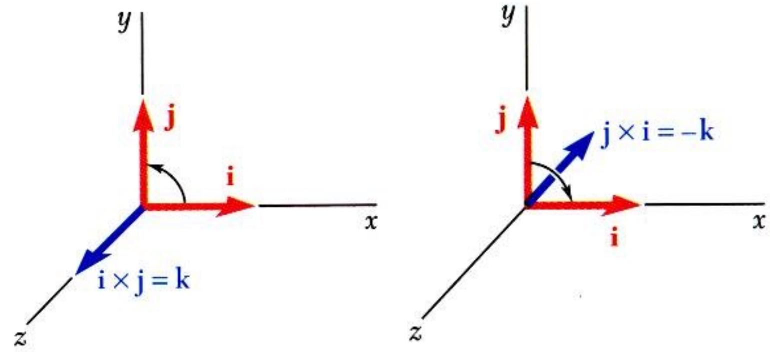
(*b*)

- Vector products:
  1. are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  2. are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  3. are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Vector Products: Rectangular Components

- Vector products of Cartesian unit vectors,

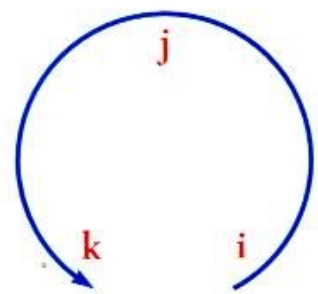
$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$



- Vector products in terms of rectangular coordinates

$$\vec{V}=\left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$
$$=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}$$
$$+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$



Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on it point of application.

- The *moment* of $\boldsymbol{F}$ about O is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

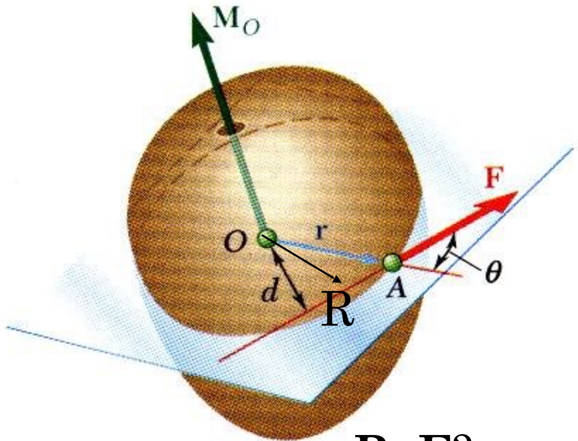
- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing O and the force $\boldsymbol{F}$.

- Magnitude of $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M_O}$.

$$M_O = rF\sin\theta = Fd$$

The sense of the moment may be determined by the right-hand rule.

- Any force $\boldsymbol{F'}$ that has the same magnitude and direction as $\boldsymbol{F}$, is *equivalent* if it also has the same line of action and therefore, produces the same moment.



(a)

(b)

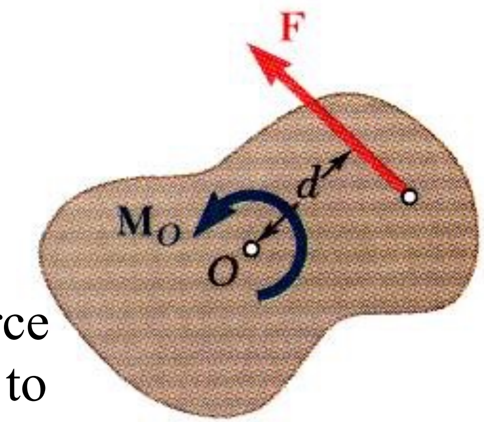
Moment of a Force About a Point

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained in the plane of the structure.

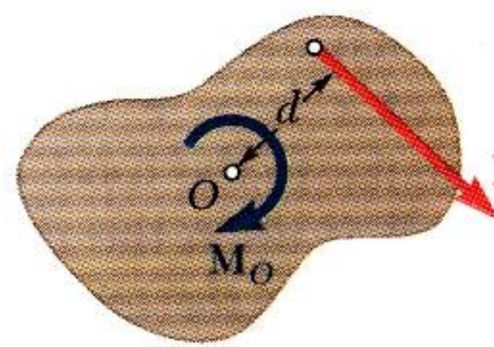
- The plane of the structure contains the point O and the force $\boldsymbol{F}$. $\boldsymbol{M}_O$, the moment of the force about O is perpendicular to the plane.

- If the force tends to rotate the structure clockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.

- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(a) $\mathbf{M}_O = +Fd$



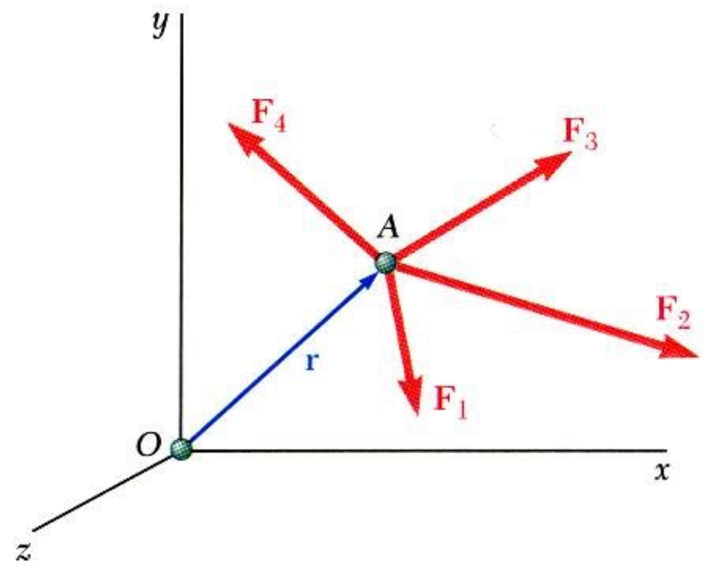
(b) $\mathbf{M}_O = -Fd$

Varignon's Theorem

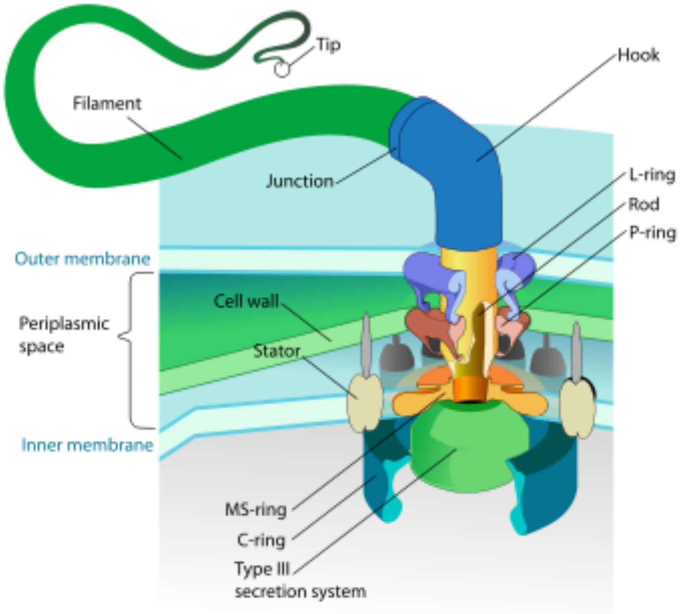
- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

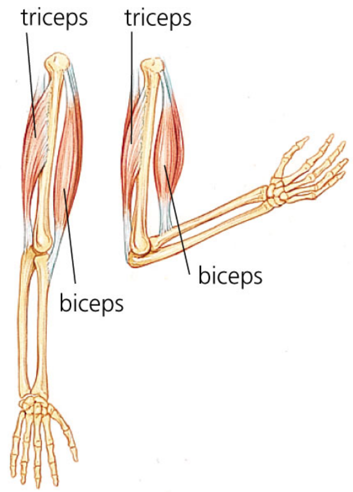
$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$

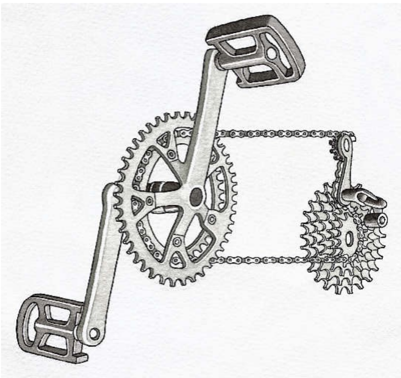
- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

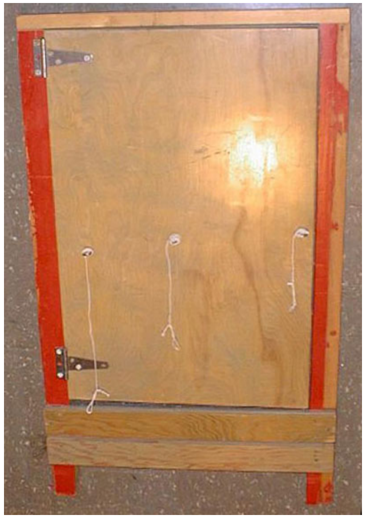


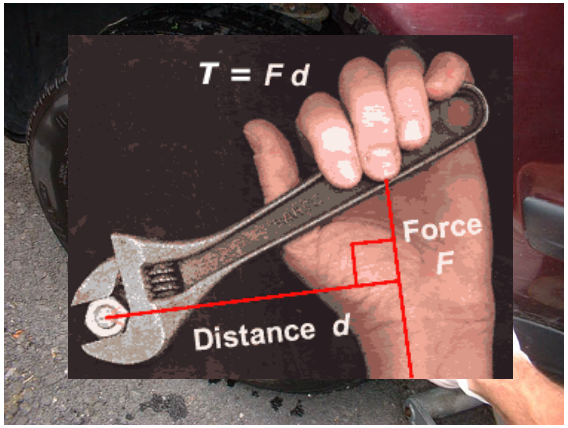
Examples of Torque in Action











Rectangular Components of the Moment of a Force

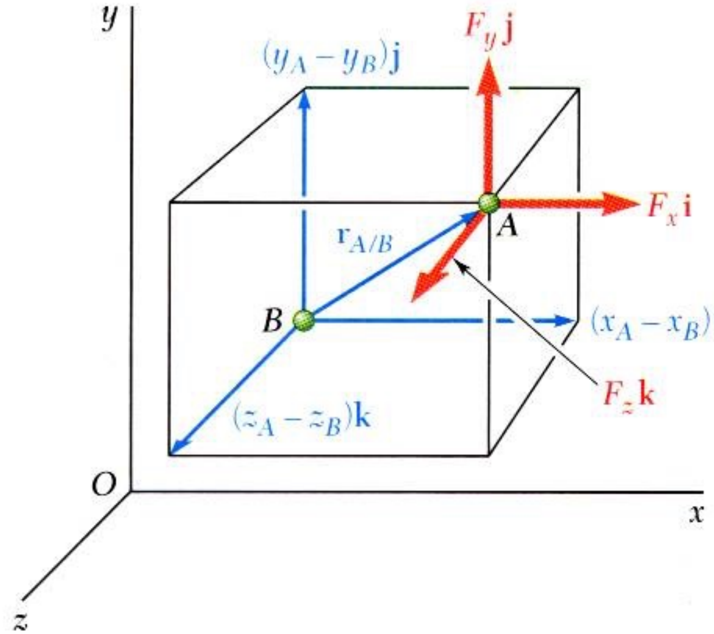
The moment of ***F*** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

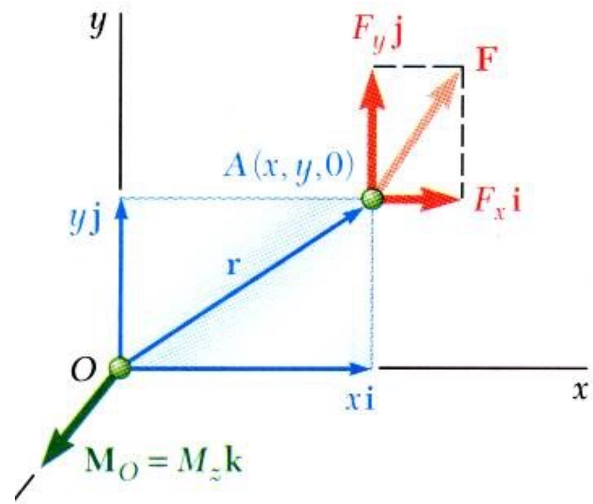
Rectangular Components of the Moment of a Force

For two-dimensional structures,

$$\vec{M}_O = \left(xF_y - yF_z\right)\vec{k}$$

$$M_O = M_Z$$

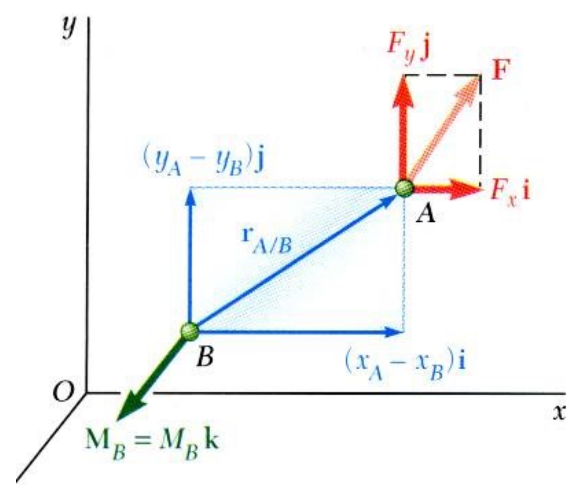
$$= xF_y - yF_z$$



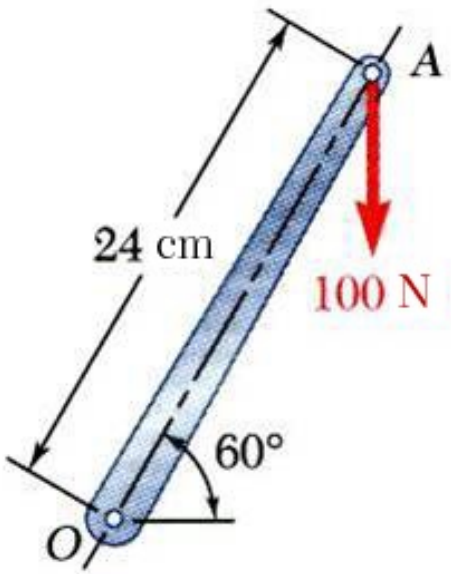
$$\vec{M}_B = \left[\left(x_A - x_B\right)F_y - \left(y_A - y_B\right)F_z\right]\vec{k}$$

$$M_B = M_Z$$

$$= \left(x_A - x_B\right)F_y - \left(y_A - y_B\right)F_z$$



Sample Problem 3.1



A 100-N vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

a) moment about *O*,

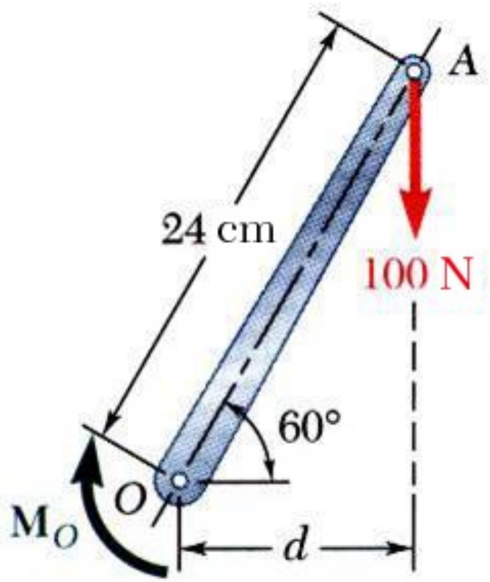
b) horizontal force at *A* which creates the same moment,

c) smallest force at A which produces the same moment,

d) location for a 240-N vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.1



a) Moment about *O* is equal to the product of the force and the perpendicular distance between the line of action of the force and *O*. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

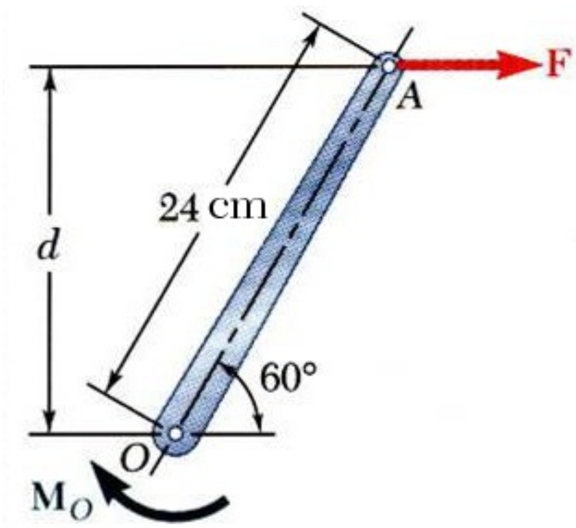
$$M_O = Fd$$

$$d = (24\,\mathrm{cm})\cos 60^\circ = 12\ \mathrm{cm}$$

$$M_O = (100\ \mathrm{N})(12\ \mathrm{cm})$$

$$\boxed{M_O = 1200\ \mathrm{N \cdot cm}}$$

Sample Problem 3.1



c) Horizontal force at A that produces the same moment,

$$d = (24\text{ cm})\sin 60° = 20.8\text{ cm}$$

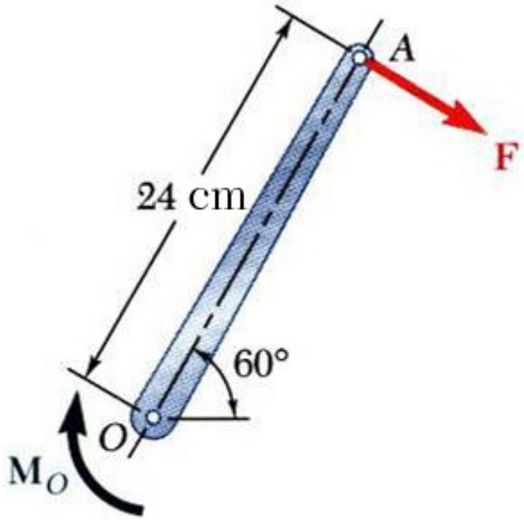
$$M_O = Fd$$

$$1200\text{ N}\cdot\text{cm} = F(20.8\text{ cm})$$

$$F = \frac{1200\text{ N}\cdot\text{cm}}{20.8\text{ cm}}$$

$$\boxed{F = 57.7\text{ N}}$$

Sample Problem 3.1



c) The smallest force A to produce the same moment occurs when the perpendicular distance is a maximum or when F is perpendicular to OA.

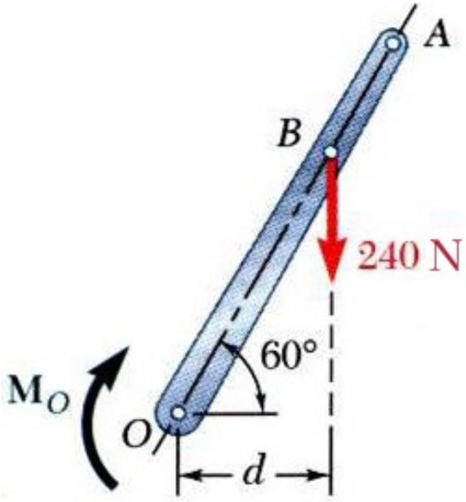
$$M_O = Fd$$

$$1200\ \mathrm{N \cdot cm} = F(24\ \mathrm{cm})$$

$$F = \frac{1200\ \mathrm{N \cdot cm}}{24\ \mathrm{cm}}$$

$$\boxed{F = 50\ \mathrm{N}}$$

Sample Problem 3.1



d) To determine the point of application of a 240 lb force to produce the same moment,

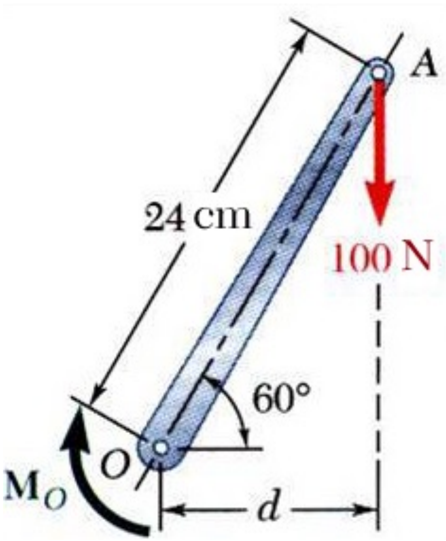
$$M_O = Fd$$

$$1200\ \mathrm{N \cdot cm} = (240\ \mathrm{N})d$$

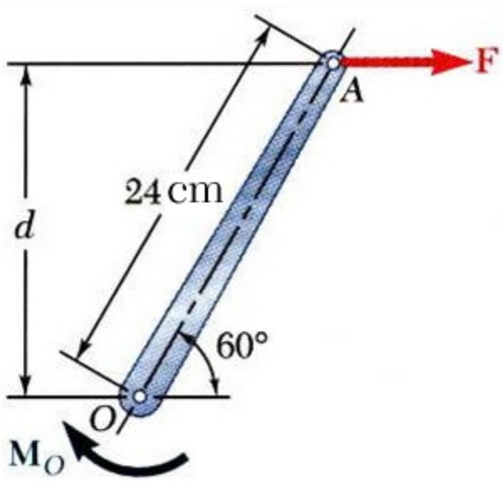
$$d = \frac{1200\ \mathrm{N \cdot cm}}{240\ \mathrm{N}} = 5\ \mathrm{cm}$$

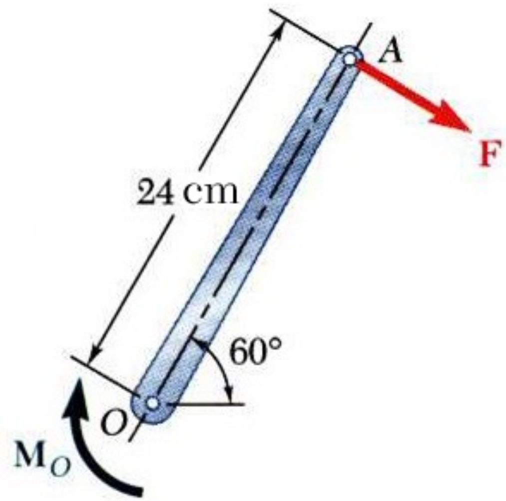
$$OB \cos 60° = 5\ \mathrm{cm} \qquad \boxed{OB = 10\ \mathrm{cm}}$$

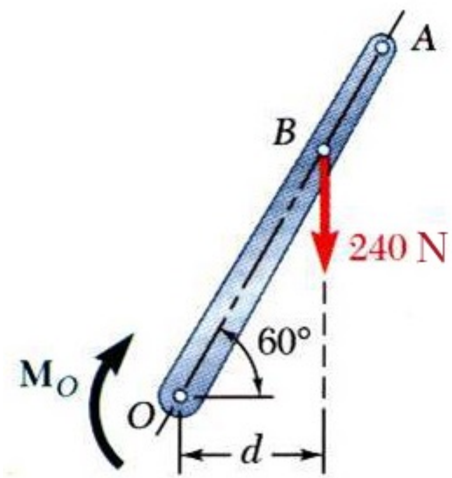
Sample Problem 3.1



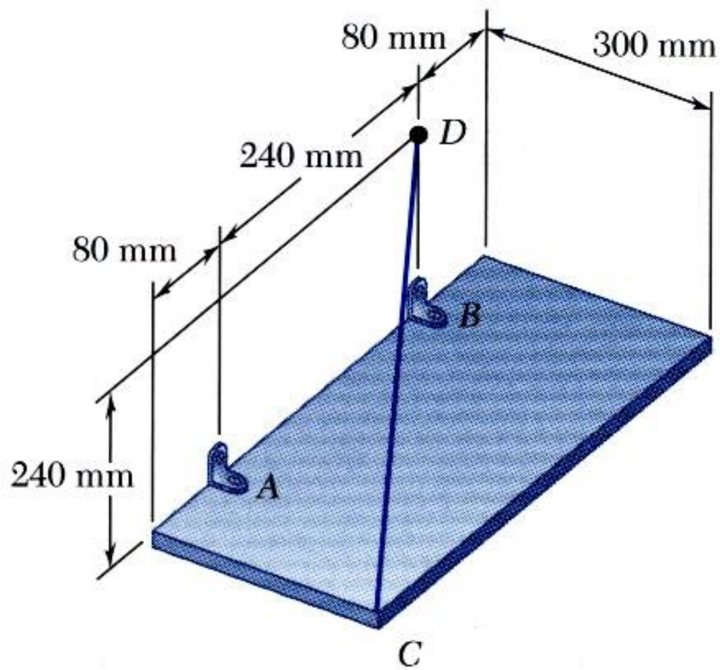
e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 N force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 N force.







Sample Problem 3.4



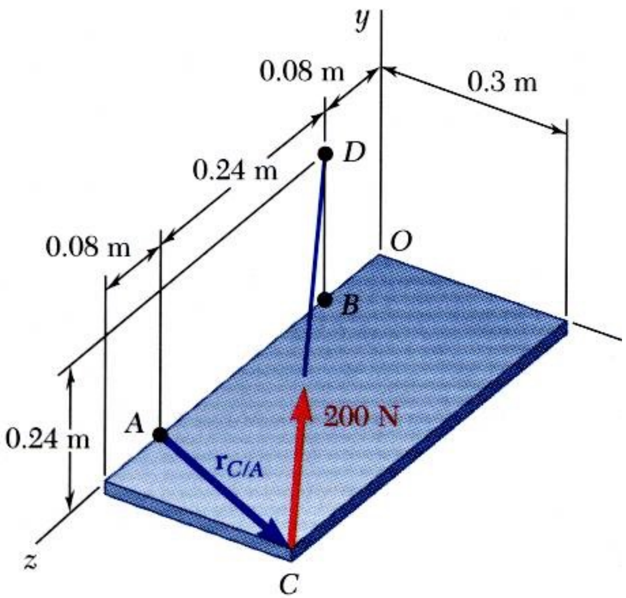
The rectangular plate is supported by the brackets at A and B and by a wire CD. Knowing that the tension in the wire is 200 N, determine the moment about A of the force exerted by the wire at C.

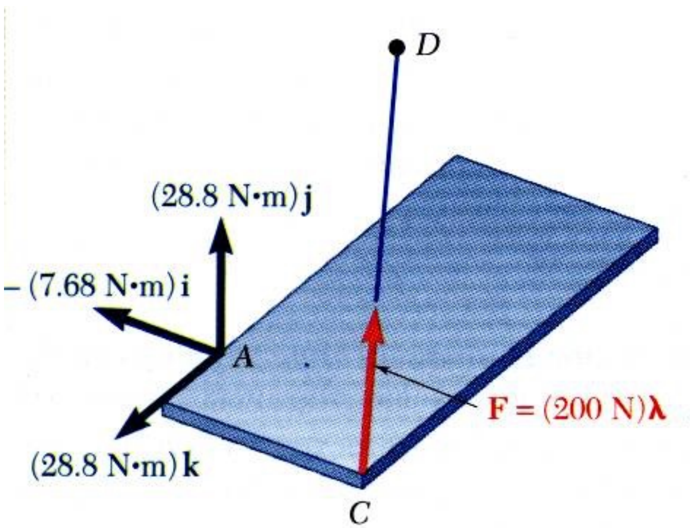
SOLUTION:

The moment M_A of the force F exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Sample Problem 3.4





SOLUTION:

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\text{ m})\vec{i} + (0.08 m)\vec{k}$$

$$\vec{F} = F\vec{\lambda} = (200\text{ N})\frac{\vec{r}_{D|C}}{r_{D|C}}$$

$$= (200\text{ N})\frac{-(0.3\text{ m})\vec{i} + (0.24\text{ m})\vec{j} - (0.32\text{ m})\vec{k}}{0.5\text{ m}}$$

$$= -(120\text{ N})\vec{i} + (96\text{ N})\vec{j} - (128\text{ N})\vec{k}$$

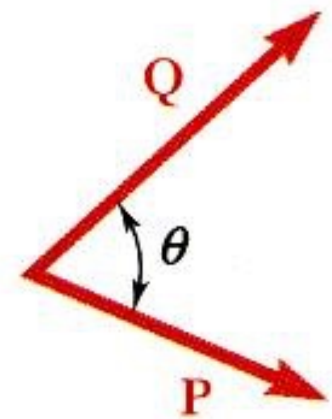
$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\vec{M}_A = -(7.68\text{ N}\cdot\text{m})\vec{i} + (28.8\text{ N}\cdot\text{m})\vec{j} + (28.8\text{ N}\cdot\text{m})\vec{k}}$$

Scalar Product of Two Vectors

- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as

  $$\vec{P} \bullet \vec{Q} = PQ\cos\theta \quad (\text{scalar result})$$



- Scalar products:
  - are commutative, $\vec{P} \bullet \vec{Q} = \vec{Q} \bullet \vec{P}$
  - are distributive, $\vec{P} \bullet \left(\vec{Q}_1 + \vec{Q}_2\right) = \vec{P} \bullet \vec{Q}_1 + \vec{P} \bullet \vec{Q}_2$
  - are not associative, $\left(\vec{P} \bullet \vec{Q}\right) \bullet \vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

$$\vec{P} \bullet \vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right) \bullet \left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$\vec{i} \bullet \vec{i} = 1 \quad \vec{j} \bullet \vec{j} = 1 \quad \vec{k} \bullet \vec{k} = 1 \quad \vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$

$$\vec{P} \bullet \vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

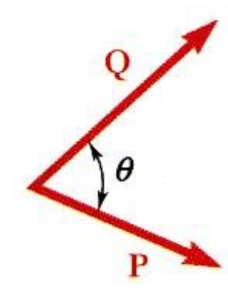
$$\vec{P} \bullet \vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Scalar Product of Two Vectors: Applications

- Angle between two vectors:

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta = P_xQ_x + P_yQ_y + P_zQ_z$$

$$\cos\theta = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ}$$

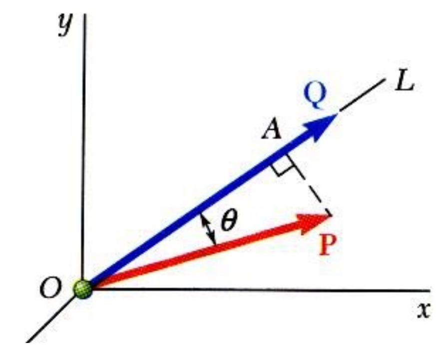


- Projection of a vector on a given axis:

$$P_{OL} = P\cos\theta = \text{projection of } P \text{ along } OL$$

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta$$

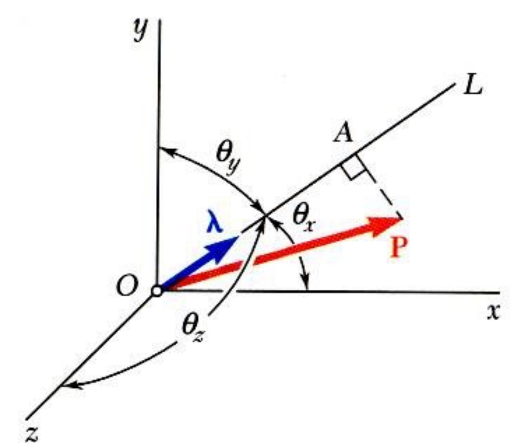
$$\frac{\vec{P} \bullet \vec{Q}}{Q} = P\cos\theta = P_{OL}$$



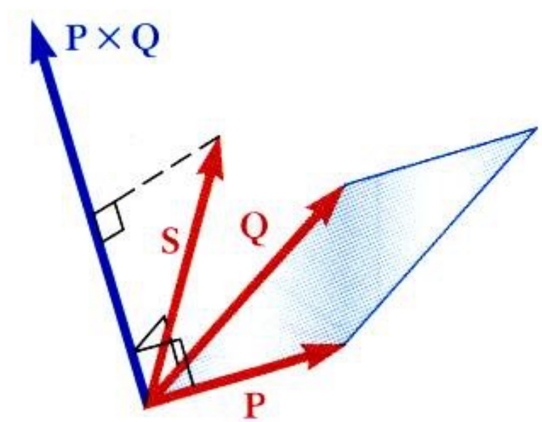
- For an axis defined by a unit vector:

$$P_{OL} = \vec{P} \bullet \vec{\lambda}$$

$$= P_x\cos\theta_x + P_y\cos\theta_y + P_z\cos\theta_z$$



Mixed Triple Product of Three Vectors



- Mixed triple product of three vectors,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \text{scalar result}$$

- The six mixed triple products formed from ***S***, ***P***, and ***Q*** have equal magnitudes but not the same sign,

$$\begin{aligned}\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) &= \vec{P} \bullet \left(\vec{Q} \times \vec{S}\right) = \vec{Q} \bullet \left(\vec{S} \times \vec{P}\right) \\ &= -\vec{S} \bullet \left(\vec{Q} \times P\right) = -\vec{P} \bullet \left(\vec{S} \times \vec{Q}\right) = -\vec{Q} \bullet \left(\vec{P} \times \vec{S}\right)\end{aligned}$$

- Evaluating the mixed triple product,

$$\begin{aligned}\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) &= S_x\left(P_y Q_z - P_z Q_y\right) + S_y\left(P_z Q_x - P_x Q_z\right) \\ &\quad + S_z\left(P_x Q_y - P_y Q_x\right) \\ &= \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}\end{aligned}$$

Some basic facts in Euclidean geometry

- Two parallel lines form a unique plane
- Three non-colinear points form a unique plane
- A line and a non-colinear point form a unique plane
- Two non-parallel but intersecting lines form unique plane

Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

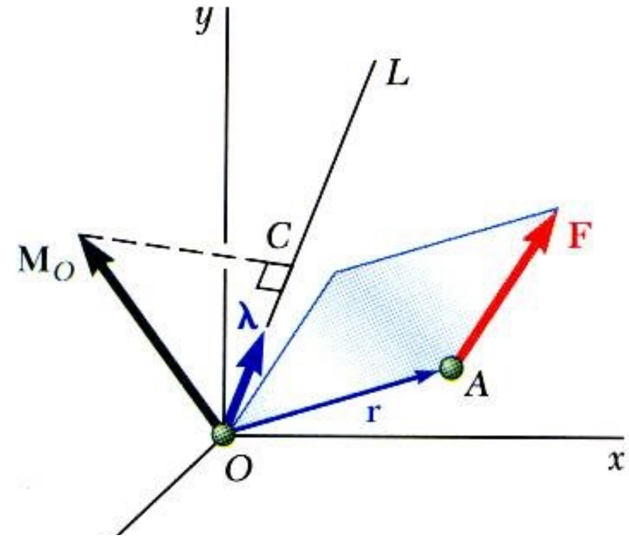
$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

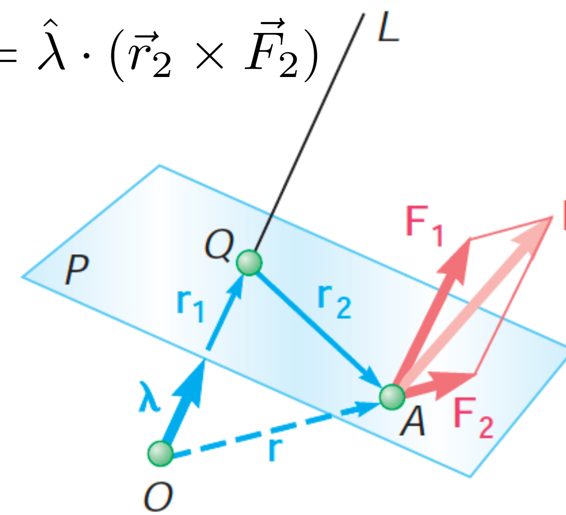
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

- Moments of $\boldsymbol{F}$ about the coordinate axes,

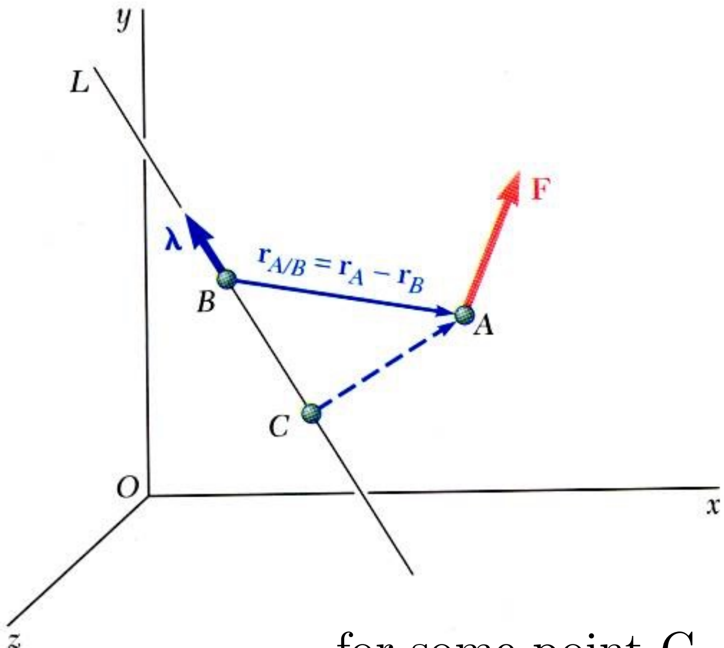
$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$



$$M_{0L} = \hat{\lambda} \cdot (\vec{r}_2 \times \vec{F}_2)$$



Moment of a Force About a Given Axis



for some point C

$$\vec{r}_{A|C} = \vec{r}_A - \vec{r}_C$$

but

$$\vec{r}_C = \vec{r}_B - \hat{\lambda} r_{BC}$$

so

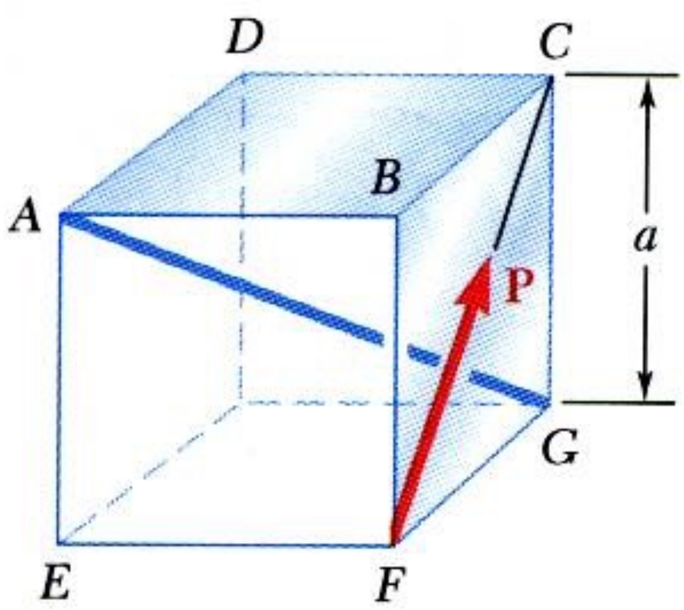
$$M_{CL} = M_{BL} + \underbrace{\hat{\lambda} \cdot (r_{BC}\hat{\lambda} \times \vec{F})}_{0}$$

- Moment of a force about an arbitrary axis,

$$M_{BL} = \vec{\lambda} \bullet \vec{M}_B$$
$$= \vec{\lambda} \bullet \left(\vec{r}_{A/B} \times \vec{F}\right)$$
$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

- **The result is independent of the point *B* along the given axis.**

Sample Problem 3.5



A cube is acted on by a force ***P*** as shown. Determine the moment of ***P***

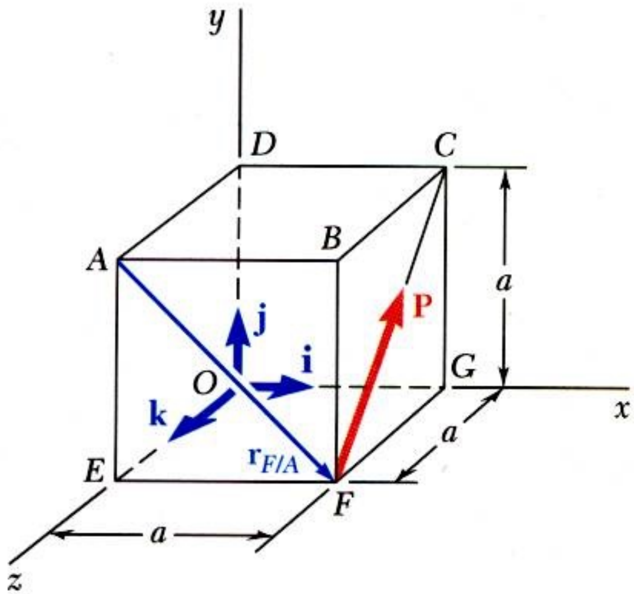
a) about *A*

b) about the edge *AB* and

c) about the diagonal *AG* of the cube.

d) Determine the perpendicular distance between *AG* and *FC*.

Sample Problem 3.5



- Moment of **P** about *A*,

$$\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$$

$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a\left(\vec{i} - \vec{j}\right)$$

$$\vec{P} = P/\sqrt{2}\left(\vec{i} + \vec{j}\right) = P/\sqrt{2}\left(\vec{i} + \vec{j}\right)$$

$$\vec{M}_A = a\left(\vec{i} - \vec{j}\right) \times P/\sqrt{2}\left(\vec{i} + \vec{j}\right)$$

$$\boxed{\vec{M}_A = \left(aP/\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)}$$

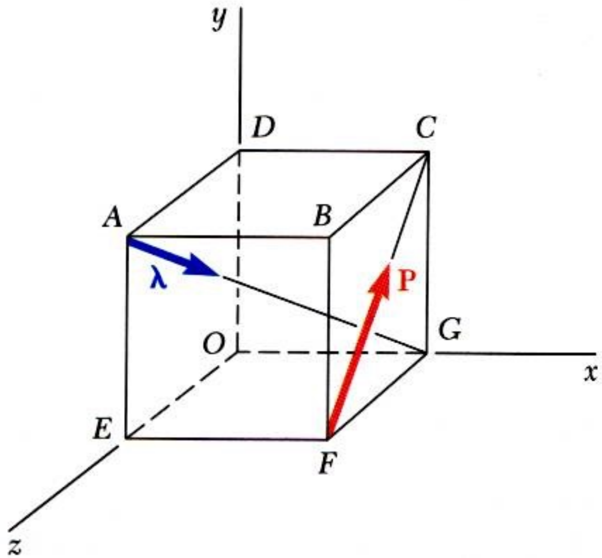
- Moment of **P** about *AB*,

$$M_{AB} = \vec{i} \bullet \vec{M}_A$$

$$= \vec{i} \bullet \left(aP/\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$\boxed{M_{AB} = aP/\sqrt{2}}$$

Sample Problem 3.5



- Moment of ***P*** about the diagonal *AG*,

$$M_{AG} = \vec{\lambda} \bullet \vec{M}_A$$

$$\vec{\lambda} = \frac{\vec{r}_{A/G}}{r_{A/G}} = \frac{a\vec{i} - a\vec{j} - a\vec{k}}{a\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right)$$

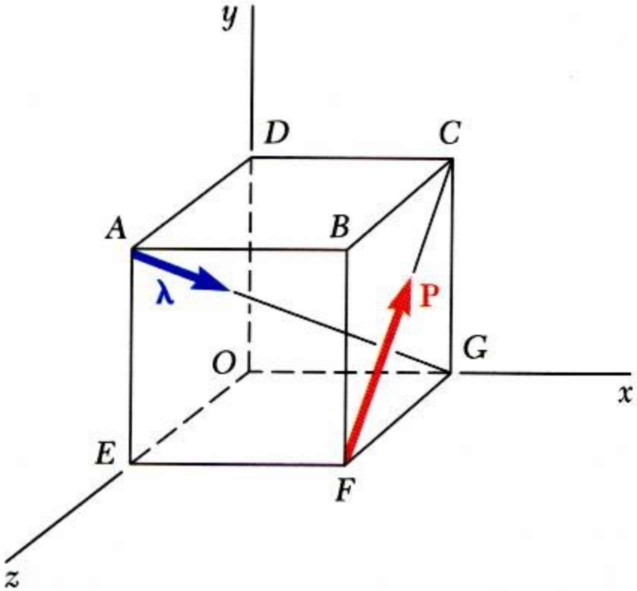
$$\vec{M}_A = \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

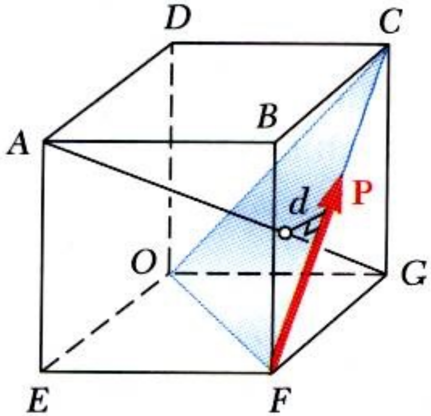
$$M_{AG} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) \bullet \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$= \frac{aP}{\sqrt{6}}(1 - 1 - 1)$$

$$\boxed{M_{AG} = -\frac{aP}{\sqrt{6}}}$$

Sample Problem 3.5





- Perpendicular distance between *AG* and *FC*,

$$\vec{P} \bullet \vec{\lambda} = \frac{P}{\sqrt{2}}\left(\vec{j} - \vec{k}\right) \bullet \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) = \frac{P}{\sqrt{6}}(0 - 1 + 1)$$
$$= 0$$

Therefore, *P* is perpendicular to *AG*.

$$\left|M_{AG}\right| = \frac{aP}{\sqrt{6}} = Pd$$

$$\boxed{d = \frac{a}{\sqrt{6}}}$$

This is straightforward currently since

General rule to get distance between lines

$$\text{Line-1} : \hat{\lambda}$$

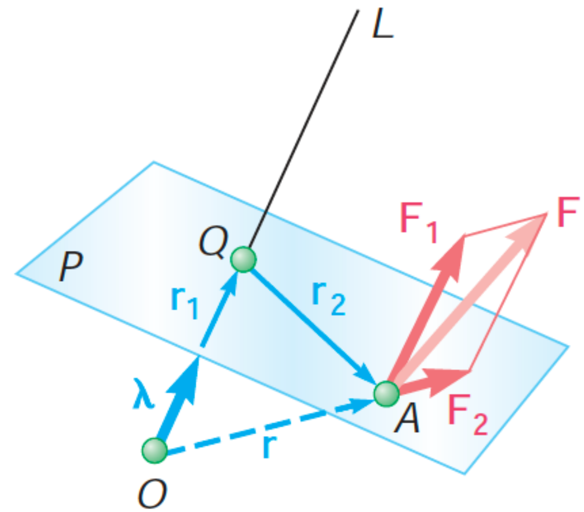
$$\text{Line-2} : \vec{F}$$

$$\text{Find} : M_\lambda$$

$$\vec{F}_\perp = \vec{F} - (\vec{F} \cdot \hat{\lambda})\hat{\lambda}$$

Obtain distance between 1 and 2 :

$$M_\lambda = |\vec{F}_\perp| d$$

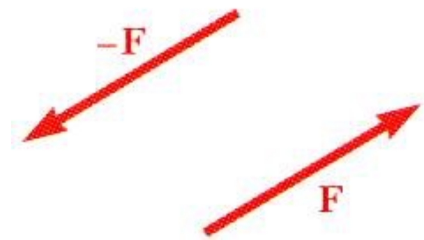


$$M_{0L} = \hat{\lambda} \cdot (\vec{r}_2 \times \vec{F}_2)$$

$$\vec{F}_2 \equiv \vec{F}_\perp$$

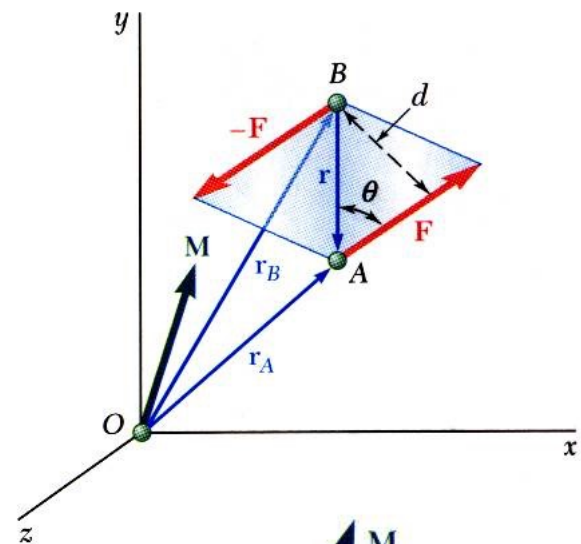
Moment of a Couple

- Two forces ***F*** and -***F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

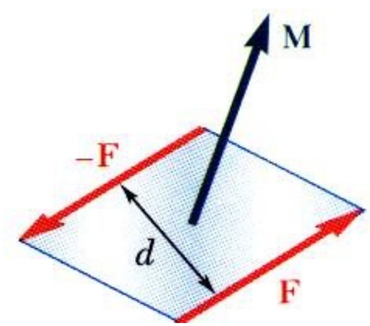


- Moment of the couple,

$$
\begin{aligned}
\vec{M} &= \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right) \\
&= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F} \\
&= \vec{r} \times \vec{F} \\
M &= rF \sin\theta = Fd
\end{aligned}
$$



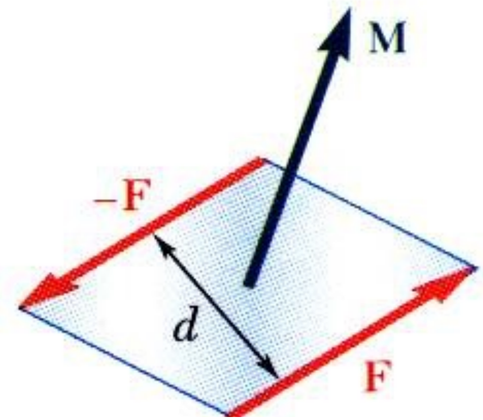
- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

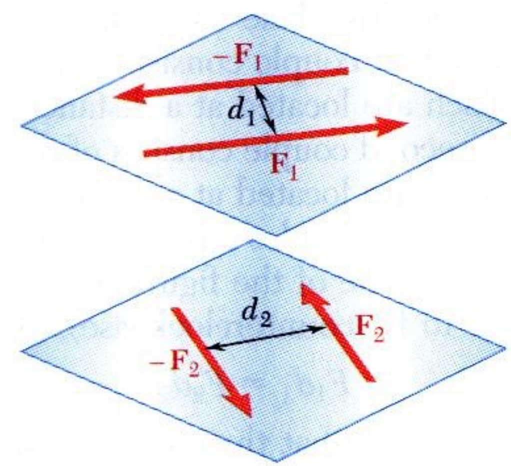


Moment of a Couple

Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.



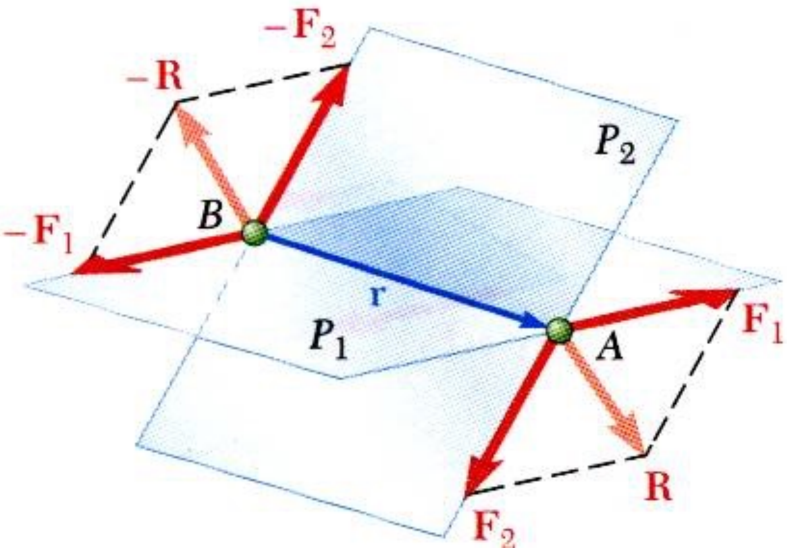


Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$

$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$



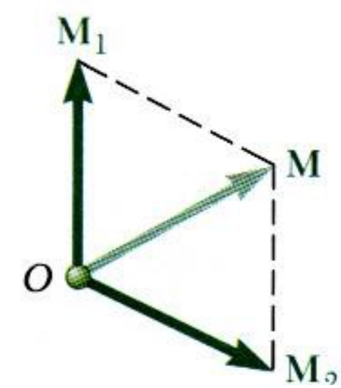
- Resultants of the vectors also form a couple

$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

- By Varignon's theorem

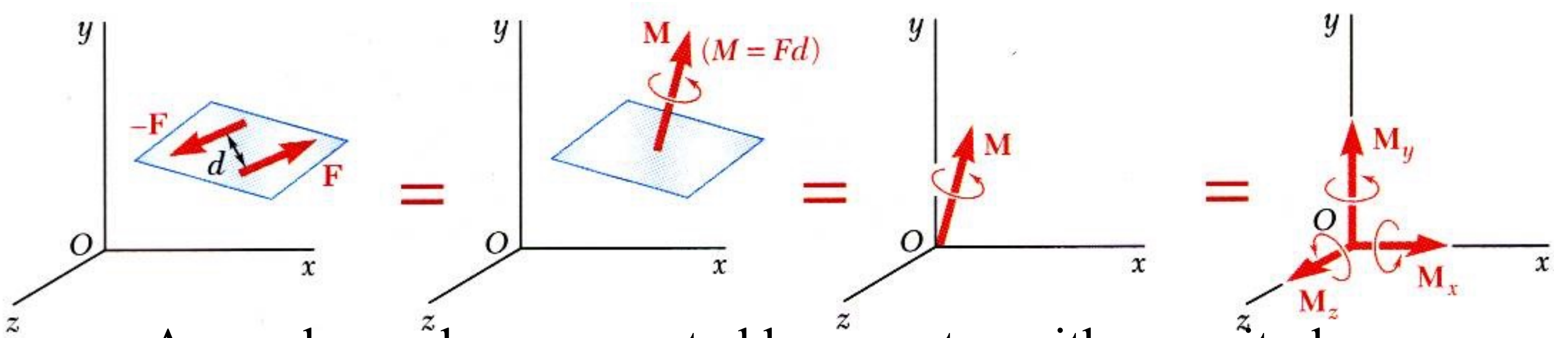
$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$

$$= \vec{M}_1 + \vec{M}_2$$



- Sum of two couples is also a couple that is equal to the vector sum of the two couples

Couples Can Be Represented by Vectors



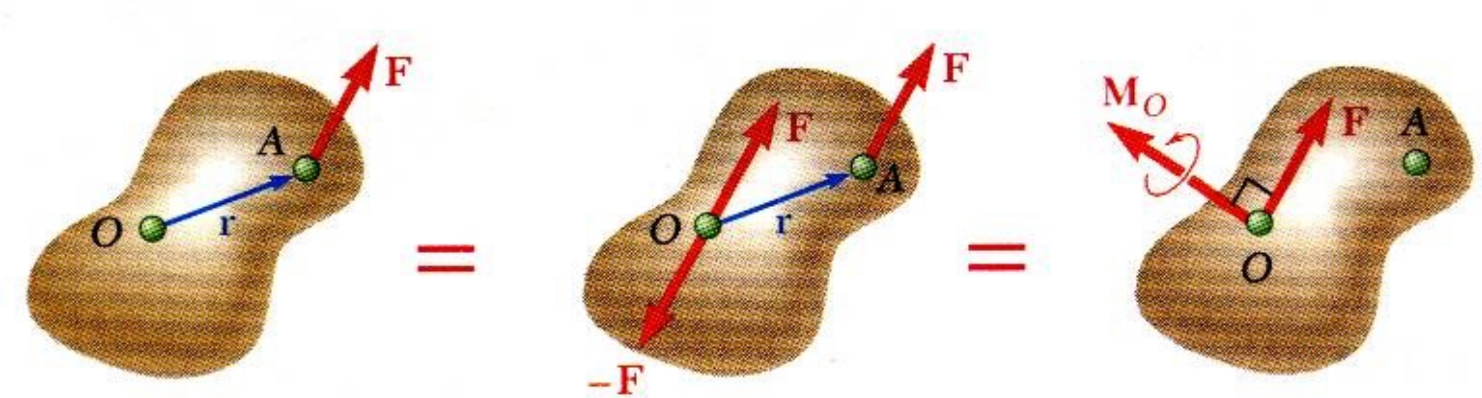
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.

- *Couple vectors* obey the law of addition of vectors.

- Couple vectors are free vectors, i.e., the point of application is not significant.

- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force at *O* and a Couple

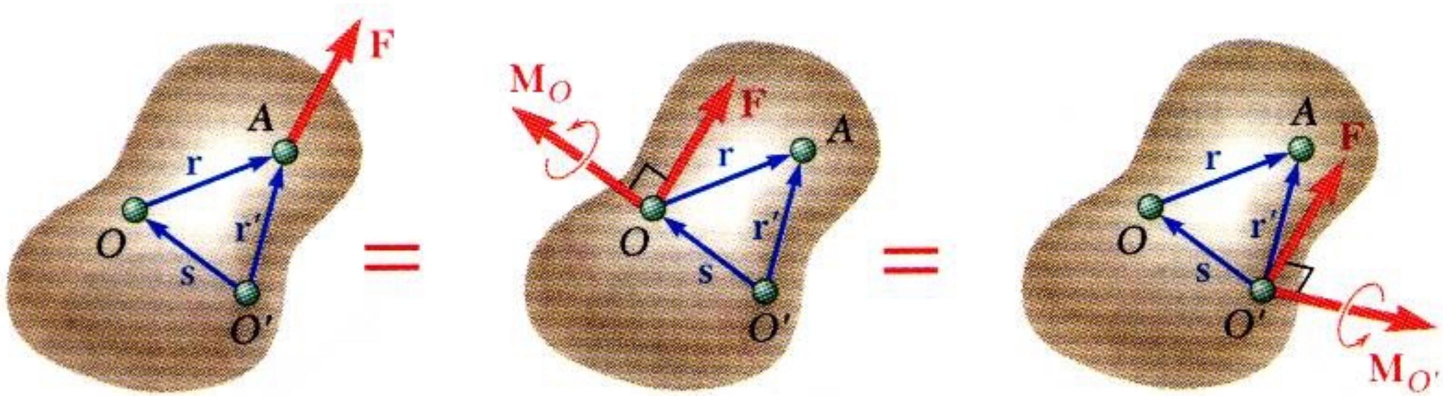


- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.

- Attaching equal and opposite force vectors at *O* produces no net effect on the body.

- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at *O* and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

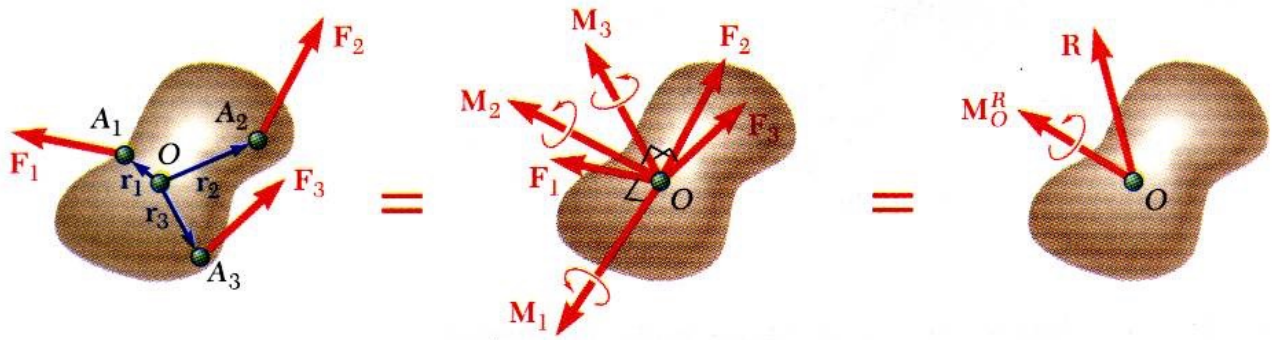
$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F}$$

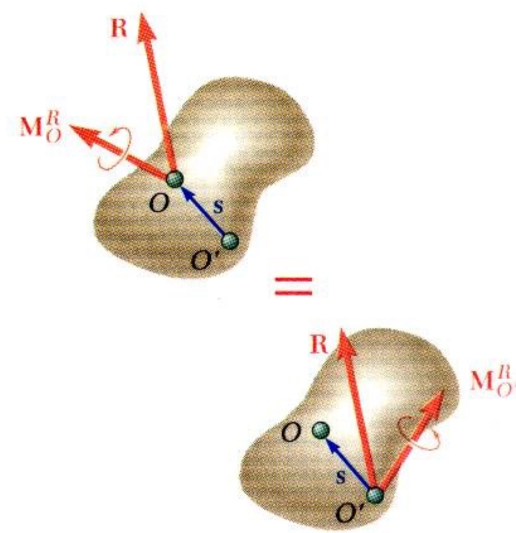
- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}\,'\times\vec{F} = (\vec{r} + \vec{s})\times\vec{F} = \vec{r}\times\vec{F} + \vec{s}\times\vec{F}$$
$$= \vec{M}_O + \vec{s}\times\vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

System of Forces: Reduction to a Force and Couple





- A system of forces may be replaced by a collection of force-couple systems acting a given point *O*
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F}\right)$$

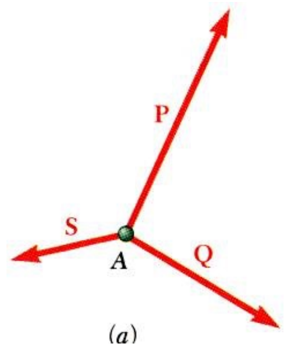
- The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

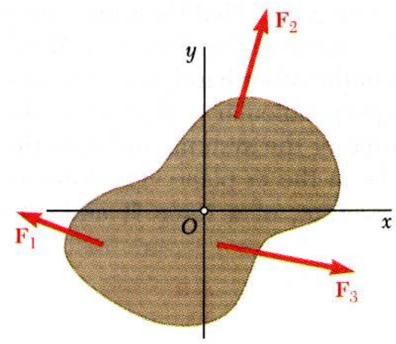
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

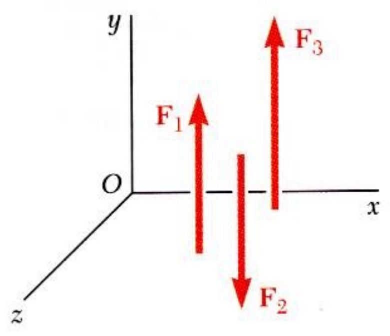
Further Reduction of a System of Forces

- If the resultant force and couple at *O* are mutually perpendicular, they can be replaced by a single force acting along a new line of action.

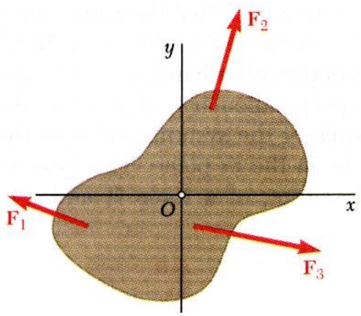


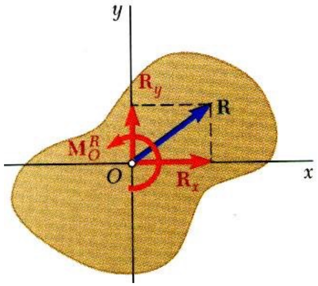
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.



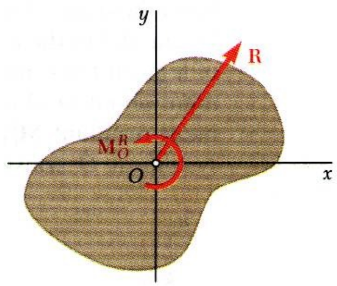


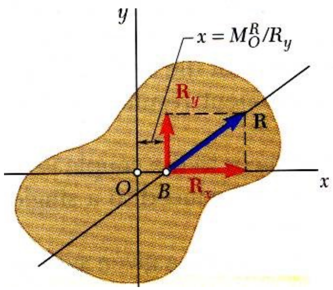
Further Reduction of a System of Forces





- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.

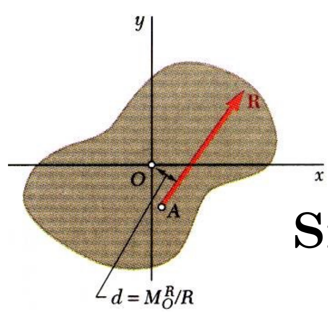




- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$

- In terms of rectangular coordinates,

$$xR_y - yR_x = M_O^R$$



Simplest Resultant