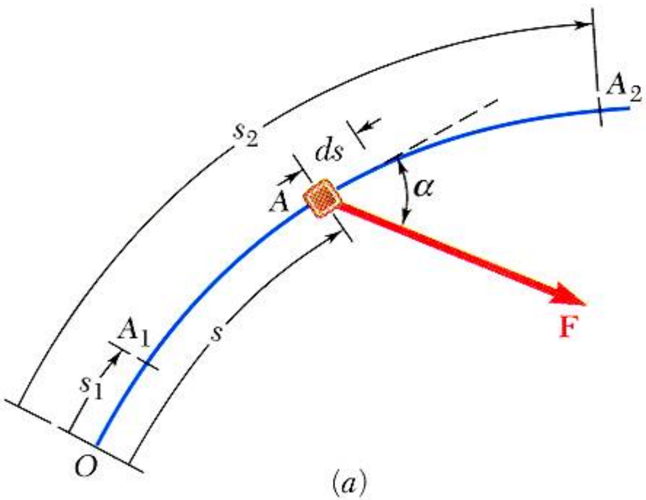
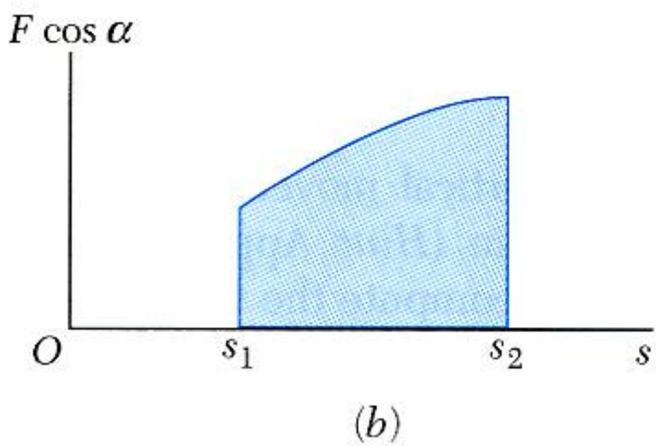
CE 102: Engineering Mechanics

Minimum Potential Energy

Work of a Force During a Finite Displacement



(a)



(b)

- Work of a force corresponding to an infinitesimal displacement,

$$dU = \vec{F}\cdot d\vec{r}$$

$$= F\,ds\cos\alpha$$

- Work of a force corresponding to a finite displacement,

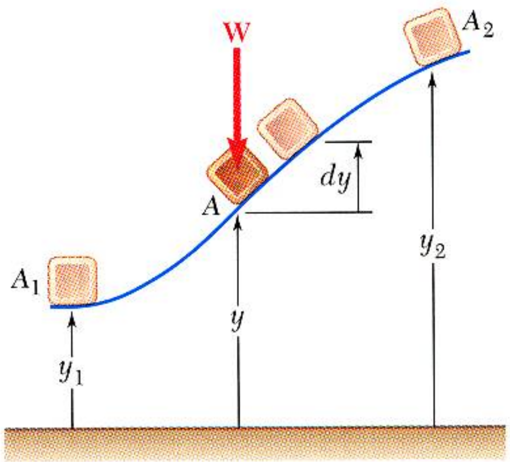
$$U_{1\to 2} = \int_{s_1}^{s_2} (F\cos\alpha)ds$$

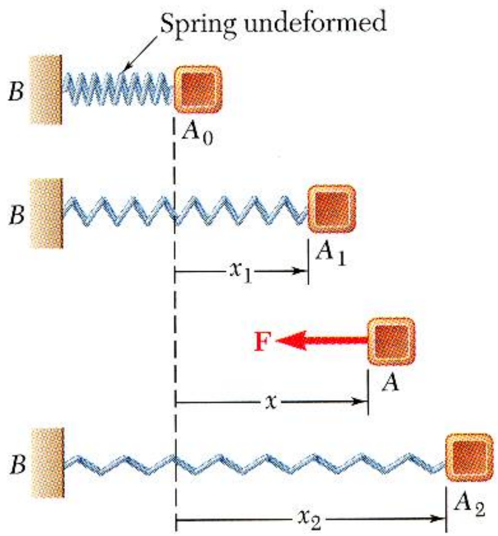
- Similarly, for the work of a couple,

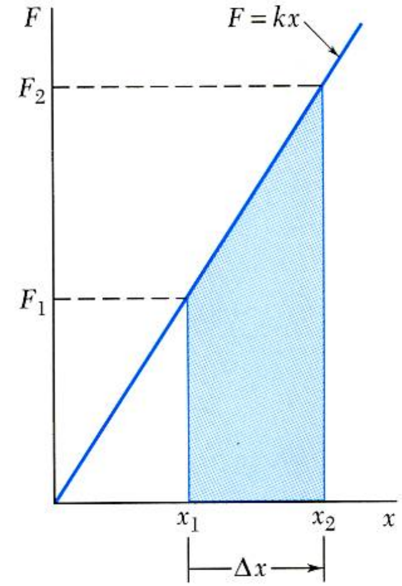
$$dU = Md\theta$$

$$U_{1\to 2} = M(\theta_2 - \theta_1)$$

Work of a Force During a Finite Displacement







Work of a weight,

$$dU = -W dy$$

$$U_{1\to 2} = -\int_{y_1}^{y_2} W dy$$

$$= W y_1 - W y_2$$

$$= -W \Delta y$$

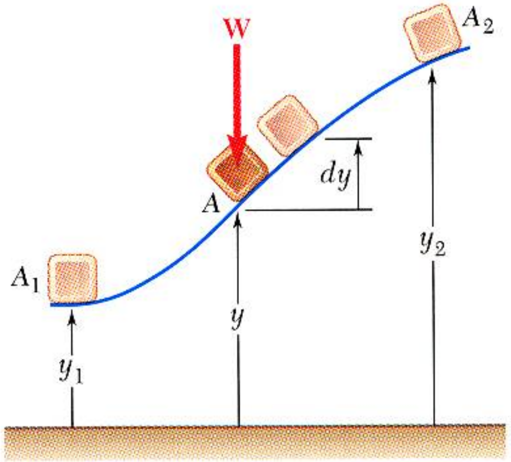
Work of a spring,

$$dU = -F dx = -(kx) dx \qquad U_{1\to 2} = -\frac{1}{2}(F_1 + F_2)\Delta x$$

$$U_{1\to 2} = -\int_{x_1}^{x_2} kx\, dx$$

$$\frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2$$

Potential Energy



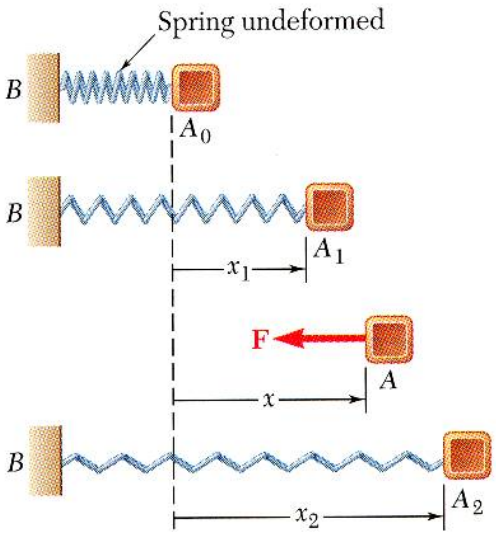
- Work of a weight

$$U_{1\to 2} = Wy_1 - Wy_2$$

The work is independent of path and depends only on

$Wy = V_g =$ *potential energy* of the body with respect to the *force of gravity* $\vec{W}$

$$U_{1\to 2} = \left(V_g\right)_1 - \left(V_g\right)_2$$



- Work of a spring,

$$U_{1\to 2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

$$\left(V_e\right)_1 - \left(V_e\right)_2$$

$V_e =$ *potential energy* of the body with respect to the *elastic force* $\vec{F}$

Potential Energy

- When the differential work is a force is given by an exact differential,

$$dU = -dV$$

$$U_{1\rightarrow 2} = V_1 - V_2$$

$$\textit{negative of change in potential energy}$$

- Forces for which the work can be calculated from a change in potential energy are *conservative forces*.

Convention: Work done on a body is *negative* of *change* in the *potential energy* of the body.

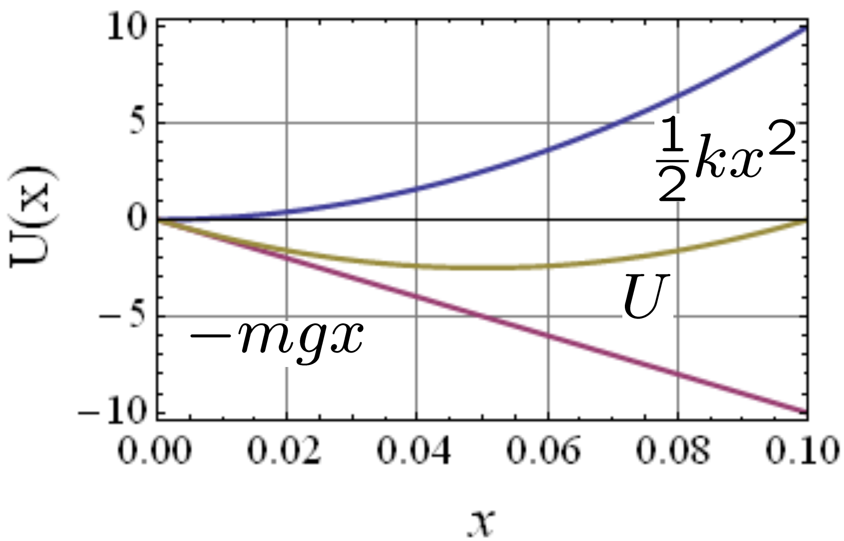
Note: It is very important to chose an *appropriate datum* and refer the potential energy *consistently* with respect to that datum

Simple Illustration

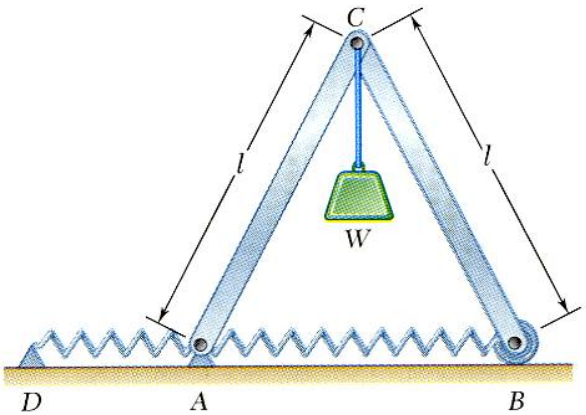
The potential energy is given by

$$U = \frac{1}{2}kx^2 - mgx$$

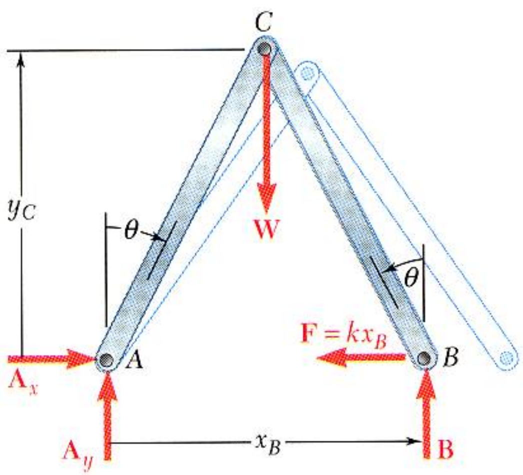
Plot of *V(x)* vs. *x* will look like



Potential Energy and Equilibrium



The un-stretched length of the spring is DA



- When the potential energy of a system is known, the principle of virtual work becomes

$$\delta U = 0 = -\delta V = -\frac{dV}{d\theta}\delta\theta$$

$$0 = \frac{dV}{d\theta}$$

- For the structure shown,

$$V = V_e + V_g = \tfrac{1}{2}kx_B^2 + Wy_C$$

$$= \tfrac{1}{2}k(2l\sin\theta)^2 + W(l\cos\theta)$$

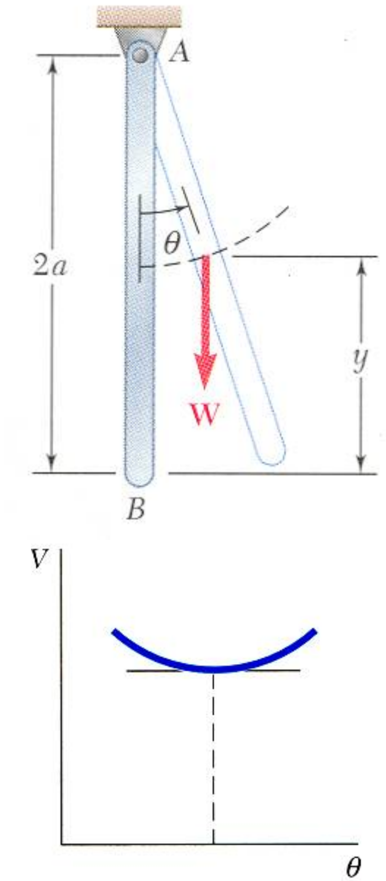
- At the position of equilibrium,

$$\frac{dV}{d\theta} = 0 = l\sin\theta(4kl\cos\theta - W)$$

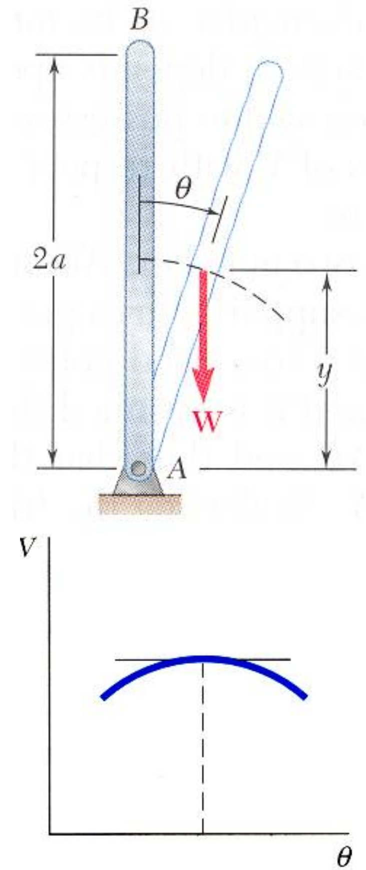
indicating two positions of equilibrium.

Stability of Equilibrium

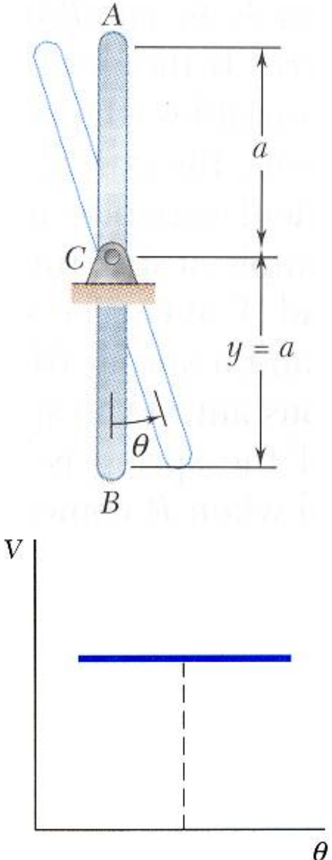
$$\frac{dV}{d\theta} = 0$$



(*a*) Stable equilibrium



(*b*) Unstable equilibrium



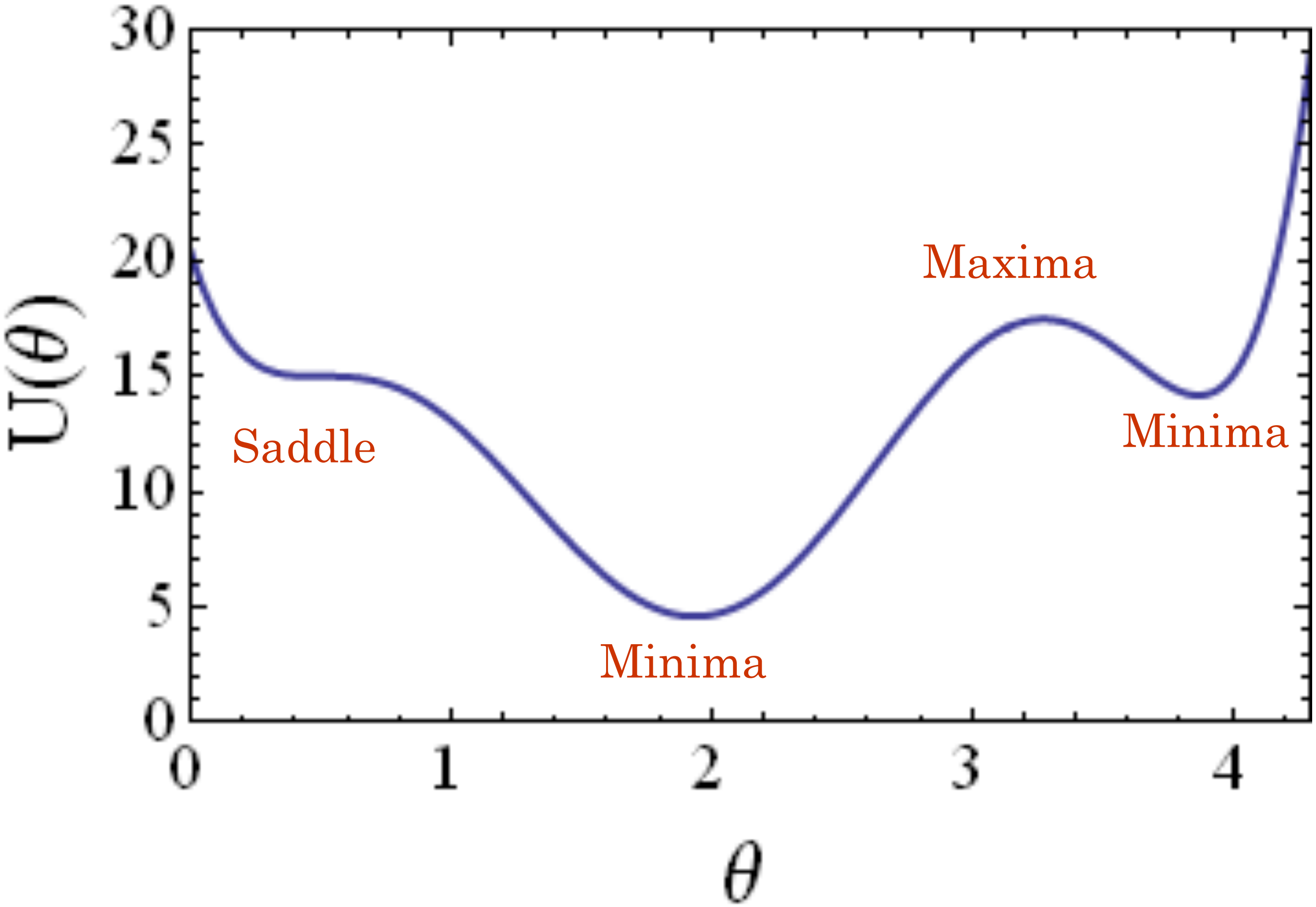
(*c*) Neutral equilibrium

$$\frac{d^2V}{d\theta^2} > 0$$

$$\frac{d^2V}{d\theta^2} > 0$$

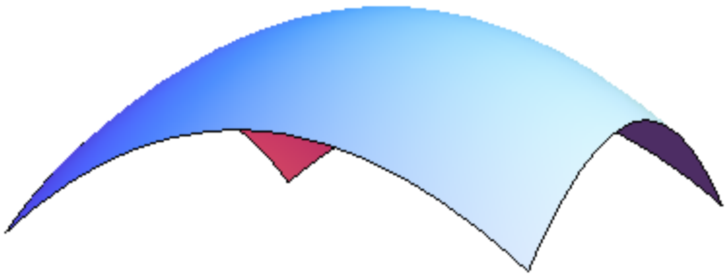
Must examine higher order derivatives.

Energy Landscape

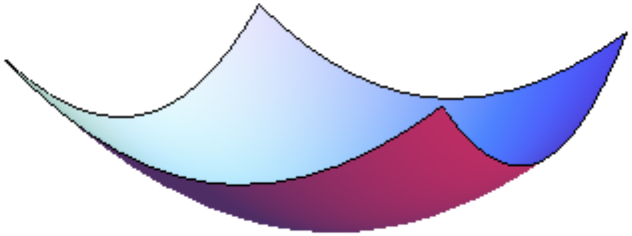


<u>Note</u>: If the saddle is wide it is practically *neutral* equilibrium

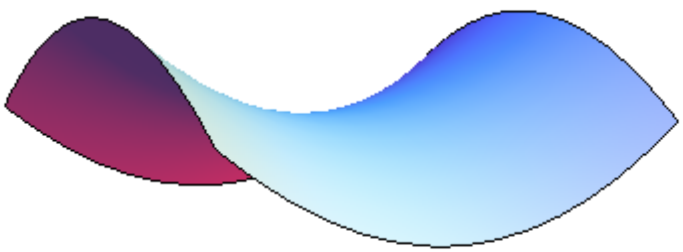
Stability in Higher Dimensions



Maxima:
Unstable

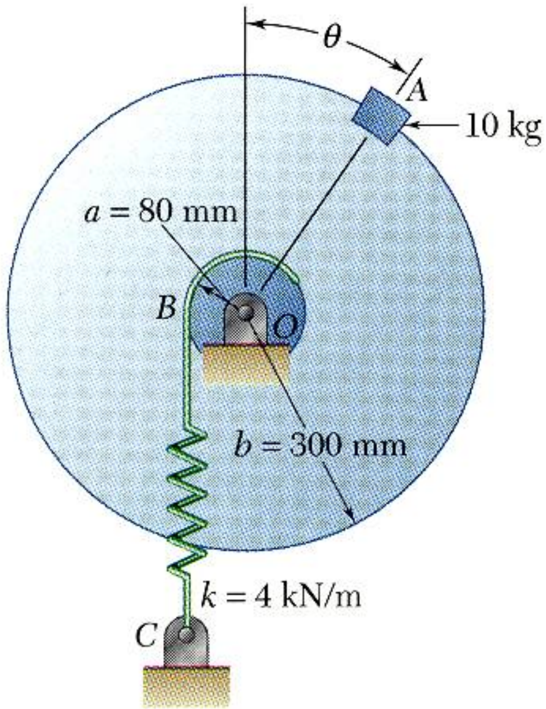


Minima:
Stable



Saddle:
Unstable

Sample Problem



Knowing that the spring *BC* is unstretched when θ = 0, determine the position or positions of equilibrium and state whether the equilibrium is stable, unstable, or neutral.

SOLUTION:

- Derive an expression for the total potential energy of the system.

$$V = V_e + V_g$$

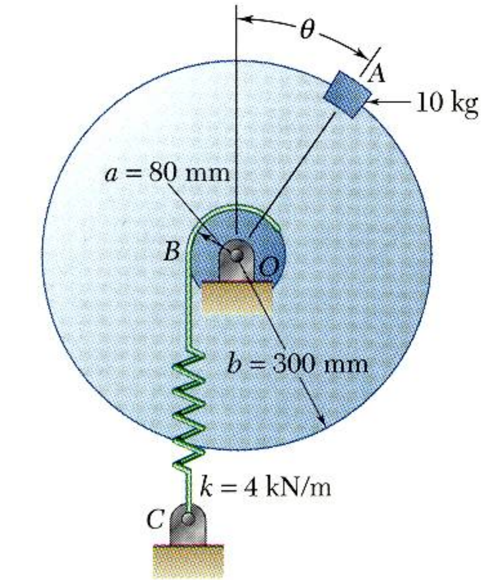
- Determine the positions of equilibrium by setting the derivative of the potential energy to zero.

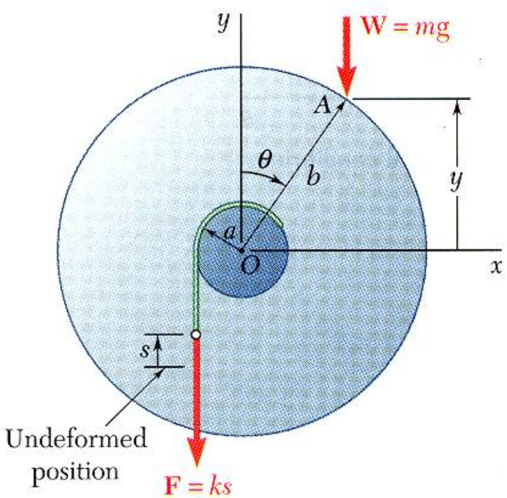
$$\frac{dV}{d\theta} = 0$$

- Evaluate the stability of the equilibrium positions by determining the sign of the second derivative of the potential energy.

$$\frac{d^2V}{d\theta^2} \overset{?}{><} 0$$

Sample Problem





SOLUTION:

- Derive an expression for the total potential energy of the system.

$$V = V_e + V_g$$

$$\frac{1}{2} ks^2 + mgy$$

$$\frac{1}{2} k(a\theta)^2 + mg(b\cos\theta)$$

- Determine the positions of equilibrium by setting the derivative of the potential energy to zero.

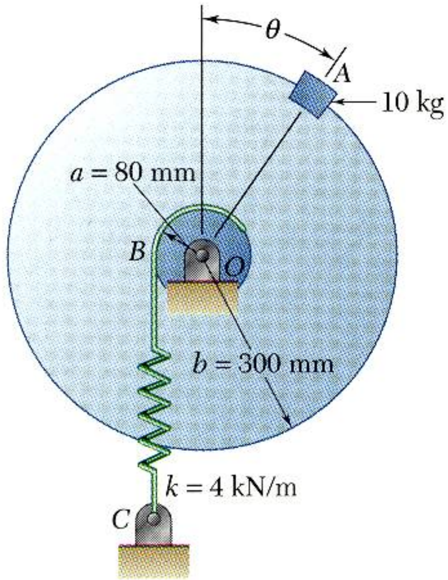
$$\frac{dV}{d\theta} = 0 = ka^2\theta - mgb\sin\theta$$

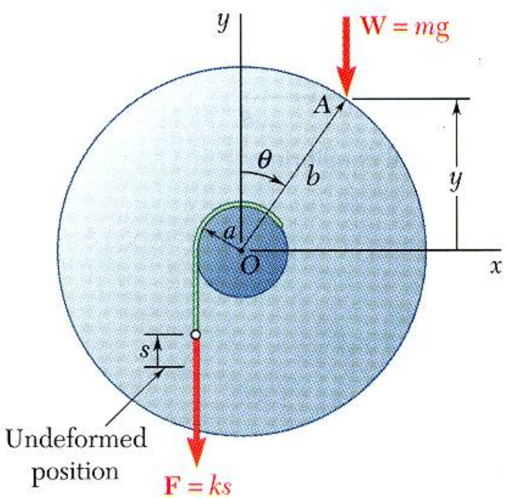
$$\sin\theta = \frac{ka^2}{mgb}\theta = \frac{(4\text{kN}/m)(0.08m)^2}{(10\text{kg})(9.81m/s^2)(0.3\text{m})}\theta$$

$$0.8699\,\theta$$

$$\theta = 0 \quad \theta = 0.902 \text{ rad} = 51.7°$$

Sample Problem





- Evaluate the stability of the equilibrium positions by determining the sign of the second derivative of the potential energy.

$$V = \frac{1}{2}k(a\theta)^2 + mg(b\cos\theta)$$

$$\frac{dV}{d\theta} = 0 = ka^2\theta - mgb\sin\theta$$

$$\theta = 0$$

$$\theta = 0.902\ \text{rad} = 51.7^\circ$$

$$\frac{d^2V}{d\theta^2} = ka^2 - mgb\cos\theta$$

$$(4\text{kN}/m)(0.08m)^2 - (10\text{kg})(9.81m/s^2)(0.3\text{m})\cos\theta$$

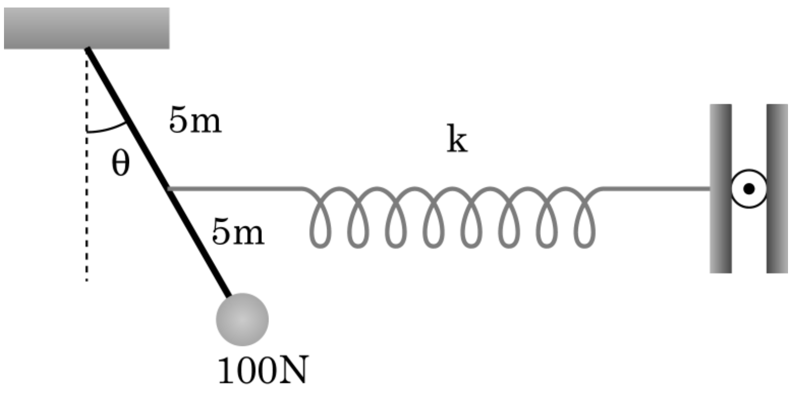
$$25.6 - 29.43\cos\theta$$

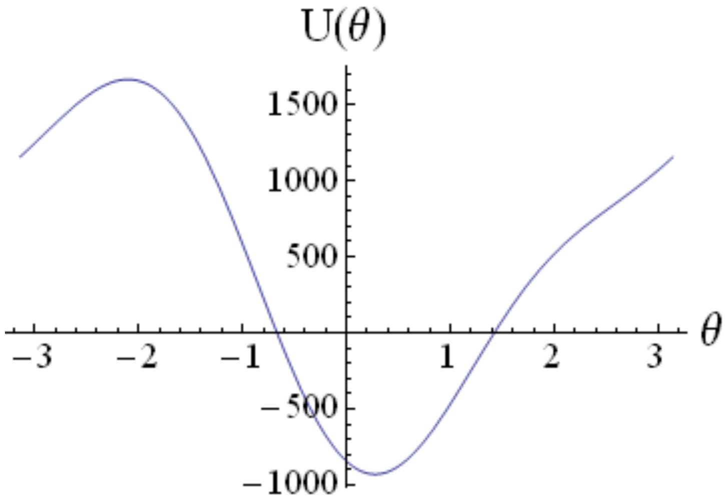
at θ = 0: $\dfrac{d^2V}{d\theta^2} = -3.83 < 0$ $\boxed{\text{unstable}}$

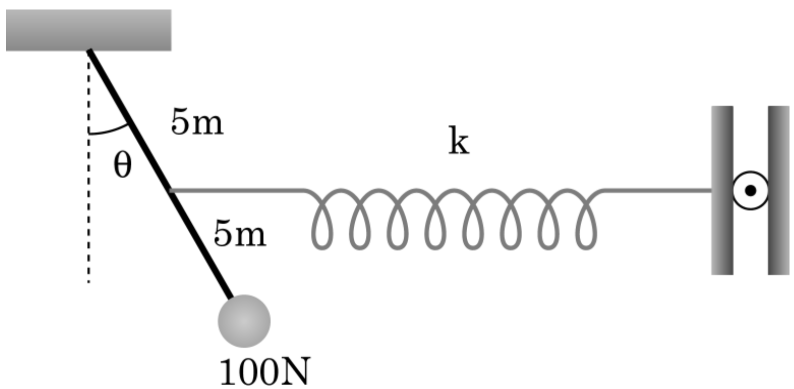
at θ = 51.7°: $\dfrac{d^2V}{d\theta^2} = +7.36 > 0$ $\boxed{\text{stable}}$

Problem 1

- The spring is un-stretched when $\theta = 30^o$. At any position of the pendulum, the spring remains horizontal. If the spring constant is k = 50 N/m, at what position will the system be in equilibrium.







Total Potential Energy (PE) :

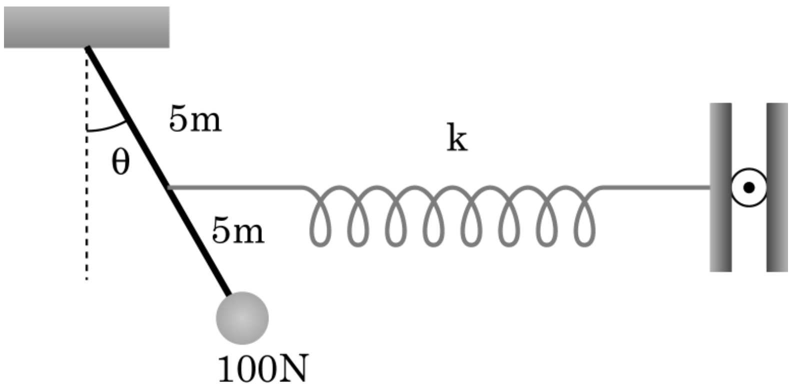
$$V = -100 * 10\cos\Theta + \left(\frac{1}{2}\right) * k(5\sin\Theta - 5\sin 30)^2$$

$for\ equilibrium$

$$\frac{\partial V}{\partial\Theta} = 1000\sin\Theta + 50(\sin\Theta - \sin 30) * 5\cos\Theta = 0$$

$$\frac{\partial^2 V}{\partial\Theta^2} = 1000\cos\Theta + 1250(\cos\Theta^2 - \sin\Theta^2 + \sin\Theta\sin 30)$$

This is not required if you are not interested in finding the nature of the equilibrium solution, i.e, stable or unstable



However, it is useful if you are solving the first equation by Newton-Raphson method

Therefore, stability part comes as a by-product

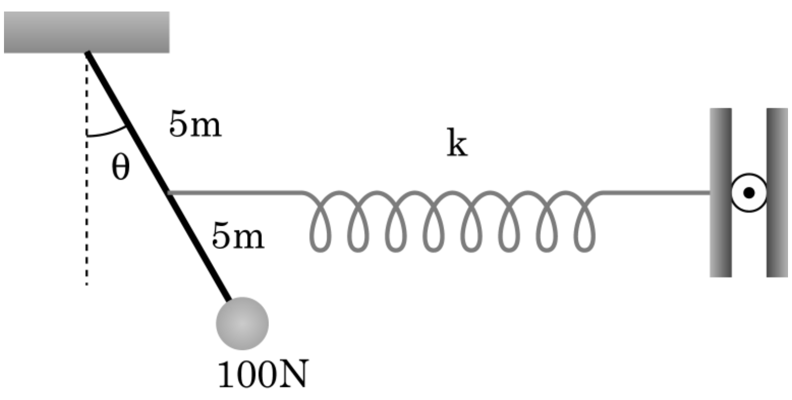
The solution is: Θ=15.842°

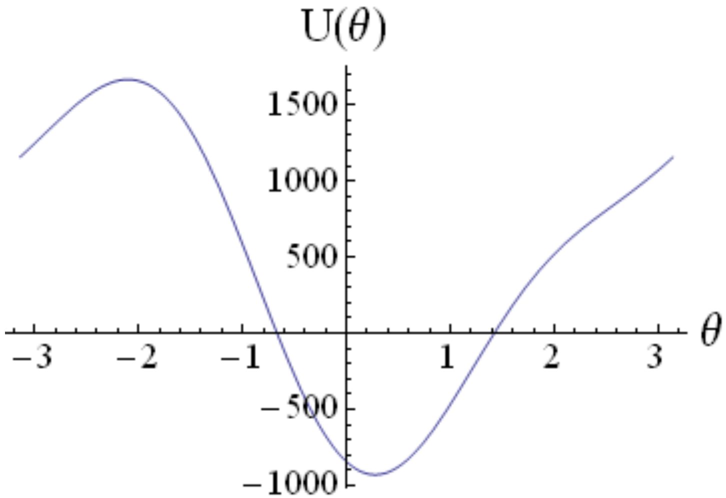
$$at\ \Theta = 15.842,$$

$$\frac{\partial^2 V}{\partial \Theta^2} = 2196 > 0 => stable\ equilibrium$$

Problem 1

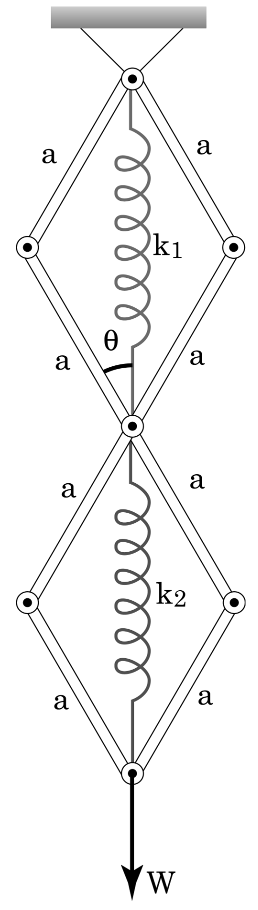
- The spring is un-stretched when $\theta = 30^o$. At any position of the pendulum, the spring remains horizontal. If the spring constant is k = 50 N/m, at what position will the system be in equilibrium.





Problem 2

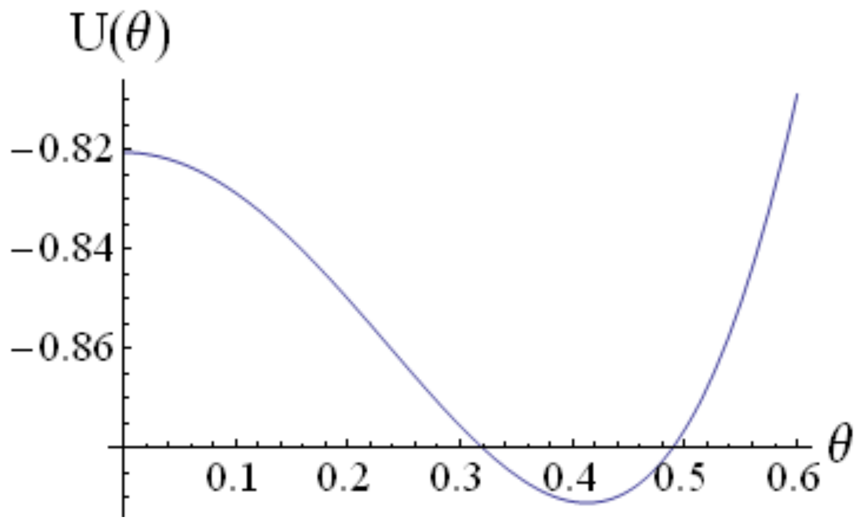
- If the springs are un-stretched when $\theta = \theta_o$, find the angle θ when the weight W is applied on the system. Use the method of minimum potential energy.



Energy in non-dimensional form

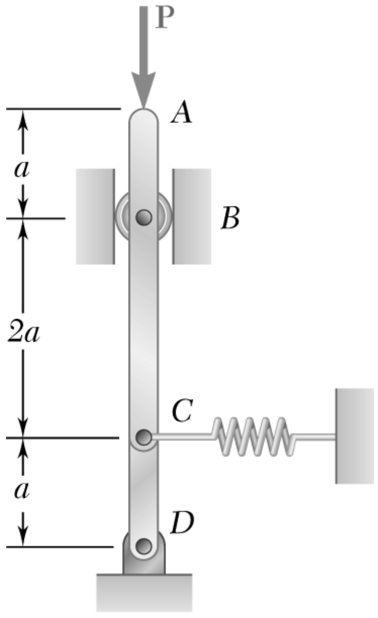
$$\frac{r}{2}\left(\left(\mathrm{Cos}[\theta]-\mathrm{Cos}[\theta o]\right)\right)^2-\mathrm{Cos}[\theta]\ (*\ r=\frac{a\ k}{W}\ *)$$

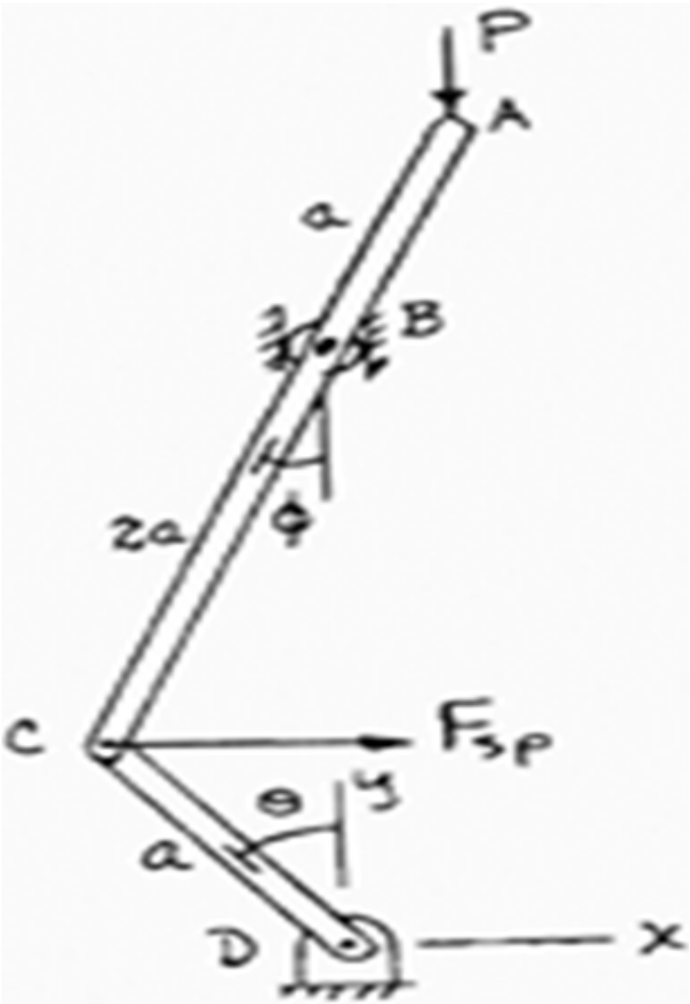
$$r_0 = 20; \theta_0 = \pi/6$$



Problem 2

- Two bars are attached to a single spring of constant k that is un-stretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.





From Geometry, x_c= -asin θ = -2asin ϕ

for small values of θ, ϕ

$\theta = 2\phi$

Or $\phi = \frac{\theta}{2}$

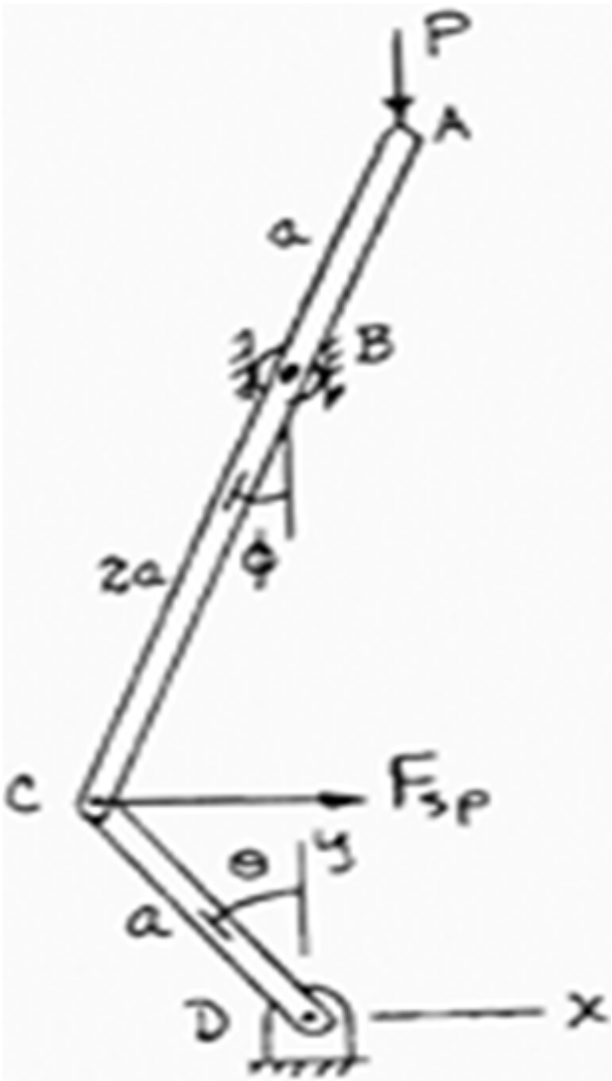
$y_a = a\cos\theta + 2a\cos\phi = a(\cos\theta + 3\cos\frac{\theta}{2})$

for spring,

$$s = x_c = -a\sin\theta$$

Potential Energy : $V = V_{sp} + V_P$

$$= \frac{k}{2}(-a\sin\theta)^2 + Pa(\cos\theta + 3\cos\frac{\theta}{2})$$



$$V = \frac{k}{2}(-a\sin\theta)^2 + Pa(\cos\theta + 3\cos\frac{\theta}{2})$$

$$\frac{dV}{d\theta} = ka^2\sin\theta\cos\theta - Pa\,(\sin\theta + 3\sin\frac{\theta}{2})$$

$$\frac{d^2V}{d\theta^2} = ka^2(-\sin^2\theta + \cos^2\theta) - Pa(\cos\theta + \frac{3}{4}\cos\frac{\theta}{2})$$

For stable equilibrium, $\frac{d^2V}{d\theta^2} > 0$

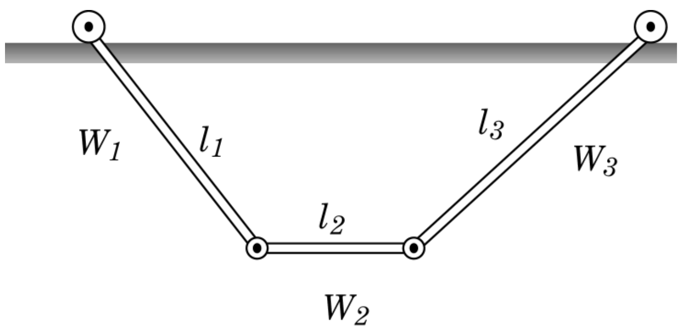
Then, with $\theta = 0$

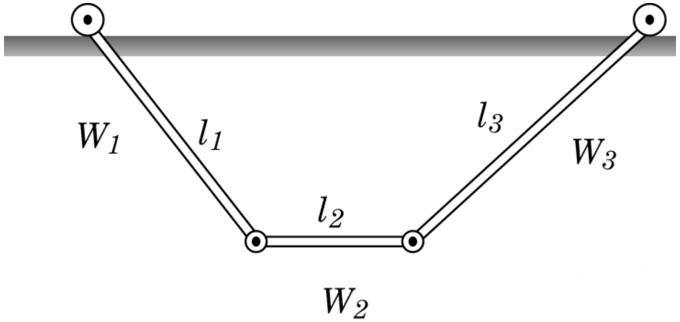
$$ka^2 - Pa\left(1 + \frac{3}{4}\right) > 0$$

$$P < \frac{4}{7}ka$$

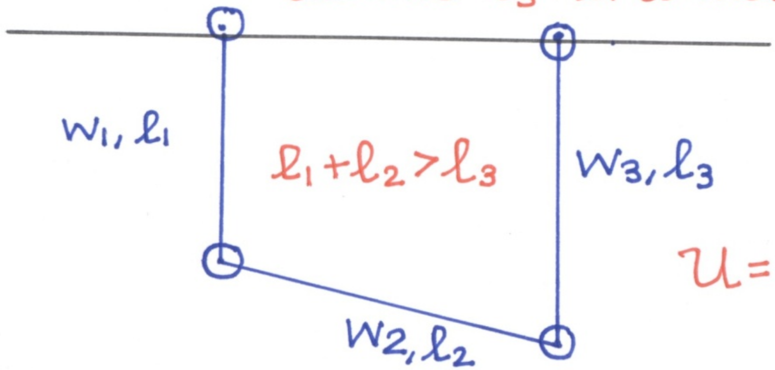
Problem 3

- Determine the angle of inclination of each linkage in the figure shown. The rollers move without friction on the support.



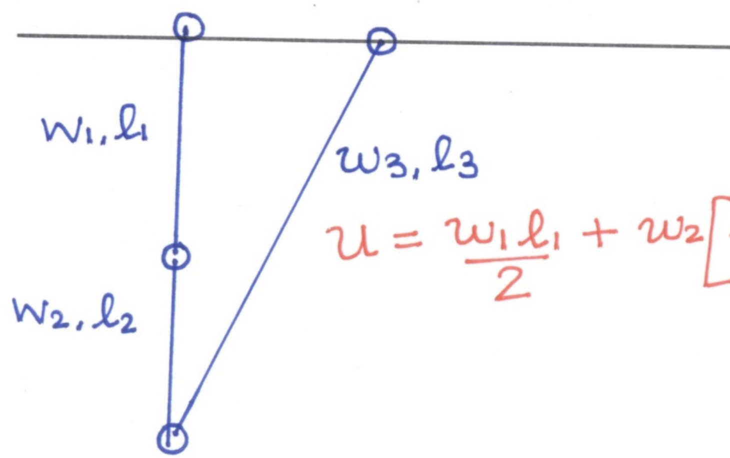


assume $l_3 > l_1$ without losing generality



$$U = \frac{w_1 l_1}{2} + \frac{w_3 l_3}{2} + w_2\left(\frac{l_1 + l_3}{2}\right)$$

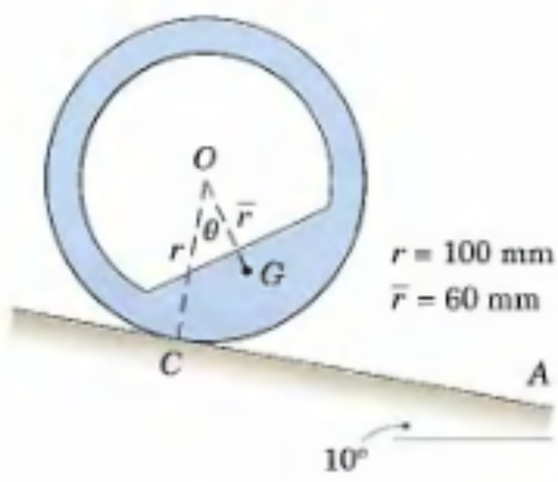
Each link should lower its CG as much as possible

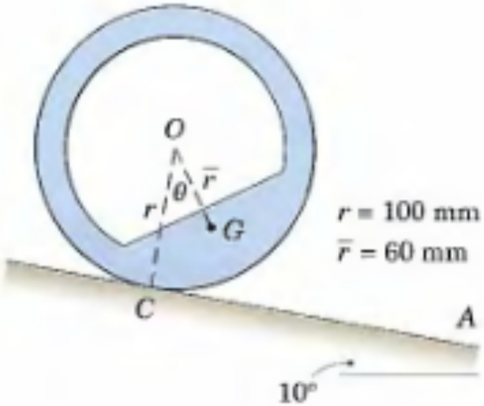


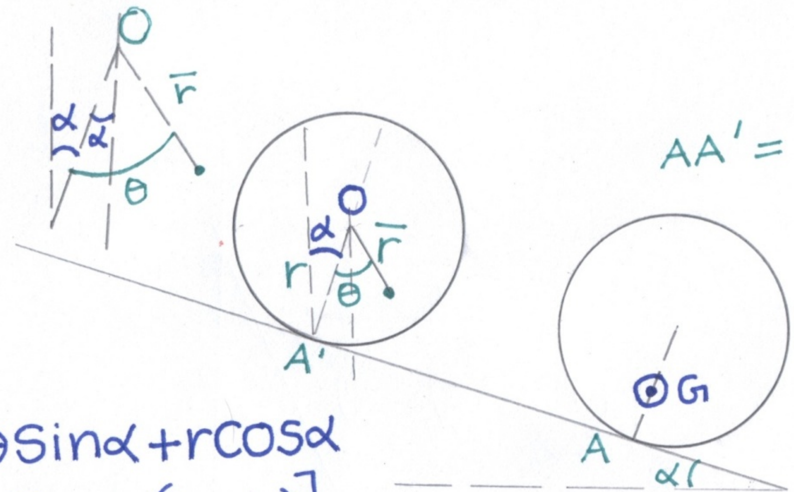
$$U = \frac{w_1 l_1}{2} + w_2\left[\frac{l_1 + l_2}{2}\right] + w_3\left[\frac{l_1 + l_2}{2}\right]$$

Problem 4

- Determine the equilibrium values of θ and the stability of equilibrium at each point for the unbalanced wheel on the 10° incline. Static friction is sufficient to prevent slipping. The mass center is at G.







$AA' = r\theta$

$$V_g = W[r\theta \sin\alpha + r\cos\alpha - \bar{r}\cos(\theta - \alpha)]$$

equilibrium

$$\frac{dV_g}{d\theta} = 0 \Rightarrow W[r\sin\alpha + \bar{r}\sin(\theta - \alpha)]$$

using $\alpha = 10,\ r = 100mm,\ \bar{r} = 60mm$

$$\frac{dV_g}{d\theta} = W[0.1\sin 10 + 0.060\sin(\theta - 10^\circ)] = 0$$

$$\Rightarrow \theta - 10^\circ = \sin^{-1}\left[\frac{-0.1}{0.06}\sin 10^\circ\right] = -16.82^\circ \text{ or } 196.8^\circ$$

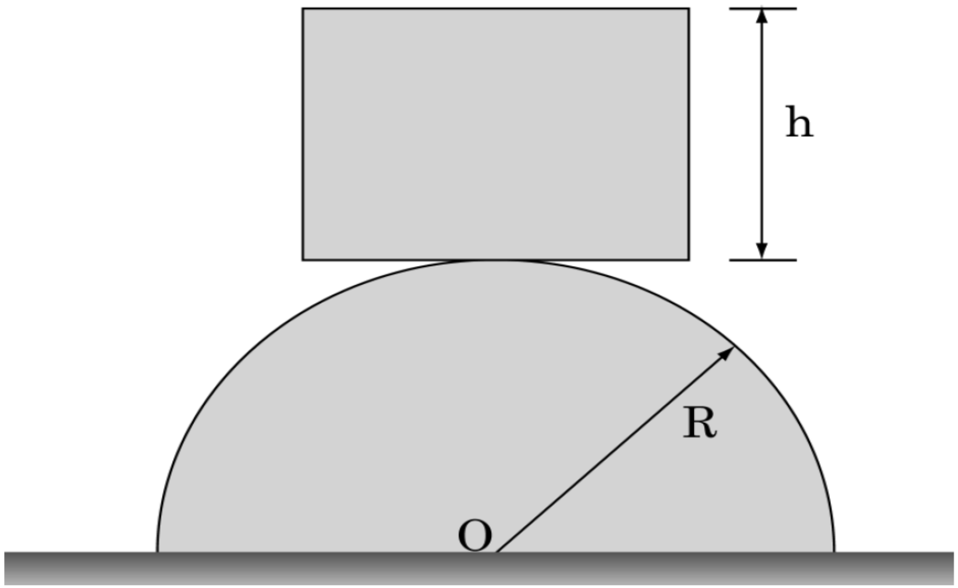
or $\theta = -6.82^\circ$ or $\theta = 207^\circ$

$$\frac{d^2V_g}{d\theta^2} = W[\bar{r}\cos(\theta - \alpha)]$$

@ $\theta = -6.82^\circ$, $= W\bar{r}\cos(-16.82^\circ) > 0$ stable

@ $\theta = 196.8$ $= W\bar{r}\cos(196.8^\circ) < 0$ unstable

Problem 5

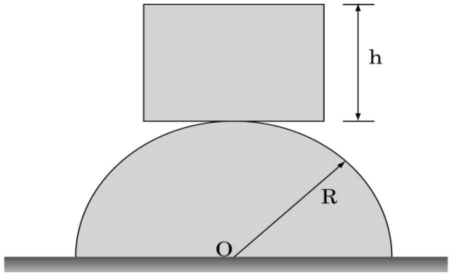


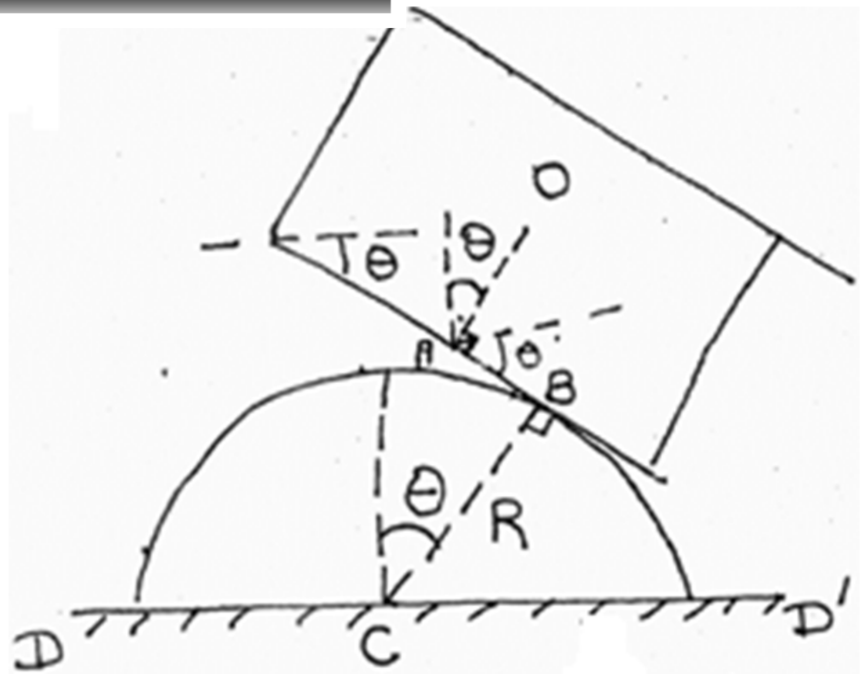
SOLUTION:

The equation of PE (V) for the rectangular mass is obtained w.r.t. the datum plane

For stable equilibrium condition $\frac{d^2V}{d\theta^2} > 0$ and unstable equilibrium condition: $\frac{d^2V}{d\theta^2} < 0$

- A rectangular uniform solid body of mass m and height h rests on a fixed cylinder with a semicircular section. Set up criteria for stable and unstable equilibrium for the rectangular body in terms of h and R for the position shown. Assume that the rectangular body does not slip on the cylinder





$$OA = \frac{h}{2}$$

$$AB = R\theta$$

PE of mass of rectangle w.r.t. datum plane DD':

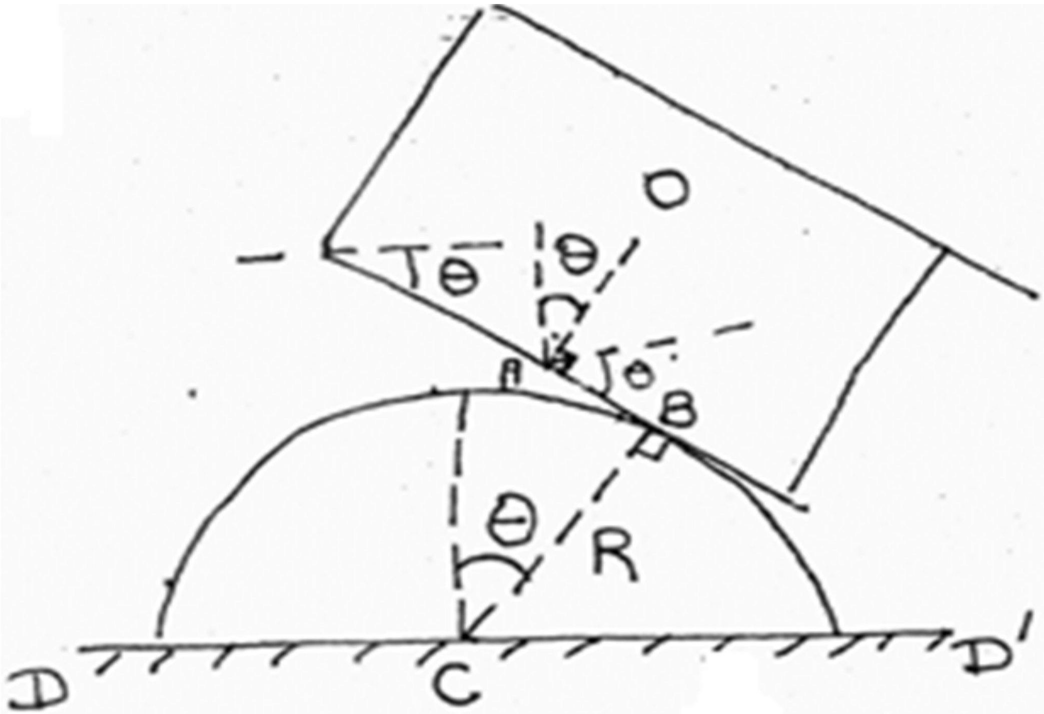
$$V = mg[CB\cos\theta + AB\sin\theta$$
$$+ OA\cos\theta]$$

$$= mg\left[R\cos\theta + R\theta\sin\theta + \frac{h}{2}\cos\theta\right]$$

For equilibrium,

$$\frac{dV}{d\theta} = 0 = mg[-R\sin\theta + R\sin\theta + R\theta\cos\theta - \frac{h}{2}\sin\theta]$$

$$= mg[R\theta\cos\theta - \frac{h}{2}\sin\theta] \Rightarrow \theta = 0 \; is\ an\ equilibrium\ position$$



For stability,

$$\frac{d^2V}{d\theta^2} = mg[R\cos\theta - R\theta\sin\theta - \frac{h}{2}\cos\theta]$$

<u>For $\theta = 0$,</u>

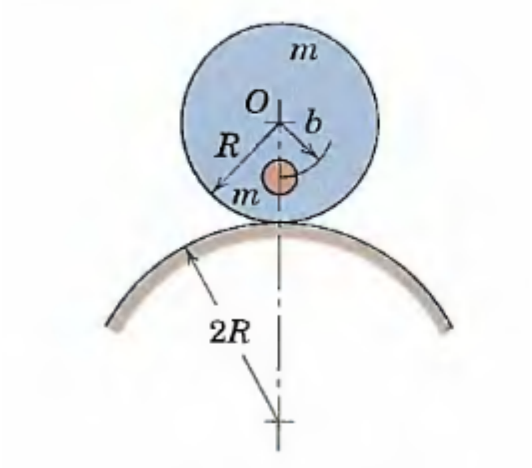
$$\frac{d^2V}{d\theta^2} = mg[R - \frac{h}{2}]$$

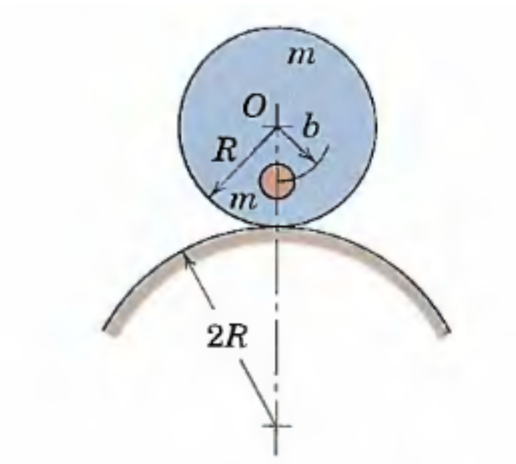
For stable equilibrium condition : $\frac{d^2V}{d\theta^2} > 0 \Rightarrow R > \frac{h}{2}$

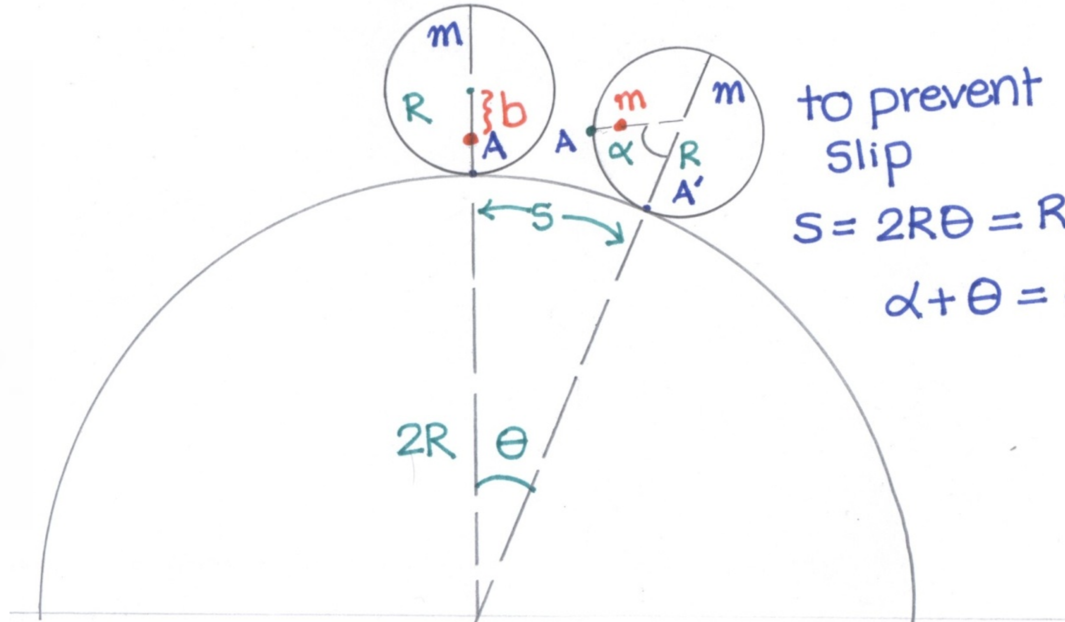
For unstable equilibrium condition : $\frac{d^2V}{d\theta^2} < 0 \Rightarrow R < \frac{h}{2}$

Problem 6

- The uniform disk of radius R and mass m rolls without slipping on the fixed cylinder surface of radius $2R$. Fastened to the disk is a lead cylinder also of mass m with its center located a distance b from the center O of the disk. Determine the minimum value of b for which the disk will remain in stable equilibrium on the cylindrical surface.





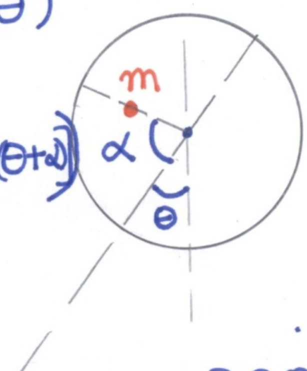


to prevent slip

$$s = 2R\theta = R\alpha$$

$$\alpha + \theta = 3\theta$$

$$V_g = mg(3R\cos\theta) + mg\left[3R\cos\theta + b\cos[\pi - (\theta+\alpha)]\right]$$



$$\Rightarrow V_g = 6mgR\cos\theta - mgb\cos 3\theta$$

$$\frac{dV_g}{d\theta} = -6mgR\sin\theta + 3mgb\sin 3\theta = 0 \ @\ \theta = 0^\circ$$

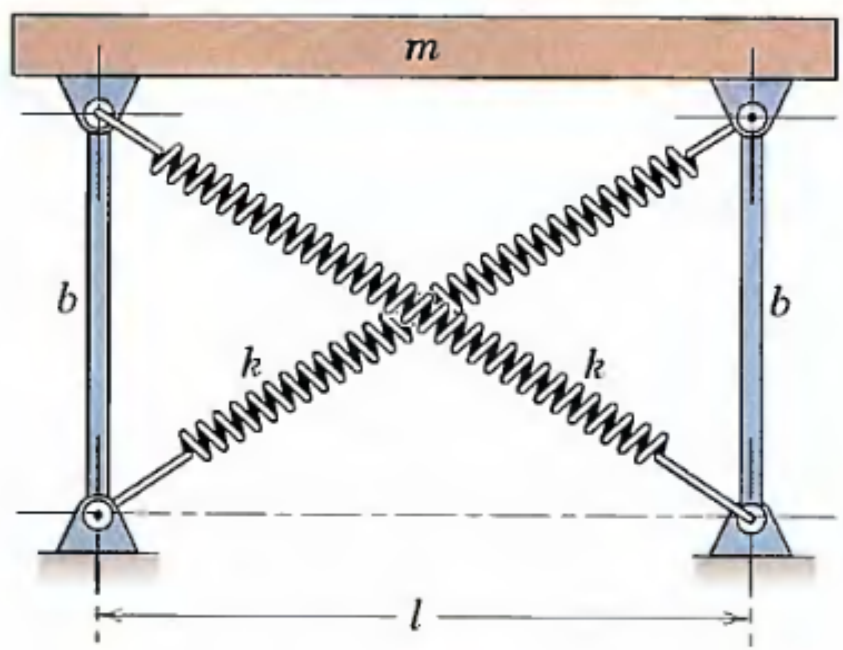
$$\frac{d^2V_g}{d\theta^2} = -6mgR\cos\theta + 9mgb\cos 3\theta \quad @\ \theta = 0$$

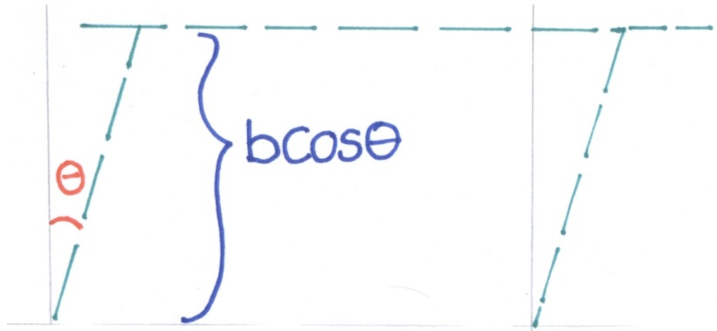
$$\frac{d^2V_g}{d\theta^2} = 3mg[-2R + 3b]$$; stable if > 0

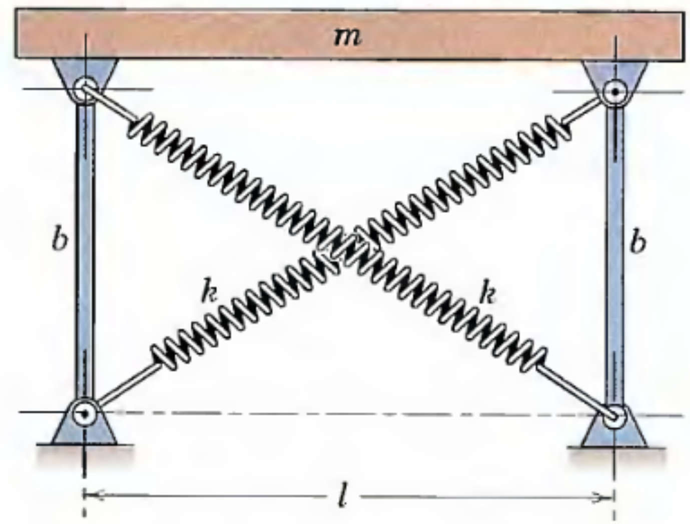
$$\Rightarrow b_{min} = \frac{2}{3}R$$

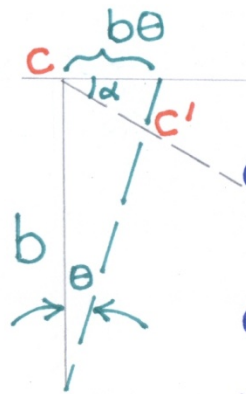
Problem-7

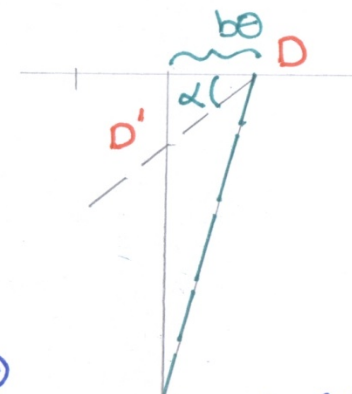
The platform of mass m is supported by equal legs and braced by the two springs as shown. If the masses of the legs and springs are negligible, design the springs by determining the minimum stiffness k of each spring which will ensure stability of the platform in the position shown. Each spring has a tensile preset deflection of Δ.











$CC' = b\cos\alpha\,\underbrace{\theta}_{\text{small}}$

comp.

$DD' = b\cos\alpha\,\theta$

ext.

$$V = V_g + V_e = mgb\cos\theta + \frac{1}{2}k(\Delta - b\theta\cos\alpha)^2 + \frac{1}{2}k(\Delta + b\theta\cos\alpha)^2$$

$$= k(\Delta^2 + b^2\theta^2\cos^2\alpha) + mgb\cos\theta$$

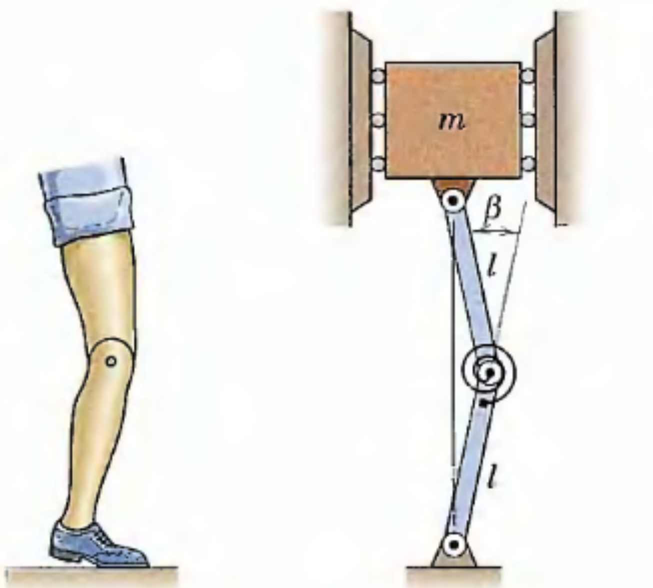
$$\frac{dV}{d\theta} = 2kb^2\theta\cos^2\alpha - mgb\sin\theta, \quad (\theta = 0,\ \text{ok}) \qquad \Bigg\| \qquad \frac{d^2V}{d\theta^2} = 2kb^2\cos^2\alpha - mgb \quad \theta = 0$$

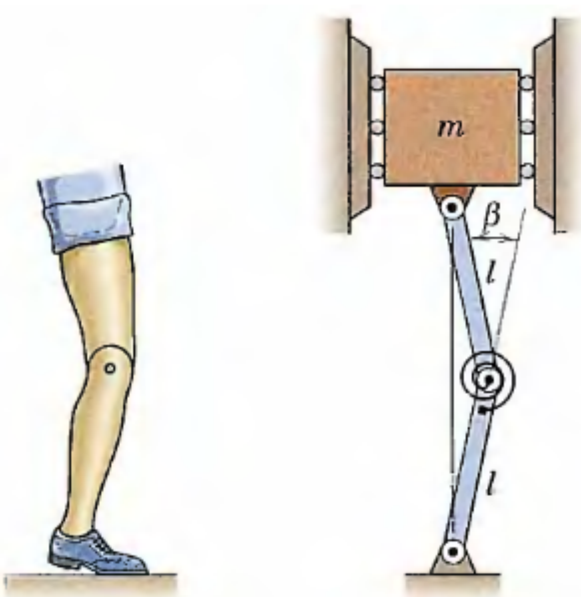
$$k_{min} = \frac{mg}{2b\cos^2\alpha} = \frac{mg}{2b}\left(1 + \frac{b^2}{l^2}\right)$$

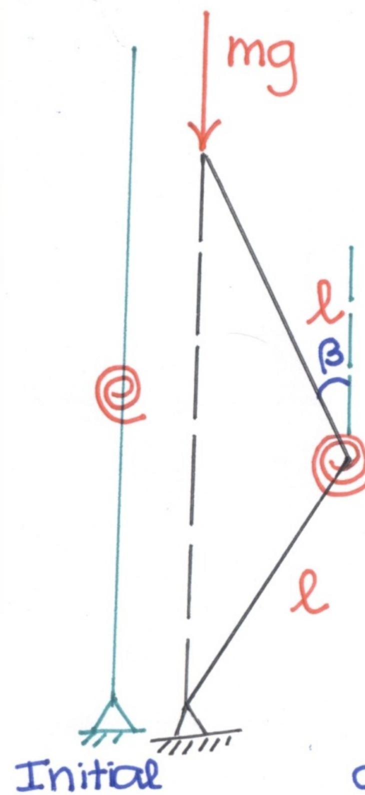
Doubts?

Problem 1

- One of the critical requirements in the design of an artificial leg for an amputee is to prevent the knee joint from buckling under load when the leg is straight. As a first approximation, simulate the artificial leg by the two light links with a torsion spring at their common joint. The spring develops a torque $M = K\beta$, which Is proportional to the angle of the bend β at the joint. Determine the minimum value of K which will ensure the stability of the knee joint for $\beta = 0$.







$$V_g = 2mgl\cos\frac{\beta}{2}$$

$$V_e = \frac{1}{2}k\beta^2$$

$$V = V_g + V_e$$

$$\frac{dV}{d\beta} = -mgl\sin\frac{\beta}{2} + k\beta$$

$$\left.\frac{dV}{d\beta}\right|_{\beta=0} = 0$$

$$\frac{d^2V}{d\beta^2} = -\frac{1}{2}mgl\cos\frac{\beta}{2} + k$$

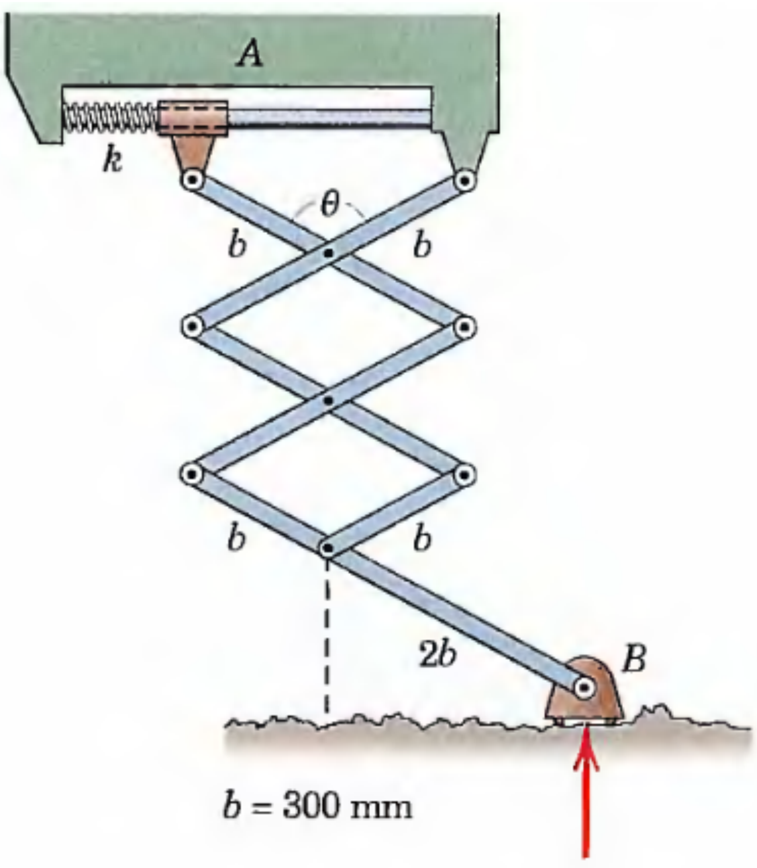
$$= -\frac{1}{2}mgl + k \quad @\ \beta = 0$$

+ve if

$$k > \frac{mgl}{2}$$

Problem-2

An exploration device, which unfolds from the body *A* of an unmanned space vehicle reseting on the moon's surface, consists of a spring-loaded pantograph with detector head *B*. It is desired to select a spring that will limit the vertical contact force *P* to 100 N in the position for which θ = 120 deg. If the mass of the arms and head is negligible, specify the necessary spring stiffness.



$$V = \frac{1}{2}k\left[0.6\left(\sin\frac{\theta}{2} - \sin 15^\circ\right)\right]^2 + \underbrace{P}_{100} * 2.1\cos\frac{\theta}{2}$$

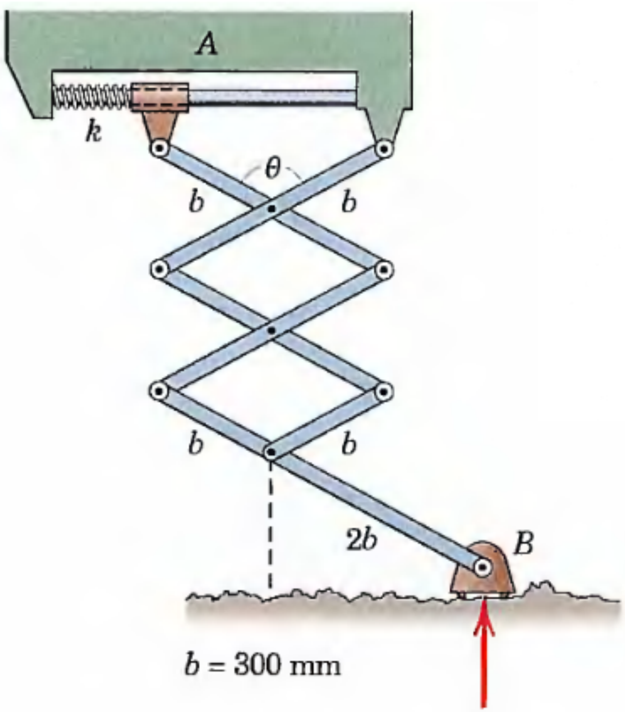
$$\frac{dV}{d\theta} = k\left[0.6^2\left(\sin\frac{\theta}{2} - \sin 15\right)\right] * \frac{1}{2}\cos\frac{\theta}{2}$$

$$- \frac{210}{2}\sin\frac{\theta}{2} = 0$$

for $\theta = 120^\circ$

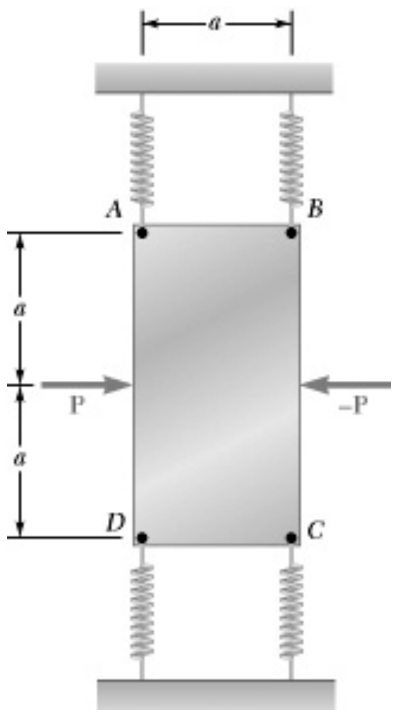
$$105\frac{\sqrt{3}}{2} = 0.18k\left[\frac{\sqrt{3}}{2} - 0.2588\right]\frac{1}{2}$$

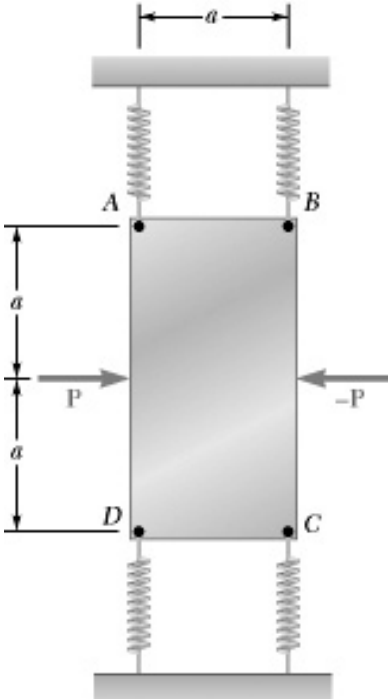
$$\Rightarrow k = 1664 \text{ N/m}$$

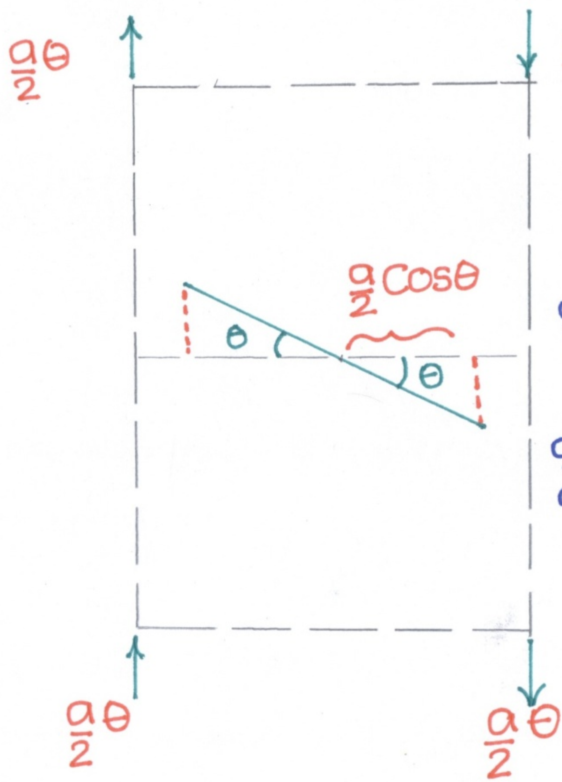


Problem 3

- The uniform plate *ABCD* of negligible mass is attached to four springs of constant k and is in equilibrium in the position shown. Knowing that the springs can act in either tension or compression and are undeformed in the given position, determine the range of values of the magnitude $\boldsymbol{P}$ of two equal and opposite horizontal forces **P** and for which the equilibrium position is stable w.r.t small rotation about the center of the plate.







$$V = \frac{Pa}{2}\cos\theta * 2 + 4 * \frac{1}{2}k\left(\frac{a\theta}{2}\right)^2$$

$$\frac{dV}{d\theta} = -Pa\sin\theta + ka^2\theta \Big\} 0 \ @ \ \theta = 0$$

$$\frac{d^2V}{d\theta^2} = -Pa\cos\theta + ka^2$$

$$= \underset{(\theta=0)}{-Pa + ka^2} > 0 \text{ for stability}$$

or

$$\underbrace{P < ka}_{\text{for stable equilibrium @ } \theta = 0}$$

Problem 4

- In the mechanism shown the spring of stiffness *k* is uncompressed when θ = 60º. Also the masses of the parts are small compared with the sum *m* of the masses of the two cylinders. The mechanism is constructed so that the arms may swing past the vertical, as seen in the right-hand side view. Determine the values of θ for equilibrium and investigate the stability of the mechanism in each position. Neglect friction.