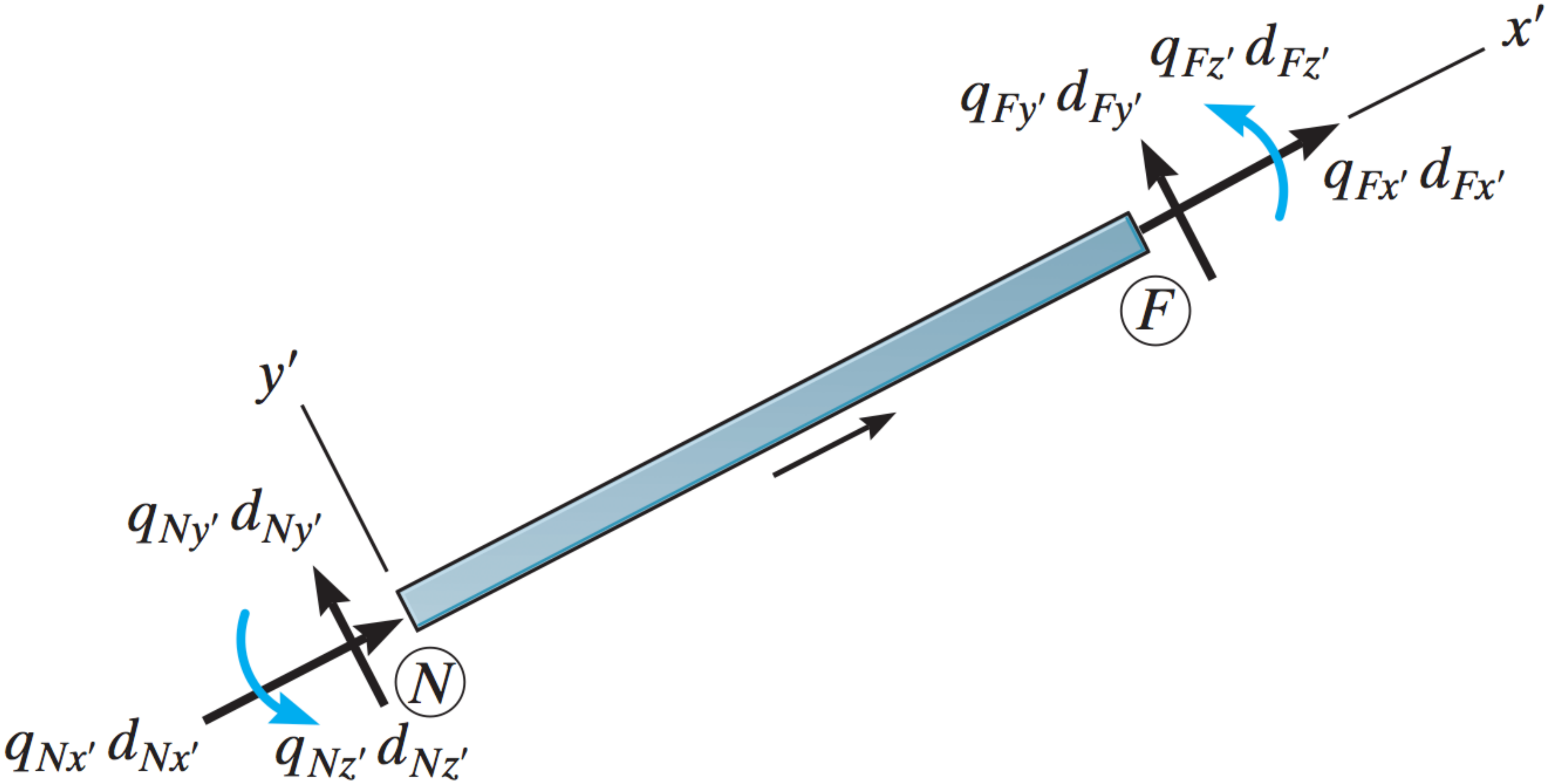
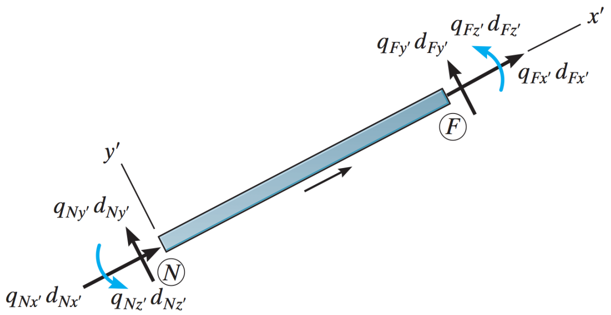
Direct Stiffness for Frames

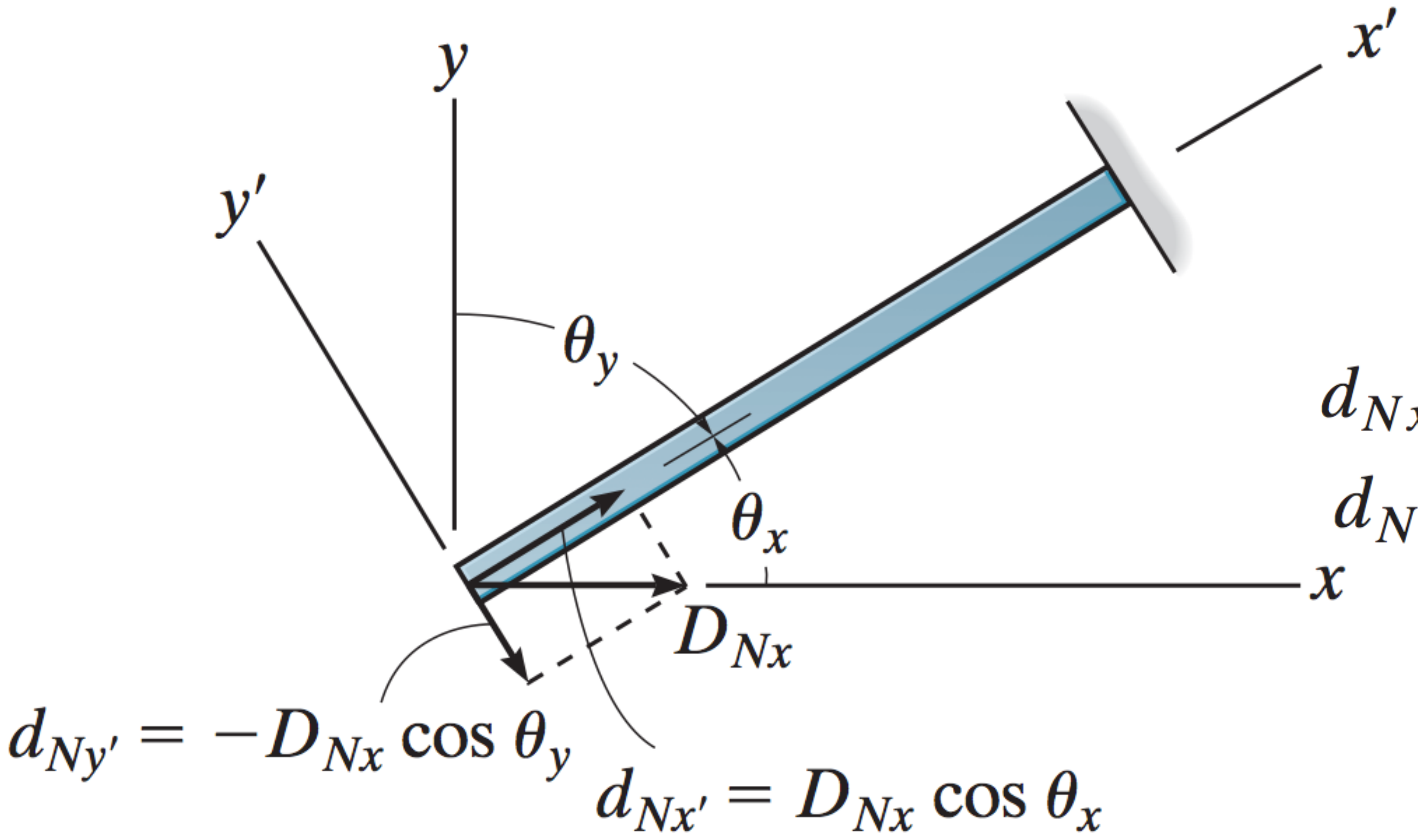


x'
$q_{Fz'}\ d_{Fz'}$
$q_{Fy'}\ d_{Fy'}$
$q_{Fx'}\ d_{Fx'}$
F
y'
$q_{Ny'}\ d_{Ny'}$
N
$q_{Nx'}\ d_{Nx'}$
$q_{Nz'}\ d_{Nz'}$

$$\mathbf{q} = \mathbf{k}'\mathbf{d}$$



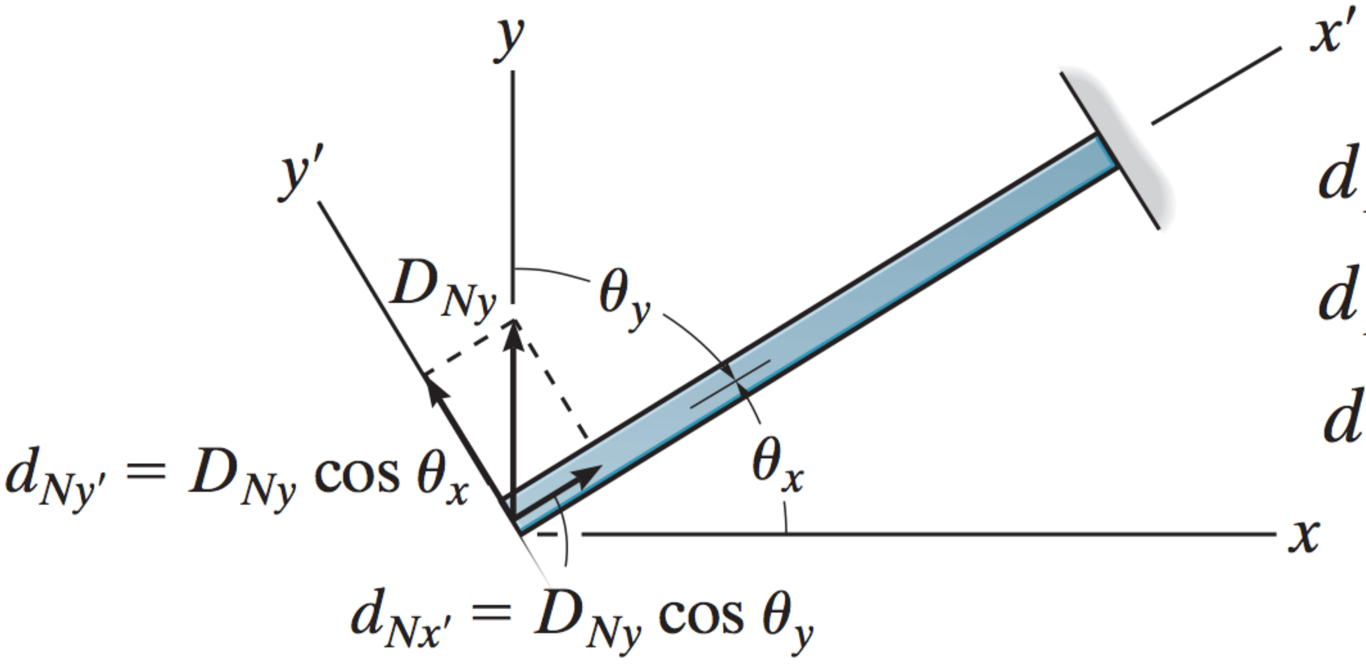
$$\begin{bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nz'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}$$



$$d_{Nx'} = D_{Nx} \cos \theta_x \qquad d_{Ny'} = -D_{Nx} \cos \theta_y$$

$$d_{Nx'} = D_{Ny} \cos \theta_y \qquad d_{Ny'} = D_{Ny} \cos \theta_x$$

$$d_{Nz'} = D_{Nz}$$



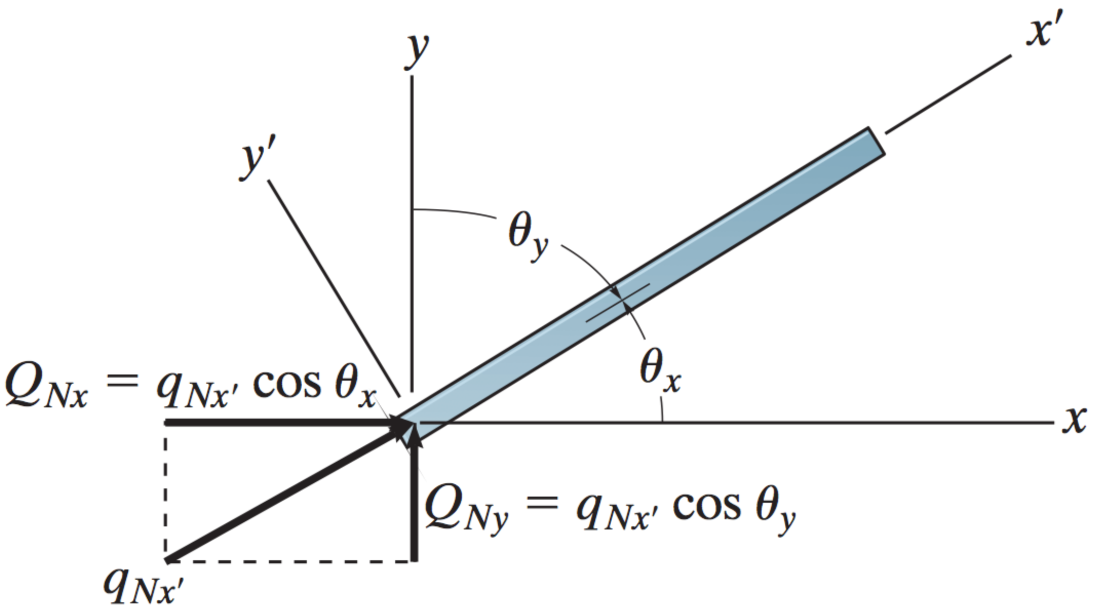
$$d_{Fx'} = D_{Fx} \cos \theta_x \qquad d_{Fy'} = -D_{Fx} \cos \theta_y$$

$$d_{Fx'} = D_{Fy} \cos \theta_y \qquad d_{Fy'} = D_{Fy} \cos \theta_x$$

$$d_{Fz'} = D_{Fz}$$

$$\begin{bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nz'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Nz} \\ D_{Fx} \\ D_{Fy} \\ D_{Fz} \end{bmatrix}$$

$$\mathbf{d} = \mathbf{T}\mathbf{D}$$



$$Q_{Nx} = q_{Nx'} \cos \theta_x \qquad Q_{Ny} = q_{Nx'} \cos \theta_y$$

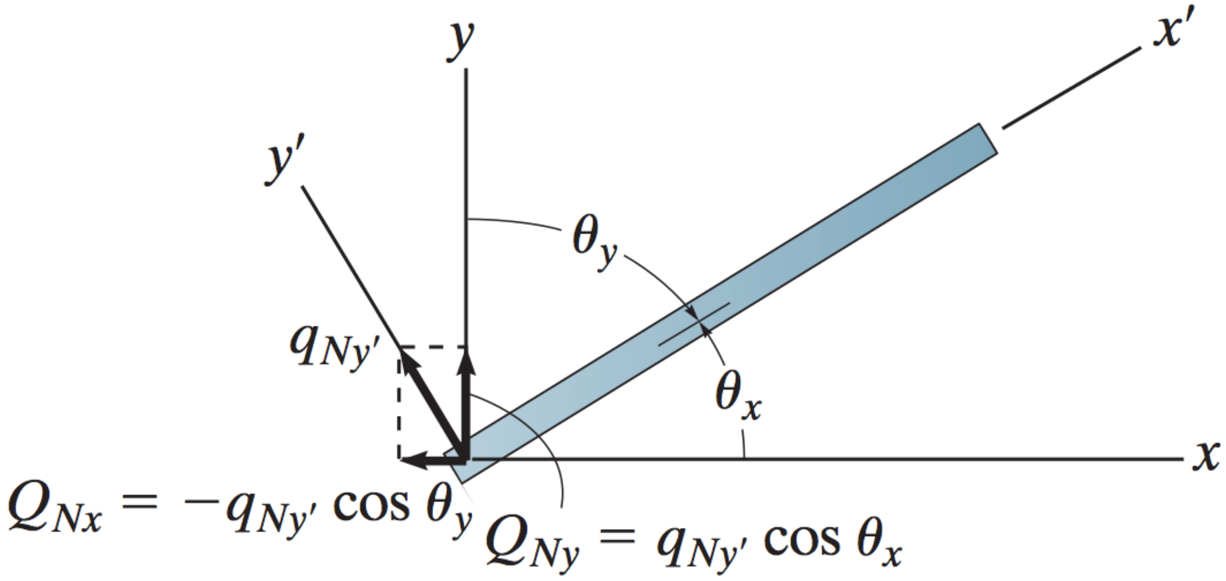
$$Q_{Nx} = -q_{Ny'} \cos \theta_y \qquad Q_{Ny} = q_{Ny'} \cos \theta_x$$

$$Q_{Nz} = q_{Nz'}$$

$$Q_{Fx} = q_{Fx'} \cos \theta_x \qquad Q_{Fy} = q_{Fx'} \cos \theta_y$$

$$Q_{Fx} = -q_{Fy'} \cos \theta_y \qquad Q_{Fy} = q_{Fy'} \cos \theta_x$$

$$Q_{Fz} = q_{Fz'}$$



$$\begin{bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Nz} \\ Q_{Fx} \\ Q_{Fy} \\ Q_{Fz} \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 & 0 & 0 & 0 \\ \lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & -\lambda_y & 0 \\ 0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix}$$

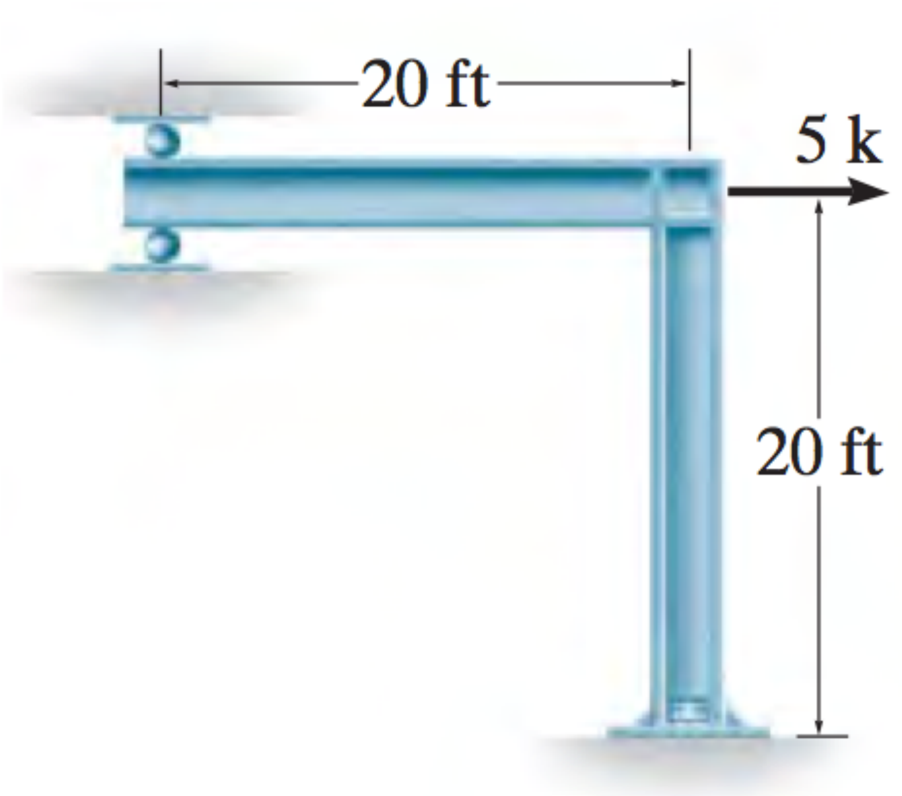
$$\mathbf{Q} = \mathbf{T}^T\mathbf{q}$$

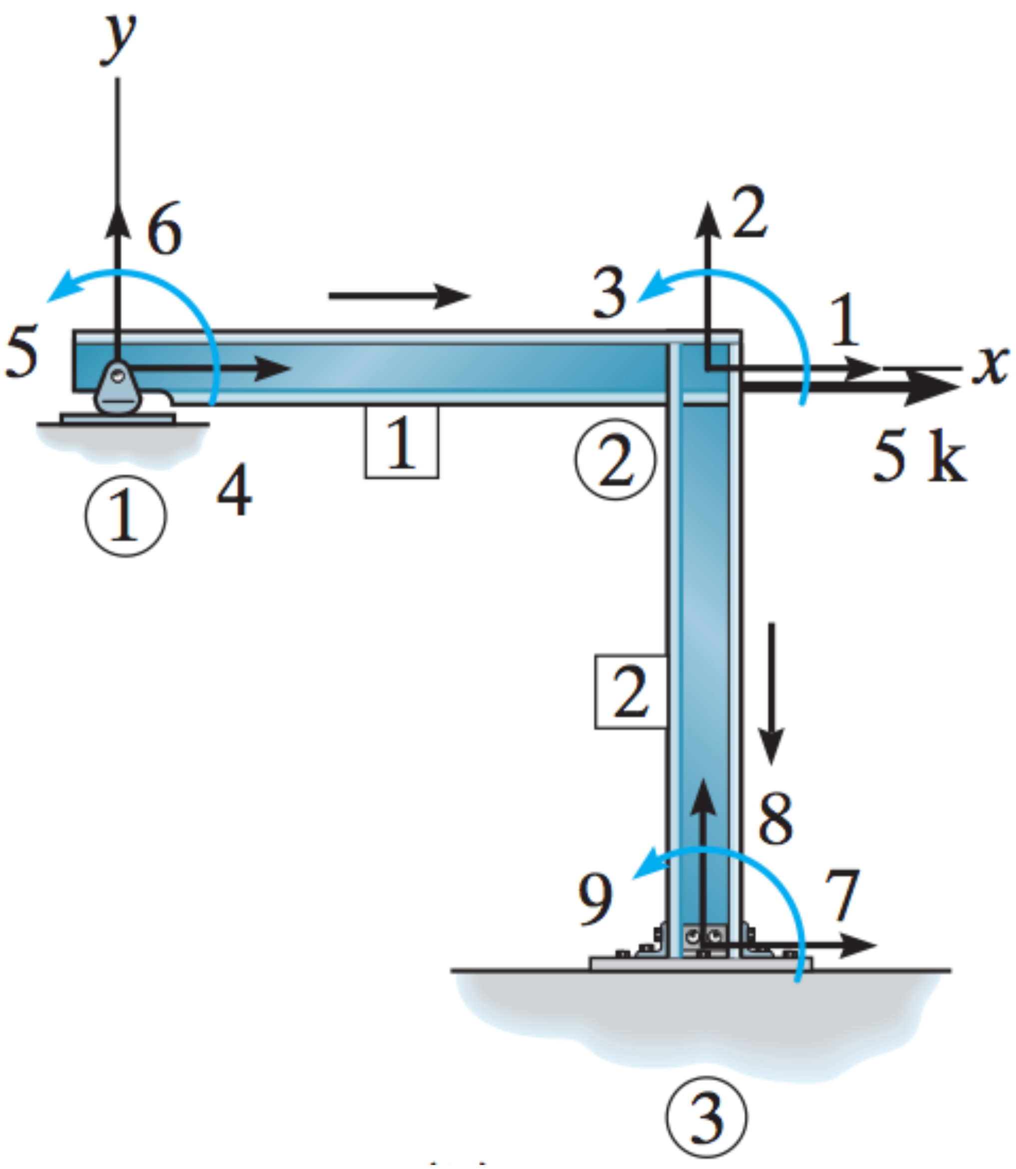
$$\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T}$$

$$\mathbf{Q} = \mathbf{k}\mathbf{D}$$

$$\mathbf{k} = \begin{array}{ccccccc}
N_x & N_y & N_z & F_x & F_y & F_z & \\
\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & N_x \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & N_y \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & N_z \\
-\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & F_x \\
-\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & F_y \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & F_z
\end{array}$$

Determine the loadings at the joints of the two-member frame shown in Fig. 16–4*a*. Take $I = 500 \text{ in}^4$, $A = 10 \text{ in}^2$, and $E = 29(10^3)$ ksi for both members.





$$\mathbf{D}_k = \begin{array}{c}\left[\begin{array}{c}0\\0\\0\\0\end{array}\right]\begin{array}{c}6\\7\\8\\9\end{array}\end{array} \qquad \mathbf{Q}_k = \begin{array}{c}\left[\begin{array}{c}5\\0\\0\\0\\0\end{array}\right]\begin{array}{c}1\\2\\3\\4\\5\end{array}\end{array}$$

$$\frac{AE}{L} = \frac{10[29(10^3)]}{20(12)} = 1208.3 \text{ k/in.}$$

$$\frac{12EI}{L^3} = \frac{12[29(10^3)(500)]}{[20(12)]^3} = 12.6 \text{ k/in.}$$

$$\frac{6EI}{L^2} = \frac{6[29(10^3)(500)]}{[20(12)]^2} = 1510.4 \text{ k}$$

$$\frac{4EI}{L} = \frac{4[29(10^3)(500)]}{20(12)} = 241.7(10^3) \text{ k} \cdot \text{in.}$$

$$\frac{2EI}{L} = \frac{2[29(10^3)(500)]}{20(12)} = 120.83(10^3) \text{ k} \cdot \text{in.}$$

Member 1:

$$\lambda_x = \frac{20 - 0}{20} = 1 \qquad \lambda_y = \frac{0 - 0}{20} = 0$$

$$\mathbf{k_1} = \begin{array}{c} \\ \left[\begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \\ 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.7(10^3) & 0 & -1510.4 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83(10^3) & 0 & -1510.4 & 241.7(10^3) \end{array} \right] \end{array} \begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}$$

Member 2:

$$\lambda_x = \frac{20 - 20}{20} = 0 \qquad \lambda_y = \frac{-20 - 0}{20} = -1$$

$$\mathbf{k_2} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccc} 12.6 & 0 & 1510.4 & -12.6 & 0 & 1510.4 \\ 0 & 1208.3 & 0 & 0 & -1208.3 & 0 \\ 1510.4 & 0 & 241.7(10^3) & -1510.4 & 0 & 120.83(10^3) \\ -12.6 & 0 & -1510.4 & 12.6 & 0 & -1510.4 \\ 0 & -1208.3 & 0 & 0 & 1208.3 & 0 \\ 1510.4 & 0 & 120.83(10^3) & -1510.4 & 0 & 241.7(10^3) \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}$$

$$\left[\begin{array}{c}5\\0\\0\\0\\0\\\hdashline Q_6\\Q_7\\Q_8\\Q_9\end{array}\right]=\begin{array}{c}\begin{array}{ccccc:cccc}1&2&3&4&5&6&7&8&9\end{array}\\\left[\begin{array}{ccccc:cccc}1220.9&0&1510.4&-1208.3&0&0&-12.6&0&1510.4\\0&1220.9&-1510.4&0&-1510.4&-12.6&0&-1208.3&0\\1510.4&-1510.4&483.3(10^3)&0&120.83(10^3)&1510.4&-1510.4&0&120.83(10^3)\\-1208.3&0&0&1208.3&0&0&0&0&0\\0&-1510.4&120.83(10^3)&0&241.7(10^3)&1510.4&0&0&0\\\hdashline 0&-12.6&1510.4&0&1510.4&12.6&0&0&0\\-12.6&0&-1510.4&0&0&0&12.6&0&-1510.4\\0&-1208.3&0&0&0&0&0&1208.3&0\\1510.4&0&120.83(10^3)&0&0&0&-1510.4&0&241.7(10^3)\end{array}\right]\end{array}\left[\begin{array}{c}D_1\\D_2\\D_3\\D_4\\D_5\\\hdashline 0\\0\\0\\0\end{array}\right]$$

Displacements and Loads. Expanding to determine the displacements yields

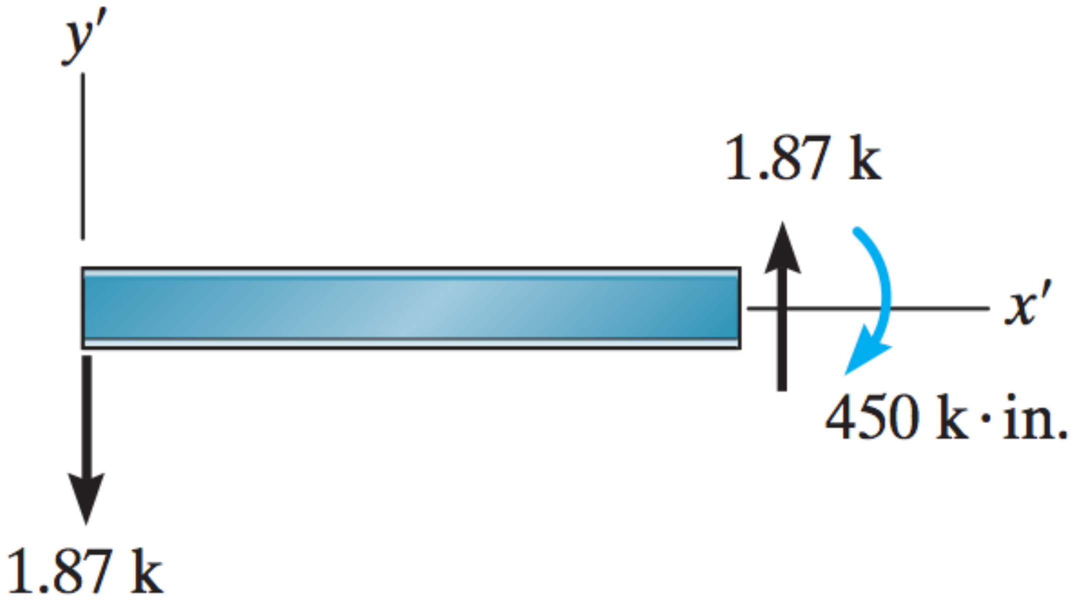
$$\begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1220.9&0&1510.4&-1208.3&0\\0&1220.9&-1510.4&0&-1510.4\\1510.4&-1510.4&483.3(10^3)&0&120.83(10^3)\\-1208.3&0&0&1208.3&0\\0&-1510.4&120.83(10^3)&0&241.7(10^3)\end{bmatrix}\begin{bmatrix}D_1\\D_2\\D_3\\D_4\\D_5\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = \begin{bmatrix} 0.696 \text{ in.} \\ -1.55(10^{-3}) \text{ in.} \\ -2.488(10^{-3}) \text{ rad} \\ 0.696 \text{ in.} \\ 1.234(10^{-3}) \text{ rad} \end{bmatrix}$$

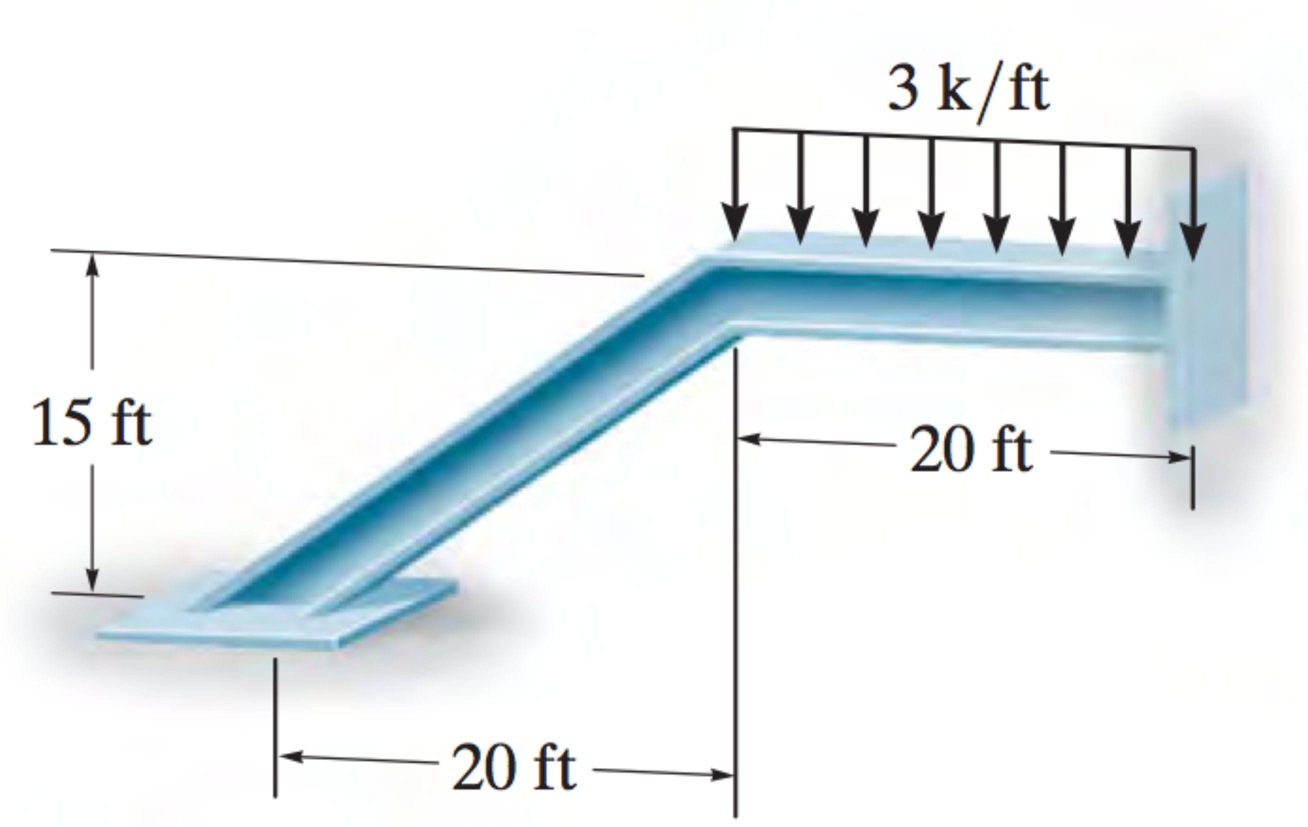
$$\begin{bmatrix} Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{bmatrix} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & -12.6 & 1510.4 & 0 & 1510.4 \\ -12.6 & 0 & -1510.4 & 0 & 0 \\ 0 & -1208.3 & 0 & 0 & 0 \\ 1510.4 & 0 & 120.83(10^3) & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} 0.696 \\ -1.55(10^{-3}) \\ -2.488(10^{-3}) \\ 0.696 \\ 1.234(10^{-3}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.87\text{ k} \\ -5.00\text{ k} \\ 1.87\text{ k} \\ 750\text{ k}\cdot\text{in.} \end{bmatrix}$$

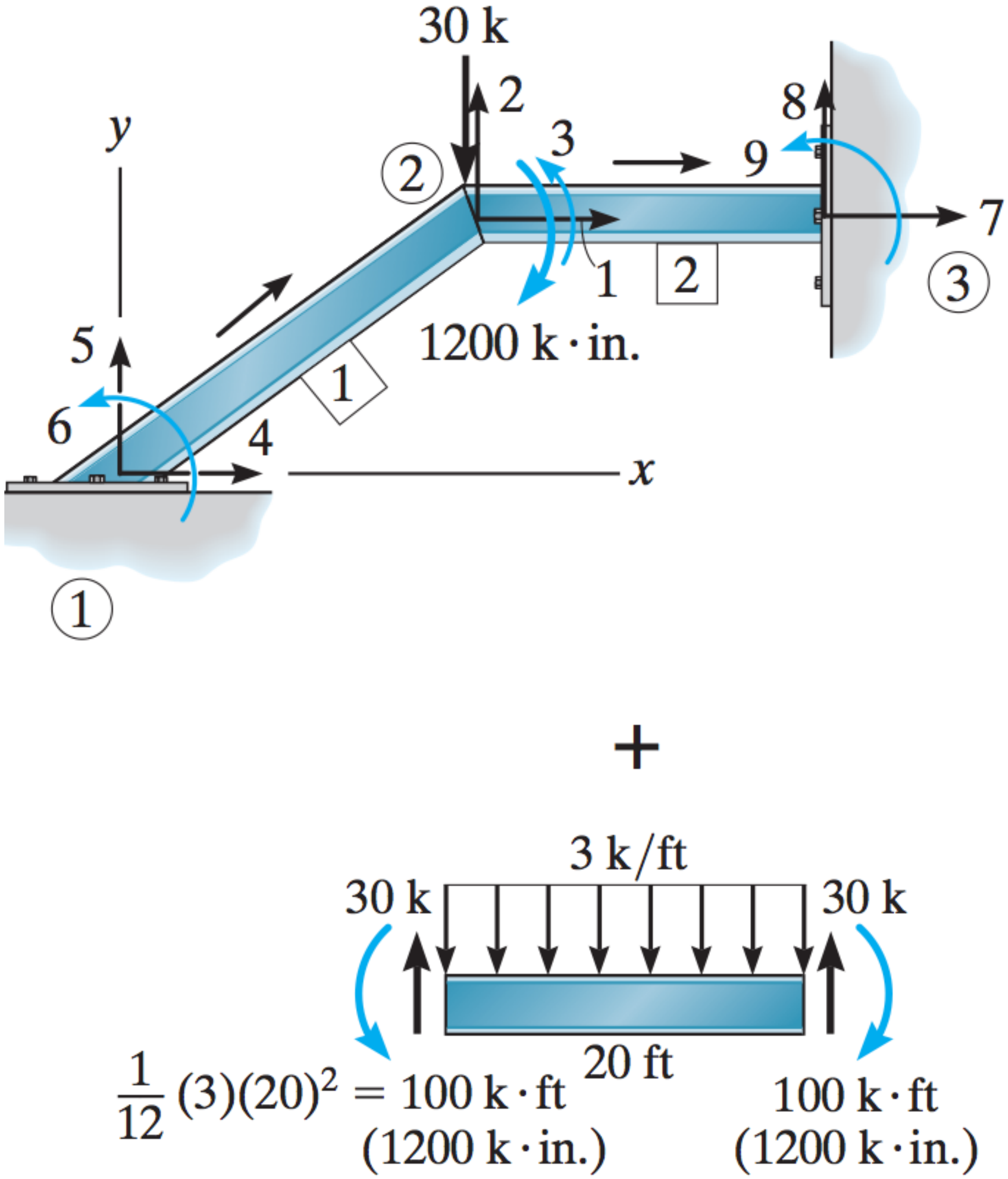
$$\mathbf{q}_1 = \mathbf{k}_1\mathbf{T}_1\mathbf{D} = \begin{array}{c} \begin{matrix} 4 & 6 & 5 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.7(10^3) & 0 & -1510.4 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83(10^3) & 0 & -1510.4 & 241.7(10^3) \end{bmatrix} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{cc} \begin{bmatrix} 0.696 \\ 0 \\ 1.234(10^{-3}) \\ 0.696 \\ -1.55(10^{-3}) \\ -2.488(10^{-3}) \end{bmatrix} & \begin{matrix} 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

$$\begin{bmatrix} q_4 \\ q_6 \\ q_5 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.87\text{ k} \\ 0 \\ 0 \\ 1.87\text{ k} \\ -450\text{ k}\cdot\text{in.} \end{bmatrix}$$



Determine the loadings at the ends of each member of the frame shown in Fig. 16–5*a*. Take $I = 600\ \text{in}^4$, $A = 12\ \text{in}^2$, and $E = 29(10^3)$ ksi for each member.





Structure Stiffness Matrix

Member 1:

$$\frac{AE}{L} = \frac{12[29(10^3)]}{25(12)} = 1160 \text{ k/in.}$$

$$\frac{12EI}{L^3} = \frac{12[29(10^3)]600}{[25(12)]^3} = 7.73 \text{ k/in.}$$

$$\frac{6EI}{L^2} = \frac{6[29(10^3)]600}{[25(12)]^2} = 1160 \text{ k}$$

$$\frac{4EI}{L} = \frac{4[29(10^3)]600}{25(12)} = 232(10^3) \text{ k} \cdot \text{in.}$$

$$\frac{2EI}{L} = \frac{2[29(10^3)]600}{25(12)} = 116(10^3) \text{ k} \cdot \text{in.}$$

$$\lambda_x = \frac{20 - 0}{25} = 0.8 \qquad \lambda_y = \frac{15 - 0}{25} = 0.6$$

Member 2:

$$\frac{AE}{L} = \frac{12[29(10^3)]}{20(12)} = 1450 \text{ k/in.}$$

$$\frac{12EI}{L^3} = \frac{12[29(10^3)]600}{[20(12)]^3} = 15.10 \text{ k/in.}$$

$$\frac{6EI}{L^2} = \frac{6[29(10^3)]600}{[20(12)]^2} = 1812.50 \text{ k}$$

$$\frac{4EI}{L} = \frac{4[29(10^3)]600}{20(12)} = 2.90(10^5) \text{ k} \cdot \text{in.}$$

$$\frac{2EI}{L} = \frac{2[29(10^3)]600}{[20(12)]} = 1.45(10^5) \text{ k} \cdot \text{in.}$$

$$\lambda_x = \frac{40 - 20}{20} = 1 \qquad \lambda_y = \frac{15 - 15}{20} = 0$$

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccccc} 745.18 & 553.09 & -696 & -745.18 & -553.09 & -696 \\ 553.09 & 422.55 & 928 & -553.09 & -422.55 & 928 \\ -696 & 928 & 232(10^3) & 696 & -928 & 116(10^3) \\ -745.18 & -553.09 & 696 & 745.18 & 553.09 & 696 \\ -553.09 & -422.55 & -928 & 553.09 & 422.55 & -928 \\ -696 & 928 & 116(10^3) & 696 & -928 & 232(10^3) \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccc} 1450 & 0 & 0 & -1450 & 0 & 0 \\ 0 & 15.10 & 1812.50 & 0 & -15.10 & 1812.50 \\ 0 & 1812.50 & 290(10^3) & 0 & -1812.50 & 145(10^3) \\ -1450 & 0 & 0 & 1450 & 0 & 0 \\ 0 & -15.10 & -1812.50 & 0 & 15.10 & -1812.50 \\ 0 & 1812.50 & 145(10^3) & 0 & -1812.50 & 290(10^3) \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \end{array}$$

$$\left[\begin{array}{c} 0 \\ -30 \\ -1200 \\ \hline Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{array}\right] = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccc|cccccc} 2195.18 & 553.09 & 696 & -745.18 & -553.09 & 696 & -1450 & 0 & 0 \\ 553.09 & 437.65 & 884.5 & -553.09 & -422.55 & -928 & 0 & -15.10 & 1812.50 \\ 696 & 884.5 & 522(10^3) & -696 & 928 & 116(10^3) & 0 & -1812.50 & 145(10^3) \\ \hline -745.18 & -553.09 & -696 & 745.18 & 553.09 & -696 & 0 & 0 & 0 \\ -553.09 & -422.55 & 928 & 553.09 & 422.55 & 928 & 0 & 0 & 0 \\ 696 & -928 & 116(10^3) & -696 & 928 & 232(10^3) & 0 & 0 & 0 \\ -1450 & 0 & 0 & 0 & 0 & 0 & 1450 & 0 & 0 \\ 0 & -15.10 & -1812.50 & 0 & 0 & 0 & 0 & 15.10 & -1812.50 \\ 0 & 1812.50 & 145(10^3) & 0 & 0 & 0 & 0 & -1812.50 & 290(10^3) \end{array}\right] \end{array} \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Displacements and Loads. Expanding to determine the displacements, and solving, we have

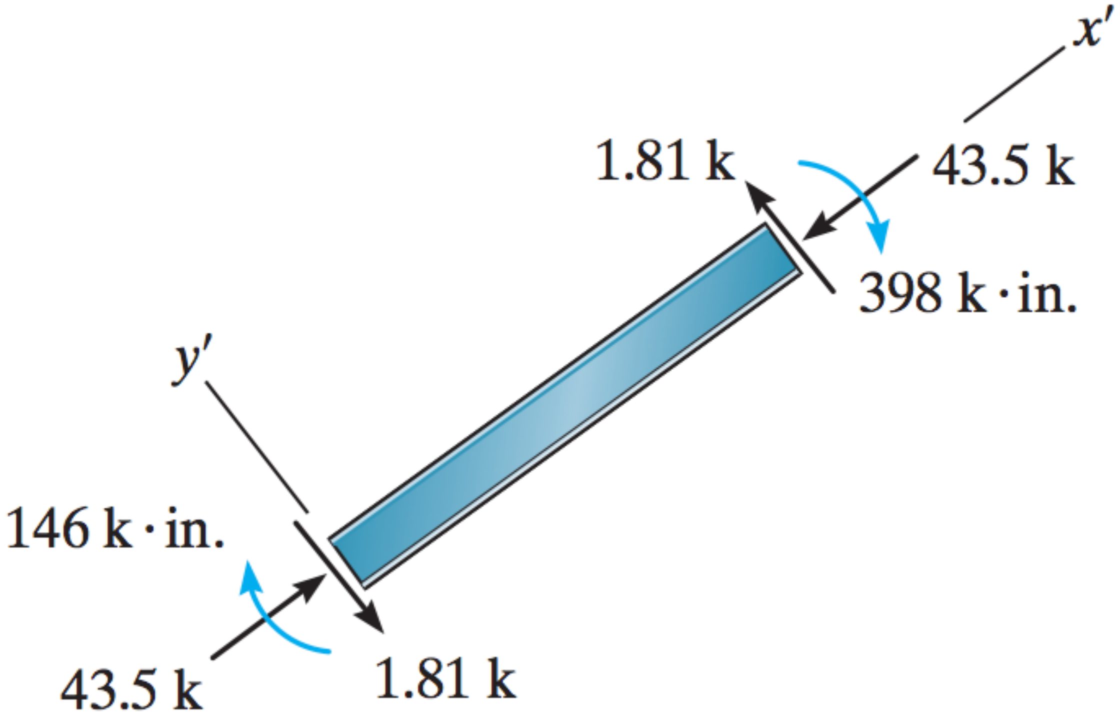
$$\begin{bmatrix} 0 \\ -30 \\ -1200 \end{bmatrix} = \begin{bmatrix} 2195.18 & 553.09 & 696 \\ 553.09 & 437.65 & 884.5 \\ 696 & 884.5 & 522(10^3) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

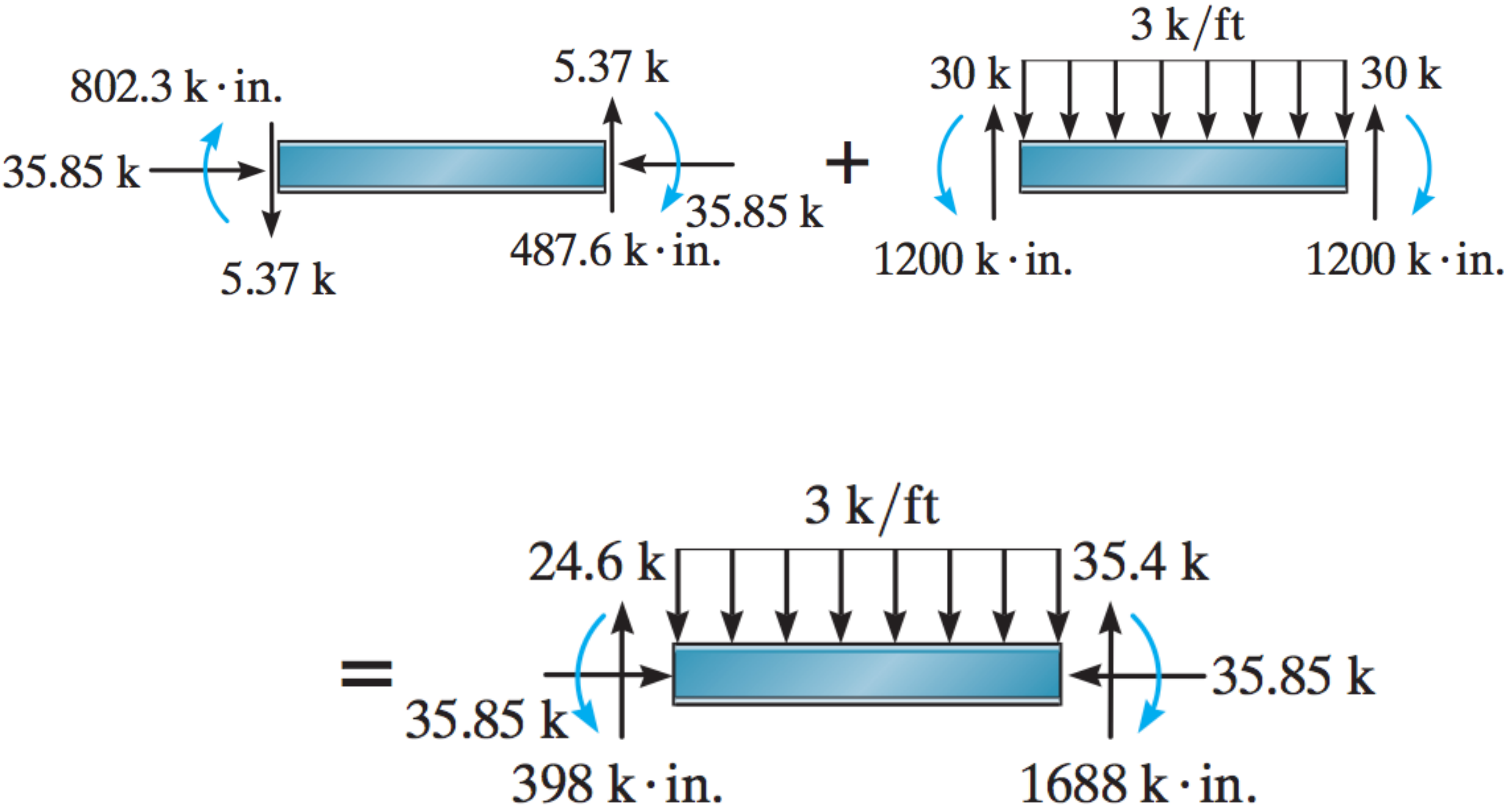
$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0.0247 \text{ in.} \\ -0.0954 \text{ in.} \\ -0.00217 \text{ rad} \end{bmatrix}$$

$$\begin{bmatrix} Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{bmatrix} = \begin{bmatrix} -745.18 & -553.09 & -696 \\ -553.09 & -422.55 & 928 \\ 696 & -928 & 116(10^3) \\ -1450 & 0 & 0 \\ 0 & -15.10 & -1812.50 \\ 0 & 1812.50 & 145(10^3) \end{bmatrix} \begin{bmatrix} 0.0247 \\ -0.0954 \\ -0.00217 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 35.85 \text{ k} \\ 24.63 \text{ k} \\ -145.99 \text{ k} \cdot \text{in.} \\ -35.85 \text{ k} \\ 5.37 \text{ k} \\ -487.60 \text{ k} \cdot \text{in.} \end{bmatrix}$$

$$
\begin{bmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} =
\begin{array}{c}
\begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
1160 & 0 & 0 & -1160 & 0 & 0 \\
0 & 7.73 & 1160 & 0 & -7.73 & 1160 \\
0 & 1160 & 232(10^3) & 0 & -1160 & 116(10^3) \\
-1160 & 0 & 0 & 1160 & 0 & 0 \\
0 & -7.73 & -1160 & 0 & 7.73 & -1160 \\
0 & 1160 & 116(10^3) & 0 & -1160 & 232(10^3)
\end{bmatrix}
\end{array}
\begin{bmatrix}
0.8 & 0.6 & 0 & 0 & 0 & 0 \\
-0.6 & 0.8 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.8 & 0.6 & 0 \\
0 & 0 & 0 & -0.6 & 0.8 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0247 \\ -0.0954 \\ -0.00217 \end{bmatrix}
\begin{array}{c} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array}
$$

$$
\begin{bmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} =
\begin{bmatrix} 43.5 \text{ k} \\ -1.81 \text{ k} \\ -146 \text{ k} \cdot \text{in.} \\ -43.5 \text{ k} \\ 1.81 \text{ k} \\ -398 \text{ k} \cdot \text{in.} \end{bmatrix}
$$




802.3 k · in.
5.37 k
35.85 k
35.85 k
5.37 k
487.6 k · in.
+
3 k/ft
30 k
30 k
1200 k · in.
1200 k · in.
=
3 k/ft
24.6 k
35.4 k
35.85 k
35.85 k
398 k · in.
1688 k · in.

Determine the support reactions at pins ① and ③. Take $E = 200$ GPa, $I = 350(10^6)$ mm^4, $A = 15(10^3)$ mm^2 for each member.

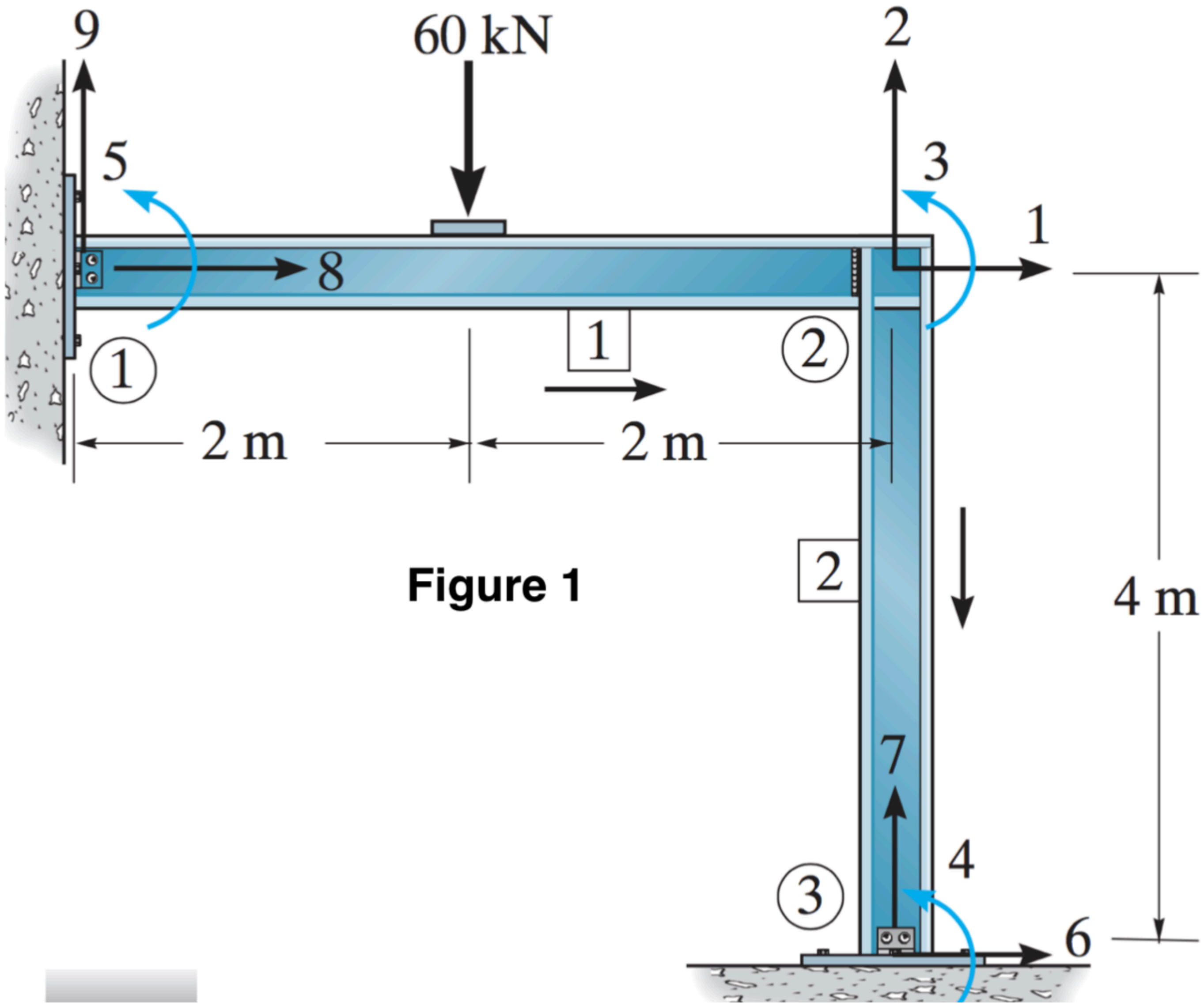


Figure 1

Member Stiffness Matrices. The origin of the global coordinate system will be set at joint ①. For member $\boxed{1}$ and $\boxed{2}$, $L = 4\text{m}$.

$$\frac{AE}{L} = \frac{0.015[200(10^9)]}{4} = 750(10^6)\ \text{N/m}$$

$$\frac{12EI}{L^3} = \frac{12[200(10^9)][350(10^{-6})]}{4^3} = 13.125(10^6)\ \text{N/m}$$

$$\frac{6EI}{L^2} = \frac{4[200(10^9)][350(10^{-6})]}{4^2} = 26.25(10^6)\ \text{N}$$

$$\frac{4EI}{L} = \frac{4[200(10^9)][350(10^{-6})]}{4} = 70(10^6)\ \text{N}\cdot\text{m}$$

$$\frac{2EI}{L} = \frac{2[200(10^9)][350(10^{-6})]}{4} = 35(10^6)\ \text{N}\cdot\text{m}$$

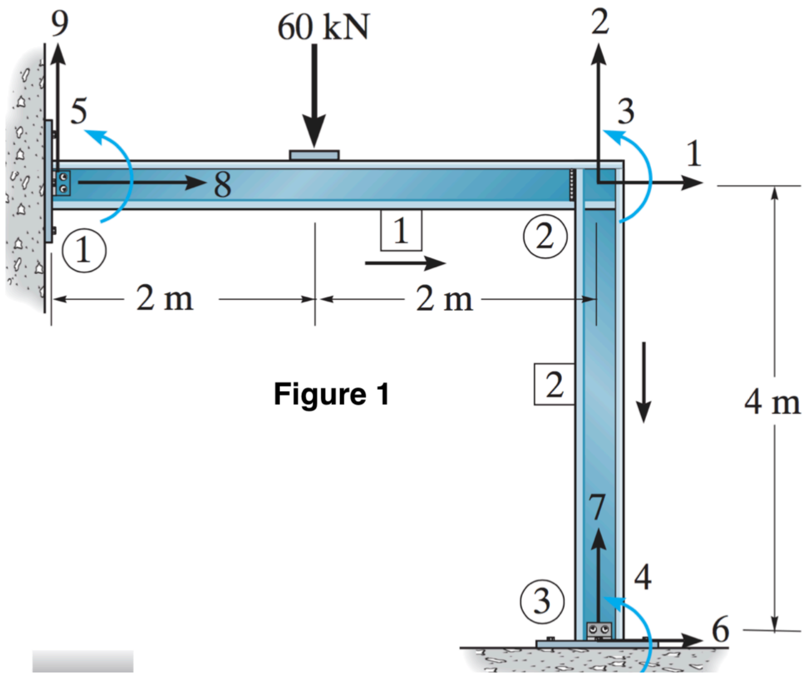


Figure 1

$$
\mathbf{k} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
\left[\begin{array}{cccccc|c}
N_x & N_y & N_z & F_x & F_y & F_z & \\
\left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & N_x \\
\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & N_y \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & N_z \\
-\left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & F_x \\
-\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & F_y \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & F_z
\end{array}\right]
$$

For member $\boxed{1}$, $\lambda_x = \dfrac{4-0}{4} = 1$ and $\lambda_y = \dfrac{0-0}{4} = 0$. Thus,

$$
\mathbf{k}_1 = \left[\begin{array}{cccccc|c}
8 & 9 & 5 & 1 & 2 & 3 & \\
750 & 0 & 0 & -750 & 0 & 0 & 8 \\
0 & 13.125 & 26.25 & 0 & -13.125 & 26.25 & 9 \\
0 & 26.25 & 70 & 0 & -26.25 & 35 & 5 \\
-750 & 0 & 0 & 750 & 0 & 0 & 1 \\
0 & -13.125 & -26.25 & 0 & 13.125 & -26.25 & 2 \\
0 & 26.25 & 35 & 0 & -26.25 & 70 & 3
\end{array}\right](10^6)
$$

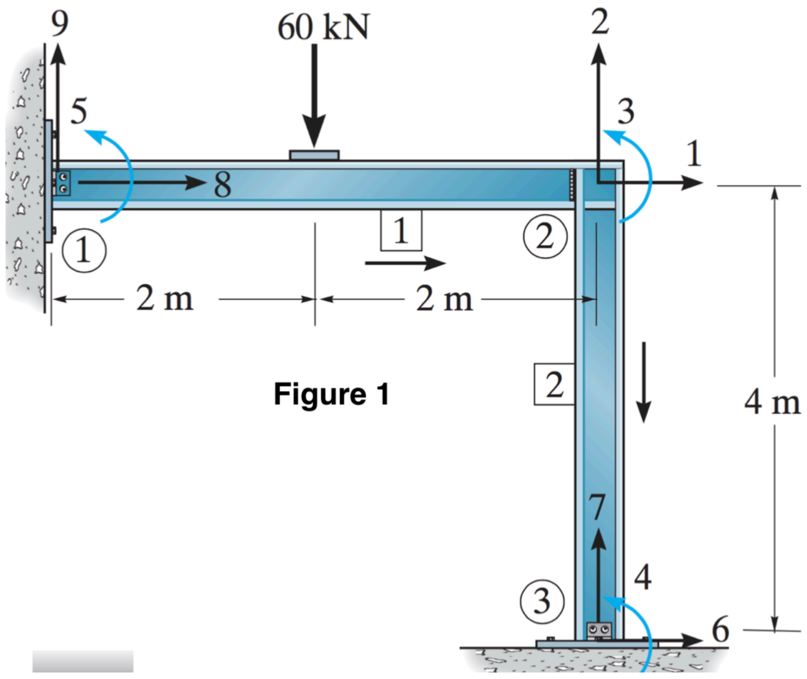


Figure 1

$$\mathbf{k} = \begin{array}{cccccc|c}
N_x & N_y & N_z & F_x & F_y & F_z & \\
\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & N_x \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & N_y \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & N_z \\
-\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & F_x \\
-\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & F_y \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & F_z
\end{array}$$

For member $\boxed{2}$, $\lambda_x = \dfrac{4 - 4}{4} = 0$, and $\lambda_y = \dfrac{-4 - 0}{4} = -1$. Thus,

$$\mathbf{k}_2 = \begin{array}{cccccc|c}
1 & 2 & 3 & 6 & 7 & 4 & \\
13.125 & 0 & 26.25 & -13.125 & 0 & 26.25 & 1 \\
0 & 750 & 0 & 0 & -750 & 0 & 2 \\
26.25 & 0 & 70 & -26.25 & 0 & 35 & 3 \\
-13.125 & 0 & -26.25 & 13.125 & 0 & -26.25 & 6 \\
0 & -750 & 0 & 0 & 750 & 0 & 7 \\
26.25 & 0 & 35 & -26.25 & 0 & 70 & 4
\end{array} (10^6)$$

Structure Stiffness Matrix. It is a 9 × 9 matrix since the highest code number is 9. Thus

$$\mathbf{K} = \begin{array}{cc} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} & \\ \left[\begin{array}{ccccccccc} 763.125 & 0 & 26.25 & 26.25 & 0 & -13.125 & 0 & -750 & 0 \\ 0 & 763.125 & -26.25 & 0 & -26.25 & 0 & -750 & 0 & -13.125 \\ 26.25 & -26.25 & 140 & 35 & 35 & -26.25 & 0 & 0 & 26.25 \\ 26.25 & 0 & 35 & 70 & 0 & -26.25 & 0 & 0 & 0 \\ 0 & -26.25 & 35 & 0 & 70 & 0 & 0 & 0 & 26.25 \\ -13.125 & 0 & -26.25 & -26.25 & 0 & 13.125 & 0 & 0 & 0 \\ 0 & -750 & 0 & 0 & 0 & 0 & 750 & 0 & 0 \\ -750 & 0 & 0 & 0 & 0 & 0 & 0 & 750 & 0 \\ 0 & -13.125 & 26.25 & 0 & 26.25 & 0 & 0 & 0 & 13.125 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \end{array} (10^6)$$

Known Nodal Loads and Deflections. The nodal load acting on the unconstrained degree of freedom (code numbers 1, 2, 3, 4, and 5) are shown in Fig. *a* and Fig. *b*.

$$\mathbf{Q}_k = \begin{bmatrix} 0 \\ -41.25(10^3) \\ 45(10^3) \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \quad \text{and} \quad \mathbf{D}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 6 \\ 7 \\ 8 \\ 9 \end{matrix}$$

Loads-Displacement Relation. Applying $\mathbf{Q} = \mathbf{K}\mathbf{D}$,

$$\left[\begin{array}{c} 0 \\ -41.25(10^3) \\ 45(10^3) \\ 0 \\ 0 \\ \hline Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{array}\right] = \left[\begin{array}{ccccc|cccc} 763.125 & 0 & 26.25 & 26.25 & 0 & -13.125 & 0 & -750 & 0 \\ 0 & 763.125 & -26.25 & 0 & -26.25 & 0 & -750 & 0 & -13.125 \\ 26.25 & -26.25 & 140 & 35 & 35 & -26.25 & 0 & 0 & 26.25 \\ 26.25 & 0 & 35 & 70 & 0 & -26.25 & 0 & 0 & 0 \\ 0 & -26.25 & 35 & 0 & 70 & 0 & 0 & 0 & 26.25 \\ \hline -13.125 & 0 & -26.25 & -26.25 & 0 & 13.125 & 0 & 0 & 0 \\ 0 & -750 & 0 & 0 & 0 & 0 & 750 & 0 & 0 \\ -750 & 0 & 0 & 0 & 0 & 0 & 0 & 750 & 0 \\ 0 & -13.125 & 26.25 & 0 & 26.25 & 0 & 0 & 0 & 13.125 \end{array}\right](10^6) \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

From the matrix partition, $\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k$,

$$0 = (763.125D_1 + 26.25D_3 + 26.25D_4)(10^6) \tag{1}$$

$$-41.25(10^3) = (763.125D_2 - 26.25D_3 - 26.25D_5)(10^6) \tag{2}$$

$$45(10^3) = (26.25D_1 - 26.25D_2 + 140D_3 + 35D_4 + 35D_5)(10^6) \tag{3}$$

$$0 = (26.25D_1 + 35D_3 + 70D_4)(10^6) \tag{4}$$

$$0 = (-26.25D_2 + 35D_3 + 70D_5)(10^6) \tag{5}$$

Solving Eqs. (1) to (5)

$$D_1 = -7.3802(10^{-6}) \qquad D_2 = -47.3802(10^{-6}) \qquad D_3 = 423.5714(10^{-6})$$

$$D_4 = -209.0181(10^{-6}) \qquad D_5 = -229.5533(10^{-6})$$

Using these results and applying $\mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k$,

$$Q_6 = (-13.125)(10^6) - 7.3802(10^{-6}) - 26.25(10^6)423.5714(10^{-6}) - 26.25(10^6) - 209.0181(10^{-6}) + 0 = -5.535 \text{ kN}$$

$$Q_7 = -750(10^6) - 47.3802(10^{-6}) + 0 = 35.535 \text{ kN}$$

$$Q_8 = -750(10^6) - 7.3802(10^{-6}) + 0 = 5.535 \text{ kN}$$

$$Q_9 = -13.125(10^6) - 47.3802(10^{-6}) + 26.25(10^6) + 423.5714(10^{-6}) + 26.25(10^6) - 229.5533(10^{-6}) + 0 = 5.715 \text{ kN}$$

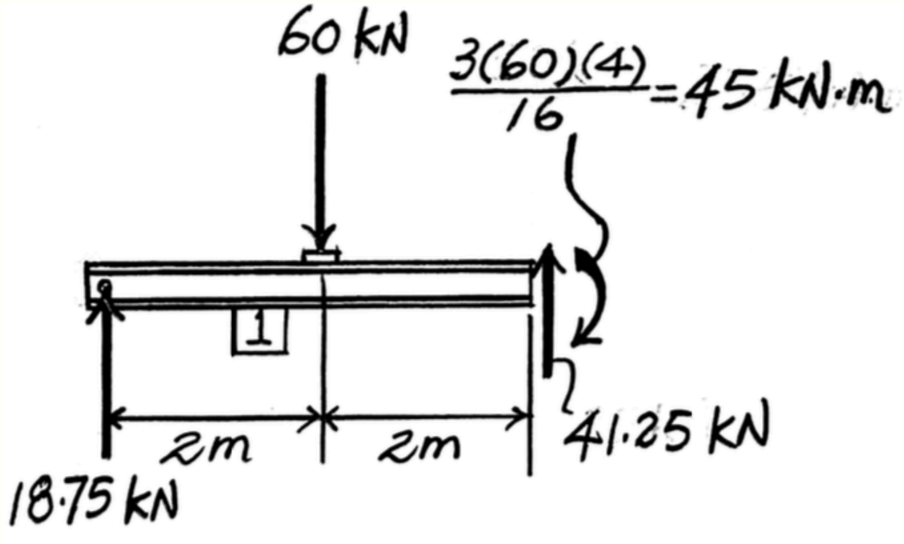
Superposition these results to those of FEM shown in Fig. *a*,

$R_6 = -5.535 \text{ kN} + 0 = 5.54 \text{ kN}$ **Ans.**

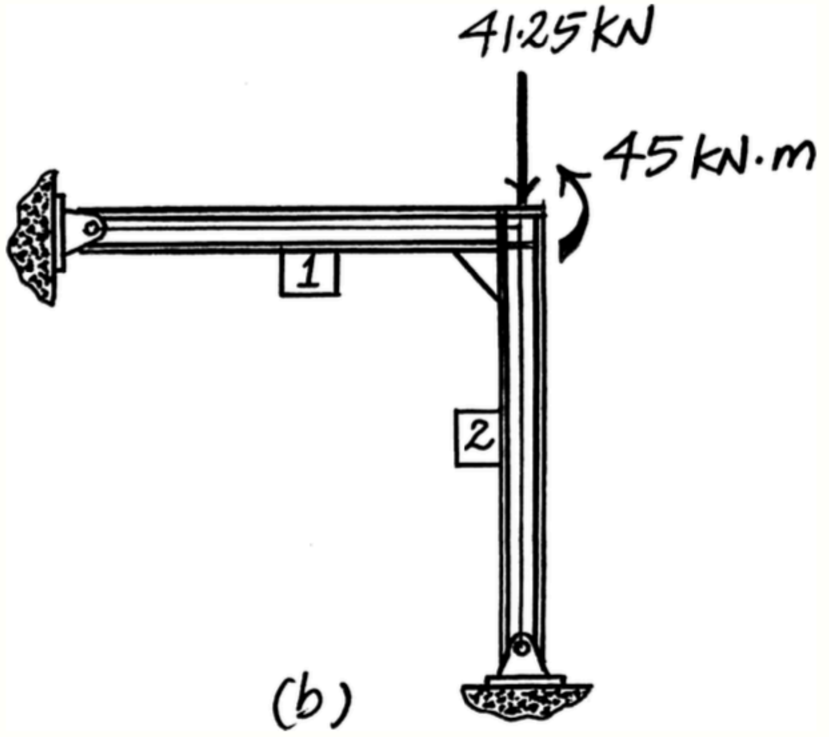
$R_7 = 35.535 + 0 = 35.5 \text{ kN}$ **Ans.**

$R_8 = 5.535 + 0 = 5.54 \text{ kN}$ **Ans.**

$R_9 = 5.715 + 18.75 = 24.5 \text{ kN}$ **Ans.**



(a)



(b)