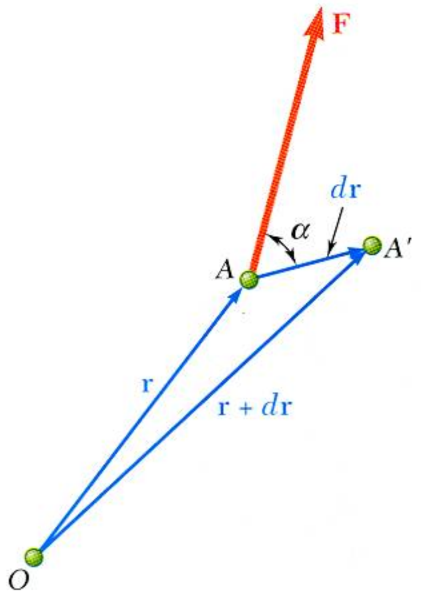
CE 102: Engineering Mechanics

Principle of Virtual Work

Work of a Force



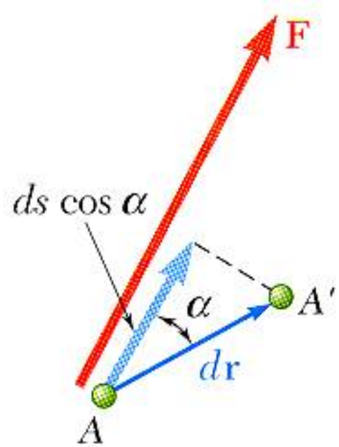
$d U = \vec{F} \cdot d\vec{r}$ = work of the force $\vec{F}$ corresponding to the displacement $d\vec{r}$

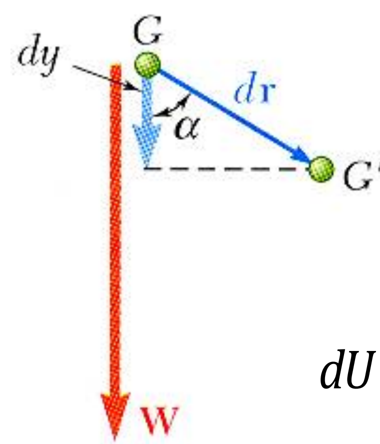
$dU = F\ ds\ \cos\alpha$

$\alpha = 0,\ dU = +F\ ds$

$\alpha = \pi,\ dU = -F\ ds$

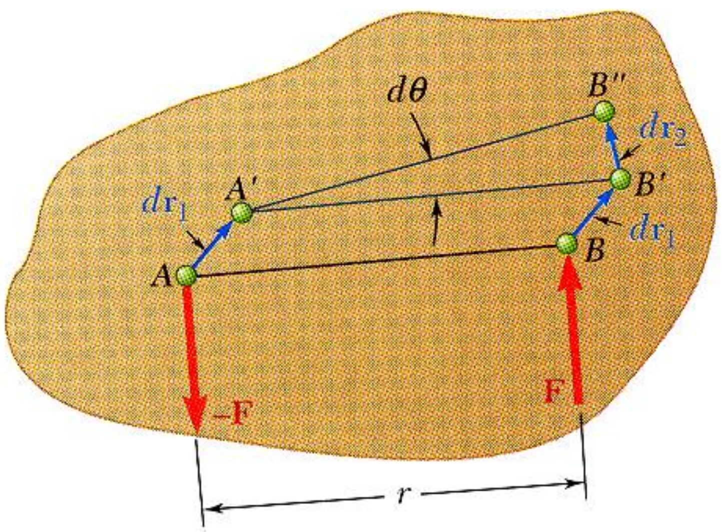
$\alpha = \frac{\pi}{2},\ dU = 0$





$dU = Wdy$

Work of a Couple



Small displacement of a rigid body:

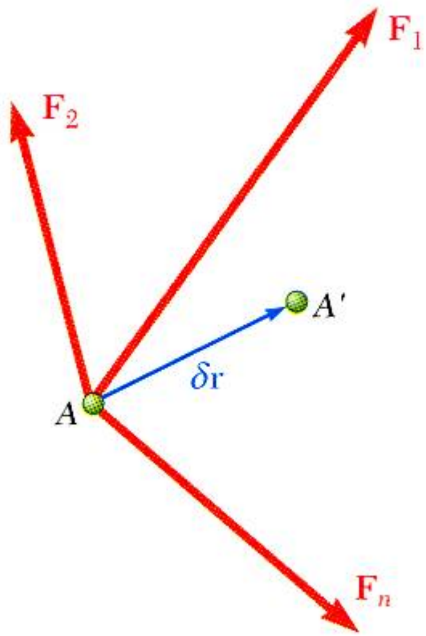
- translation to $A'B'$
- rotation of B' about A' to B''

$$W = -\vec{F}\cdot d\vec{r}_1 + \vec{F}\cdot\left(d\vec{r}_1 + d\vec{r}_2\right)$$

$$\vec{F}\cdot d\vec{r}_2 = F\ ds_2 = F\ r d\theta$$

$$M\ d\theta$$

Principle of Virtual Work



- *Imagine* the small *virtual displacement* of particle which is acted upon by several forces.
- The corresponding *virtual work*,

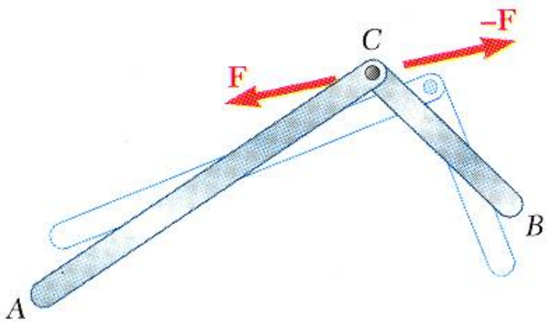
$$\delta U = \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + \vec{F}_3 \cdot \delta\vec{r} = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot \delta\vec{r}$$

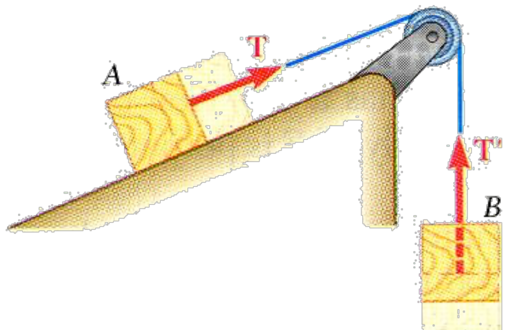
$$\vec{R} \cdot \delta\vec{r}$$

Principle of Virtual Work:

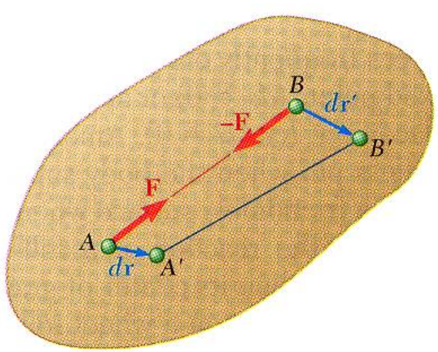
- If a particle is in equilibrium, the total virtual work of forces acting on the particle is zero for any virtual displacement.
- If a rigid body is in equilibrium, the total virtual work of external forces acting on the body is zero for any virtual displacement of the body.
- If a system of connected rigid bodies remains connected during the virtual displacement, only the work of the external forces need be considered.

Work of a Force





(b)



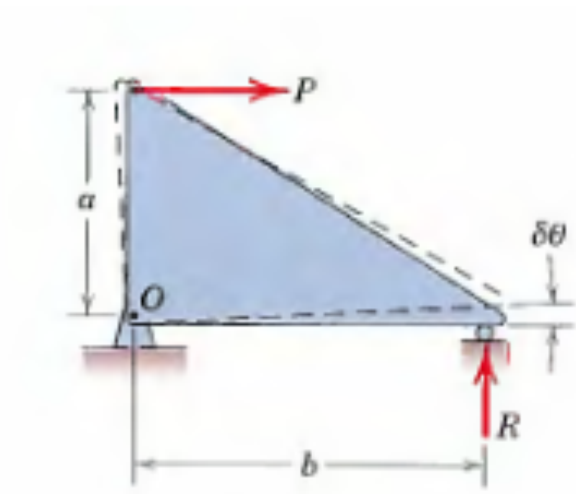
Forces which do no work:

- reaction at a frictionless pin due to rotation of a body around the pin
- reaction at a frictionless surface due to motion of a body along the surface
- weight of a body with cg moving horizontally
- friction force on a wheel moving without slipping

Sum of work done by several forces may be zero:

- bodies connected by a frictionless pin
- bodies connected by an inextensible cord
- internal forces holding together parts of a rigid body

Virtual Work for a rigid body



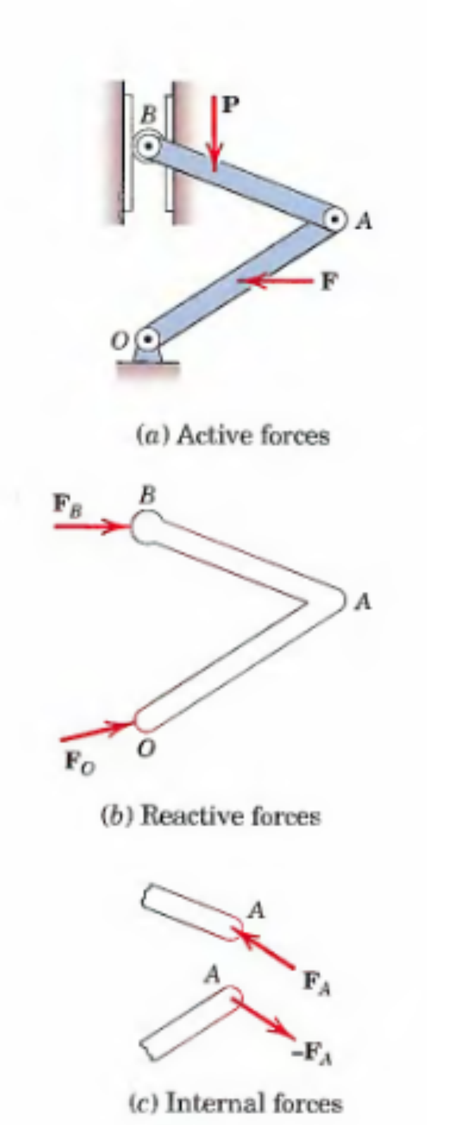
$Pa\delta\theta + Rb\delta\theta = 0$

This is equivalent to taking moment about 0.

Similarly by *virtual translations in the x and y directions we can obtain $\Sigma F_x = 0$ and $\Sigma F_y = 0$.*

Note that internal forces do *not perform work due to cancellation from equal and opposite forces*

Different types of forces



(*a*) Active forces

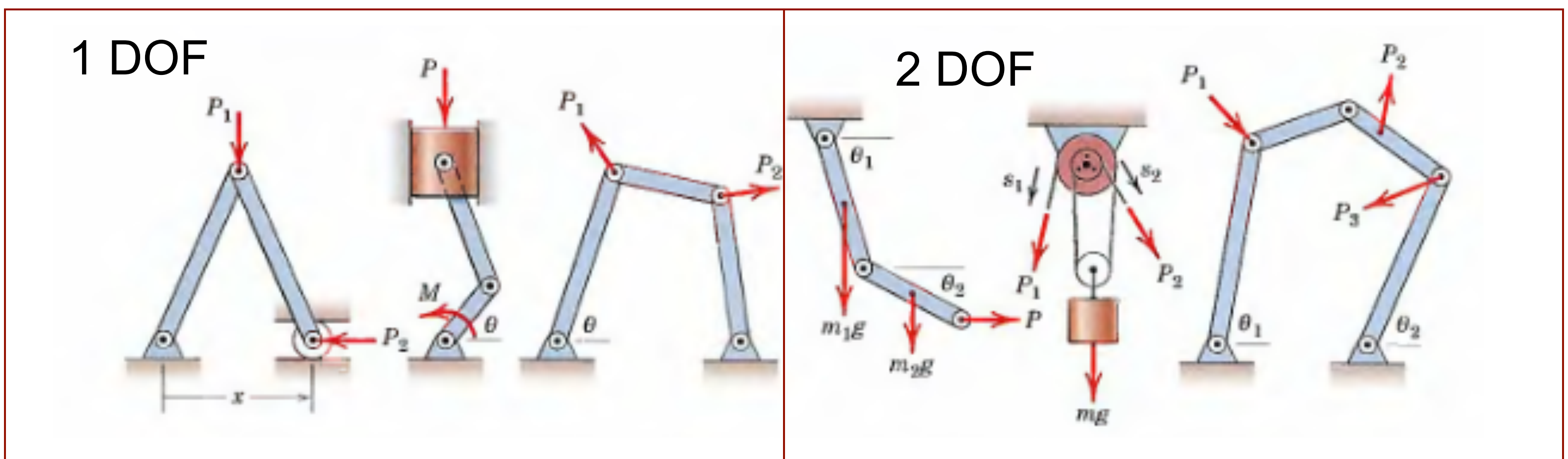
(*b*) Reactive forces

(*c*) Internal forces

- Forces that do work are called *active* force.
- Reactive and internal forces do not do any work.
- Virtual displacements are to be given carefully so that the active forces are only the known forces and the *forces* we are *interested* in obtaining
- Similar to FBD we draw active force diagram (AFD).

Degrees of Freedom

- DOF in this context is the total number of independent coordinates required to specify the complete location of every member of the structure.
- For VW method in this course we will use only 1-DOF systems.



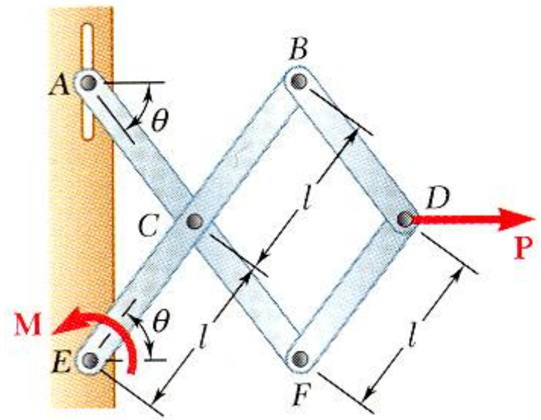
To summarize

- ***Principle of virtual work***
  - **The virtual work done by external active forces on an ideal mechanical system in equilibrium is zero for any all virtual displacements consistent with the constraints.**
- Ideal system:
  - All surfaces, joints etc. are frictionless.
  - We will deal with ideal system in this course.
- Consistent with constraints:
  - The *virtual displacement* should be such that they should not do allow the *non-active* forces to do any work.

Why principle of Virtual Work

- For complex mechanisms (we will solve some problems) we do not need to dis-member the system.
- We obtain the *active unknown force* in one shot without bothering about the *reactive* forces.
- Such type of analysis will be a stepping stone to VW analysis using *deformations* when you study *Solid Mechanics*, *Structural Mechanics etc*. not to mention powerful *Approximate methods* like the *Finite Element Method*.

Sample Problem 10.1



Determine the magnitude of the couple **M** required to maintain the equilibrium of the mechanism.

SOLUTION:

- Apply the principle of virtual work

$$\delta U = 0 = \delta U_M + \delta U_P$$

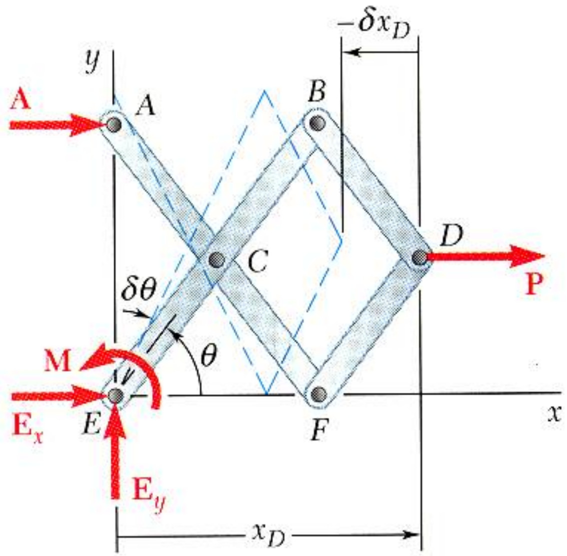
$$0 = M\delta\theta + P\delta x_D$$

$$x_D = 3l\cos\theta$$

$$\delta x_D = -3l\sin\theta\delta\theta$$

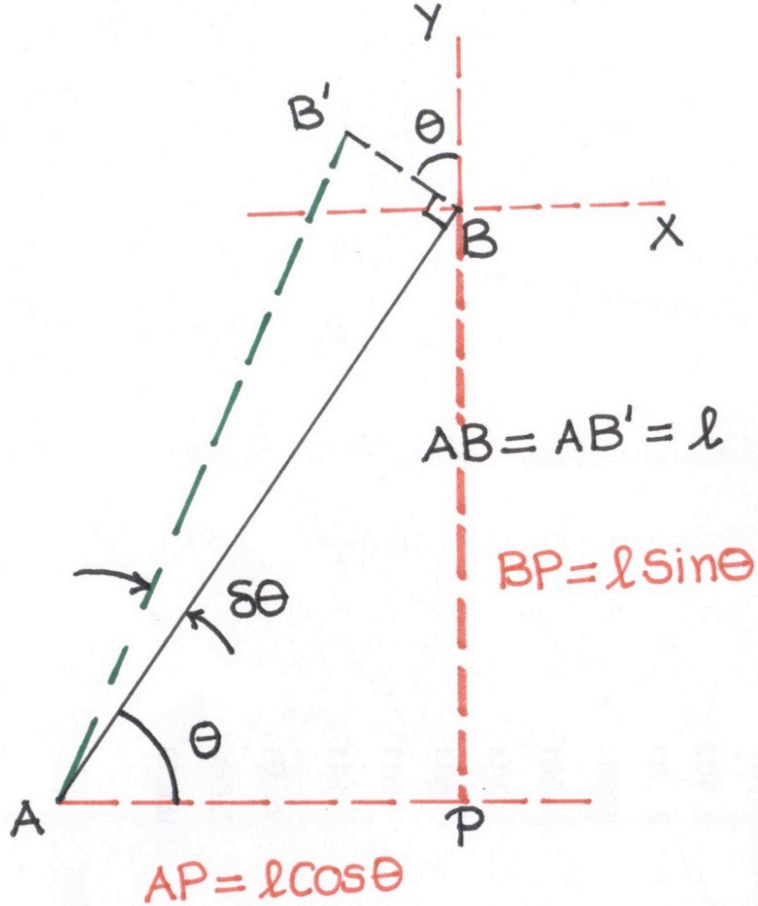
$$0 = M\delta\theta + P(-3l\sin\theta\delta\theta)$$

$$\boxed{M = 3Pl\sin\theta}$$



- Note that no support reactions were needed to solve the problem, nor was it necessary to take apart the machine at any connection. A clear and accurate FBD is still highly recommended, however.

A generic observation for rotation of rigid bodies



$BB' = \ell\delta\theta = \delta$

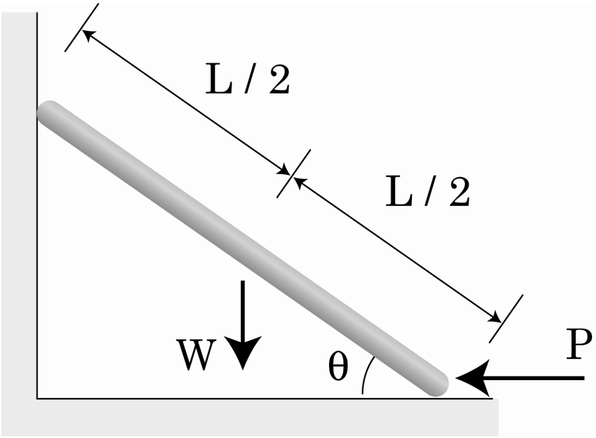
$\delta_x = \delta \sin\theta = \ell \sin\theta \, \delta\theta \leftarrow$

$\delta_y = \delta \cos\theta = \ell \cos\theta \, \delta\theta \uparrow$

Displacement in a general direction is the projection in the perpendicular direction times $\delta\theta$

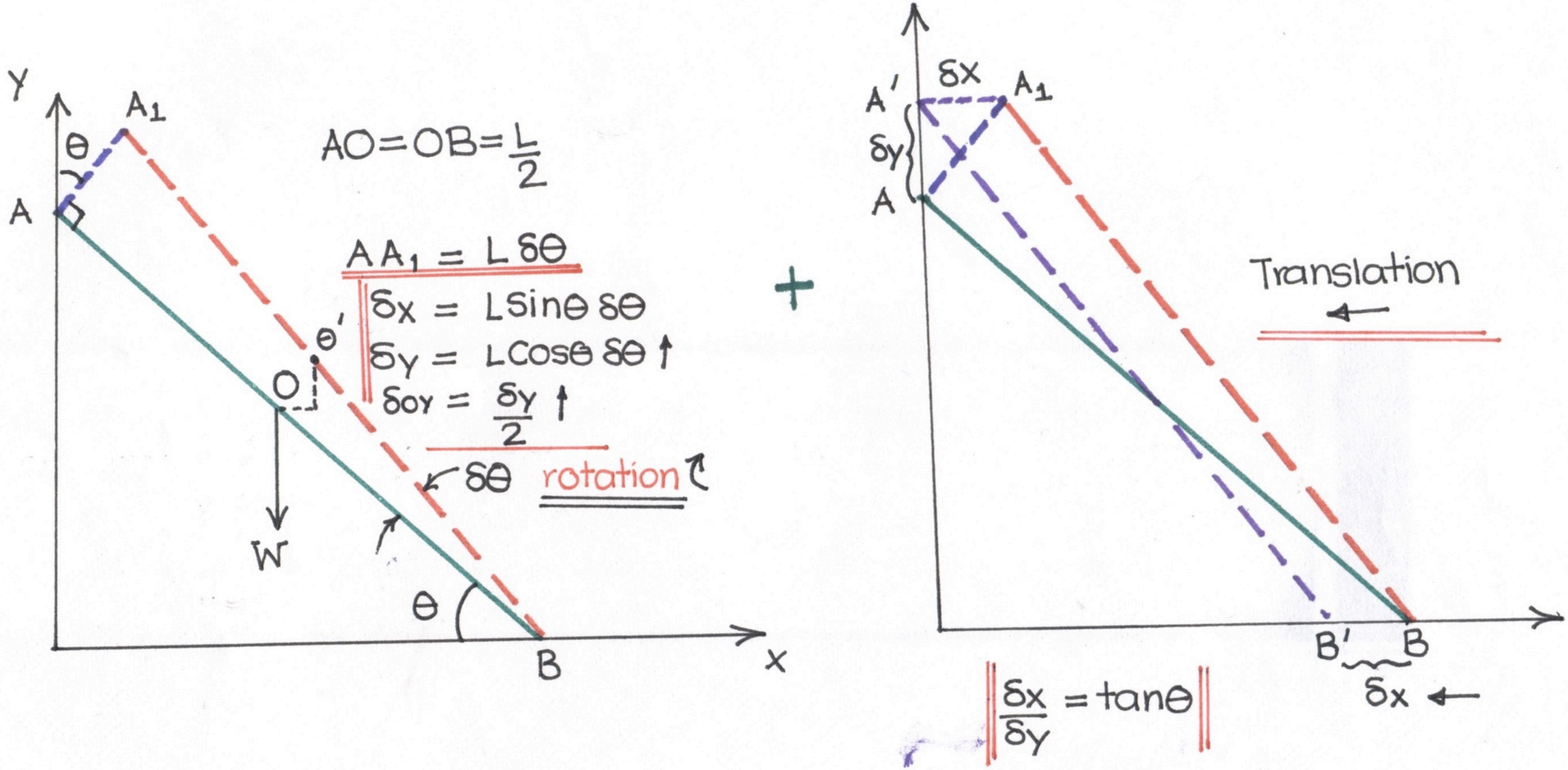
Problem 1

- Assuming frictionless contacts, determine the magnitude of P for equilibrium



This problem will be referred to as the ***Ladder*** problem

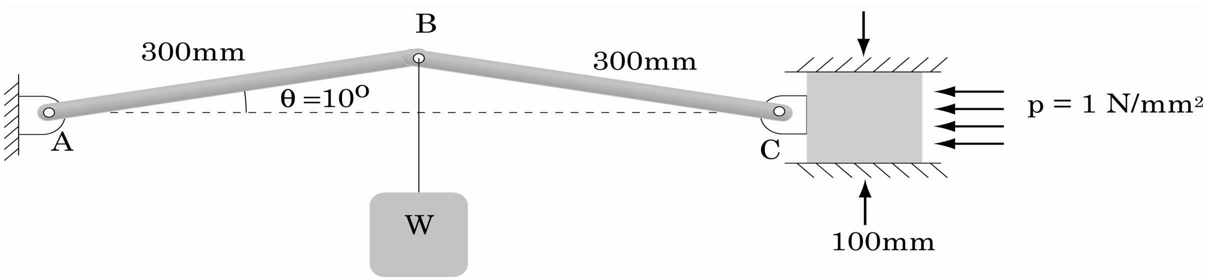
Problem-1



Y
A₁
θ
A
AO=OB=L/2
AA₁ = L δθ
δx = Lsinθ δθ
δy = Lcosθ δθ ↑
δoy = δy/2 ↑
θ'
O
W
δθ
rotation
θ
B
x
+
δx
A'
A₁
δy
A
Translation
B'
B
δx
δx/δy = tanθ

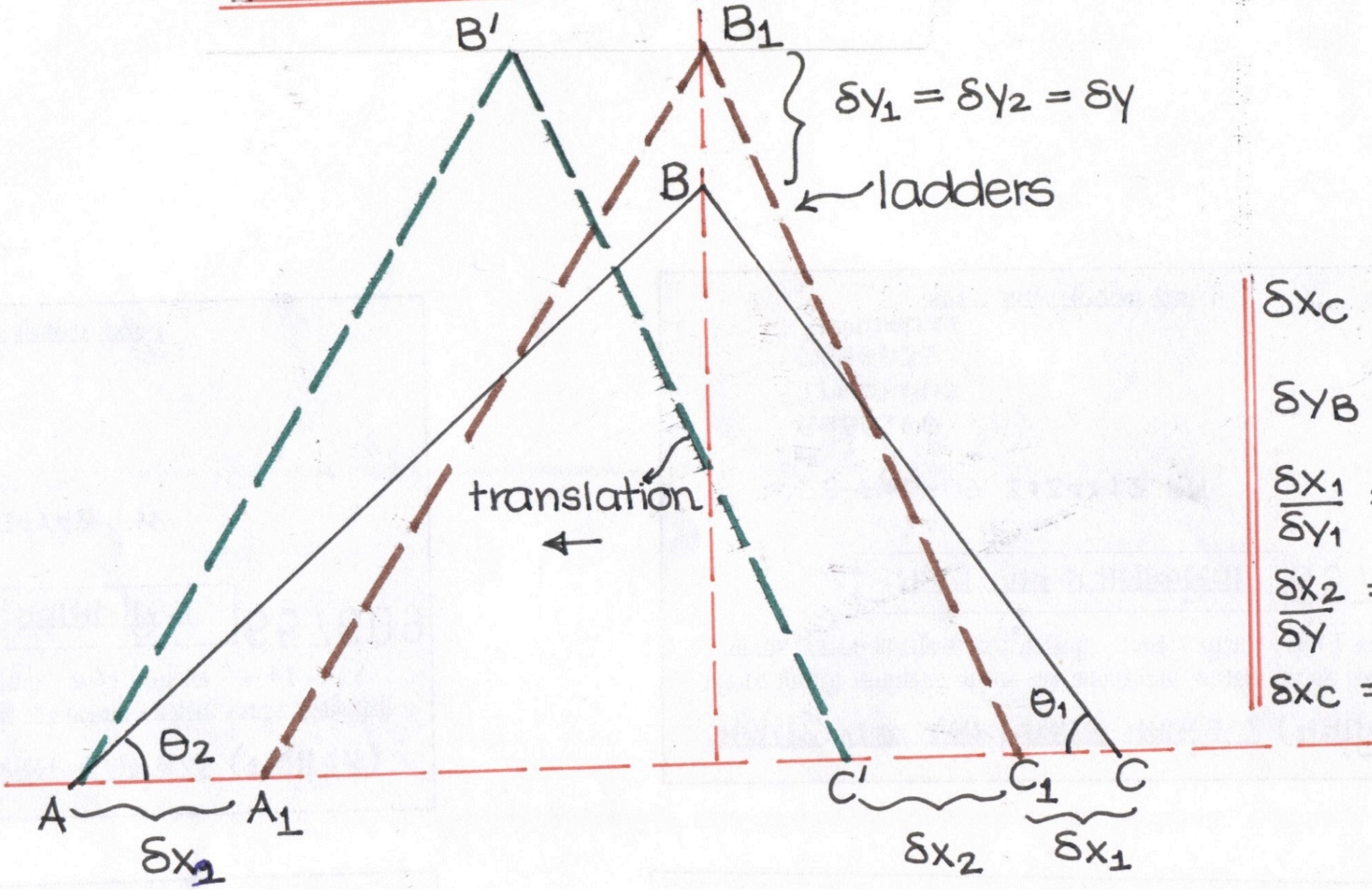
Problem 2

- The pressure *p* driving a piston of diameter 100 mm is 1 N/mm2. At the configuration shown, what weight *W* will the system hold if friction is neglected



This along with the ***ladder*** problem forms a framework for many other problems

Problem-2



$$\delta x_C = \delta x_1 + \delta x_2 \leftarrow$$

$$\delta y_B = \delta y_1 = \delta y_2 = \delta y \uparrow$$

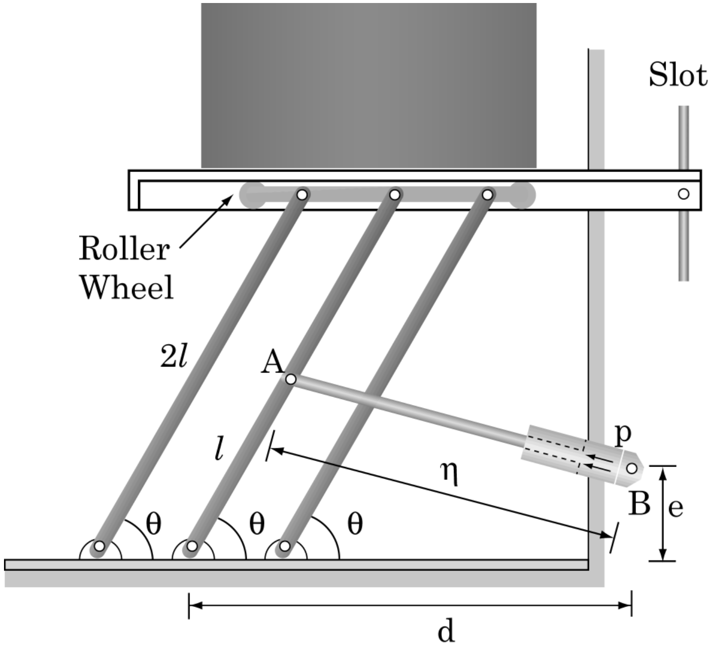
$$\frac{\delta x_1}{\delta y_1} = \tan\theta_1$$

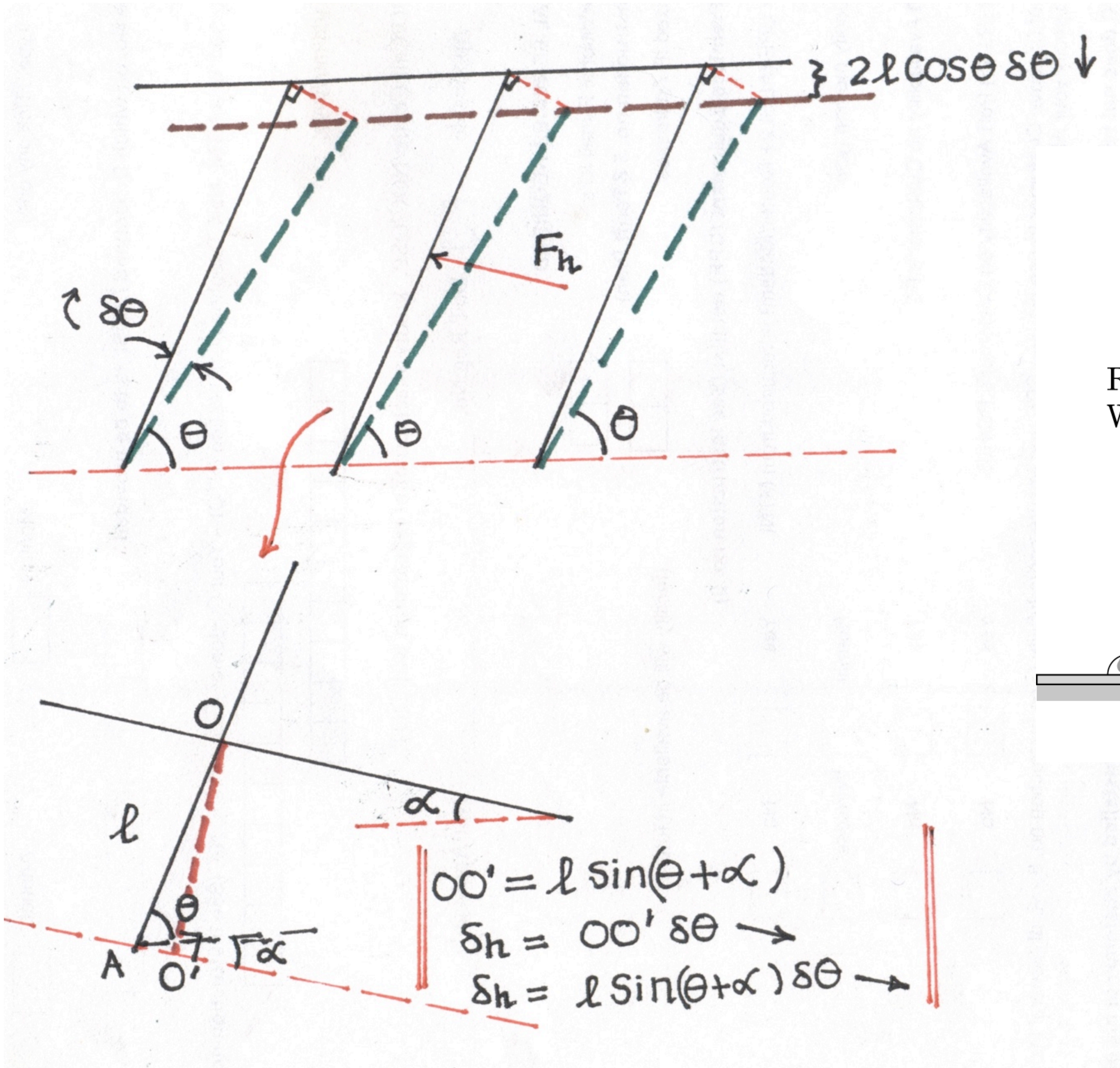
$$\frac{\delta x_2}{\delta y} = \tan\theta_2$$

$$\delta x_C = (\tan\theta_1 + \tan\theta_2)\,\delta y \leftarrow$$

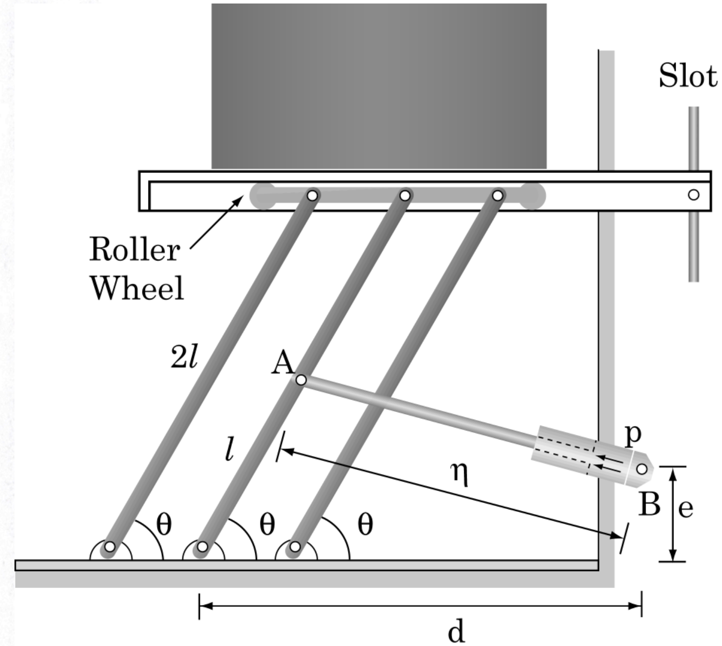
Problem 3

- A hydraulic lift platform for loading trucks supports a weight W of 5000N. Only one side of the system has been shown; the other side is identical. If the diameter of the piston in the cylinder (two) is 40 mm, what pressure p is needed to support W when $\theta = 60^{o}$. Assume $l = 240$ mm, $d = 600$ mm, and $e = 100$ mm. Neglect friction everywhere



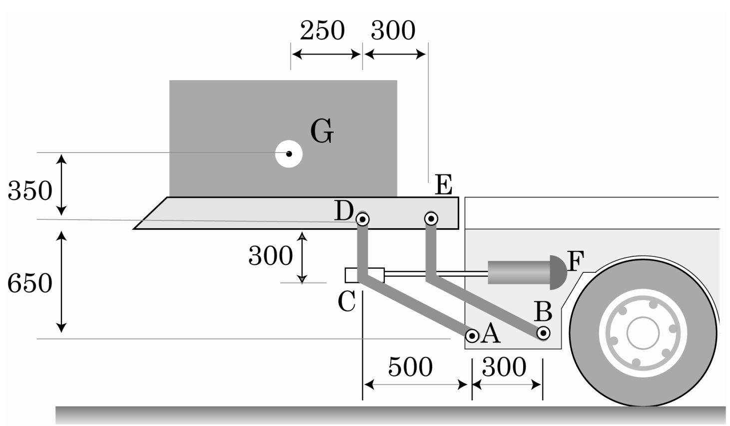


$$OO' = \ell \sin(\theta + \alpha)$$
$$\delta h = OO' \, \delta\theta \rightarrow$$
$$\delta h = \ell \sin(\theta + \alpha)\, \delta\theta \rightarrow$$

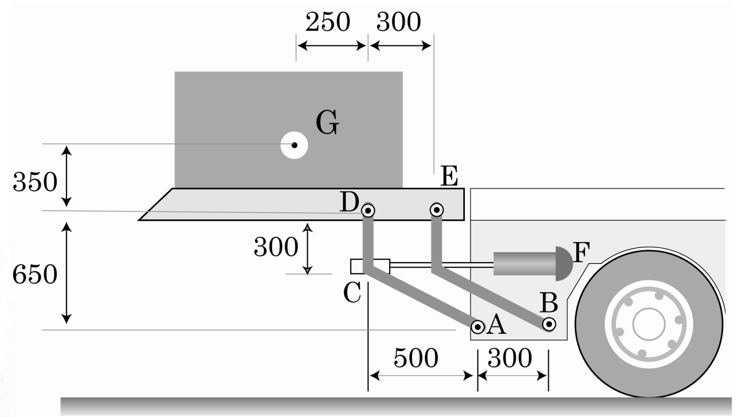


Problem 4

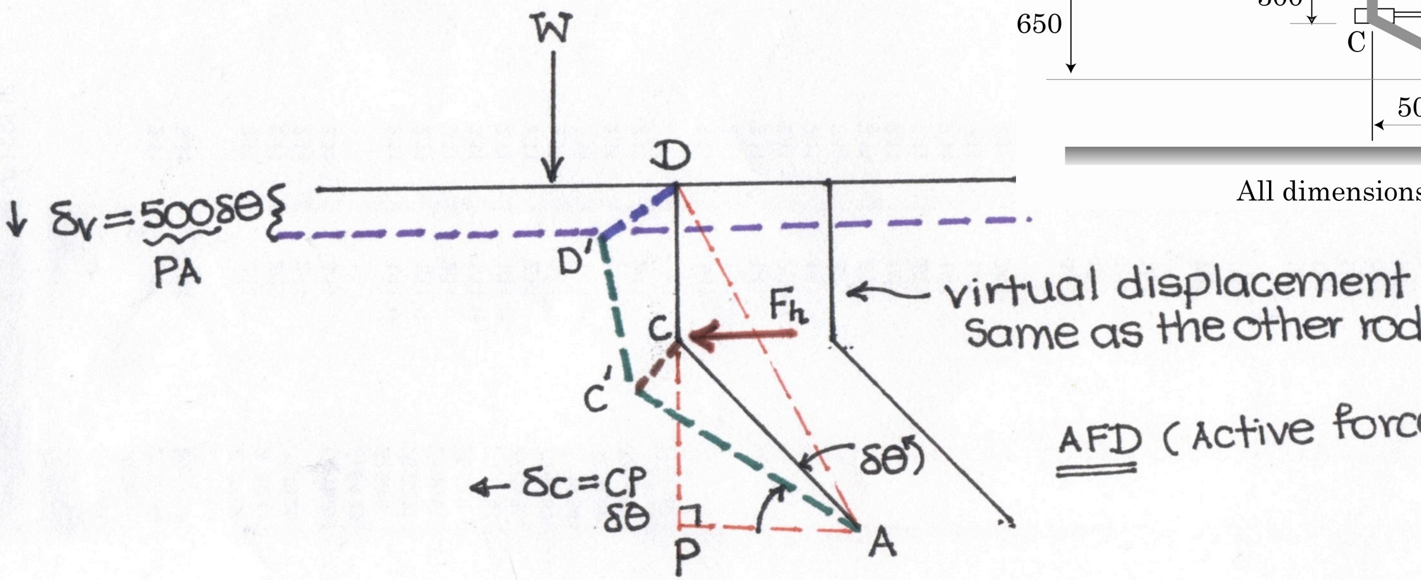
- A power-operated loading platform designed for the back of a truck is shown. The position of the platform is controlled by the hydraulic cylinder, which applies force at *C*. The links are pivoted to the truck frame at *A*, *B*, and *F*. Determine the force *P* supplied by the cylinder in order to support the platform in the position shown. The mass of the platform and the links may be neglected compared with that of the 250 kg crate with center of mass at *G*.



All dimensions in mm



All dimensions in mm



AFD (Active force diagram)

$PA = 500\ mm$

$CP = (650 - 300) = 350\ mm$

$$W\delta_v + F_h\delta_c = 0$$

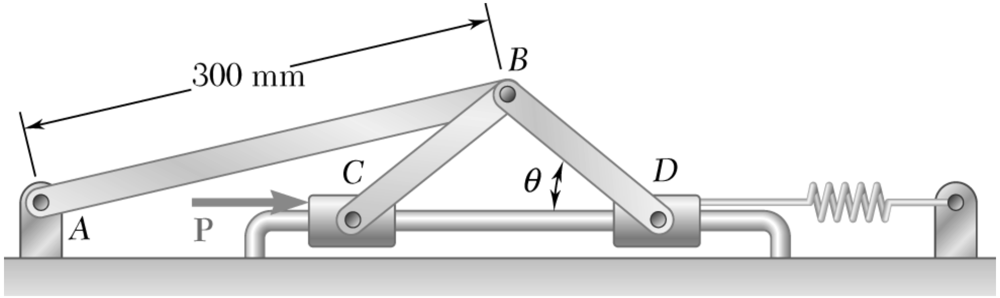
$$(W \times 500 + F_h \times 350)\,\delta\theta = 0$$

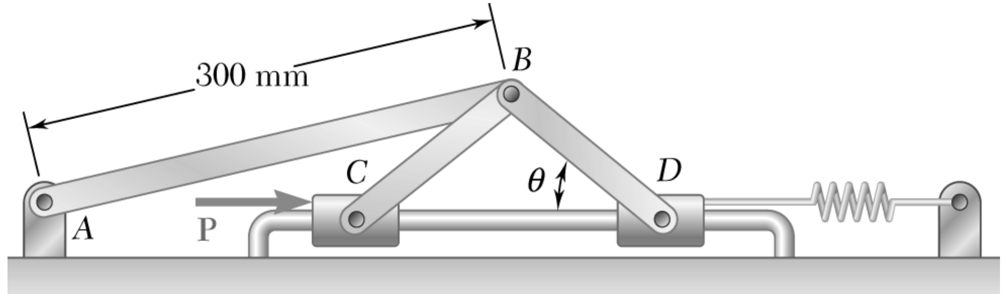
$$\Rightarrow \quad F_h = -\frac{250 \times 500 \times 9.81}{350}\ N = -3500\ N$$

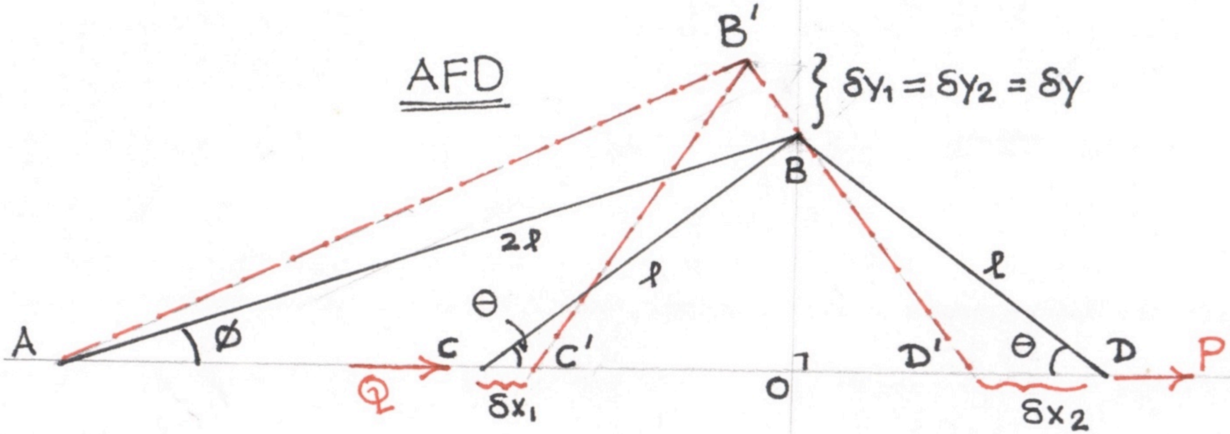
$$F_h = 3.5\,kN\ (T)$$

Problem 5

- A force **P** is applied to slider *C* as shown. The constant of the spring is 1.6 kN/m, and the spring is un-stretched when member *BD* is horizontal. Neglecting friction between the slider and the guide rod and knowing that *BC* = *BD* = 150 mm, determine the magnitude of **P** so that when the system is in equilibrium.







$$P = \left(3\ell - [2\ell\cos\phi + \ell\cos\theta]\right)k$$

$$\leftarrow \delta x_2 = (\tan\theta + \tan\phi)\,\delta y$$

$$\leftarrow \delta x_1 = \delta(\tan\phi + \tan(-\theta)) = (\tan\phi - \tan\theta)\,\delta y$$

$$\delta U = -Q\,\delta x_1 - P\,\delta x_2 = 0$$

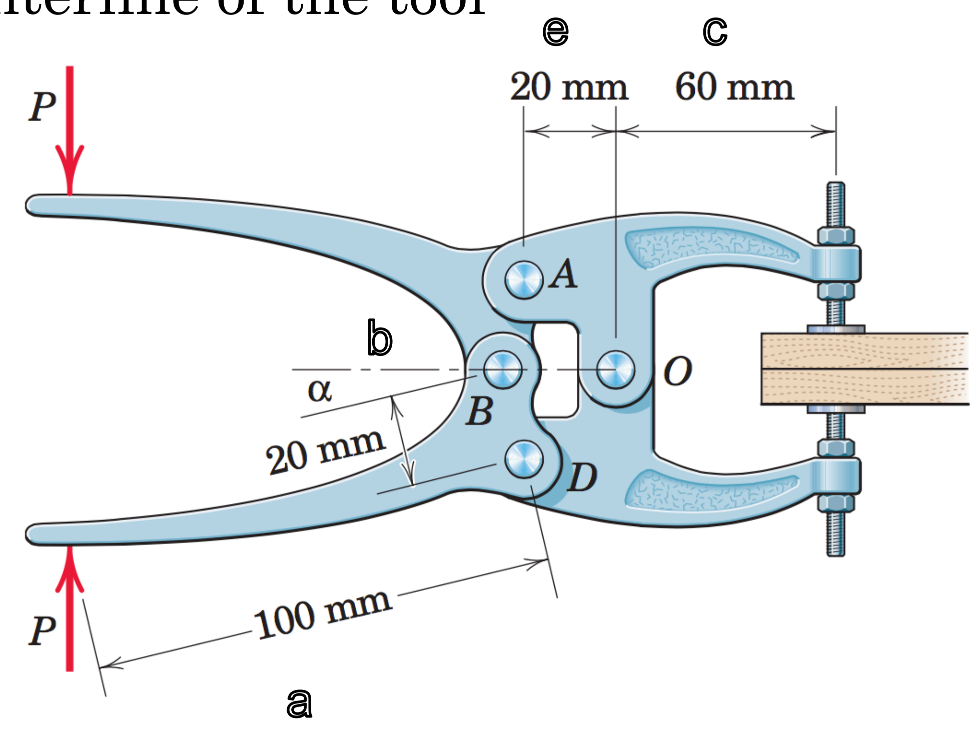
$$\Rightarrow -Q(\tan\phi - \tan\theta) - P(\tan\phi + \tan\theta) = 0$$

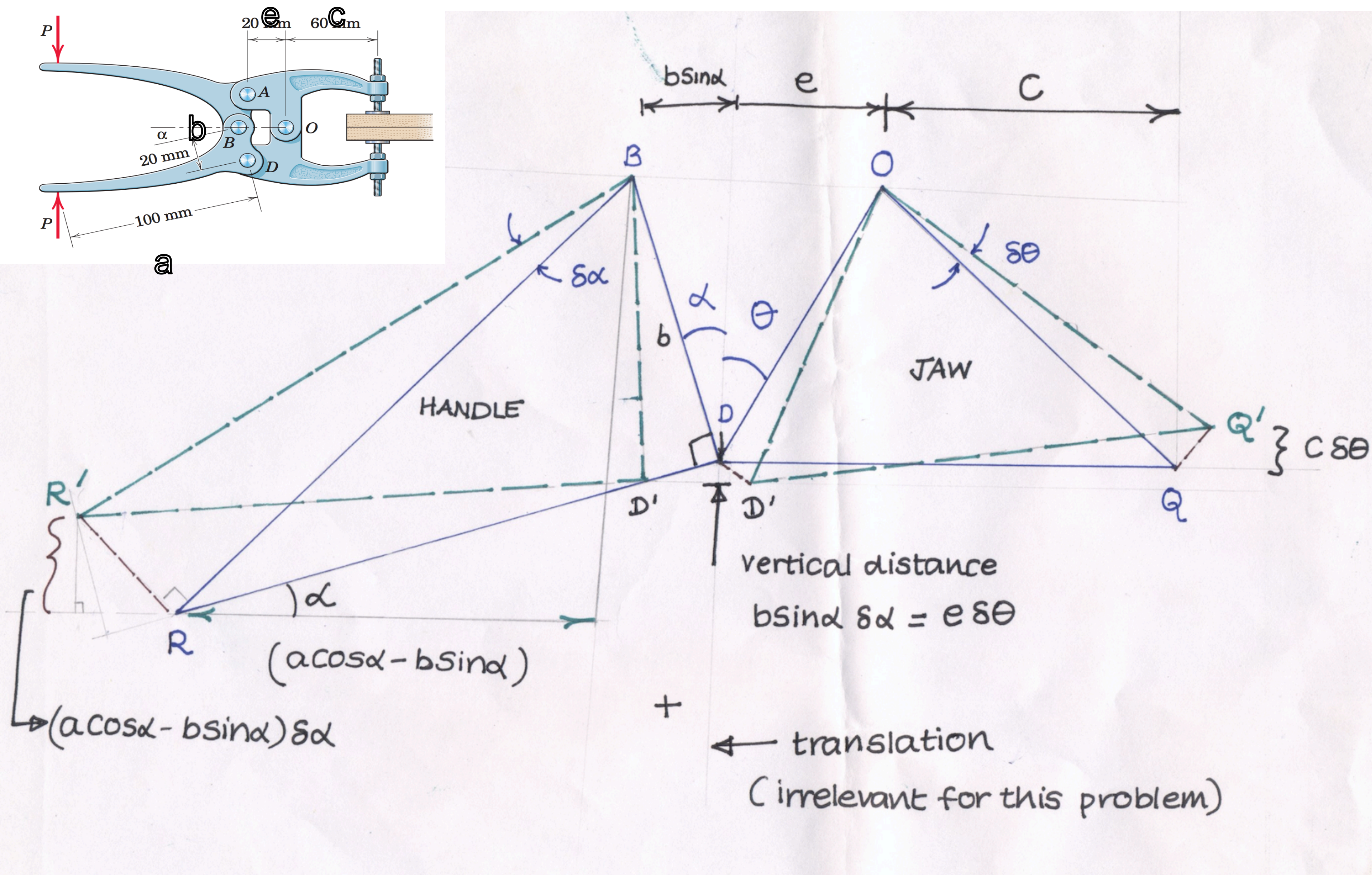
$$\Rightarrow Q = P\,\frac{\tan\theta + \tan\phi}{\tan\theta - \tan\phi}$$

$$2\ell\sin\phi = \ell\sin\theta \Rightarrow \sin\phi = \frac{\sin\theta}{2}.$$

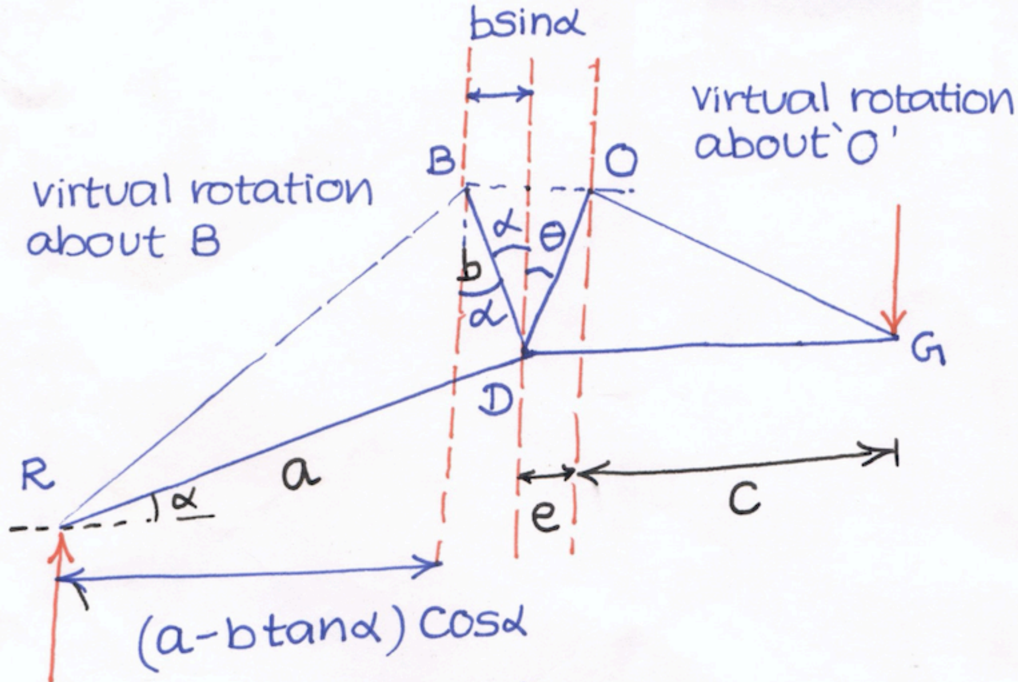
Pliers problem

- The toggle pliers are used for a variety of clamping purposes. For the handle position given by $\alpha = 10\text{deg}$ and for a handle grip $P = 150\text{N}$, calculate the clamping force C produced. Note that pins A and D are symmetric about the horizontal centerline of the tool





20 mm
60 mm
e
c
P
A
B
O
D
b
α
20 mm
100 mm
P
a
bsinα
e
c
B
O
δα
δθ
α
θ
b
HANDLE
JAW
D
Q'
c δθ
R'
D'
D'
Q
R
α
(acosα - bsinα)
(acosα - bsinα) δα
vertical distance
bsinα δα = e δθ
+
← translation
(irrelevant for this problem)



Only relevant portions for virtual displacement are shown

$d = (a\cos\alpha - b\sin\alpha)$

$$\downarrow \delta_D = b\sin\alpha\,\delta\alpha = \downarrow\delta_D = e\,\delta\theta$$

$$\Rightarrow b\sin\alpha\,\delta\alpha = e\,\delta\theta$$

$$\Rightarrow \delta\theta = \frac{b}{e}\sin\alpha\,\delta\alpha$$

$$\uparrow \delta_G = c\,\delta\theta = \frac{cb}{e}\sin\alpha\,\delta\alpha$$

$$\uparrow \delta_R = d\,\delta\theta = (a\cos\alpha - b\sin\alpha)\,\delta\alpha$$

PVW:-

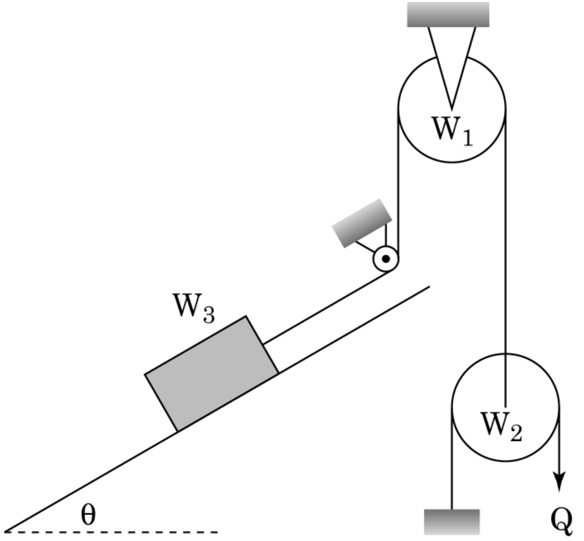
$$\delta W = 0 \Rightarrow P\,\delta_{G} - C\,\delta_{R} = 0$$

$$\Rightarrow \left\{P\left[a\cos\alpha - b\sin\alpha\right] - C\,\frac{cb}{e}\sin\alpha\right\}\delta\alpha = 0$$

$$\Rightarrow \boxed{C = \frac{Pe}{c}\left[\frac{a}{b}\cot\alpha - 1\right]}$$

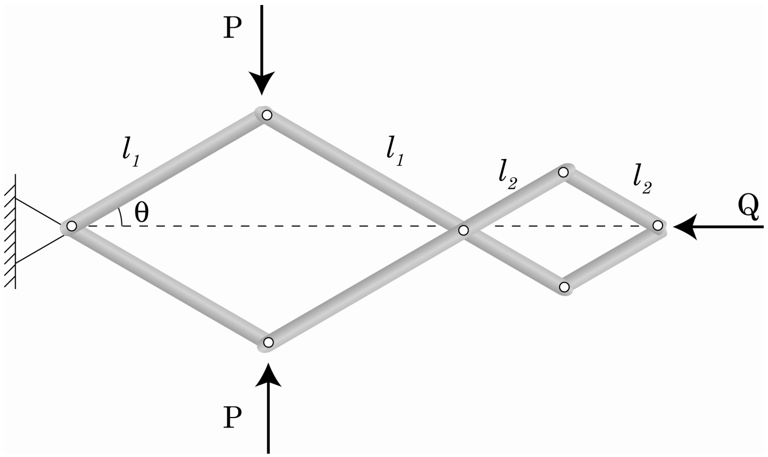
Problem 6

- Determine Q for equilibrium for the system shown. The pulleys are frictionless and have masses W_1 and W_2. The sliding body has mass W_3.



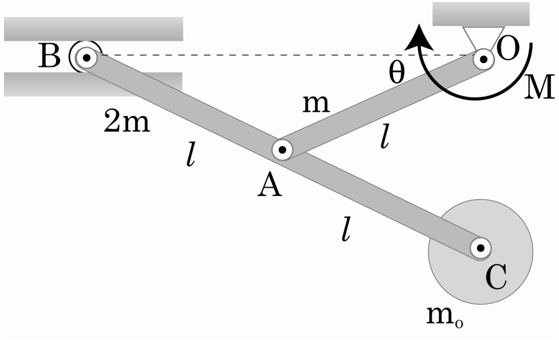
Problem 7

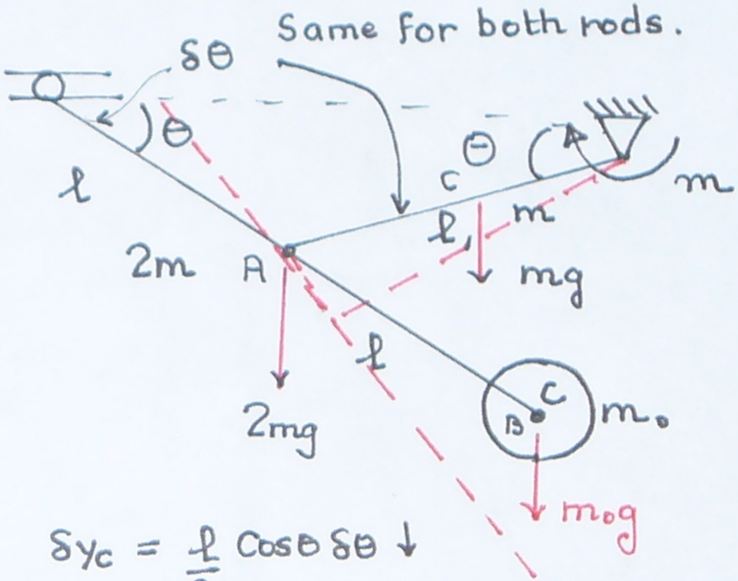
- Find the force Q required to maintain the system in equilibrium.

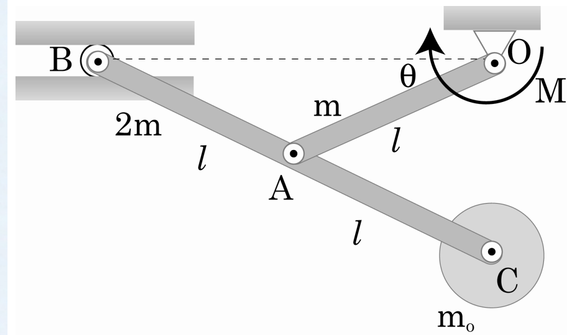


Problem 8

- Determine the couple *M* which must be applied at *O* in order to support the mechanism in the position $\theta = 30$o. The masses of the disk at *C*, bar *OA*, and bar *BC* are m_o, m, and 2m respectively.







$$\delta y_C = \frac{\ell}{2} \cos\theta \, \delta\theta \downarrow$$

$$\delta y_A = \ell \cos\theta \, \delta\theta \downarrow$$

$$\delta y_C = 2\ell \cos\theta \, \delta\theta \downarrow$$

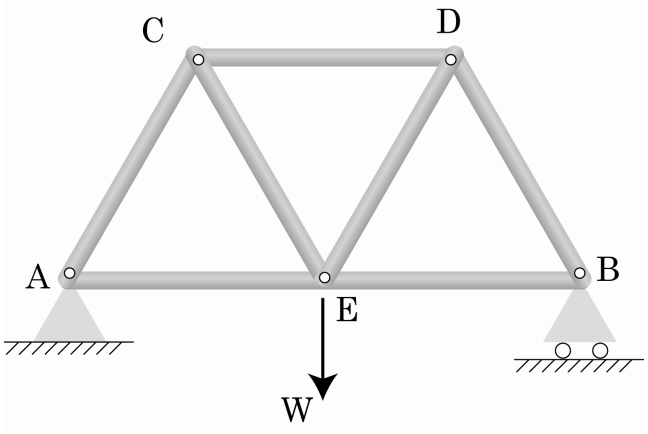
$$\delta U = \Big[2mg * \ell\cos\theta + \frac{\ell}{2}\cos\theta * mg + 2\ell\cos\theta \, m_0 g - M \Big] \delta\theta$$

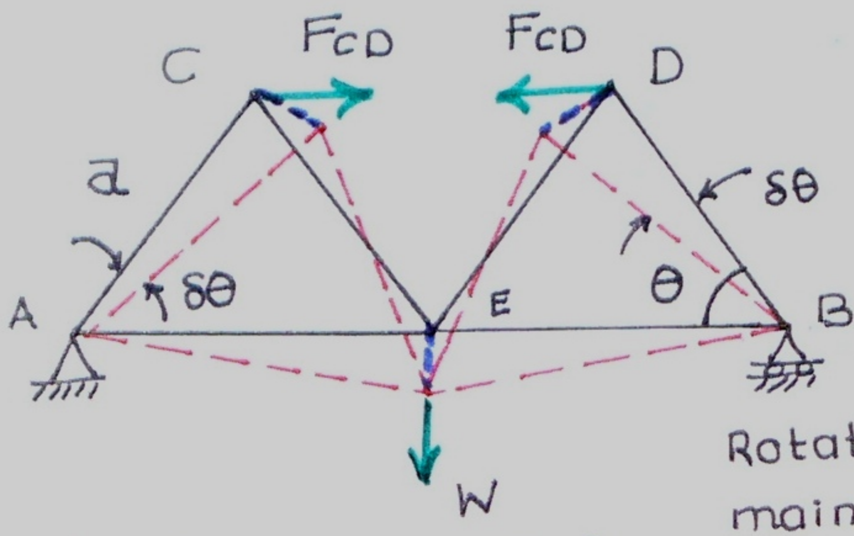
$$\Rightarrow M = mg\ell\cos\theta \left[2 + \frac{1}{2} + 2\frac{m_0}{m} \right]$$

$$M = mg\ell\cos\theta \left[\frac{5}{2} + 2\frac{m_0}{m} \right]$$

Problem 9

- Determine force in member CD by using the method of virtual work.





Rotation δθ on both sides to maintain compatibility at E

all angles are $\theta = 60°$

$\rightarrow \delta_C = a \sin\theta \, \delta\theta$

$\leftarrow \delta_D = a \sin\theta \, \delta\theta$

$\downarrow \delta_E = a \, \delta\theta$

$\delta U = F_{CD} * \delta_C + F_{CD} * \delta_D + W * \delta_E = 0$

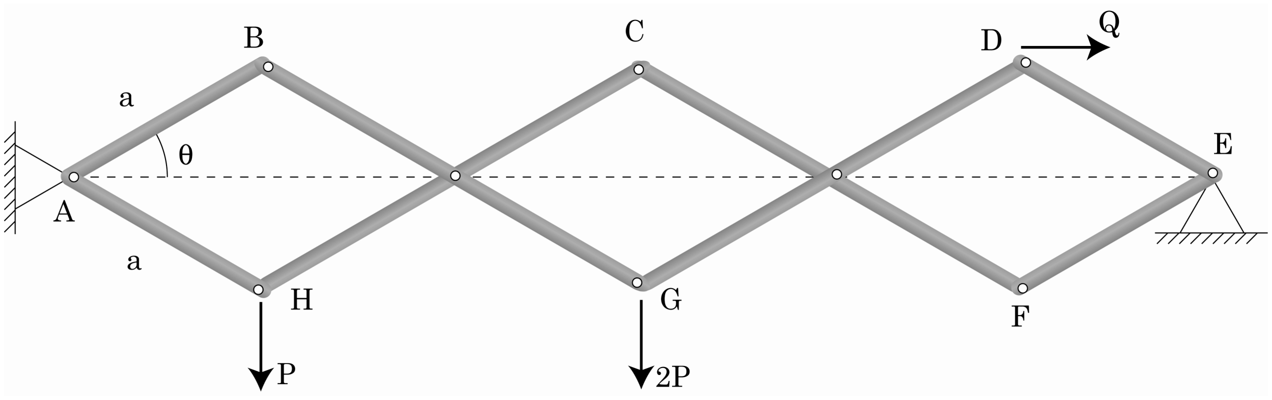
$\Rightarrow [2 F_{CD} \, a \sin\theta + W a] \, \delta\theta = 0$

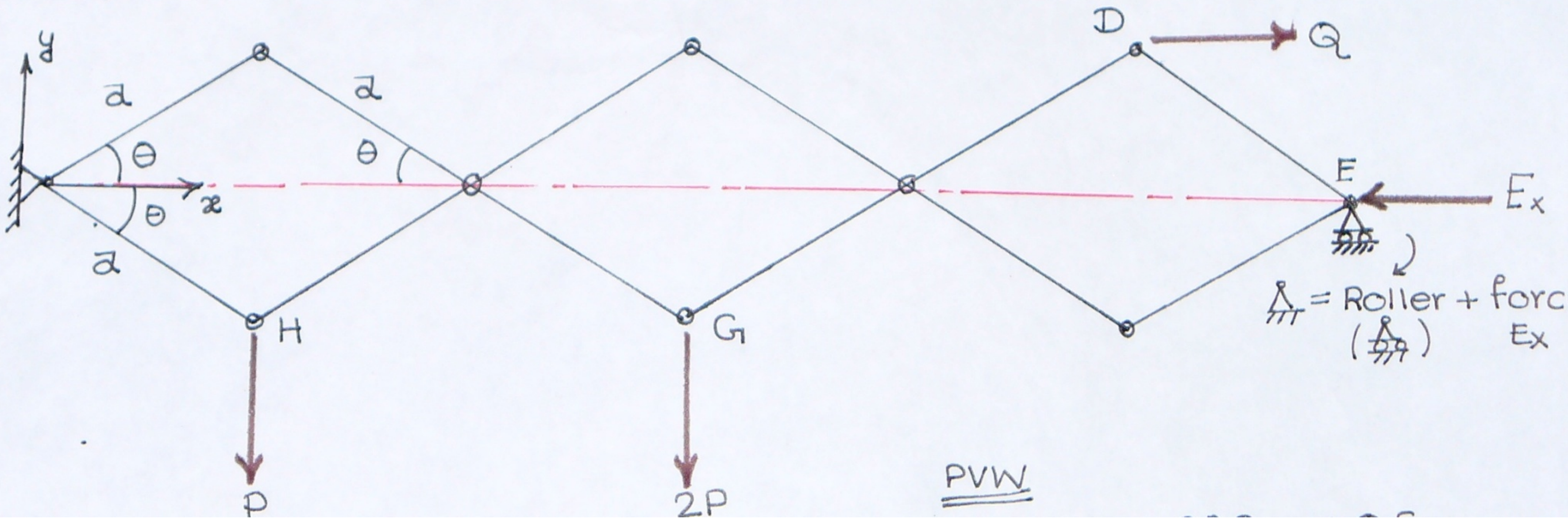
$\Rightarrow \quad F_{CD} = -\dfrac{W}{2 \sin\theta} \quad (C)$

Additional Problems

Problem 1

- Find reaction at E by using method of virtual work

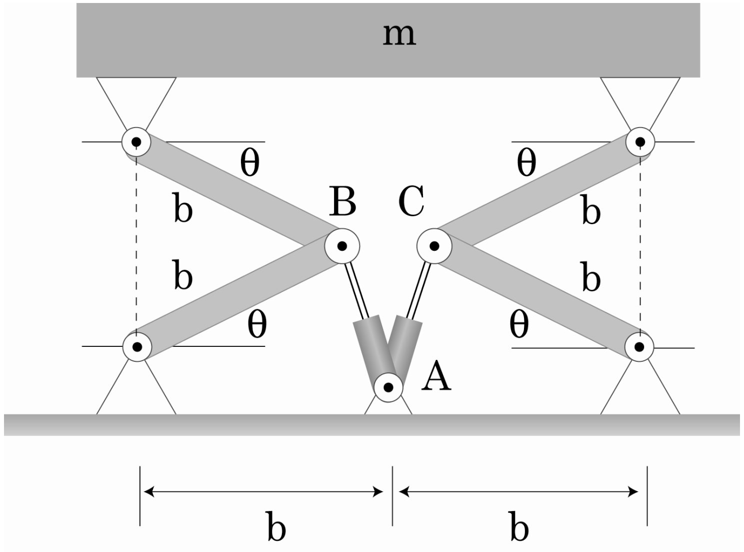


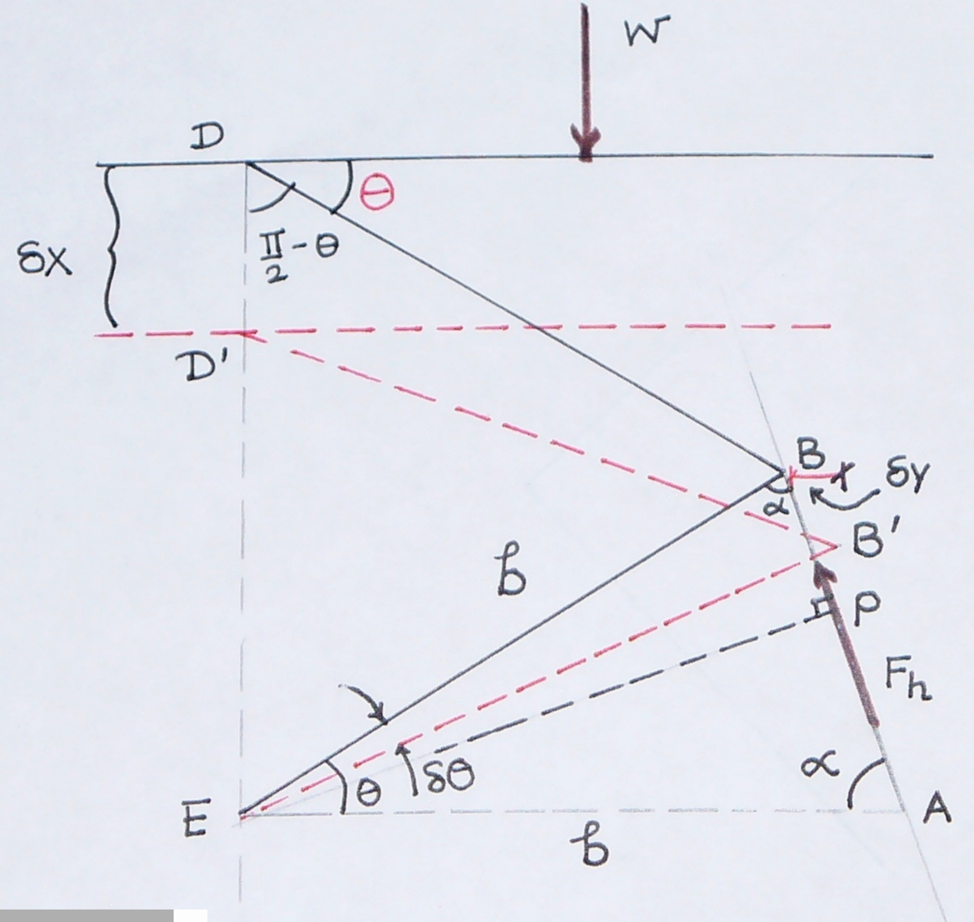


PVW

Problem 2

- The elevation of the platform of mass m supported by the four identical links is controlled by the hydraulic cylinders AB and AC which are pivoted at point A. Determine the compression P in each of the cylinders required to support the platform for the specified angle θ





$EA = EB = b$

In $\Delta\, EBA$

$\angle EBA = \angle EAB = \alpha$

$\alpha = \frac{\pi}{2} - \frac{\theta}{2}$

$EP = b \sin\alpha$

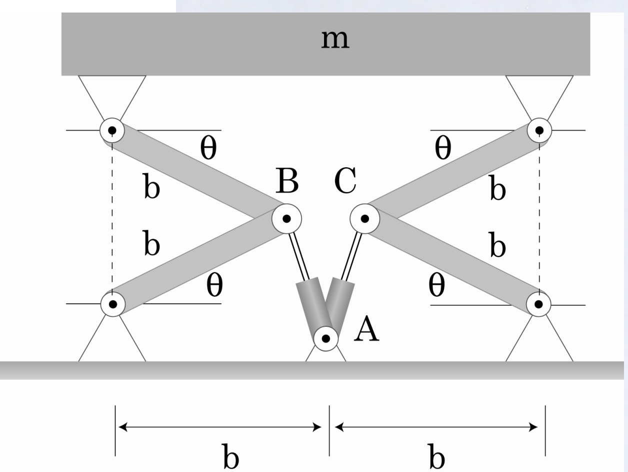
$= b \cos\frac{\theta}{2}$

δ_{BA} along BA is

$\downarrow \delta_{BA} = b \cos\frac{\theta}{2}\, \delta\theta$

$\downarrow \delta_D = \delta x$

$= \frac{2}{\tan\theta}\, \delta y$



$\delta y = b \sin\theta\, \delta\theta \rightarrow$

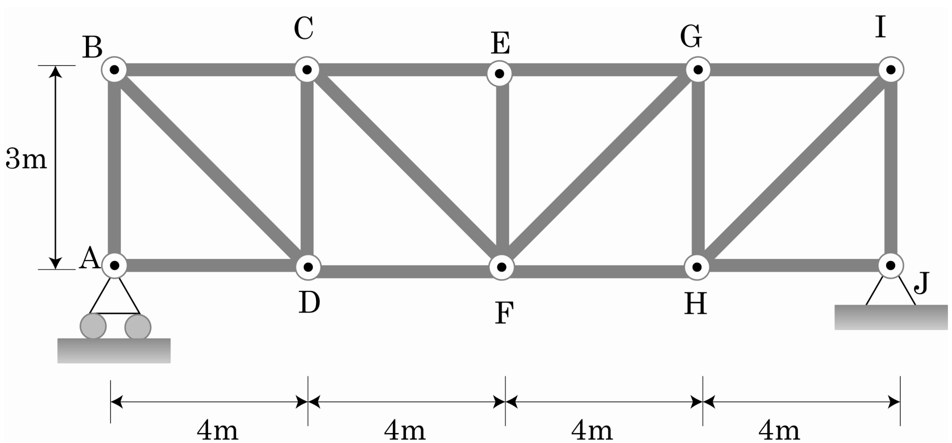
$\downarrow \delta_D = 2 \cos\theta\, b\, \delta\theta$

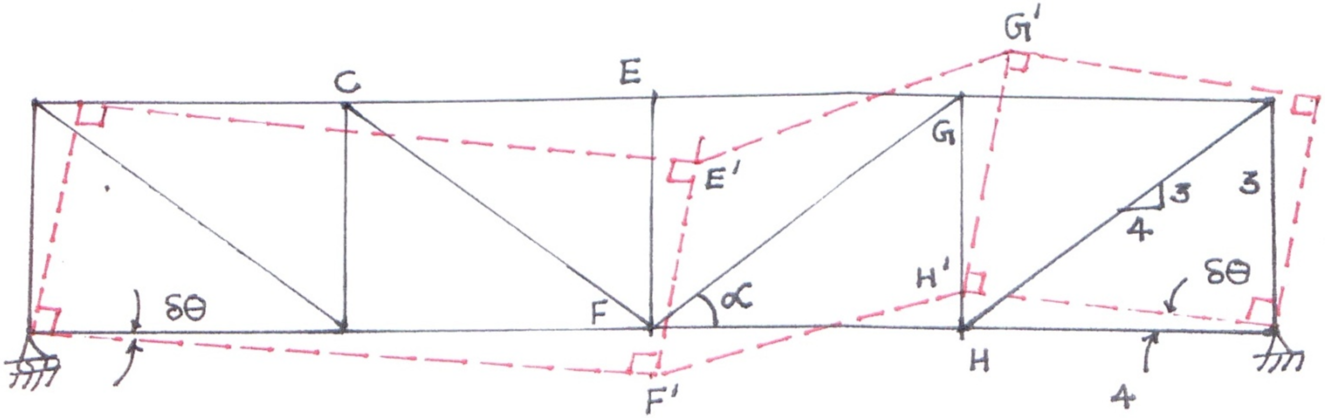
PVW: $\delta U = -2 F_h \delta_{BA} + W \delta_D = 0$

$\Rightarrow F_h = \frac{W \cos\theta}{\cos\theta/2}$

Problem 3

- Determine the vertical movement of joint C, if the member FG is lengthened by 50 mm





Note that in order to make $\delta_E \rightarrow \; = \; \delta_G \rightarrow$ (so that l(GE) remains constant in this deformation)
$\delta\theta$ on both sides is same

We will now obtain $\delta\theta$

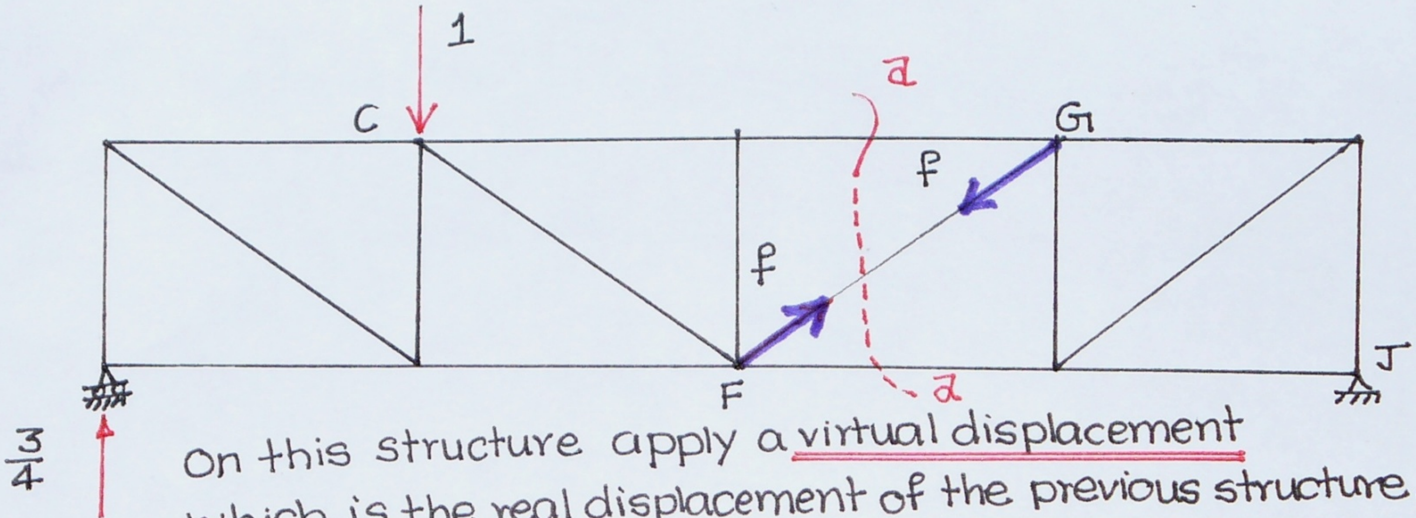
displacement of F along FG ↙ δ_F is $2 \times 4 \sin\alpha * \delta\theta$

$$= 8 \times \frac{3}{5}\,\delta\theta = \frac{24}{5}\,\delta\theta$$

" " G " " ↗ δ_G is $= 8 \times \frac{3}{5}\,\delta\theta = \frac{24}{5}\,\delta\theta$

now change in length of FG $=$ ↙$\delta_F + \delta_G$↗ $= \frac{48}{5}\,\delta\theta = \delta$

$\Rightarrow \delta\theta = \frac{5\delta}{12 \times 4} \Rightarrow \delta_C \downarrow = 4\,\delta\theta = \frac{5\,\delta}{12} = 20.833\,\text{mm} \downarrow$



On this structure apply a <u>virtual displacement</u> which is the <u>real displacement of the previous structure.</u>

$$\delta U = 1 * \delta_C + f\,\delta F - f\,\delta G$$

$$= \delta_C + f(\delta F - \delta G)$$

($\delta F - \delta G = -\delta$)

$$= \delta_C - f\delta$$

$$\Rightarrow f\downarrow \delta_C = f\delta.$$

To obtain f : take section a-a.

FBD on the left.

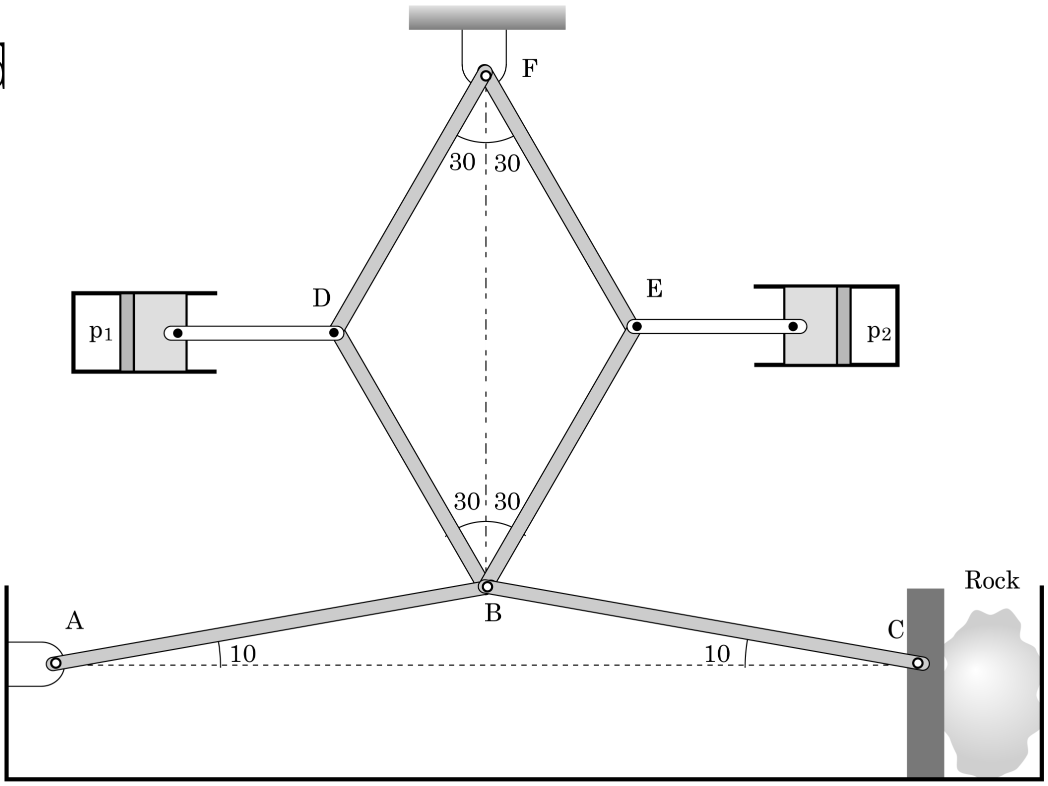
$$\Sigma F\uparrow = f * \frac{3}{5} + \frac{3}{4} - 1 = 0$$

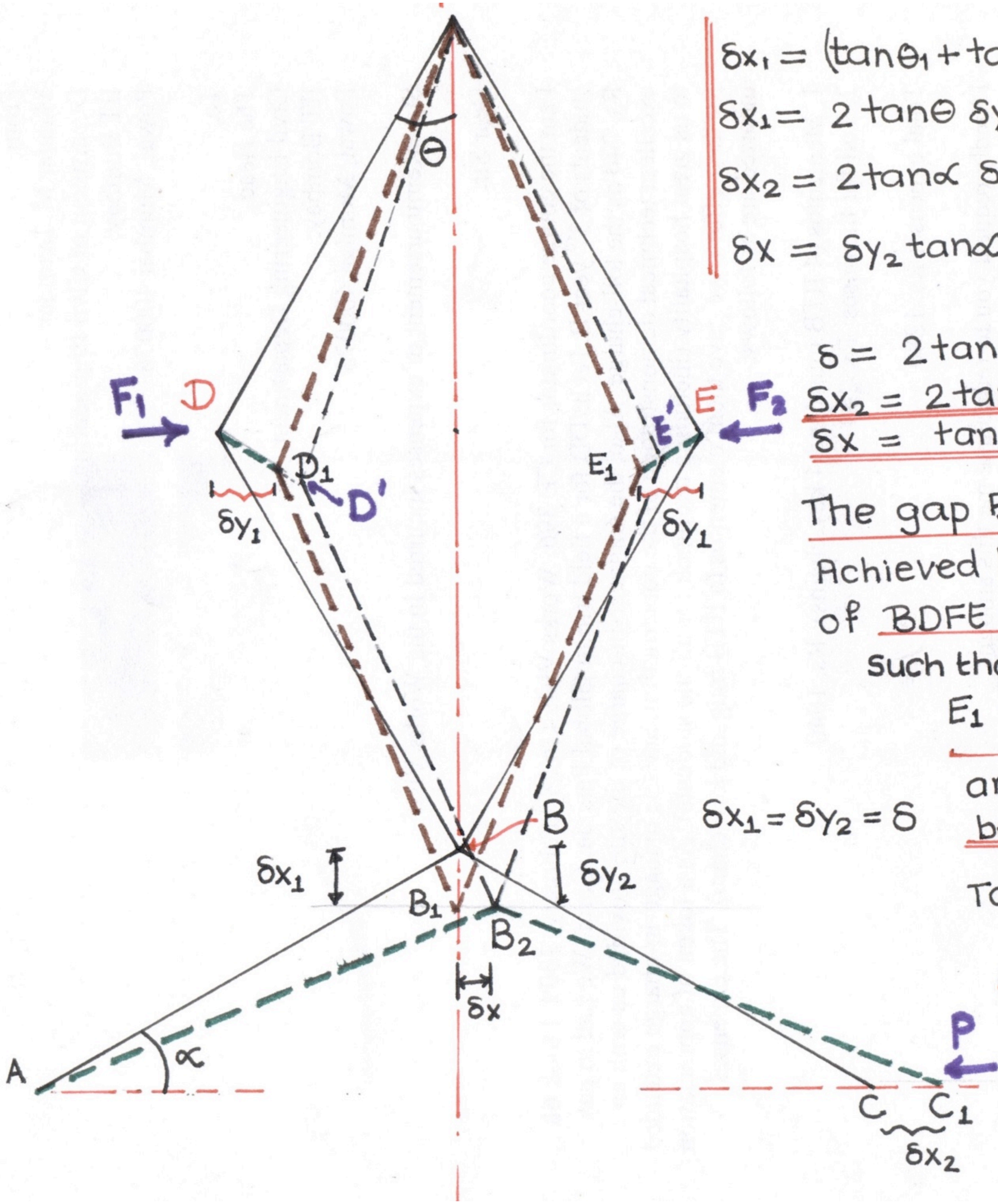
$$\Rightarrow \quad f * \frac{3}{5} = \frac{1}{4} \Rightarrow f = \frac{5}{12}\,(T)$$

$$\downarrow \delta_C = \frac{5}{12} * 50 = 20.8\,\text{mm}\downarrow$$

Problem 4

- Find the force delivered at C in the horizontal direction to crush the rock. Pressure p_1= 100 M-Pa and p_2= 60 M-Pa (measured above atmospheric pressure). The diameters of pistons are 100mm each. Neglect the weight of the rods.





$$\delta x_1 = (\tan\theta_1 + \tan\theta_2)\,\delta y_1$$

$$\delta x_1 = 2\tan\theta\,\delta y_1 = 2\tan\theta\,\delta y_1 = \delta$$

$$\delta x_2 = 2\tan\alpha\,\delta y_2 = 2\tan\alpha\,\delta$$

$$\delta x = \delta y_2 \tan\alpha = \delta \tan\alpha$$

$\delta = 2\tan\theta\,\delta y_1$; $\delta y_1 = \dfrac{\delta}{2\tan\theta}$

$\delta x_2 = 2\tan\alpha\,\delta$

$\delta x = \tan\alpha\,\delta$

$\delta x_1 = \delta y_2 = \delta$

The gap $B_1 B_2$ has to be filled

Achieved by rigid body rotation of BDFE about F

Such that horizontal distance between E_1 and E' (D_1 and D') is $\dfrac{\delta x}{2}$

and thus between B_1 and B_2 becomes zero.

Total virtual displacements

$\rightarrow \delta_D = \delta y_1 + \dfrac{\delta x}{2} = \dfrac{\delta}{2\tan\theta} + \dfrac{\delta\tan\alpha}{2}$

$\leftarrow \delta_E = \delta y_1 - \dfrac{\delta x}{2} = \dfrac{\delta}{2\tan\theta} - \dfrac{\delta\tan\alpha}{2}$

$\rightarrow \delta_C = \delta x_2 = 2\tan\alpha\,\delta$

PVW

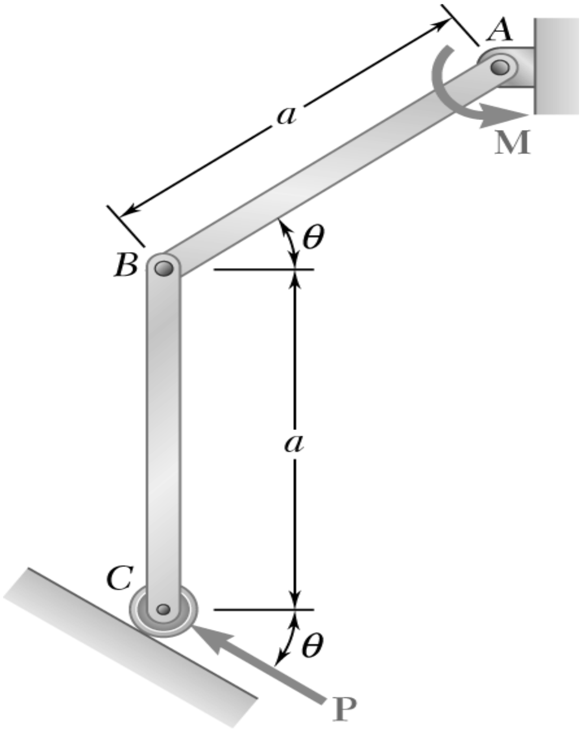
$\delta U = -P\delta_C + F_1\,\delta_D + F_2\,\delta_C = 0$

$$\delta U = \left[-P \times 2\tan\alpha + F_1 \left[\frac{1}{2\tan\theta} + \frac{\tan\alpha}{2} \right] + F_2 \left[\frac{1}{2\tan\theta} - \frac{\tan\alpha}{2} \right] \right] \delta^{\cdot} = 0$$

$$\Rightarrow P = \frac{(F_1 + F_2)}{4\tan\theta\tan\alpha} + \frac{F_1 - F_2}{4}$$

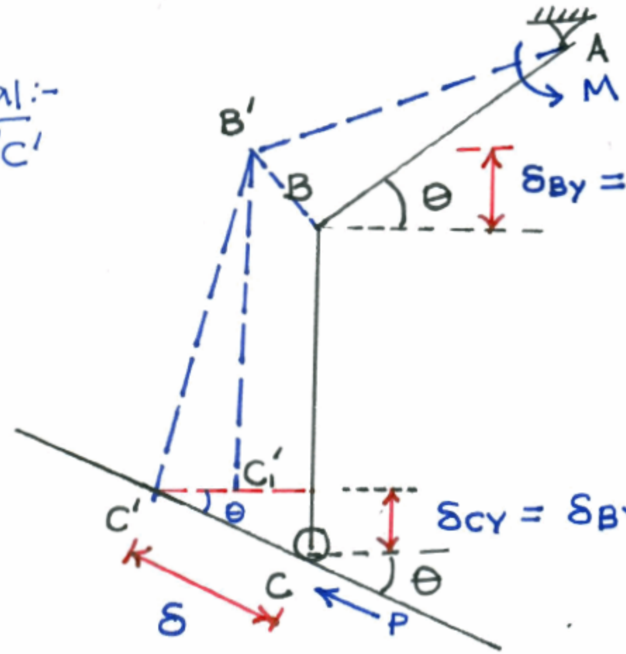
Extra Problem

- Find the moment ***M*** required to hold the system in equilibrium.
- Take $P = 135$ N, $a = 600$mm, and θ = 40 deg.



5)

Final :- $AB'C'$



$\delta_{By} = a\cos\theta\,\delta\theta \uparrow$

$\delta_{cy} = \delta_{By} = a\cos\theta\,\delta\theta \uparrow$

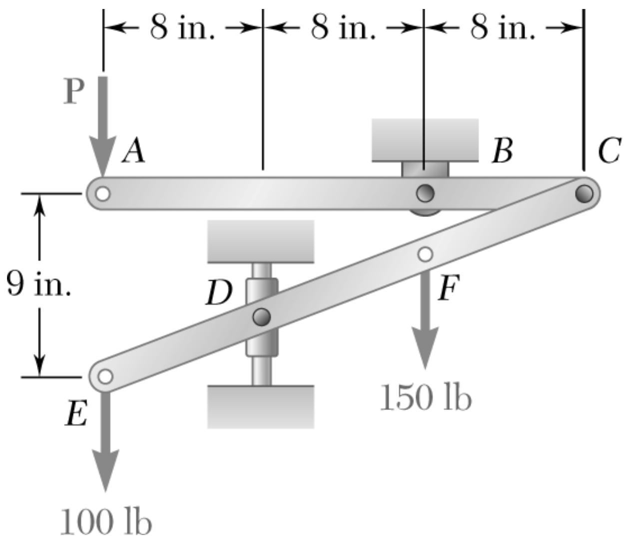
$$\delta = \frac{\delta_{cy}}{\sin\theta} = \frac{a}{\tan\theta}\,\delta\theta \leftarrow$$

$$\frac{\delta}{\delta\theta} = \frac{a}{\tan\theta}$$

PVW

$$-M\delta\theta + P\delta = 0$$

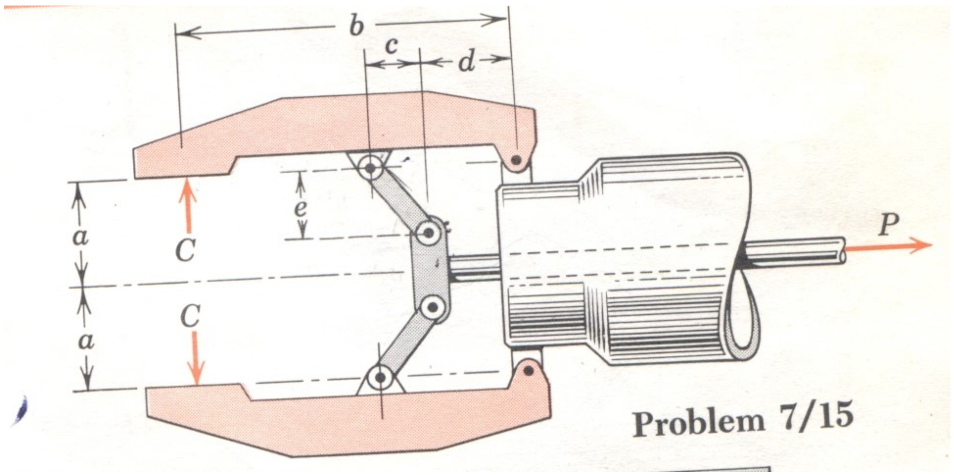
$$\Rightarrow M = \frac{P\delta}{\delta\theta} = \frac{Pa}{\tan\theta} = 96.53 \text{ N-m} \circlearrowright$$



Problem 11

- The two-bar linkage shown is supported by a pin and bracket at B and a collar at D that slides freely on a vertical rod. Determine the force **P** required to maintain the equilibrium of the linkage.

Problem 12



- The claw of the remote-action actuator develops a clamping force C as a result of the tension P in the control rod. Express C in terms of P for the configuration shown where the jaws are parallel.