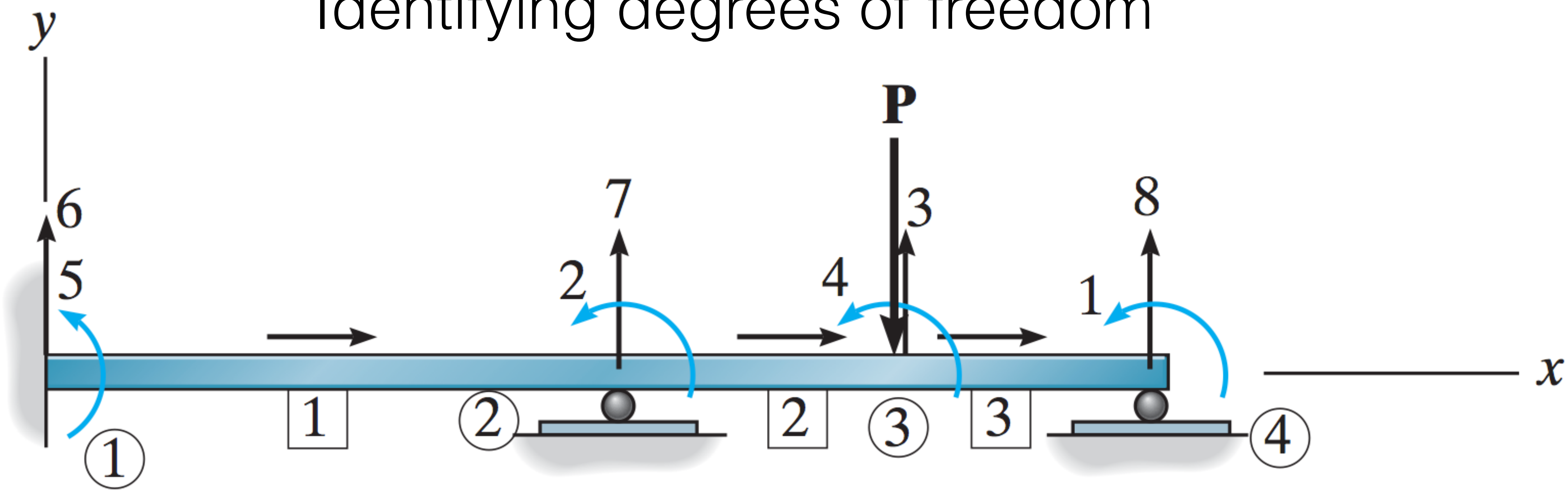
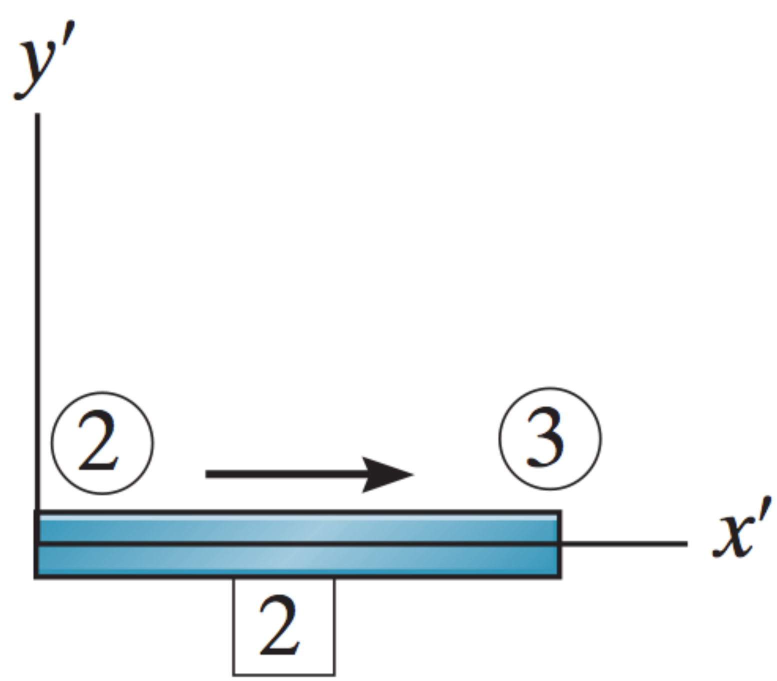
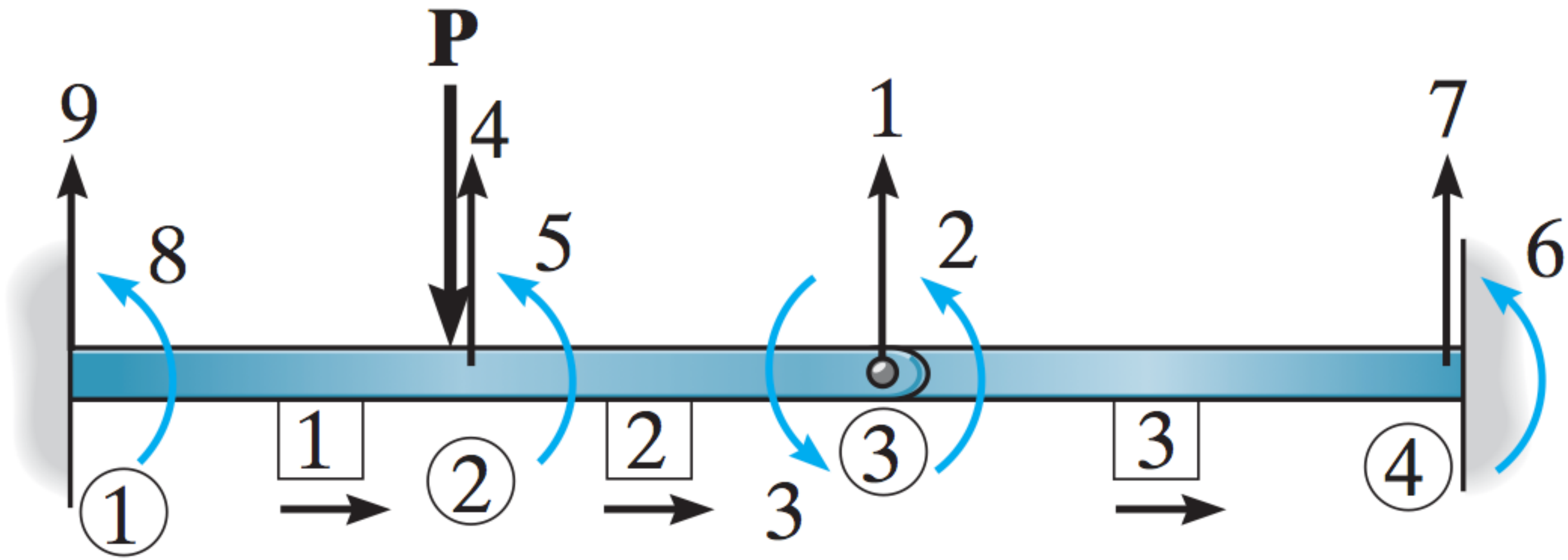
Stiffness matrix approach for beams

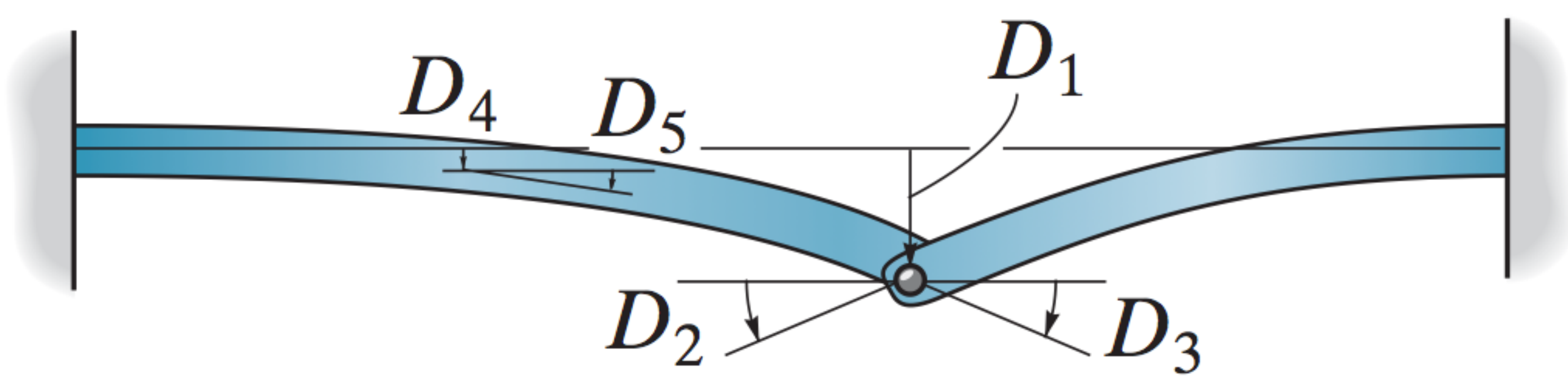
Identifying degrees of freedom



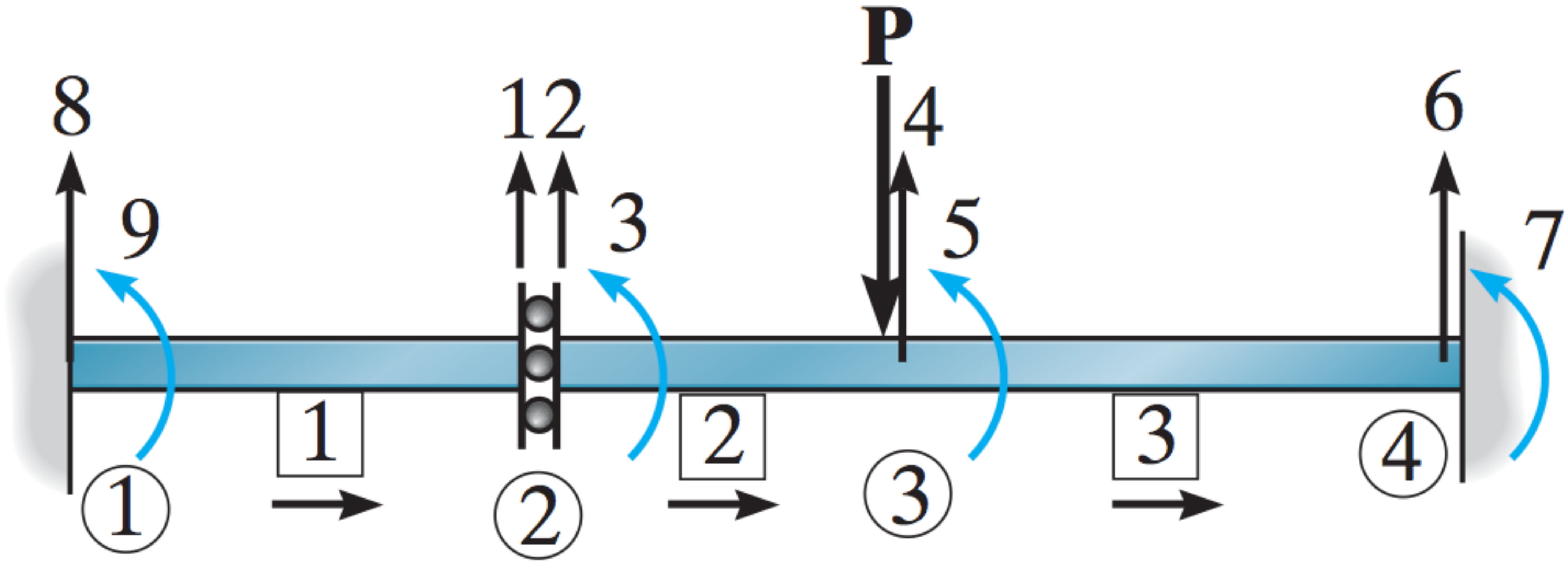




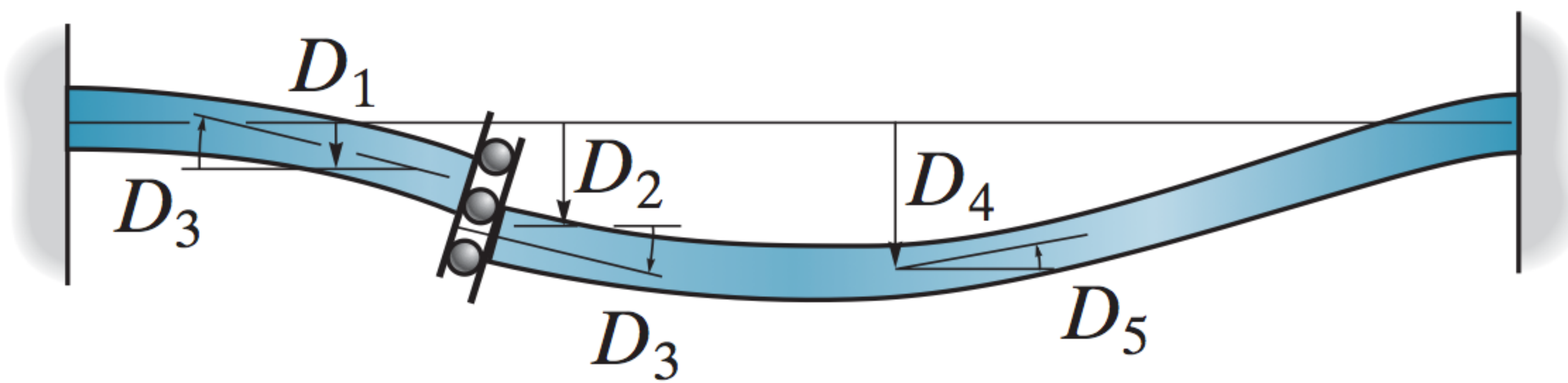
(a)



(b)

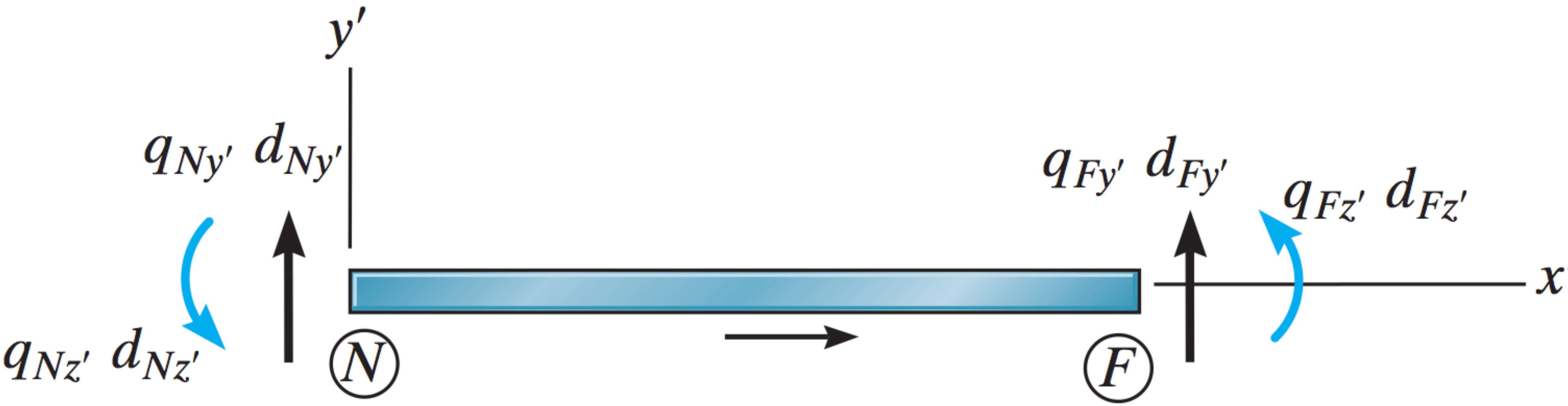


(a)



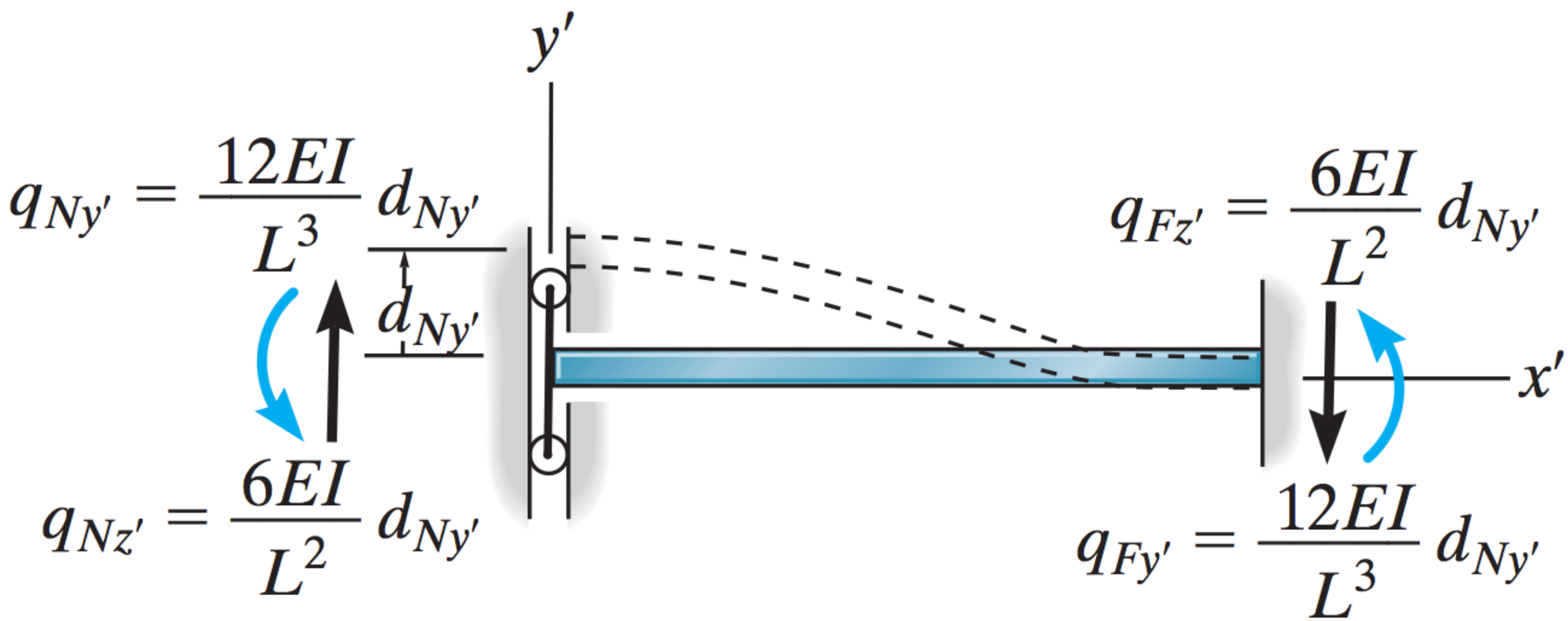
(b)

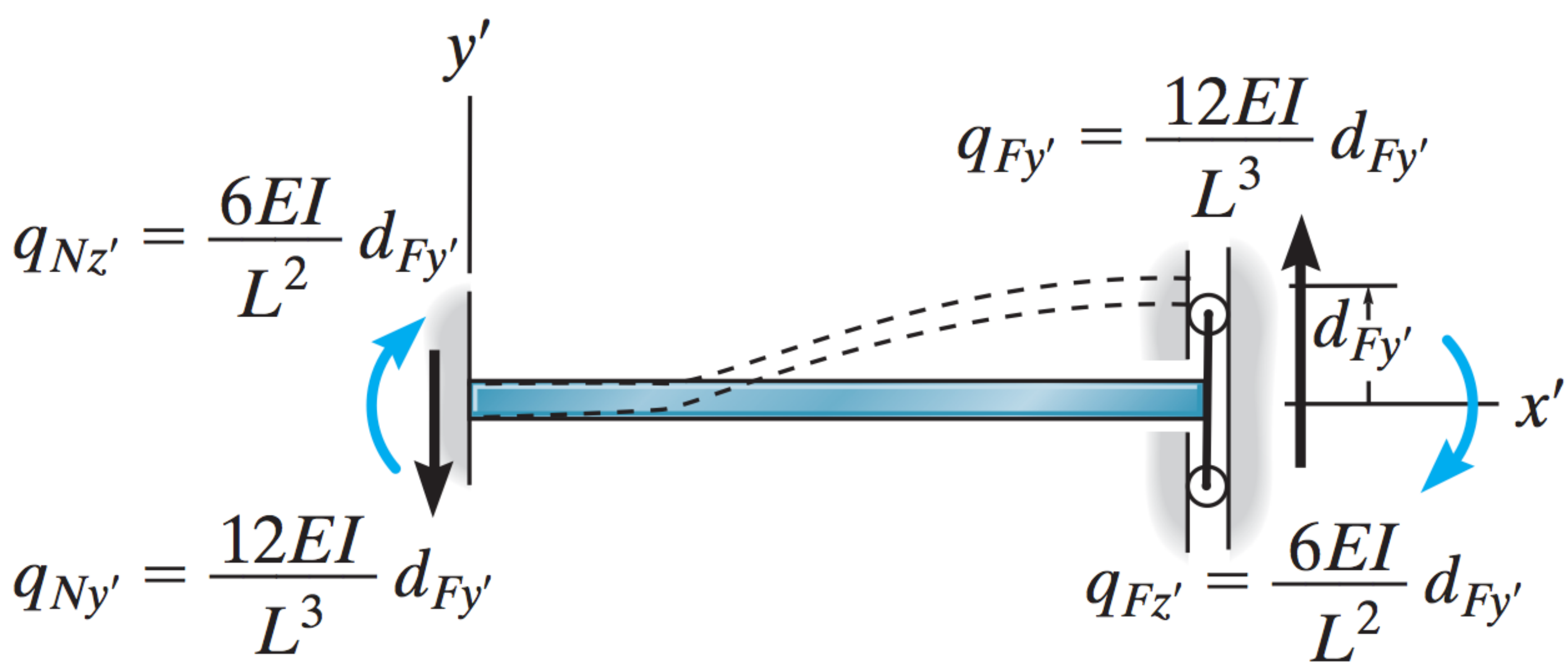
Sign convention for developing stiffness matrix



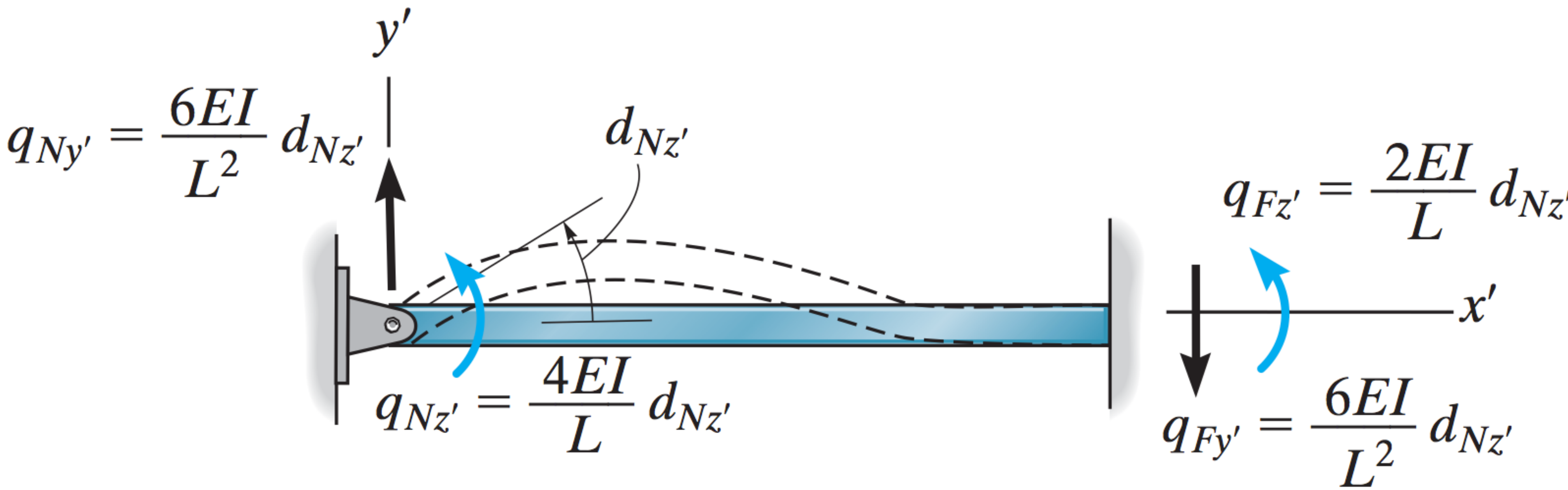
positive sign convention

y’ displacements

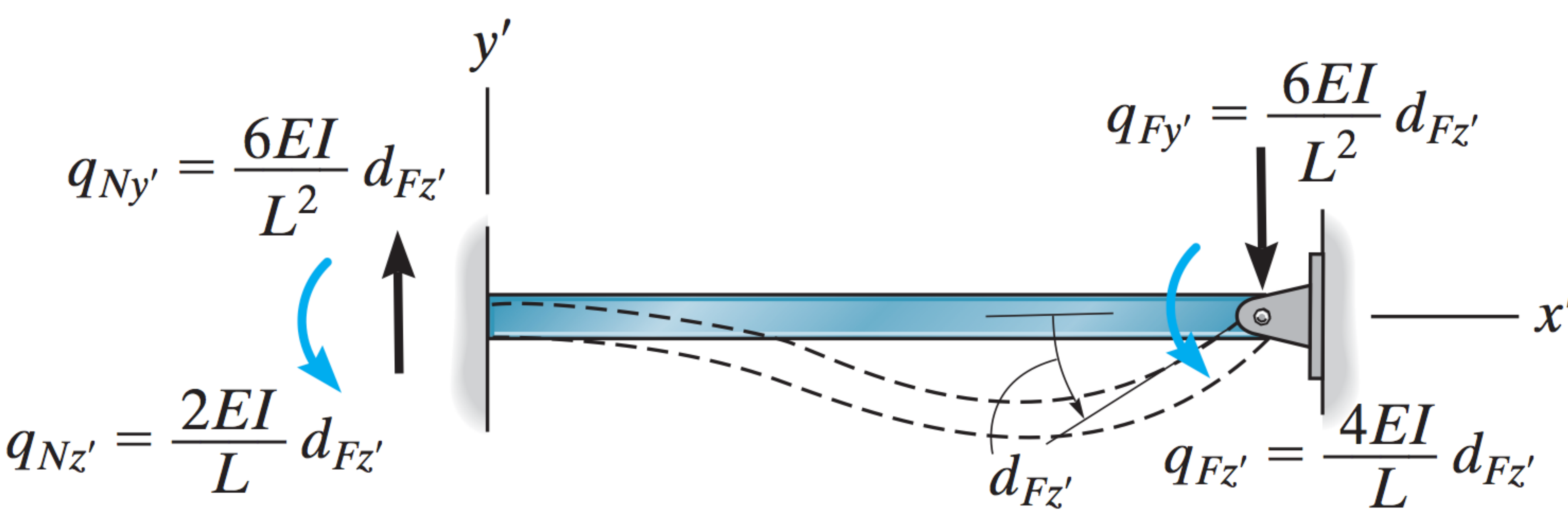




z' rotations



(a)



$$
\begin{bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} \qquad N_{y'} & \qquad\quad N_{z'} & \qquad\quad F_{y'} & \qquad\quad F_{z'} \end{array} \\
\begin{bmatrix}
\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} & -\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} \\[2ex]
\dfrac{6EI}{L^2} & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{2EI}{L} \\[2ex]
-\dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\[2ex]
\dfrac{6EI}{L^2} & \dfrac{2EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{4EI}{L}
\end{bmatrix}
\end{array}
\begin{bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}
$$

$$\mathbf{q} = \mathbf{k}\mathbf{d}$$

$$\mathbf{q} = \mathbf{k}\mathbf{d}$$

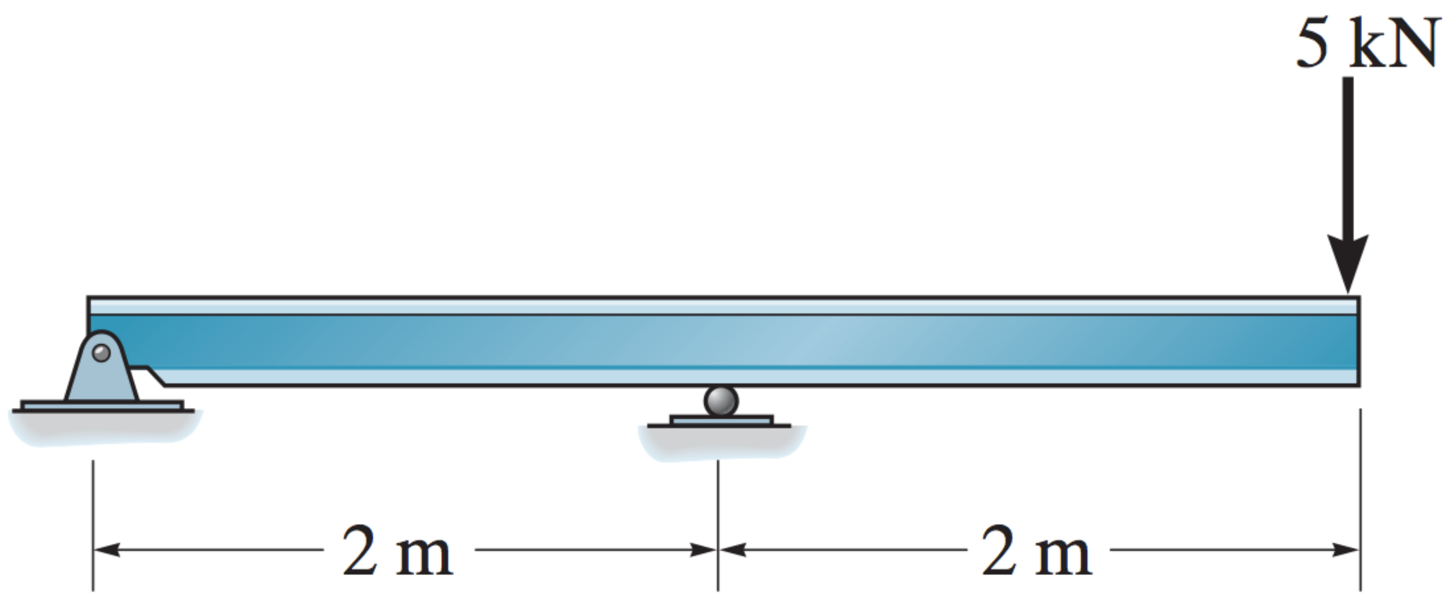
$$\left[\begin{array}{c}\mathbf{Q}_k \\ \hdashline \mathbf{Q}_u\end{array}\right] = \left[\begin{array}{c:c}\mathbf{K}_{11} & \mathbf{K}_{12} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22}\end{array}\right]\left[\begin{array}{c}\mathbf{D}_u \\ \hdashline \mathbf{D}_k\end{array}\right]$$

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k$$

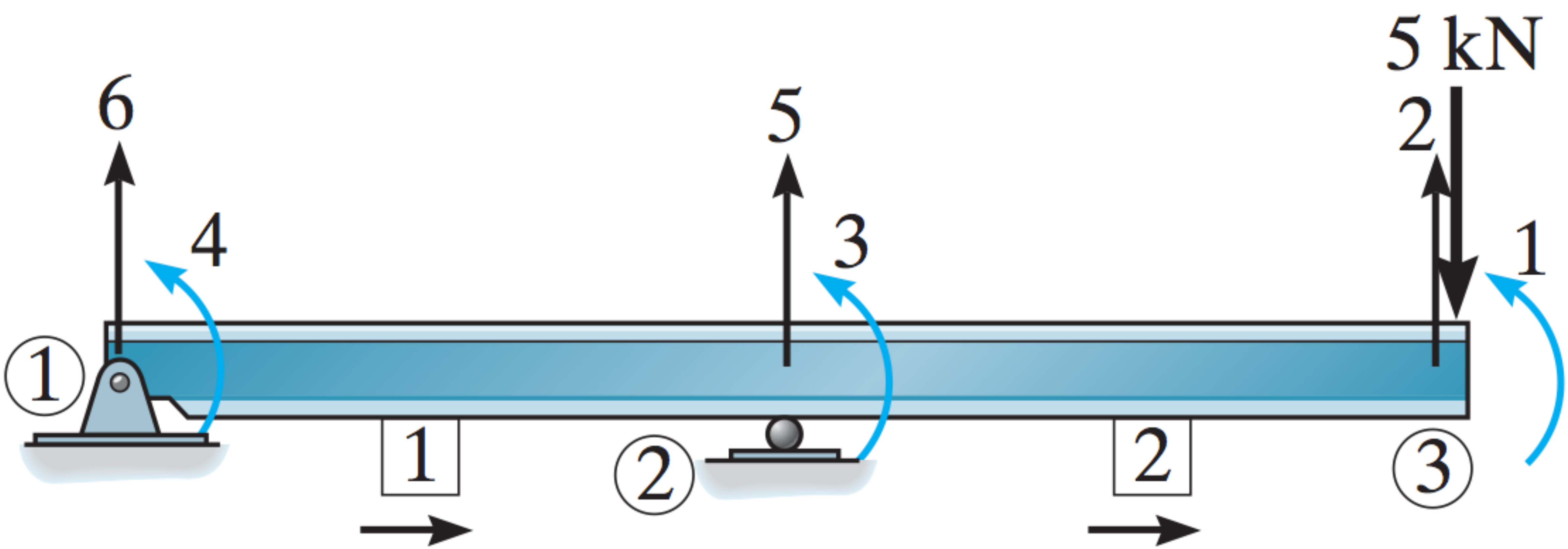
$$\mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k$$

Problem 1

Determine the reactions at the supports of the beam shown in Fig. 15–8*a*. *EI* is constant.



$$\mathbf{Q}_k = \begin{array}{c} \begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \mathbf{D}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{c} 5 \\ 6 \end{array}$$



$$
\mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 6 & 4 & 5 & 3 \end{array} \\ \left[\begin{array}{rrrr} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 4 \\ 5 \\ 3 \end{array}
$$

$$
\mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 5 & 3 & 2 & 1 \end{array} \\ \left[\begin{array}{rrrr} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 3 \\ 2 \\ 1 \end{array}
$$

$$\mathbf{Q} = \mathbf{K}\mathbf{D}$$

$$\begin{array}{c} \\ \left[\begin{array}{c} 0 \\ -5 \\ 0 \\ 0 \\ \hdashline Q_5 \\ Q_6 \end{array}\right] \end{array} = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc:cc} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ \hdashline 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{array}\right] \end{array} \begin{array}{c} \\ \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \hdashline 0 \\ 0 \end{array}\right] \end{array}$$

$$0 = 2D_1 - 1.5D_2 + D_3 + 0$$

$$-\frac{5}{EI} = -1.5D_1 + 1.5D_2 - 1.5D_3 + 0$$

$$0 = D_1 - 1.5D_2 + 4D_3 + D_4$$

$$0 = 0 + 0 + D_3 + 2D_4$$

$$D_1 = -\frac{16.67}{EI}$$

$$D_2 = -\frac{26.67}{EI}$$

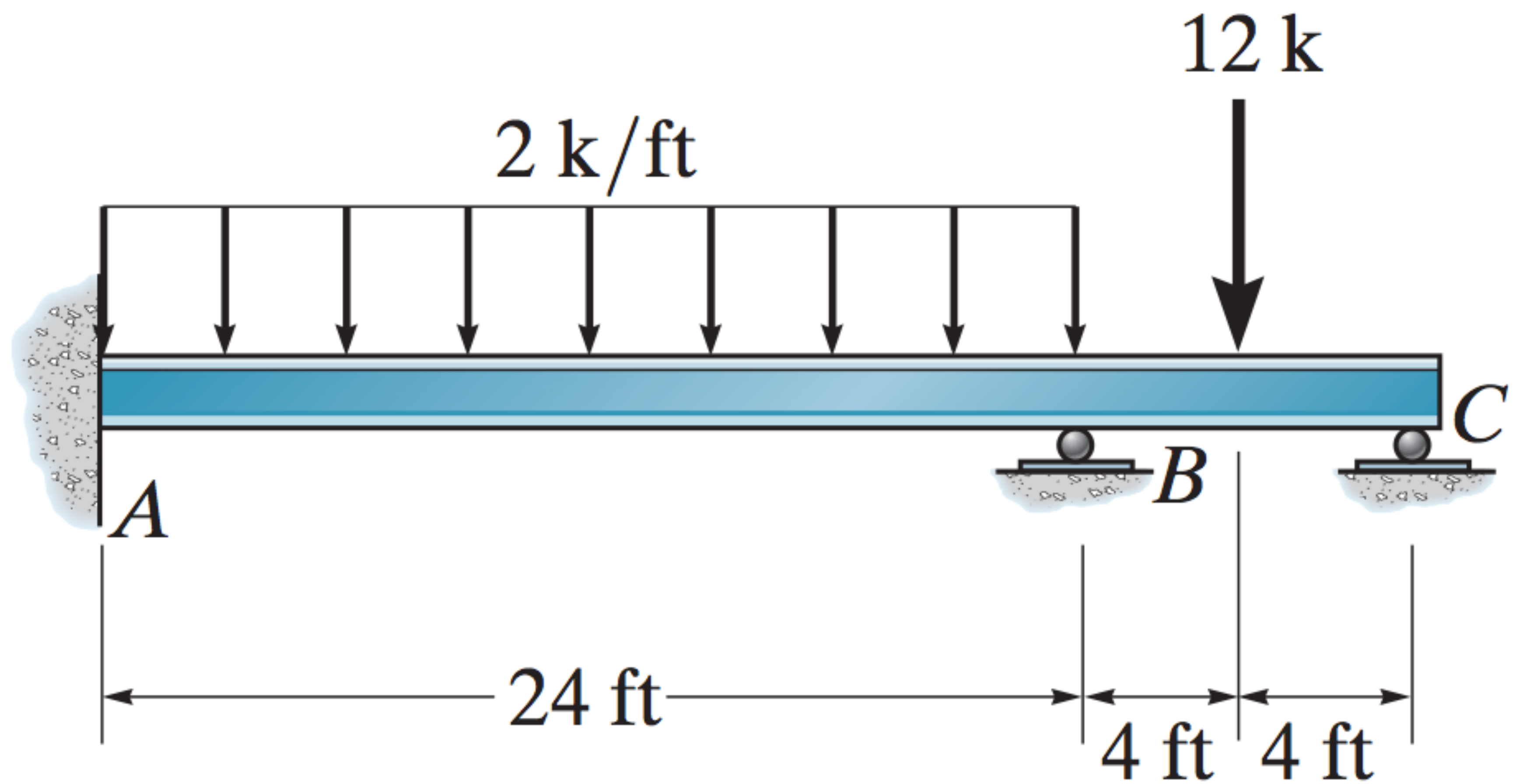
$$D_3 = -\frac{6.67}{EI}$$

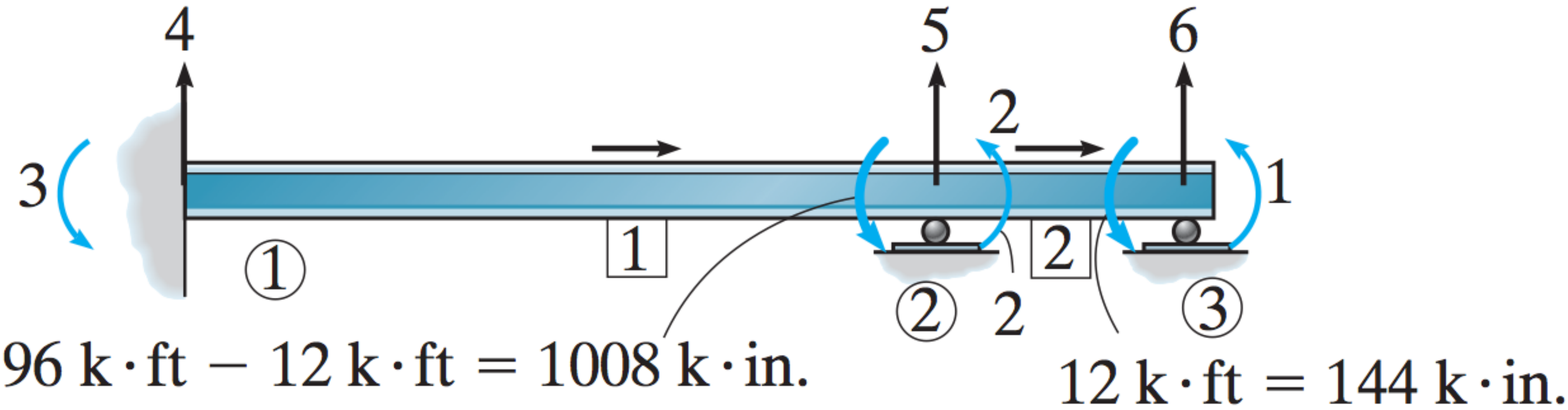
$$D_4 = \frac{3.33}{EI}$$

$$
\begin{aligned}
Q_5 &= 1.5EI\left(-\frac{16.67}{EI}\right) - 1.5EI\left(-\frac{26.67}{EI}\right) + 0 - 1.5EI\left(\frac{3.33}{EI}\right) \\
&= 10 \text{ kN}
\end{aligned}
$$

$$
\begin{aligned}
Q_6 &= 0 + 0 + 1.5EI\left(-\frac{6.67}{EI}\right) + 1.5EI\left(\frac{3.33}{EI}\right) \\
&= -5 \text{ kN}
\end{aligned}
$$

Determine the moment developed at support A of the beam shown in Fig. 15–11a. Assume the roller supports can pull down or push up on the beam. Take $E = 29(10^3)$ ksi, $I = 510$ in^4.




4
5
6
2
3
1
1
2
①
②
2
③
96 k · ft − 12 k · ft = 1008 k · in.
12 k · ft = 144 k · in.

$$\frac{12EI}{L^3} = \frac{12(29)(10^3)(510)}{[24(12)]^3} = 7.430$$

$$\frac{6EI}{L^2} = \frac{6(29)(10^3)(510)}{[24(12)]^2} = 1069.9$$

$$\frac{4EI}{L} = \frac{4(29)(10^3)(510)}{24(12)} = 205\,417$$

$$\frac{2EI}{L} = \frac{2(29)(10^3)(510)}{24(12)} = 102\,708$$

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccc} 4 & 3 & 5 & 2 \end{array} \\ \left[\begin{array}{cccc} 7.430 & 1069.9 & -7.430 & 1069.9 \\ 1069.9 & 205\,417 & -1069.9 & 102\,708 \\ -7.430 & -1069.9 & 7.430 & -1069.9 \\ 1069.9 & 102\,708 & -1069.9 & 205\,417 \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 3 \\ 5 \\ 2 \end{array}$$

$$\frac{12EI}{L^3} = \frac{12(29)(10^3)(510)}{[8(12)]^3} = 200.602 \qquad \frac{4EI}{L} = \frac{4(29)(10^3)(510)}{8(12)} = 616\,250$$

$$\frac{6EI}{L^2} = \frac{6(29)(10^3)(510)}{[8(12)]^2} = 9628.91 \qquad \frac{2EI}{L} = \frac{2(29)(10^3)(510)}{8(12)} = 308\,125$$

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 5 & 2 & 6 & 1 \end{array} \\ \left[\begin{array}{cccc} 200.602 & 9628.91 & -200.602 & 9628.91 \\ 9628.91 & 616\,250 & -9628.91 & 308\,125 \\ -200.602 & -9628.91 & 200.602 & -9628.91 \\ 9628.91 & 308\,125 & -9628.91 & 616\,250 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 2 \\ 6 \\ 1 \end{array}$$

$$\mathbf{Q} = \mathbf{K}\mathbf{D}$$

$$
\begin{bmatrix} 144 \\ 1008 \\ \hdashline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad & \quad 6 \quad \end{array} \\
\left[\begin{array}{cc:cccc}
616\,250 & 308\,125 & 0 & 0 & 9628.91 & -9628.91 \\
308\,125 & 821\,667 & 102\,708 & 1069.9 & 8559.01 & -9628.91 \\ \hdashline
0 & 102\,708 & 205\,417 & 1069.9 & -1069.9 & 0 \\
0 & 1069.9 & 1069.9 & 7.430 & -7.430 & 0 \\
9628.91 & 8559.01 & -1069.9 & -7.430 & 208.03 & -200.602 \\
-9628.91 & -9628.91 & 0 & 0 & -200.602 & 200.602
\end{array}\right]
\end{array}
\begin{bmatrix} D_1 \\ D_2 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$144 = 616\,250D_1 + 308\,125D_2$$

$$1008 = 308\,125D_1 + 821\,667D_2$$

$$D_1 = -0.4673(10^{-3}) \text{ in.}$$

$$D_2 = 1.40203(10^{-3}) \text{ in.}$$

$$Q_3 = 0 + 102\,708(1.40203)(10^{-3}) = 144 \text{ k} \cdot \text{in.} = 12 \text{ k} \cdot \text{ft}$$

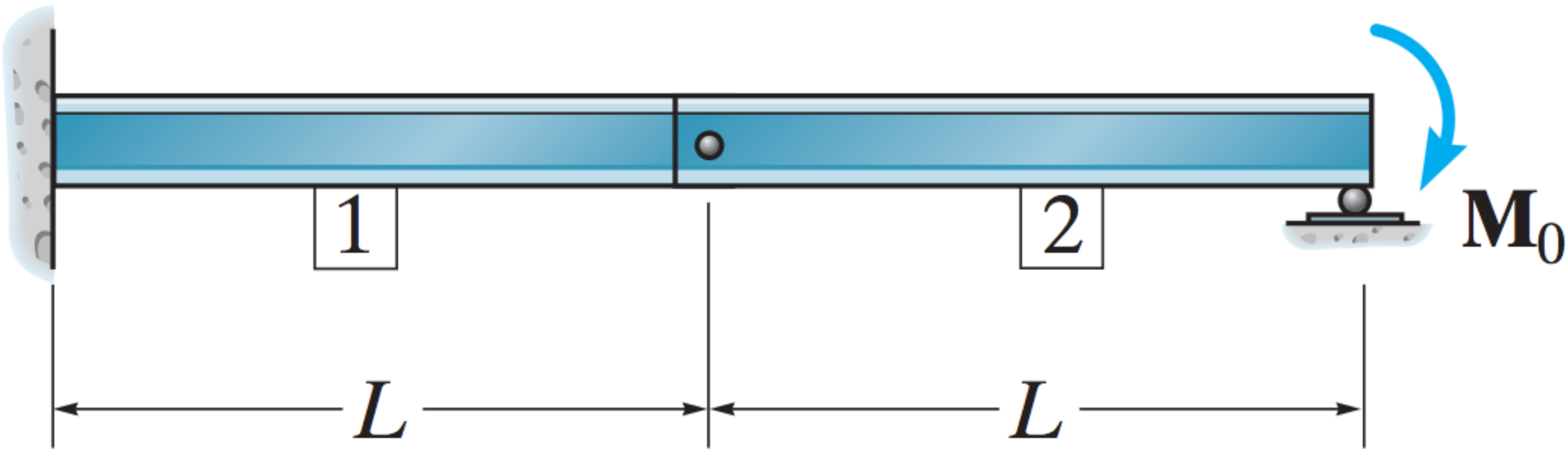
The actual moment at A must include the fixed-supported *reaction* of $+96 \text{ k} \cdot \text{ft}$ shown in Fig. 15–11*c*, along with the calculated result for Q_3. Thus,

$$M_{AB} = 12 \text{ k} \cdot \text{ft} + 96 \text{ k} \cdot \text{ft} = 108 \text{ k} \cdot \text{ft} \circlearrowleft \qquad \textit{Ans.}$$

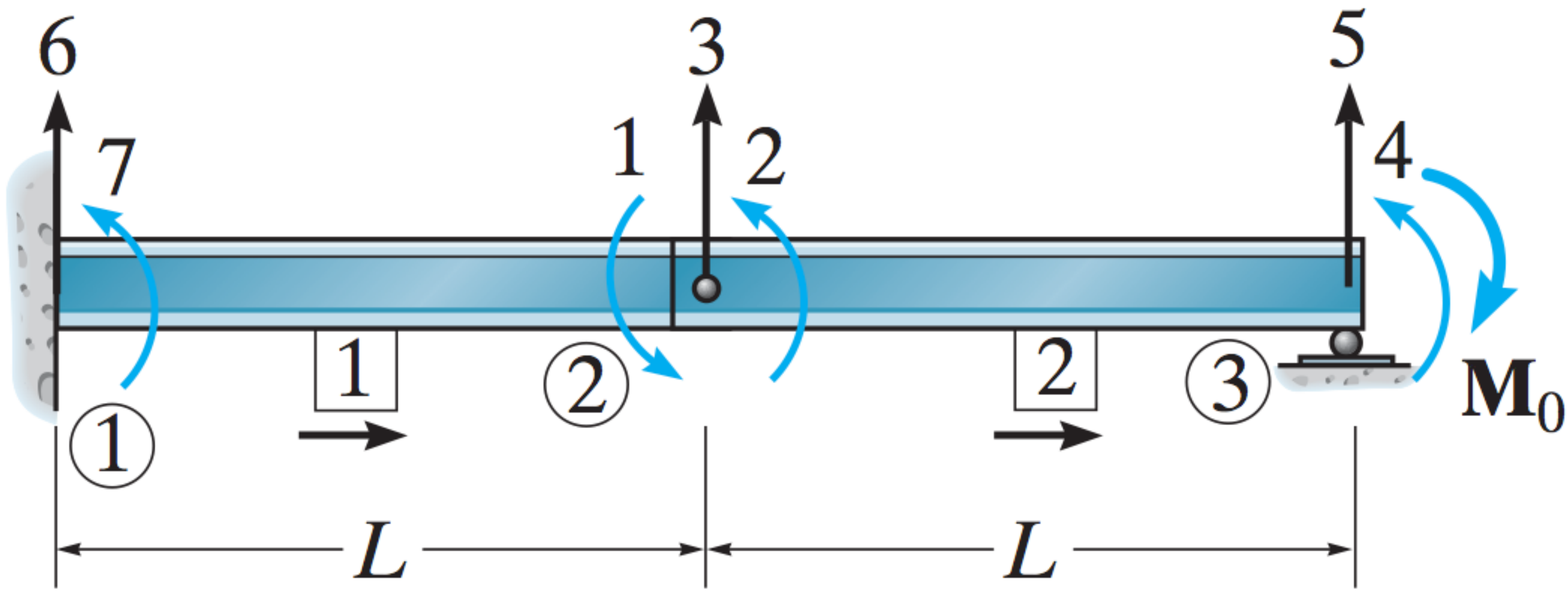
Although not required here, we can determine the internal moment and shear at B by considering, for example, member 1, node 2, Fig. 15–11b. The result requires expanding

$$\mathbf{q}_1 = \mathbf{k}_1\mathbf{d} + (\mathbf{q}_0)_1$$

$$\begin{bmatrix} q_4 \\ q_3 \\ q_5 \\ q_2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 4 & 3 & 5 & 2 \end{array} \\ \begin{bmatrix} 7.430 & 1069.9 & -7.430 & 1069.9 \\ 1069.9 & 205\,417 & -1069.9 & 102\,708 \\ -7.430 & -1069.9 & 7.430 & -1069.9 \\ 1069.9 & 102\,708 & -1069.9 & 205\,417 \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.40203 \end{bmatrix} (10^{-3}) + \begin{bmatrix} 24 \\ 1152 \\ 24 \\ -1152 \end{bmatrix}$$



Determine the internal shear and moment in member 1 of the compound beam shown in Fig. 15–9*a*. *EI* is constant.



$$\mathbf{Q}_k = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M_0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \mathbf{D}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 7 \end{array}$$

$$
\mathbf{k}_1 = EI \begin{array}{c c} \begin{matrix} \quad 6 \quad & \quad 7 \quad & \quad 3 \quad & \quad 1 \quad \end{matrix} & \\ \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} & \begin{matrix} 6 \\ 7 \\ 3 \\ 1 \end{matrix} \end{array}
$$

$$
\mathbf{k}_2 = EI \begin{array}{c c} \begin{matrix} \quad 3 \quad & \quad 2 \quad & \quad 5 \quad & \quad 4 \quad \end{matrix} & \\ \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} & \begin{matrix} 3 \\ 2 \\ 5 \\ 4 \end{matrix} \end{array}
$$

$$
\begin{array}{cccccc}
 & & & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} & & \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix} 0 \\ 0 \\ 0 \\ -M_0 \\ \hdashline Q_5 \\ Q_6 \\ Q_7 \end{bmatrix} &
= EI &
\left[\begin{array}{cccc:ccc}
\frac{4}{L} & 0 & -\frac{6}{L^2} & 0 & 0 & \frac{6}{L^2} & \frac{2}{L} \\
0 & \frac{4}{L} & \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & 0 & 0 \\
-\frac{6}{L^2} & \frac{6}{L^2} & \frac{24}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & -\frac{12}{L^3} & -\frac{6}{L^2} \\
0 & \frac{2}{L} & \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & 0 & 0 \\
\hdashline
0 & -\frac{6}{L^2} & -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & 0 & 0 \\
\frac{6}{L^2} & 0 & -\frac{12}{L^3} & 0 & 0 & \frac{12}{L^3} & \frac{6}{L^2} \\
\frac{2}{L} & 0 & -\frac{6}{L^2} & 0 & 0 & \frac{6}{L^2} & \frac{4}{L}
\end{array}\right] &
\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \hdashline 0 \\ 0 \\ 0 \end{bmatrix} &
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\end{array}
$$

$$
\begin{aligned}
0 &= \frac{4}{L}D_1 - \frac{6}{L^2}D_3 \\
0 &= \frac{4}{L}D_2 + \frac{6}{L^2}D_3 + \frac{2}{L}D_4 \\
0 &= -\frac{6}{L^2}D_1 + \frac{6}{L^2}D_2 + \frac{24}{L^3}D_3 + \frac{6}{L^2}D_4 \\
-M_0 &= \frac{2}{L}D_2 + \frac{6}{L^2}D_3 + \frac{4}{L}D_4
\end{aligned}
$$

$$D_1 = \frac{M_0 L}{2EI}$$

$$D_2 = -\frac{M_0 L}{6EI}$$

$$D_3 = \frac{M_0 L^2}{3EI}$$

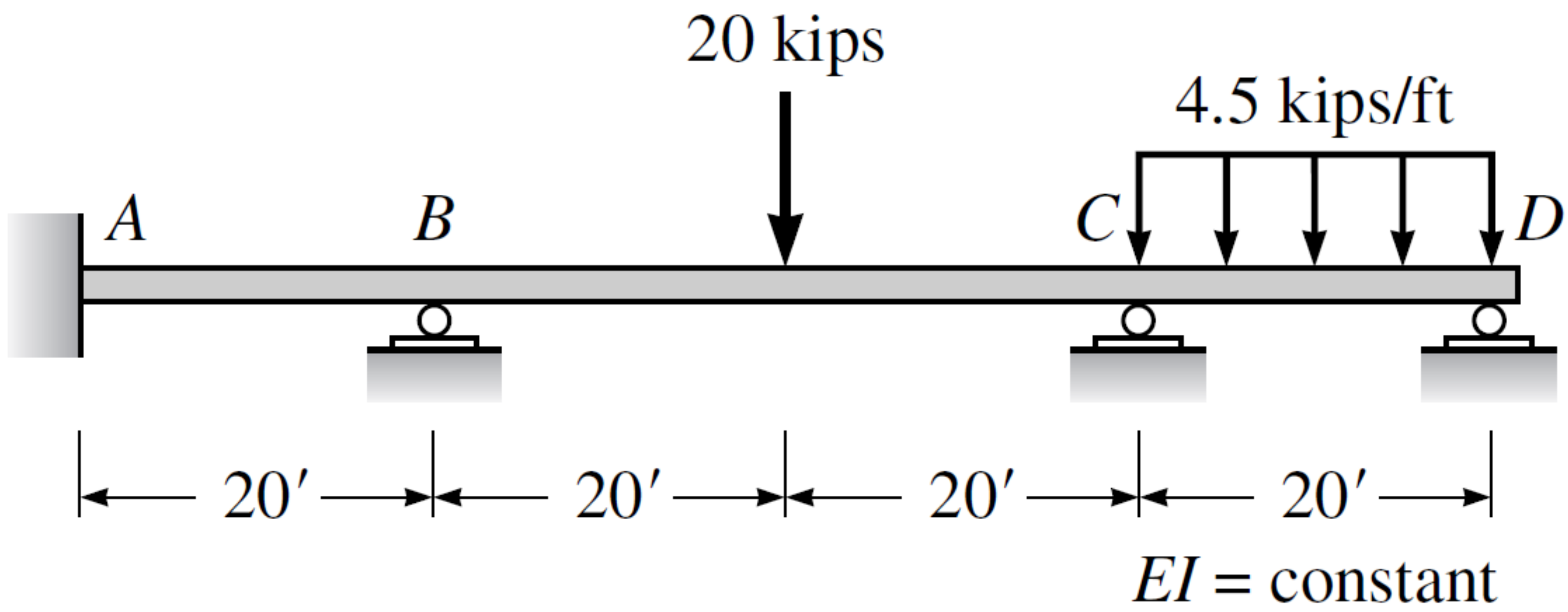
$$D_4 = -\frac{2M_0 L}{3EI}$$

$$Q_5 = -\frac{6EI}{L^2}\left(-\frac{M_0 L}{6EI}\right) - \frac{12EI}{L^3}\left(\frac{M_0 L^2}{3EI}\right) - \frac{6EI}{L^2}\left(-\frac{2M_0 L}{3EI}\right)$$

$$Q_5 = \frac{M_0}{L}$$

Example 18.2

Construct the bending moment diagram for the three-span continuous beam shown in Figure 18.4*a*. The beam, which has a constant flexural rigidity *EI*, supports a 20-kip concentrated load acting at the center of span *BC*. In addition, a uniformly distributed load of 4.5 kips/ft acts over the length of span *CD*.



Example 18.2 Solution

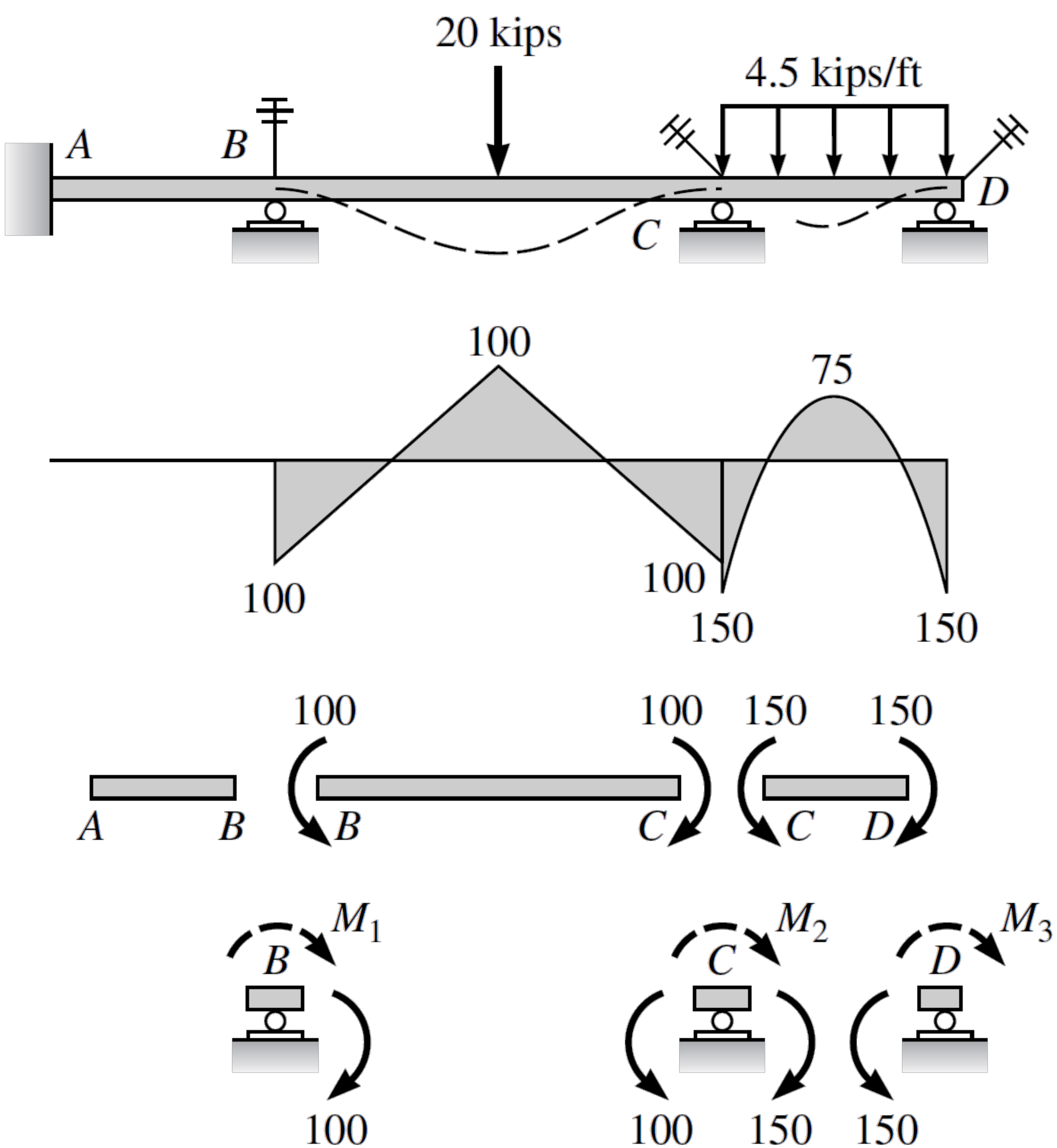
Curved arrows indicate the positive direction of the unknown joint rotations at *B*, *C*, and *D*

- An inspection of the structure indicates that the degree of kinematic indeterminacy is three. The positive directions selected for the three degrees of freedom (rotations at joints *B*, *C*, and *D*) are shown.

- **Step 1: Analysis of the Restrained Structure** Considering moment equilibrium, compute the restraining moments as follows:

Joint B:	$M_1 + 100 = 0$	$M_1 = -100 \text{ kip} \cdot \text{ft}$
Joint C:	$-100 + M_2 + 150 = 0$	$M_2 = -50 \text{ kip} \cdot \text{ft}$
Joint D:	$-150 + M_3 = 0$	$M_3 = 150 \text{ kip} \cdot \text{ft}$

Example 18.2 Solution (continued)

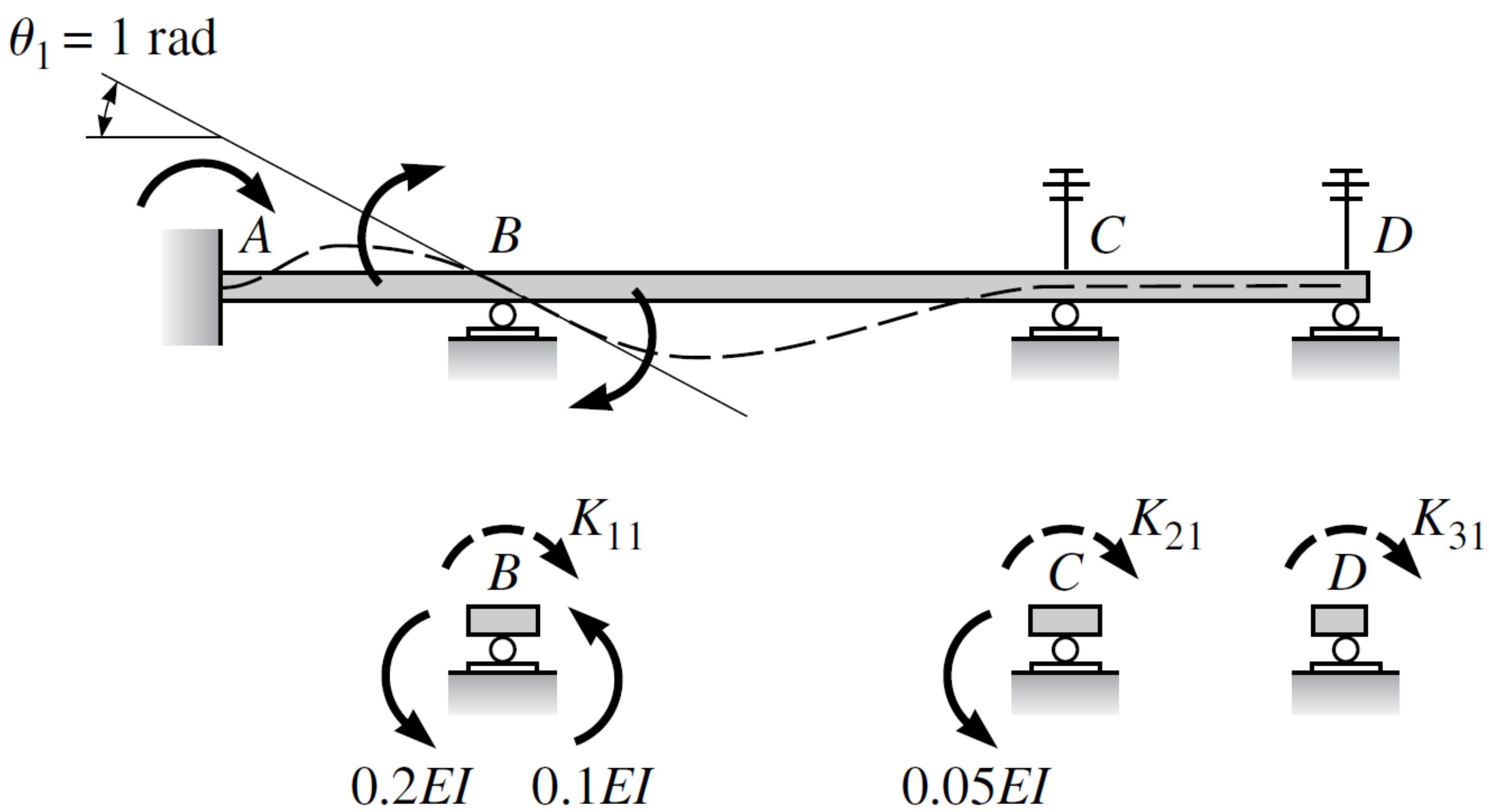


- Reversing the sign of the restraining moments, construct the force vector **F**:

$$\mathbf{F} = \begin{bmatrix} 100 \\ 50 \\ -150 \end{bmatrix} \text{kip} \cdot \text{ft}$$

Moments induced in the restrained structure by the applied loads; bottom figures show the moments acting on free-body diagrams of the clamped joints

Example 18.2 Solution (continued)



Stiffness coefficients produced by a unit rotation of joint *B* with joints *C* and *D* restrained

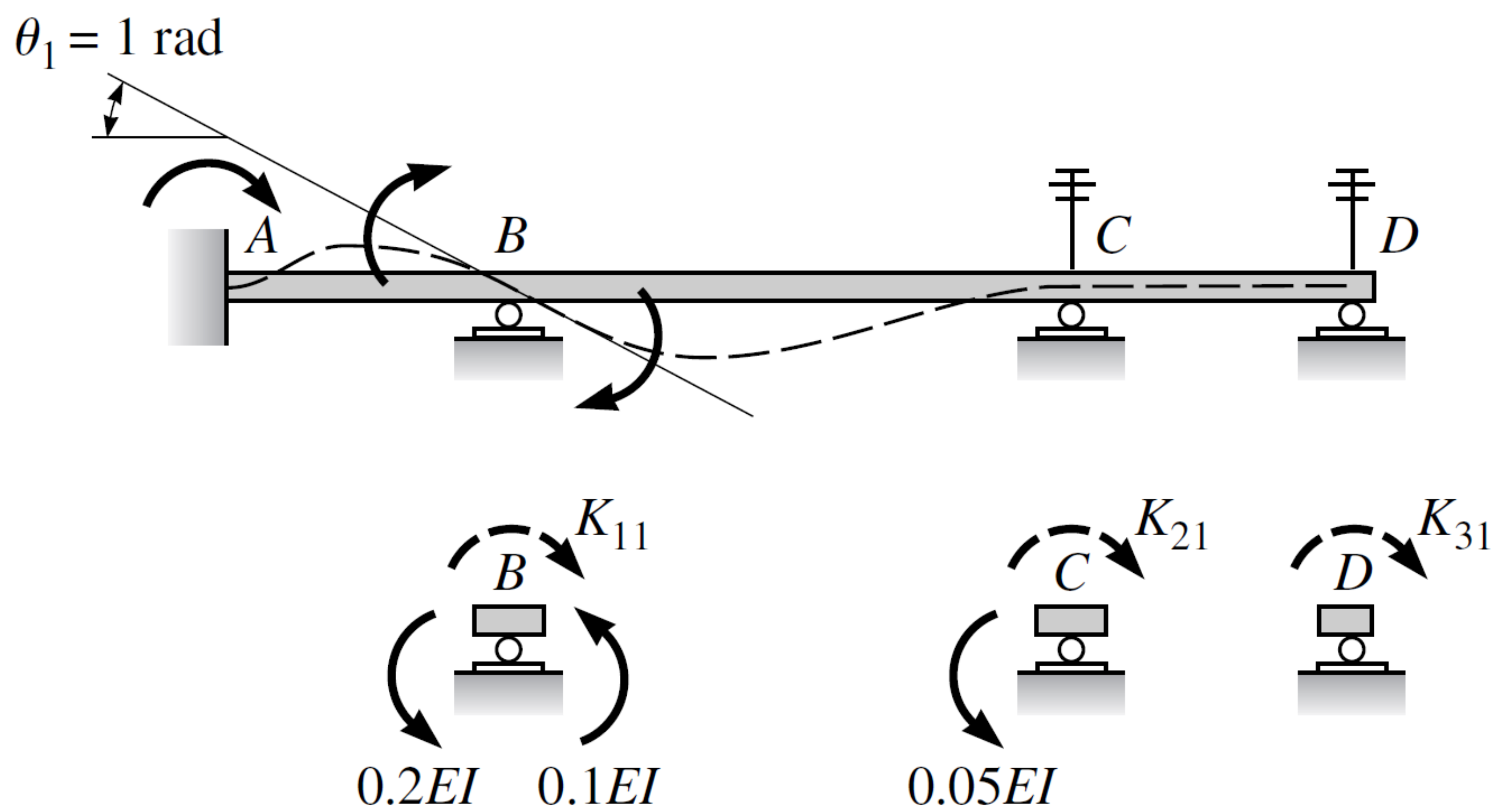
- **Step 2: Assembly of the Structure Stiffness Matrix** The elements of the structure stiffness matrix are readily calculated from the free-body diagrams of the joints. Summing moments,

$$-0.2EI - 0.1EI + K_{11} = 0 \qquad \text{and} \qquad K_{11} = 0.3EI$$

$$-0.05EI + K_{21} = 0 \qquad \text{and} \qquad K_{21} = 0.05EI$$

$$K_{31} = 0 \qquad \text{and} \qquad K_{31} = 0$$

Example 18.2 Solution (continued)

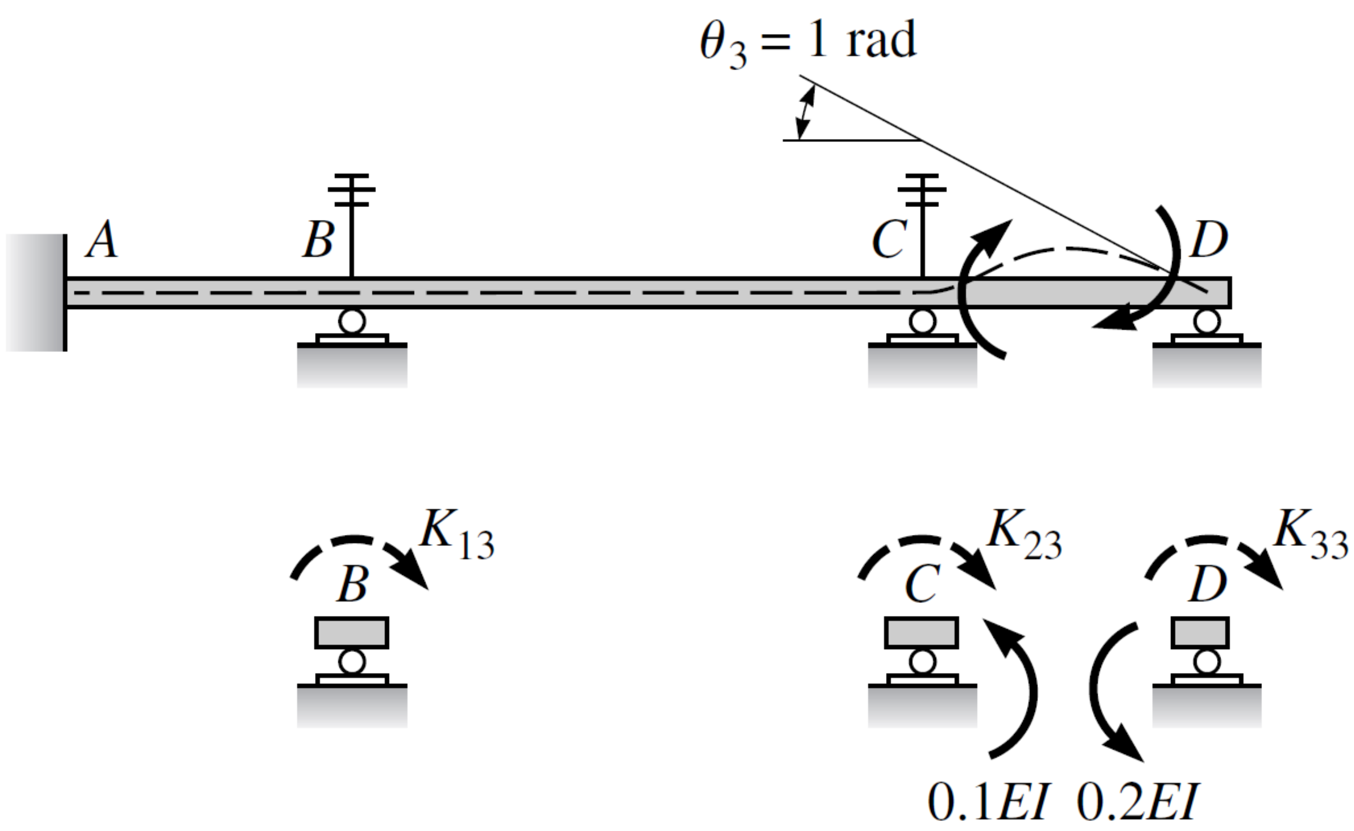


Stiffness coefficients produced by a unit rotation of joint *C* with joints *B* and *D* restrained

- From Figure 18.4*e*,

$$
\begin{aligned}
-0.05EI + K_{12} &= 0 \qquad \text{and} \qquad K_{12} = 0.05EI \\
-0.1EI - 0.2EI + K_{22} &= 0 \qquad \text{and} \qquad K_{22} = 0.3EI \\
-0.1EI + K_{32} &= 0 \qquad \text{and} \qquad K_{32} = 0.1EI
\end{aligned}
$$

Example 18.2 Solution (continued)



Stiffness coefficients produced by a unit rotation of joint *D* with joints *B* and *C* restrained

- From Figure 18.4*f*,

$$K_{13} = 0 \quad \text{and} \quad K_{13} = 0$$

$$-0.1EI + K_{23} = 0 \quad \text{and} \quad K_{23} = 0.1EI$$

$$-0.2EI + K_{33} = 0 \quad \text{and} \quad K_{33} = 0.2EI$$

Example 18.2 Solution (continued)

- Arranging these stiffness coefficients in matrix form, produce the following structure stiffness matrix **K**:

$$\mathbf{K} = EI\begin{bmatrix} 0.3 & 0.05 & 0 \\ 0.05 & 0.3 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}$$

From Betti's law, the structure stiffness matrix **K** is symmetric.

- **Step 3: Solution of Equation 18.1** Substituting the previously calculated values of **F** and **K**(given by Equations 18.19 and 18.20) into Equation 18.1 gives

$$EI\begin{bmatrix} 0.3 & 0.05 & 0 \\ 0.05 & 0.3 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \\ -150 \end{bmatrix}$$

Example 18.2 Solution (continued)

- Solving Equation 18.21, compute

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 258.6 \\ 448.3 \\ -974.1 \end{bmatrix}$$

- **Step 4: Evaluation of the Effect of Joint Displacements** The moments produced by the actual joint rotations are determined by multiplying the moments produced by the unit displacements by the actual displacements and superimposing the results. For example, the end moments in span *BC* are

$$M''_{BC} = \theta_1(0.1EI) + \theta_2(0.05EI) + \theta_3(0) = 48.3 \text{ kip} \cdot \text{ft}$$

$$M''_{CB} = \theta_1(0.05EI) + \theta_2(0.1EI) + \theta_3(0) = 57.8 \text{ kip} \cdot \text{ft}$$

Example 18.2 Solution (continued)

- For an n degree of freedom structure add n appropriately scaled unit cases. Evaluate these moments in one step by using the individual member rotational stiffness matrices. For example, in span *BC*, substitute the end rotations θ_1 and θ_2 (given by Equation 18.22) into Equation 18.5 with L = 40 ft

$$\begin{bmatrix} M''_{BC} \\ M''_{CB} \end{bmatrix} = \frac{2EI}{40}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{EI}\begin{bmatrix} 258.6 \\ 448.3 \end{bmatrix} = \begin{bmatrix} 48.3 \\ 57.8 \end{bmatrix}$$

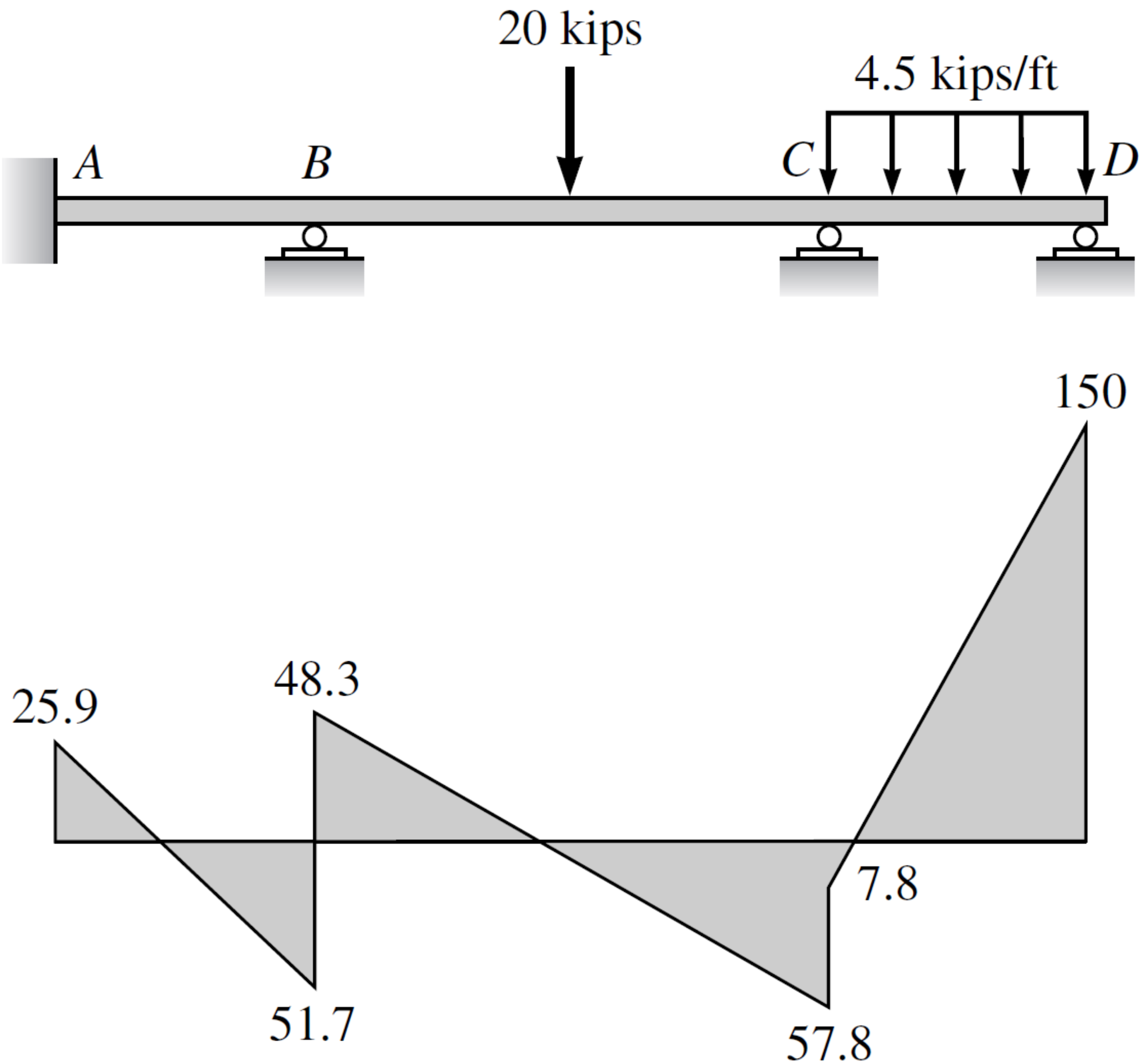
- Proceeding in a similar manner for spans *AB* and *CD*,

$$\begin{bmatrix} M''_{AB} \\ M''_{BA} \end{bmatrix} = \frac{2EI}{20}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{EI}\begin{bmatrix} 0 \\ 258.6 \end{bmatrix} = \begin{bmatrix} 25.9 \\ 51.7 \end{bmatrix}$$

$$\begin{bmatrix} M''_{CD} \\ M''_{DC} \end{bmatrix} = \frac{2EI}{20}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{EI}\begin{bmatrix} 448.3 \\ -974.1 \end{bmatrix} = \begin{bmatrix} -7.8 \\ -150.0 \end{bmatrix}$$

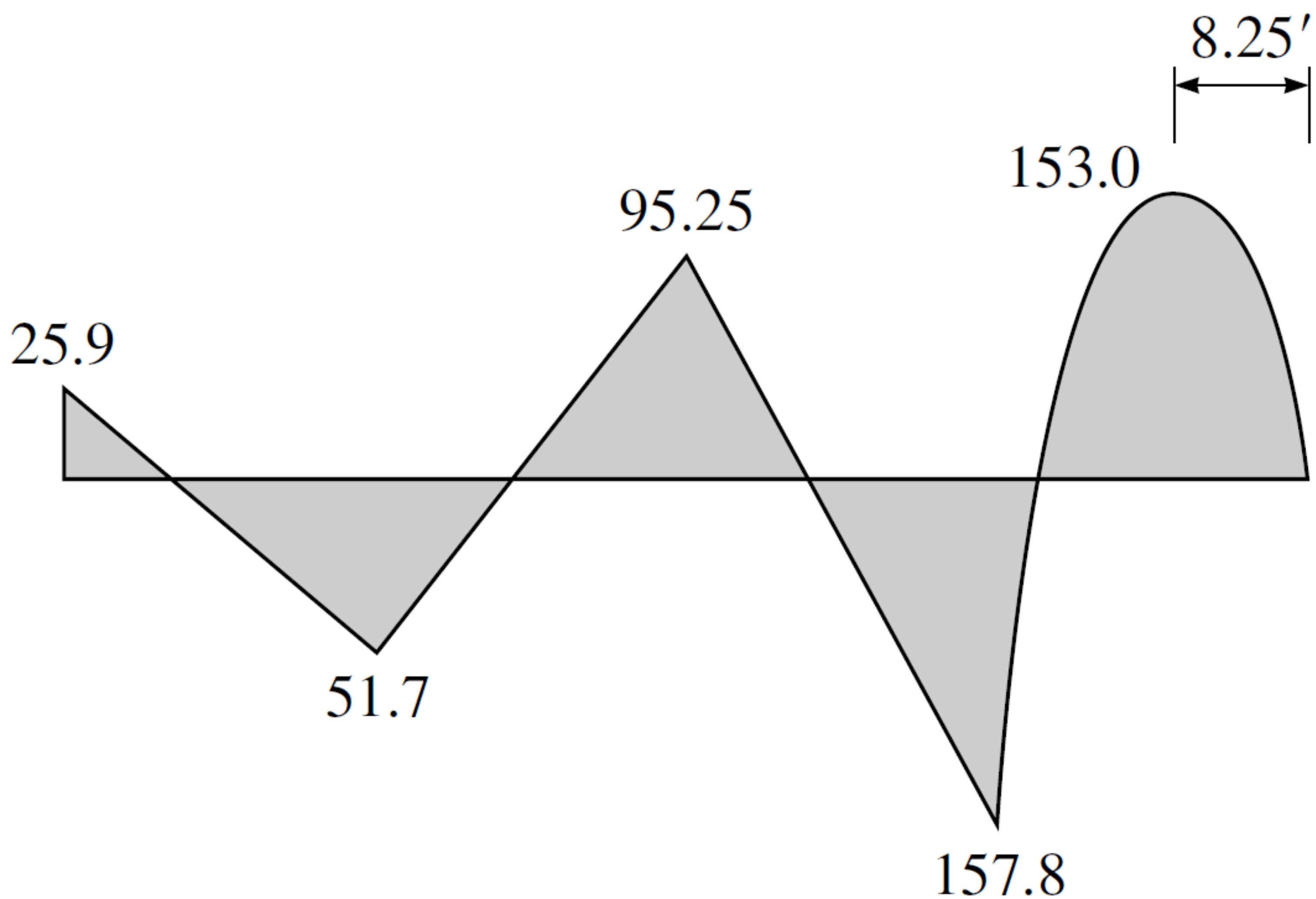
Example 18.2 Solution (continued)

- **Step 5: Calculation of Final Results** The complete solution is obtained by adding the results from the restrained case in Figure 18.4*c* to those produced by the joint displacements in Figure 18.4*g*.



Moments produced by actual joint rotations

Example 18.2 Solution (continued)



Final moment diagrams (in units of kip-ft)