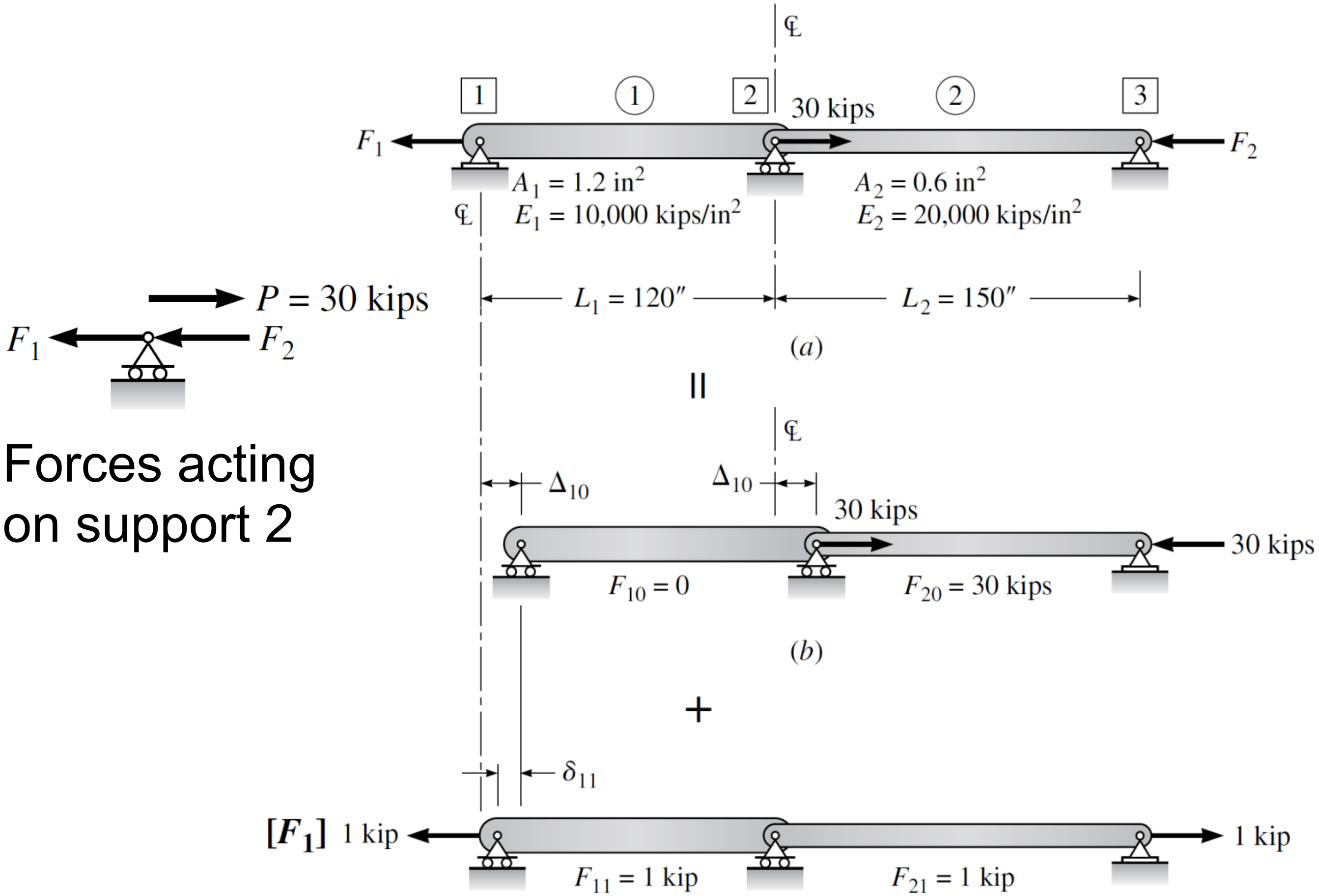
§16.2 Comparison between Flexibility and Stiffness Methods



Details of the structure

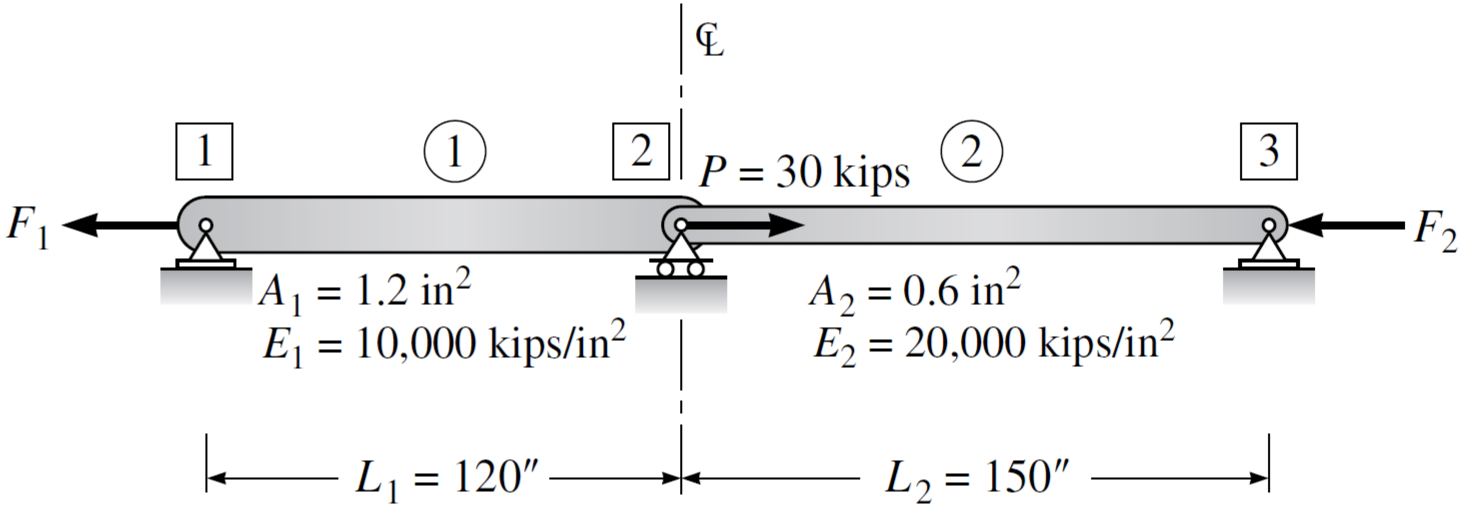
Forces acting on support 2

Load applied to the released structure

Redundant F_1 applied to the released structure at joint 1

Figure 16.1 Analysis by the flexibility method

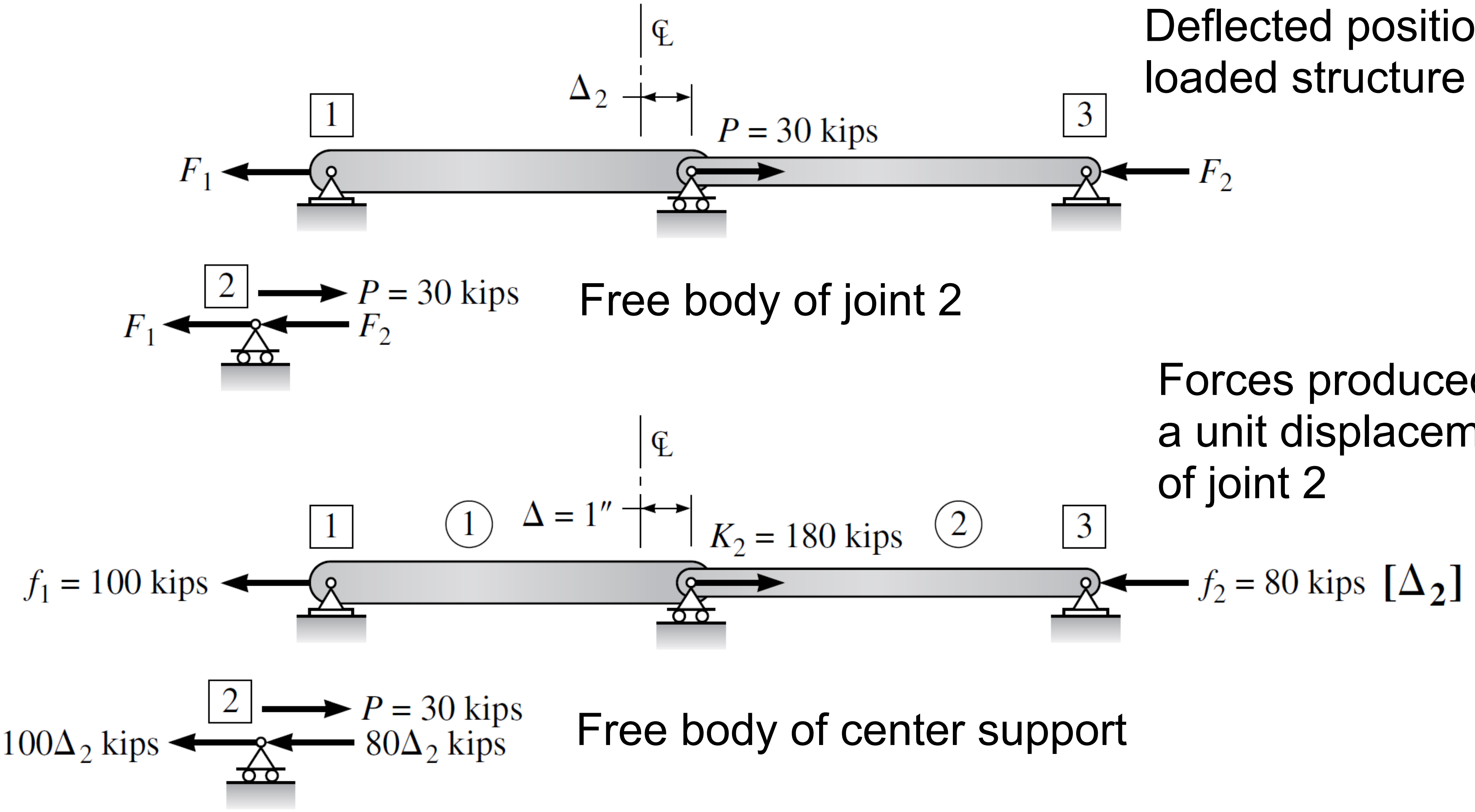
§16.2 Comparison between Flexibility and Stiffness Methods



Structure kinematically indeterminate to first degree

Figure 16.2

§16.2 Comparison between Flexibility and Stiffness Methods



Deflected position of loaded structure

Free body of joint 2

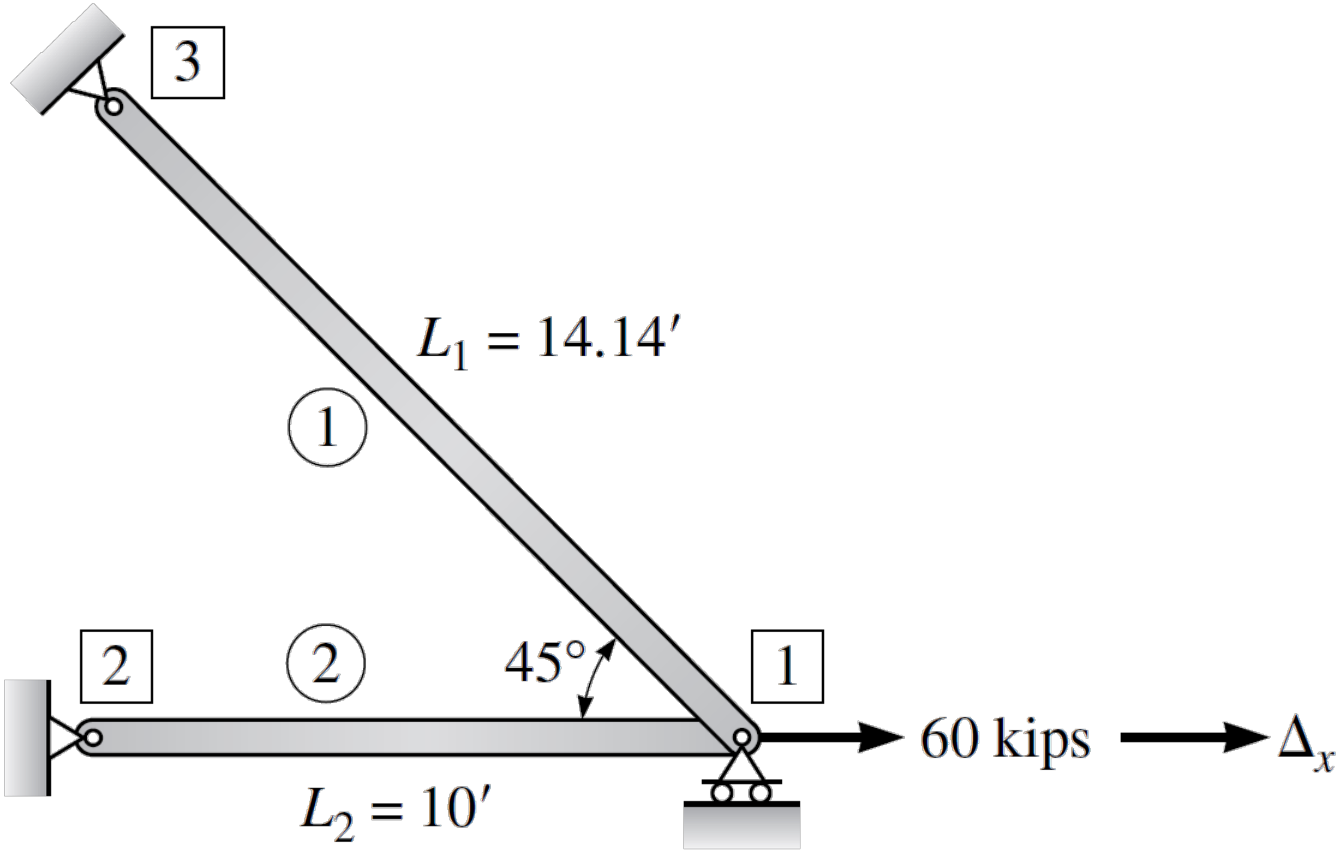
Forces produced by a unit displacement of joint 2

Free body of center support

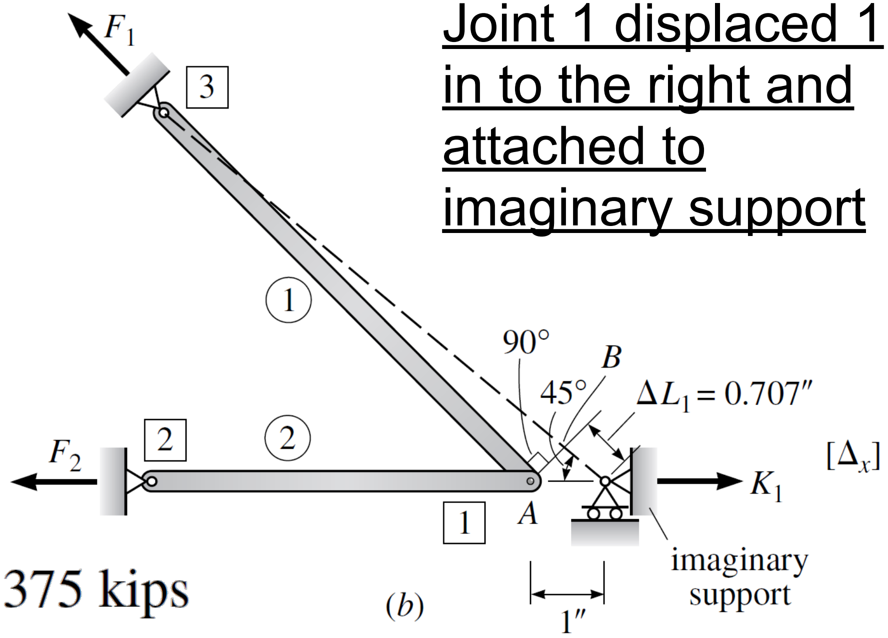
Figure 16.2 (continued)

Example 16.2

The pin-connected bars in Figure 16.5*a* are connected at joint 1 to a roller support. Determine the force in each bar and the magnitude of the horizontal displacement Δ_x of joint 1 produced by the 60-kip force. Area of bar 1 = 3 in^2, area of bar 2 = 2 in^2, and E = 30,000 kips/in^2.



Example 16.2 Solution



Joint 1 displaced 1 in to the right and attached to imaginary support

- Compute ΔL_1:

$$\Delta L_1 = (1\text{ in})(\cos 45°) = 0.707\text{ in}$$

- Use Equation 16.1 to compute the force in each bar.

$$F_1^{\text{JD}} = \frac{A_1 E \Delta L_1}{L_1} = \frac{3(30{,}000)(0.707)}{14.14 \times 12} = 375\text{ kips}$$

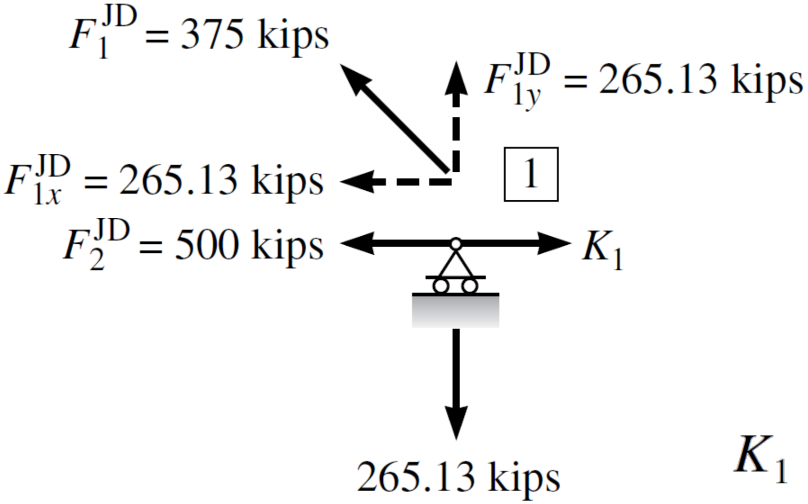
$$F_2^{\text{JD}} = \frac{A_2 E \Delta L_2}{L_2} = \frac{2(30{,}000)(1)}{10 \times 12} = 500\text{ kips}$$

- Compute the horizontal and vertical components of F_1.

$$F_{1x}^{\text{JD}} = F_1^{\text{JD}}(\cos 45°) = 375(0.707) = 265.13\text{ kips}$$

$$F_{1y}^{\text{JD}} = F_1^{\text{JD}}(\sin 45°) = 375(0.707) = 265.13\text{ kips}$$

Example 16.2 Solution (continued)



Forces at joint 1 produced by a 1-in horizontal displacement

- To evaluate *K₁*, sum forces applied to the pin (Figure 16.5*c*) in the horizontal direction.

$$\Sigma F_x = 0$$

$$K_1 - F_{1x}^{JD} - F_2^{JD} = 0$$

$$K_1 = F_{1x}^{JD} + F_2^{JD} = 265.13 + 500 = 765.13 \text{ kips}$$

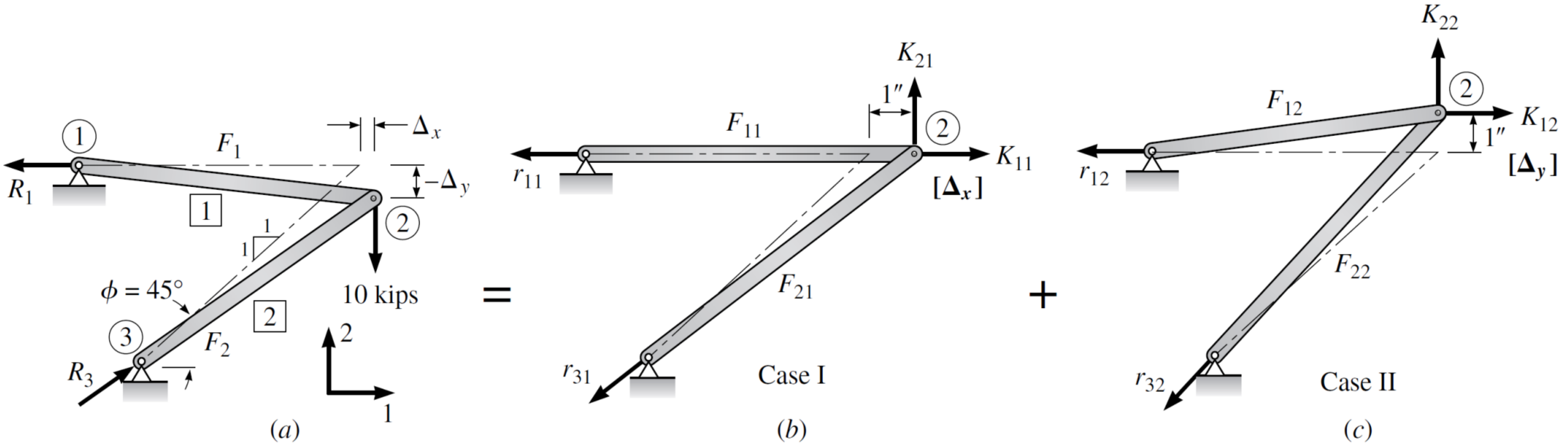
- Multiply the force K_1 in Figure 16.5*c* by Δ_x, the actual displacement and consider the horizontal force equilibrium at joint 2.

$$K_1\,\Delta_x - 60 = 0$$

$$765.13\,\Delta_x - 60 = 0$$

$$\Delta_x = 0.0784 \text{ in}$$

§17.1 Introduction



Horizontal and vertical displacements Δ_x and Δ_y produced by the 10-kip load at joint 2; initially bar 1 is horizontal: bar 2 slopes upward at 45°

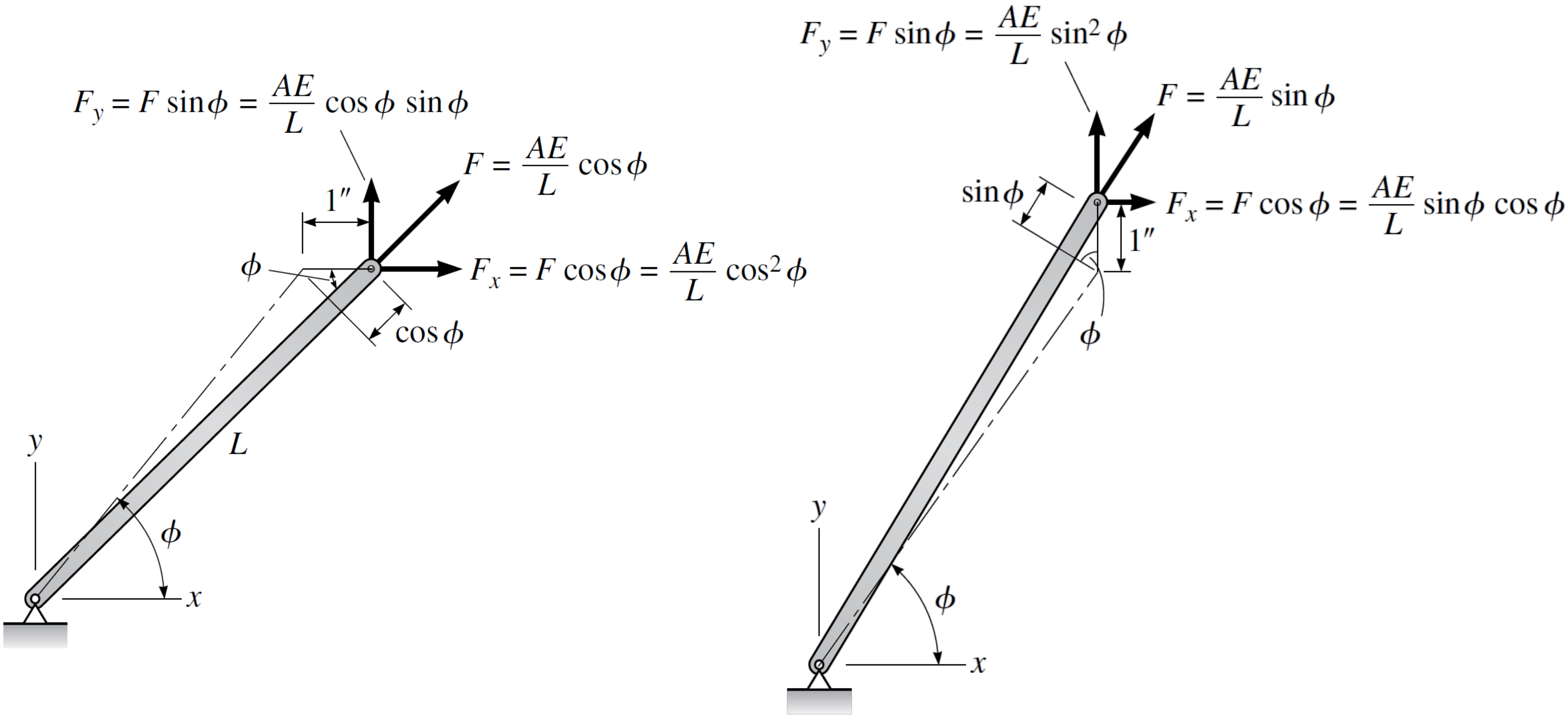
Forces (stiffness coefficients) K_{21} and K_{11} required to produce a unit horizontal displacement of joint 2

Forces K_{22} and K_{12} required to produce a unit vertical displacement of joint 2

Figure 17.1

-

§17.1 Introduction

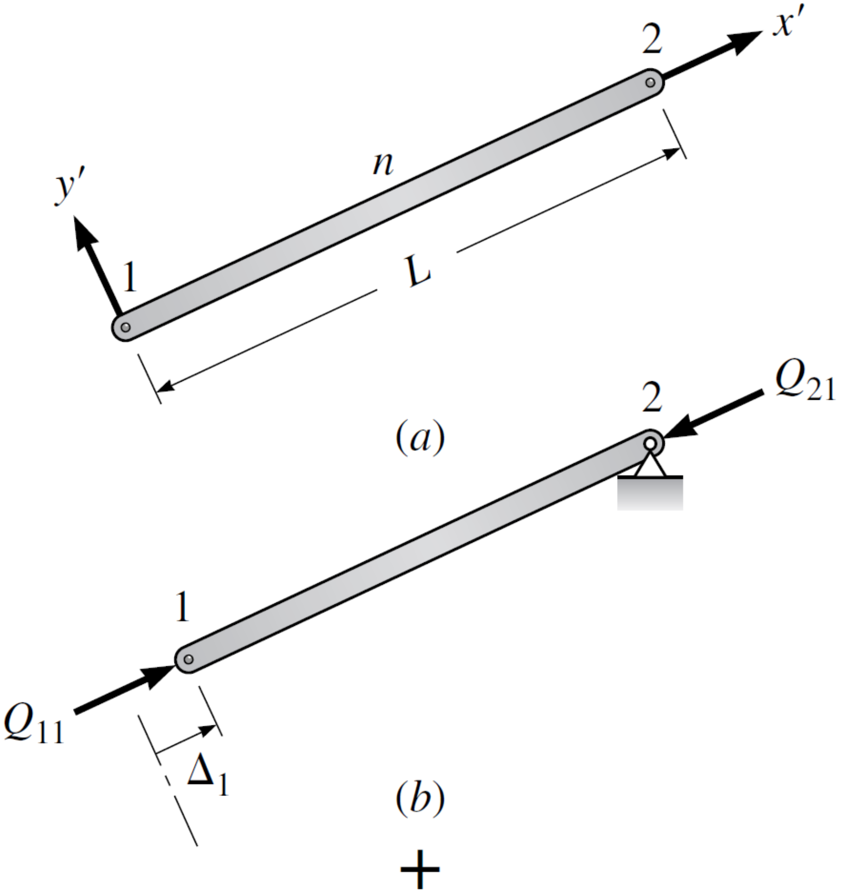


Forces created by a unit horizontal displacement

Forces created by a unit vertical displacement

Figure 17.2 Stiffness coefficients for an axially loaded bar with area *A*, length *L*, and modulus of elasticity *E*

§17.3 Construction of a Member Stiffness Matrix for an Individual Truss Bar

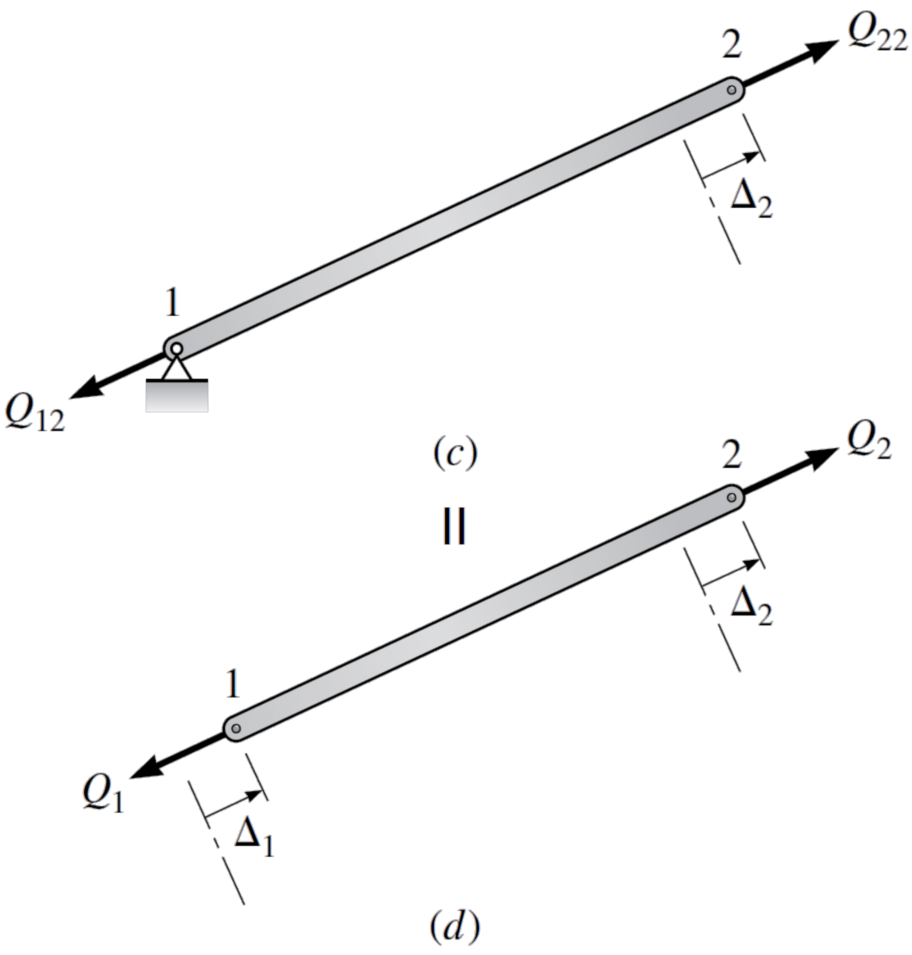


Bar showing local coordinate system with origin at node 1

Displacement introduced at node 1 with node 2 restrained

Figure 17.3 Stiffness coefficients for an axially loaded bar

§17.3 Construction of a Member Stiffness Matrix for an Individual Truss Bar



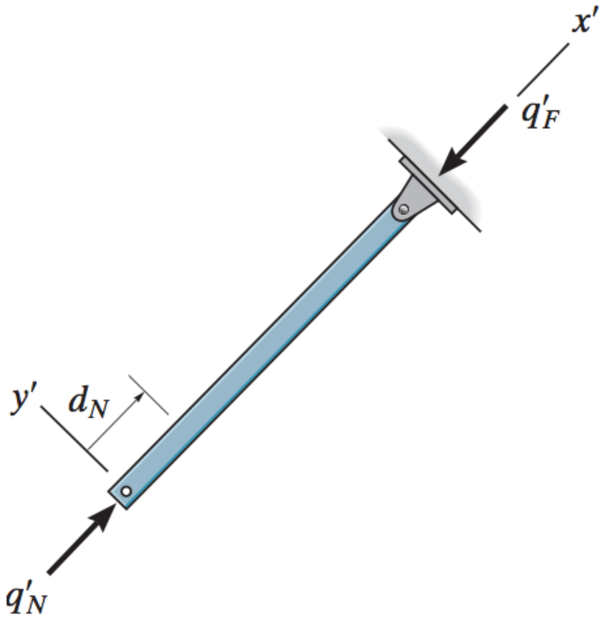
(*c*)

(*d*)

Displacement introduced at node 2 with node 1 restrained

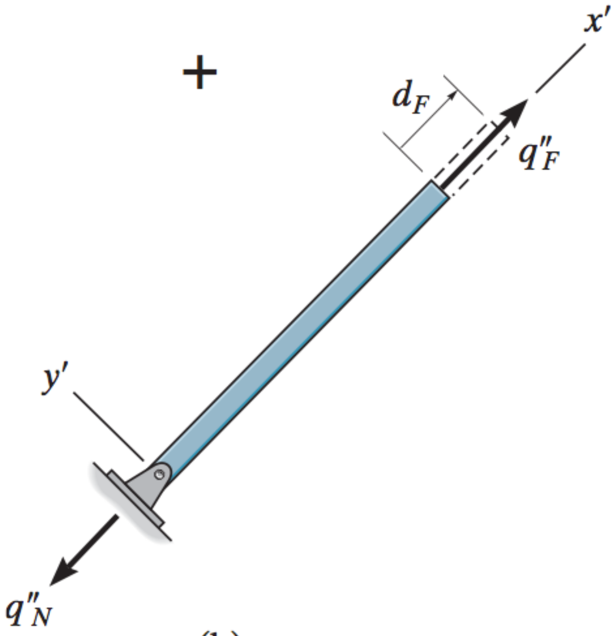
End forces and displacements of the actual bar produced by superposition of *(b)* and *(c)*

Figure 17.3 Stiffness coefficients for an axially loaded bar (continued)



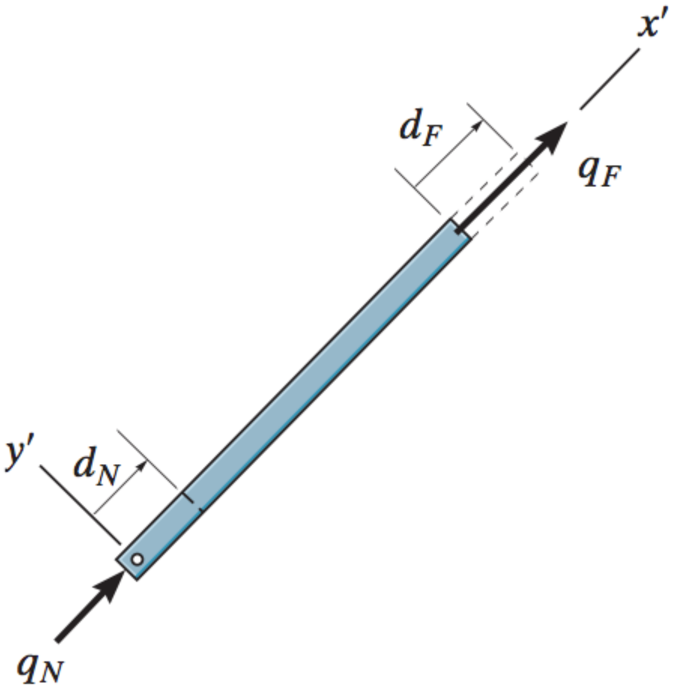
(a)

+



(b)

||

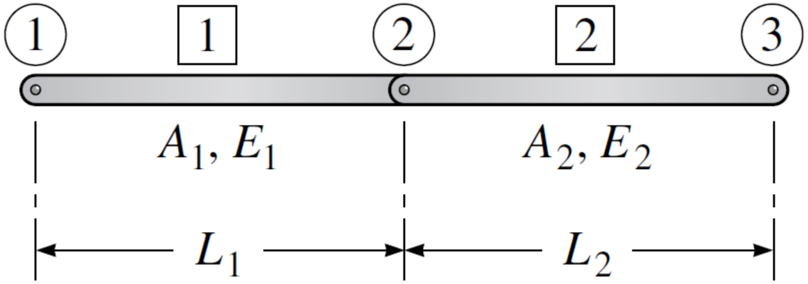


$$\begin{bmatrix} q_N \\ q_F \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} d_N \\ d_F \end{bmatrix}$$

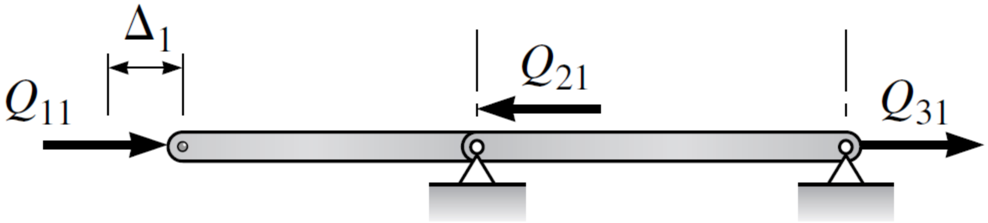
$$\boxed{\mathbf{q} = \mathbf{k}'\mathbf{d}}$$

$$\mathbf{k}' = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

§17.4 Assembly of the Structure Stiffness Matrix



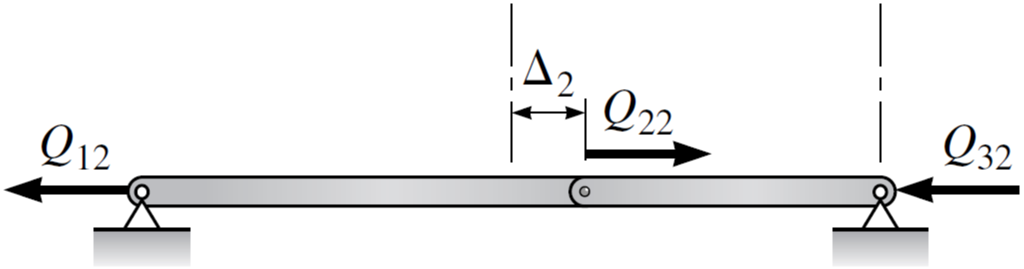
Properties of two-bar system



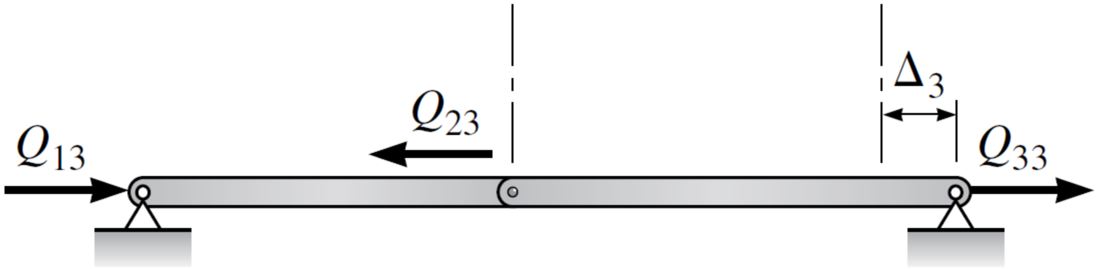
Node forces produced by a positive displacement Δ_1 of joint 1 with nodes 2 and 3 restrained

Figure 17.4 Loading conditions used to generate the structure stiffness matrix

§17.4 Assembly of the Structure Stiffness Matrix



Node forces produced by a positive displacement of node 2 with nodes 1 and 3 restrained

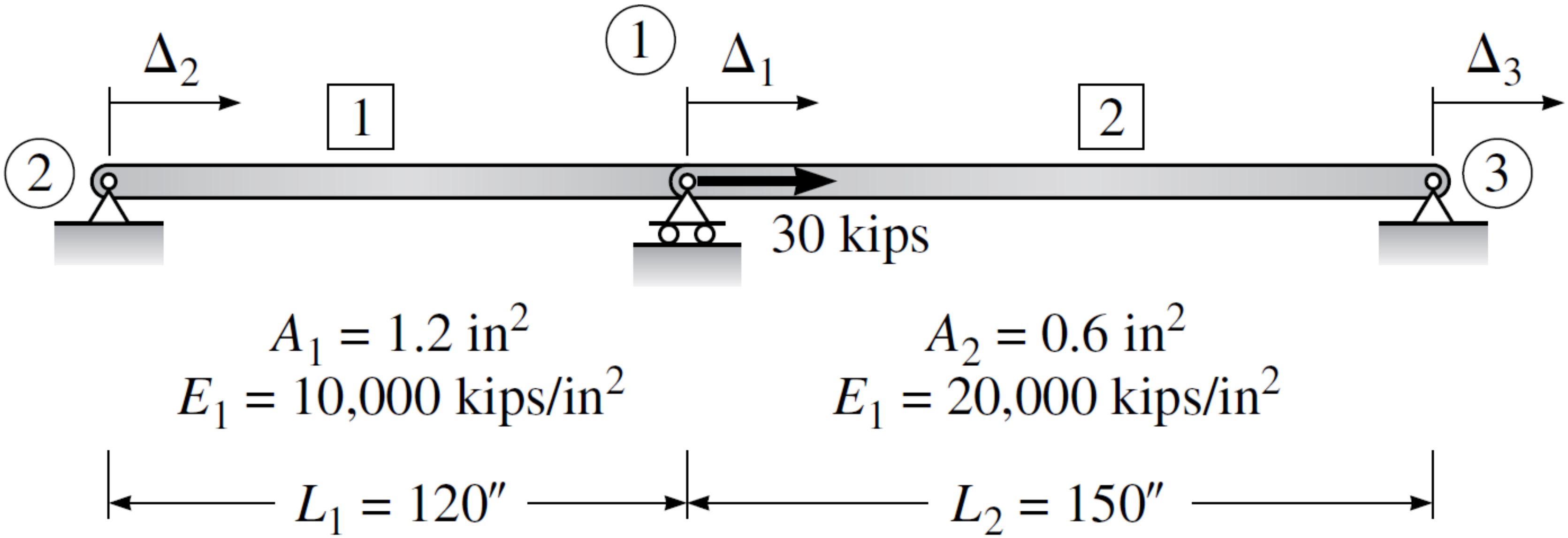


Node forces produced by a positive displacement of node 3 with nodes 1 and 2 restrained

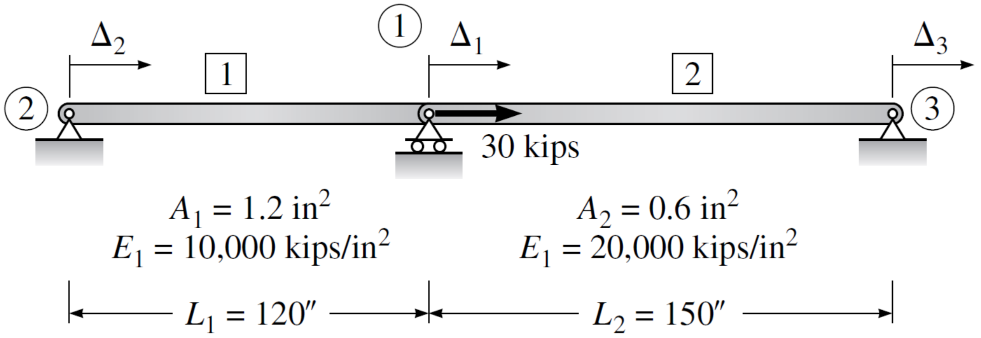
Figure 17.4 Loading conditions used to generate the structure stiffness matrix (continued)

Example 17.1

Determine the joint displacements and reactions for the structure in Figure 17.5 by partitioning the structure stiffness matrix.



Example 17.1 Solution



- Compute the stiffness *k* = *AE/L* for each member.

$$k_1 = \frac{1.2(10{,}000)}{120} = 100\ \text{kips/in}$$

$$k_2 = \frac{0.6(20{,}000)}{150} = 80\ \text{kips/in}$$

- Evaluate member stiffness matrices, using Equation 17.19. Because the local coordinate system of each bar coincides with the global coordinate system, **k'** = **k**.

$$\mathbf{k}_1 = k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{cc} 1 & 2 & \\ \left[\begin{array}{cc} 100 & -100 \\ -100 & 100 \end{array}\right. & & \left.\begin{array}{c} \end{array}\right] \end{array}$$

Example 17.1 Solution (continued)

- Set up the structure stiffness matrix **K** by combining terms of the member stiffness matrices $\mathbf{k}_1$ and $\mathbf{k}_2$. Establish Equation 17.30 as follows:

$$\left[\begin{array}{c} Q_1 = 30 \\ \hdashline Q_2 \\ Q_3 \end{array}\right] = \begin{array}{c} \begin{array}{ccc} 1 & \quad\quad\quad 2 & \quad\quad 3 \end{array} \\ \left[\begin{array}{c:cc} 100 + 80 & -100 & -80 \\ \hdashline -100 & 100 & 0 \\ -80 & 0 & 80 \end{array}\right] \end{array} \left[\begin{array}{c} \Delta_1 \\ \hdashline \Delta_2 = 0 \\ \Delta_3 = 0 \end{array}\right]$$

- Partition the matrices as indicated by Equation 17.30 and solve for Δ_1 using Equation 17.35. Since each submatrix contains one element, Equation 17.35 reduces to a simple algebraic equation.

$$\mathbf{\Delta}_f = \mathbf{K}_{11}^{-1}\mathbf{Q}_f$$

$$\Delta_1 = \frac{1}{180}(30) = \frac{1}{6}\text{ in}$$

Example 17.1 Solution (continued)

- Solve for the reactions, using Equation 17.36.

$$\mathbf{Q}_s = \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{Q}_f$$

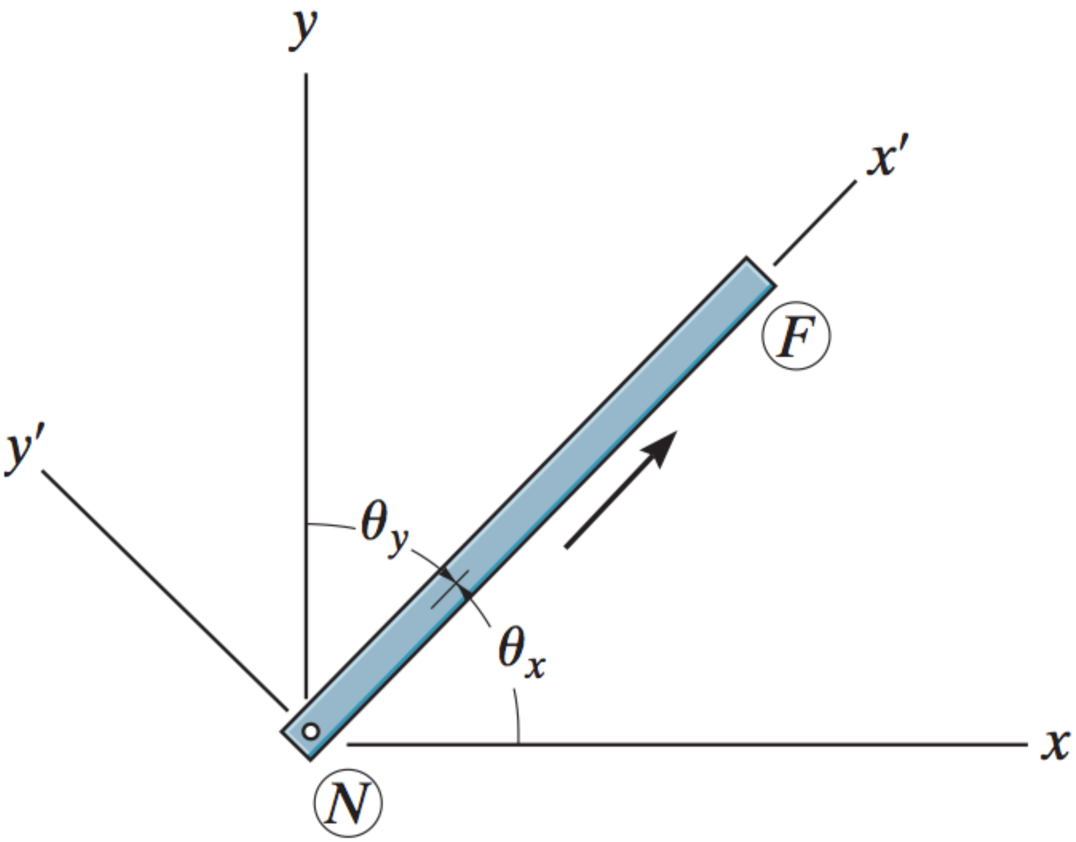
$$\begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} -100 \\ -80 \end{bmatrix}\begin{bmatrix} \dfrac{1}{180} \end{bmatrix}[30] = \begin{bmatrix} -16.67 \\ -13.33 \end{bmatrix}$$

where $$Q_2 = \frac{1}{180}(-100)30 = -16.67 \text{ kips}$$

$$Q_3 = \frac{1}{180}(-80)30 = -13.33 \text{ kips}$$

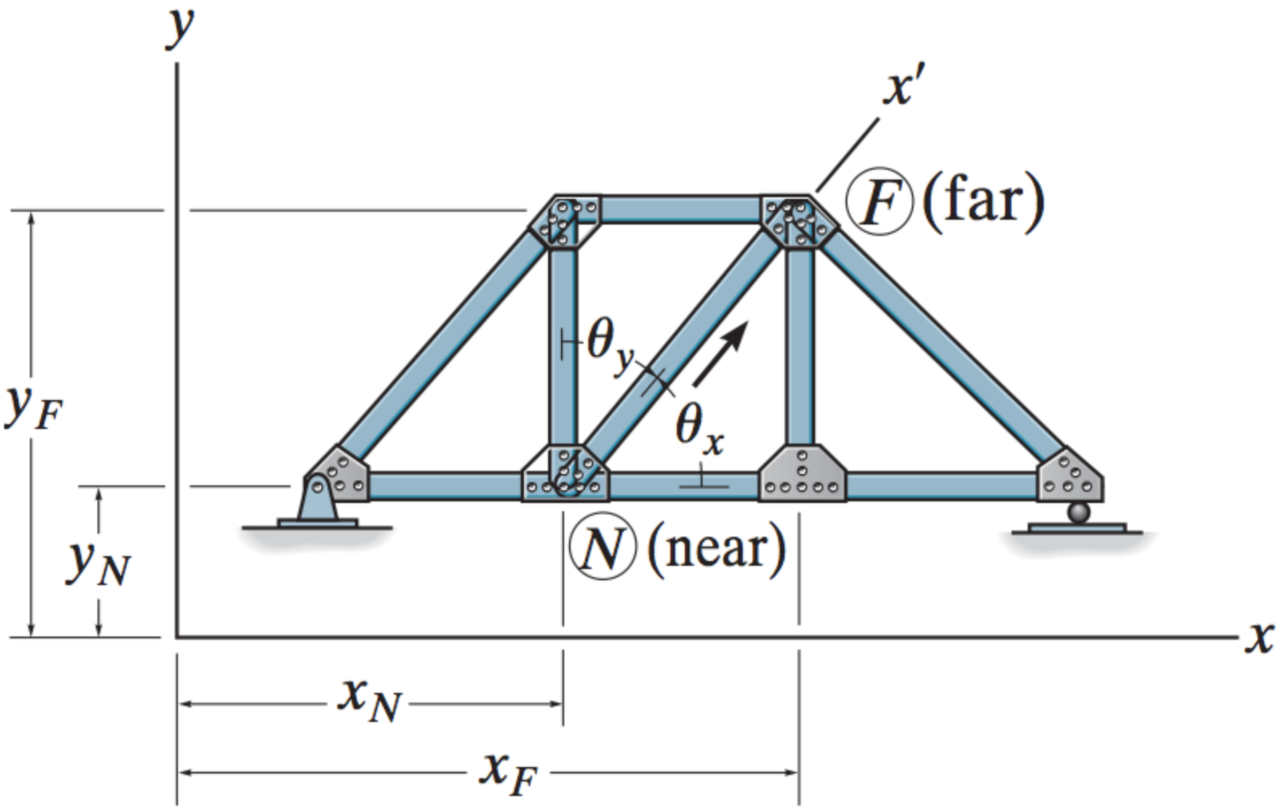
- Therefore, the reactions at joints 2 and 3 are -16.67 and -13.33 kips, respectively. The minus signs indicate that the forces act to the left.

Local to Global



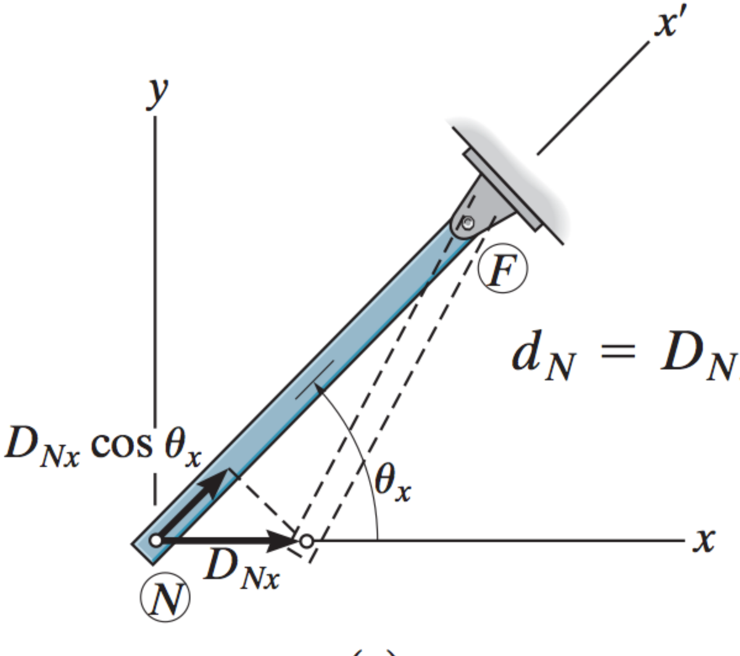
-

Local to Global



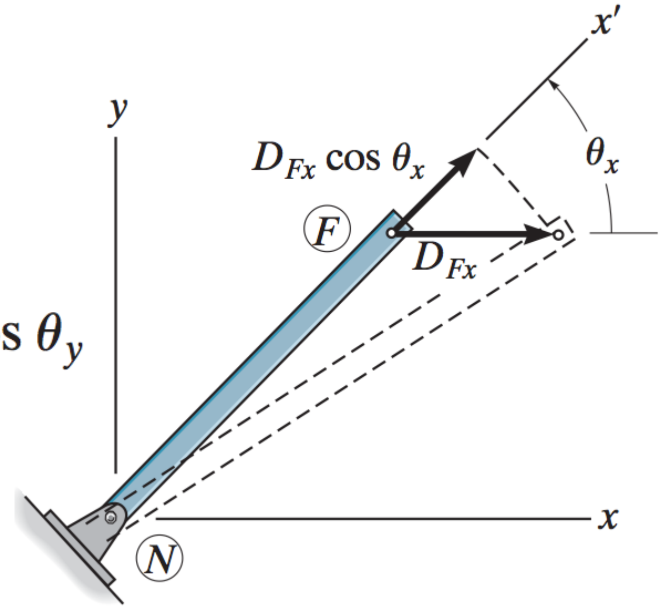
$$\lambda_x = \cos\theta_x = \frac{x_F - x_N}{L} = \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

$$\lambda_y = \cos\theta_y = \frac{y_F - y_N}{L} = \frac{y_F - y_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

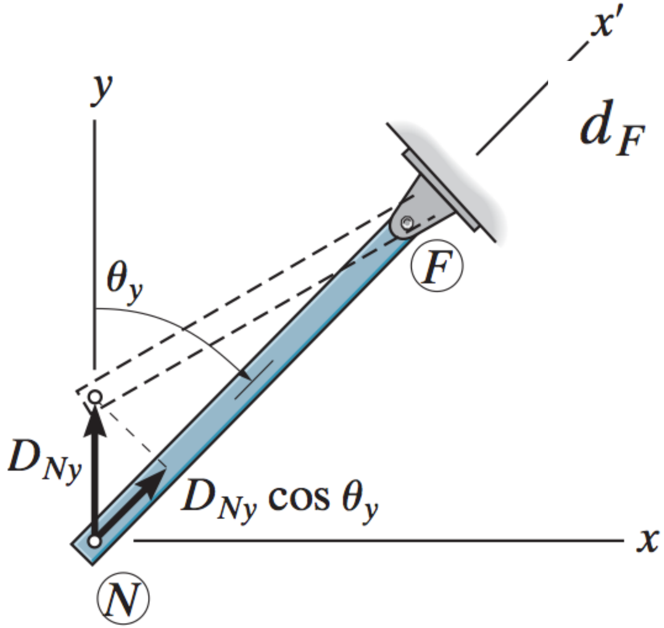


(a)

$$d_N = D_{Nx}\cos\theta_x + D_{Ny}\cos\theta_y$$

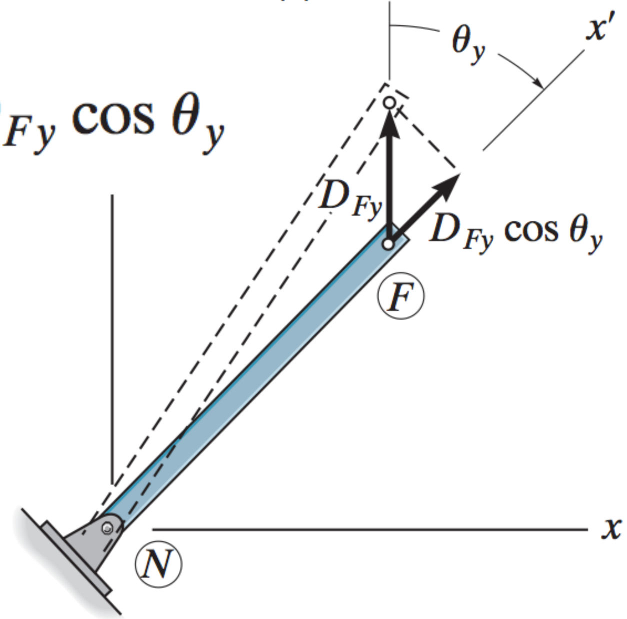


(c)



(b)

$$d_F = D_{Fx}\cos\theta_x + D_{Fy}\cos\theta_y$$



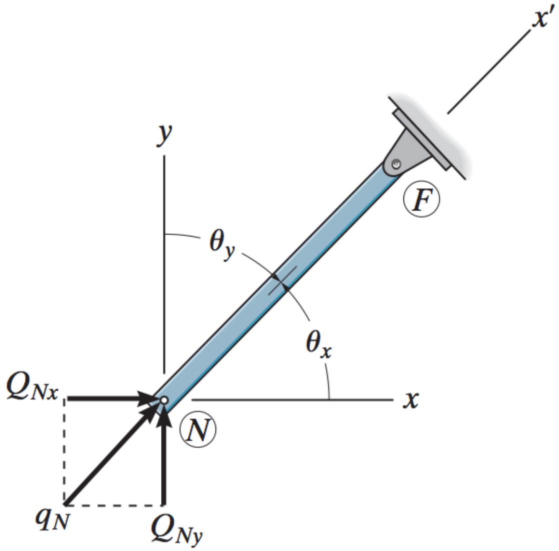
Displacement Transformation

$$\begin{bmatrix} d_N \\ d_F \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix} \begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{bmatrix}$$

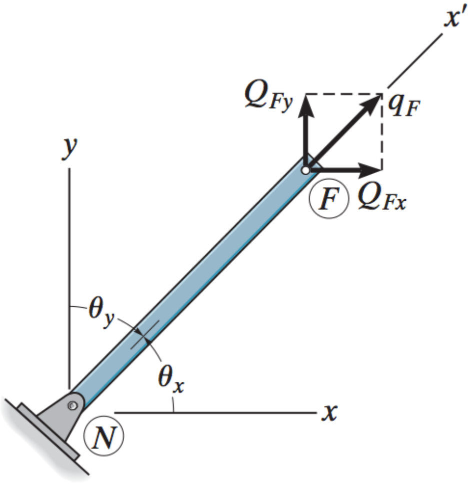
$$\boxed{\mathbf{d} = \mathbf{TD}}$$

$$\mathbf{T} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

Force Transformation



(a)



$$Q_{Nx} = q_N \cos\theta_x \qquad Q_{Ny} = q_N \cos\theta_y$$

$$Q_{Fx} = q_F \cos\theta_x \qquad Q_{Fy} = q_F \cos\theta_y$$

$$Q_{Nx} = q_N\lambda_x \qquad Q_{Ny} = q_N\lambda_y$$

$$Q_{Fx} = q_F\lambda_x \qquad Q_{Fy} = q_F\lambda_y$$

$$\begin{bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx} \\ Q_{Fy} \end{bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \begin{bmatrix} q_N \\ q_F \end{bmatrix}$$

Force Transformation

$$\boxed{\mathbf{Q} = \mathbf{T}^T \mathbf{q}}$$

$$\mathbf{T}^T = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix}$$

Global Transformation

$$\boxed{\mathbf{q} = \mathbf{k}'\mathbf{T}\mathbf{D}}$$

$$\mathbf{Q} = \mathbf{T}^T\mathbf{k}'\mathbf{T}\mathbf{D}$$

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{D}}$$

$$\boxed{\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T}}$$

Global Transformation

$$\mathbf{k} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

$$\mathbf{k} = \frac{AE}{L} \begin{array}{cc} \begin{array}{cccc} N_x & N_y & F_x & F_y \end{array} & \\ \left[\begin{array}{cccc} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{array} \right] & \begin{array}{c} N_x \\ N_y \\ F_x \\ F_y \end{array} \end{array}$$

Truss Analysis

$$\mathbf{Q} = \mathbf{K}\mathbf{D}$$

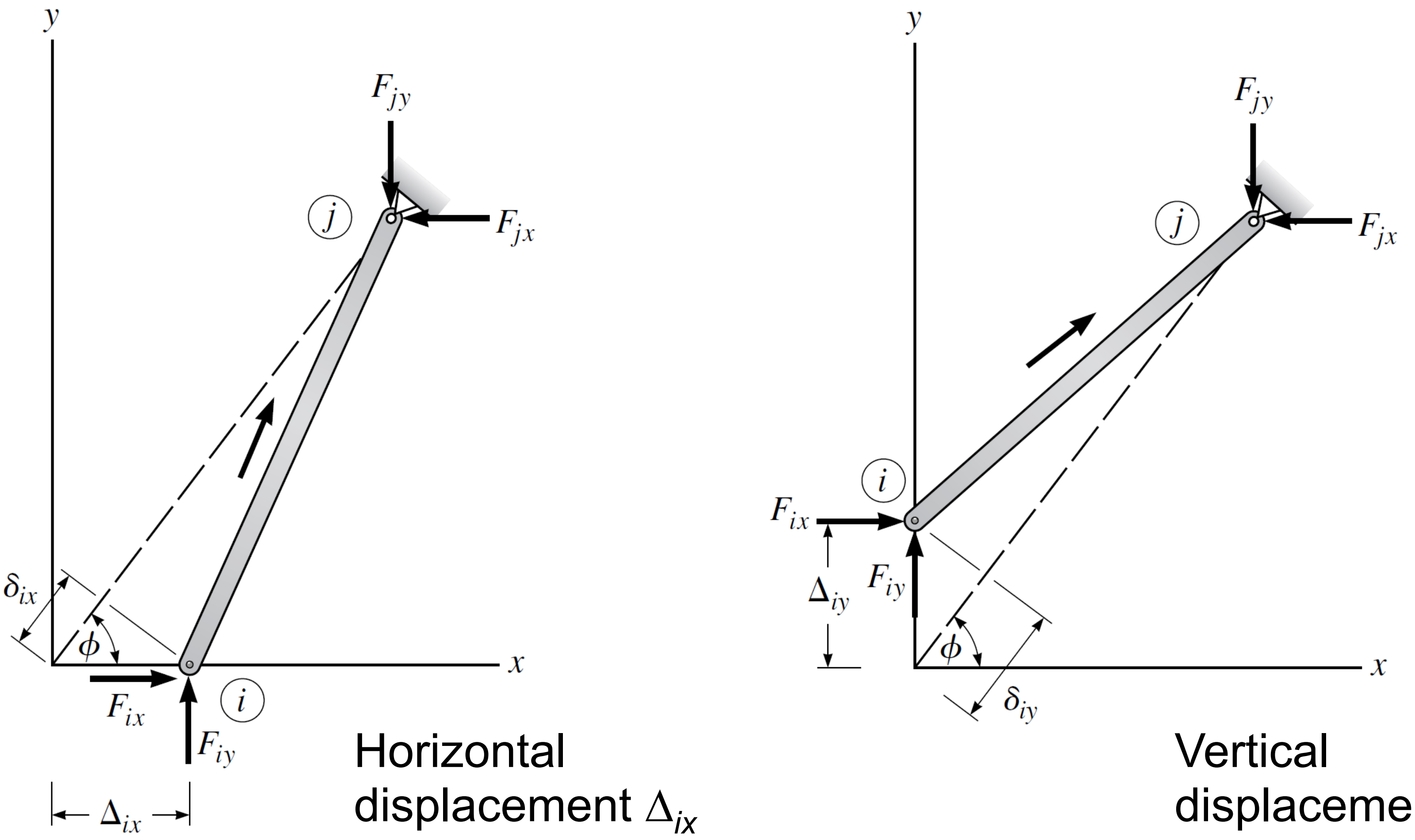
$$\left[\begin{array}{c} \mathbf{Q}_k \\ \hdashline \mathbf{Q}_u \end{array}\right] = \left[\begin{array}{c:c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{D}_u \\ \hdashline \mathbf{D}_k \end{array}\right]$$

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u$$

$$\mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u$$

$$q_F = \frac{AE}{L}\begin{bmatrix} -\lambda_x & -\lambda_y & \lambda_x & \lambda_y \end{bmatrix}\begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{bmatrix}$$

§17.6 Member Stiffness Matrix of an Inclined Truss Bar

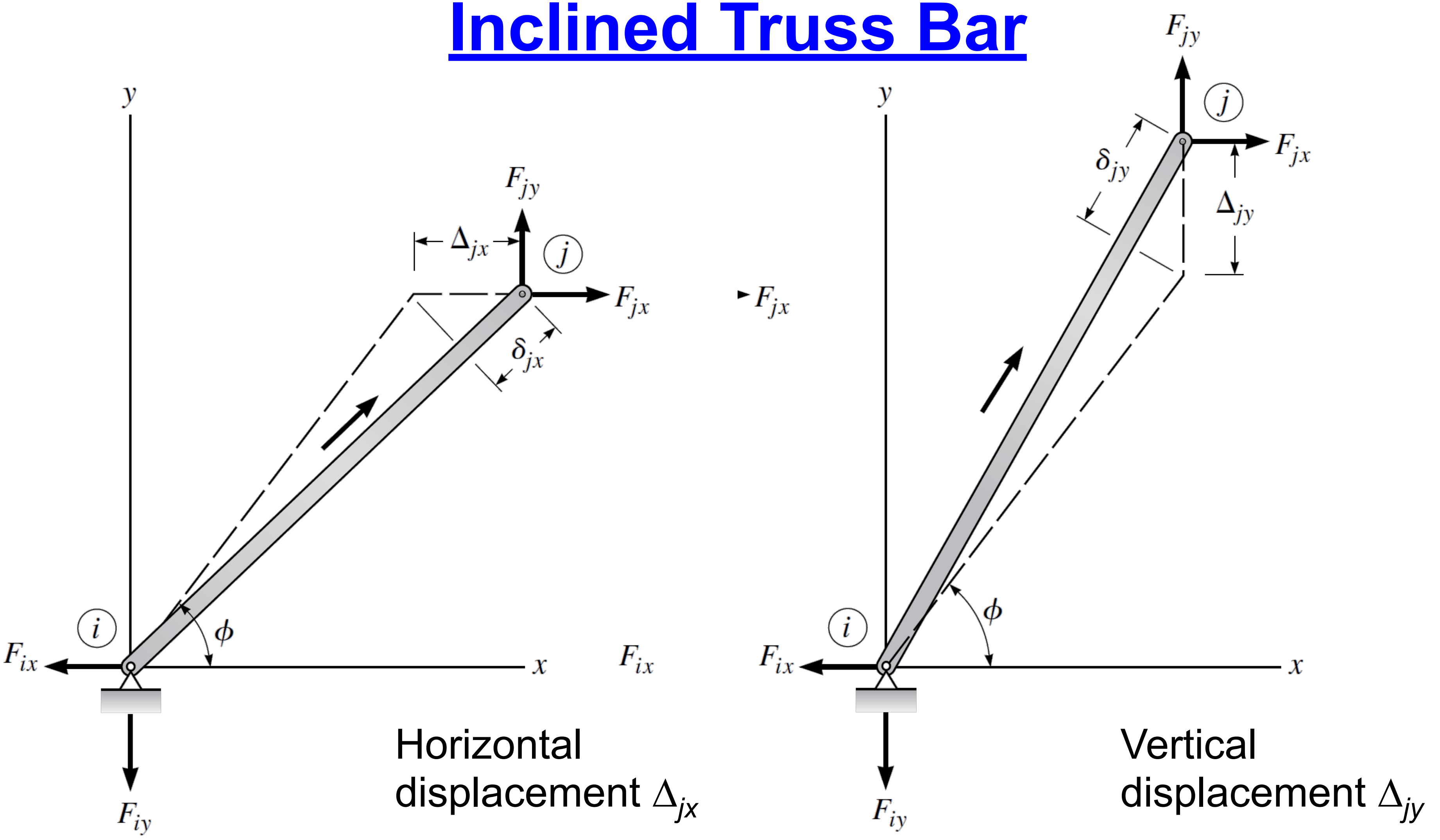


Horizontal displacement Δ_{ix}

Vertical displacement Δ_{iy}

Figure 17.6

§17.6 Member Stiffness Matrix of an Inclined Truss Bar



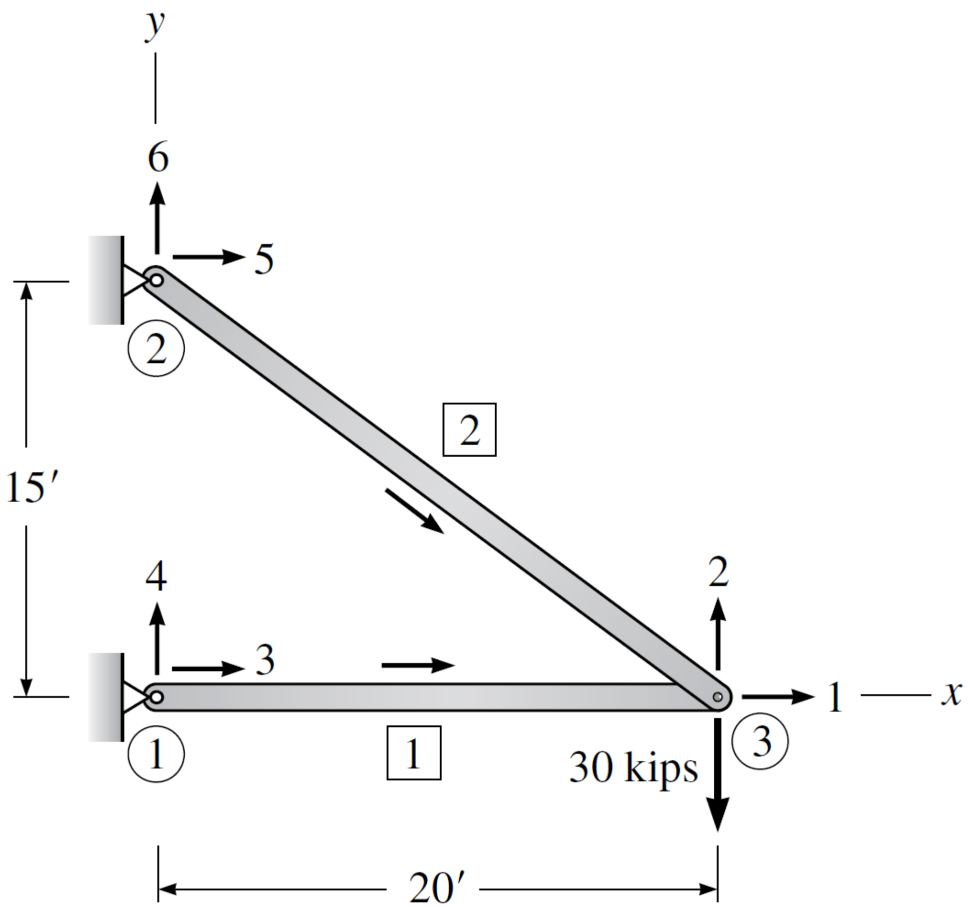
Horizontal displacement Δ_{jx}

Vertical displacement Δ_{jy}

Figure 17.6 (continued)

Example 17.2

Determine the joint displacements and bar forces of the truss in Figure17.7 by the direct stiffness method. Member properties: A_1 = 2 in^2, A_2 = 2.5 in^2, and E = 30,000 kips/in^2.



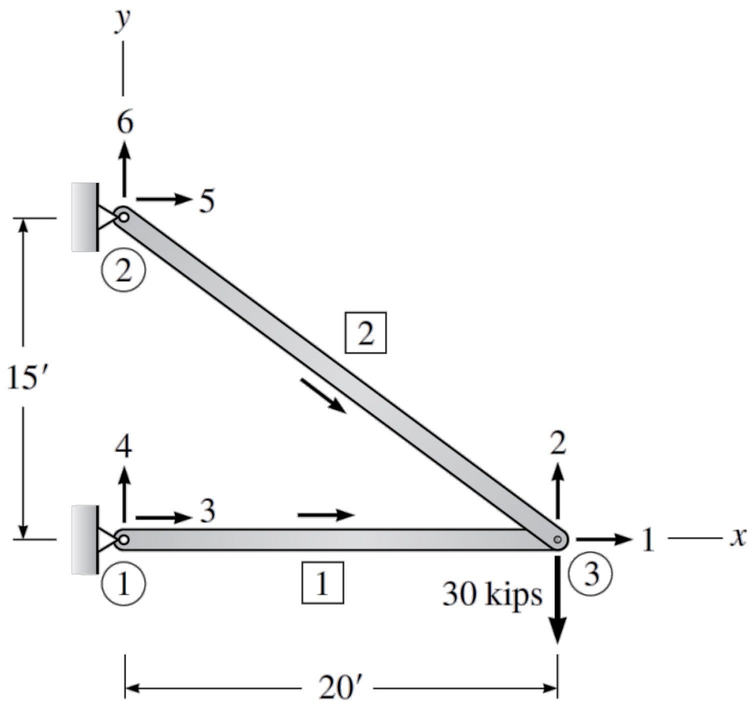
Example 17.2 Solution

- Construct member stiffness matrices. For member 1, joint 1 is the near joint and joint 3 is the far joint. Compute the sine and cosine of the slope angle with Equation 17.37.

$$\cos\phi = \frac{x_j - x_i}{L} = \frac{20 - 0}{20} = 1 \quad \text{and} \quad \sin\phi = \frac{y_i - y_i}{L} = \frac{0 - 0}{20} = 0$$

$$\frac{AE}{L} = \frac{2(30{,}000)}{20(12)} = 250 \text{ kips/in}$$

$$\mathbf{k}_1 = 250 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



Example 17.2 Solution (continued)

- For member 2, joint 2 is the near joint and joint 3 is the far joint:

$$\cos\phi = \frac{20 - 0}{25} = 0.8 \qquad \sin\phi = \frac{0 - 15}{25} = -0.6$$

$$\frac{AE}{L} = \frac{2.5(30{,}000)}{25(12)} = 250 \text{ kips/in}$$

$$\mathbf{k}_2 = 250 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array}$$

Example 17.2 Solution (continued)

- Set up the matrices for the force-displacement relationship of Equation 17.30 (that is, **Q** = **K**Δ). The structure stiffness matrix is assembled by inserting the elements of the member stiffness matrices $\mathbf{k}_1$ and $\mathbf{k}_2$ into the appropriate rows and columns.

$$\left[\begin{array}{l} Q_1 = 0 \\ Q_2 = -30 \\ \hline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array}\right] = 250 \left[\begin{array}{cc|cccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1.64 & -0.48 & -1 & 0 & 0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ \hline -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{array}\right] \left[\begin{array}{l} \Delta_1 \\ \Delta_2 \\ \hline \Delta_3 = 0 \\ \Delta_4 = 0 \\ \Delta_5 = 0 \\ \Delta_6 = 0 \end{array}\right]$$

Example 17.2 Solution (continued)

- Partition the matrices and solve for the unknown displacements Δ_1 and Δ_2 by using Equation 17.33.

$$\mathbf{Q}_f = \mathbf{K}_{11}\boldsymbol{\Delta}_f$$

$$\begin{bmatrix} 0 \\ -30 \end{bmatrix} = 250\begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

- Solving for the displacements gives

$$\boldsymbol{\Delta}_f = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} -0.16 \\ -0.547 \end{bmatrix}$$

- Substitute the values of 1 and 2 into Equation 17.34 and solve for the support reactions $\boldsymbol{Q_s}$.

$$\mathbf{Q}_s = \mathbf{K}_{21}\boldsymbol{\Delta}_f$$

$$\begin{bmatrix} Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = 250\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -0.64 & 0.48 \\ 0.48 & -0.36 \end{bmatrix}\begin{bmatrix} -0.16 \\ -0.547 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \\ -40 \\ 30 \end{bmatrix}$$

Example 17.2 Solution (continued)

- Compute member end displacements *D* in terms of member coordinates with Equation 17.51. For bar 1, *i* = joint 1 and *j* = joint 3, cos *Φ* = 1, and sin *Φ* = 0.

$$\begin{bmatrix} \delta_1 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta_3 = 0 \\ \Delta_4 = 0 \\ \Delta_1 = -0.16 \\ \Delta_2 = -0.547 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.16 \end{bmatrix} \qquad \textbf{Ans.}$$

- Substituting these values of *δ* into Equation 17.53, we compute the bar force in member 1 as

$$F_{13} = 250[0 \ -0.16]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -40 \text{ kips (compression)} \qquad \textbf{Ans.}$$

Example 17.2 Solution (continued)

- For bar 2, *i* = joint 2 and *j* = joint 3, cos *Φ* = 0.8, and sin *Φ*= 0.6.

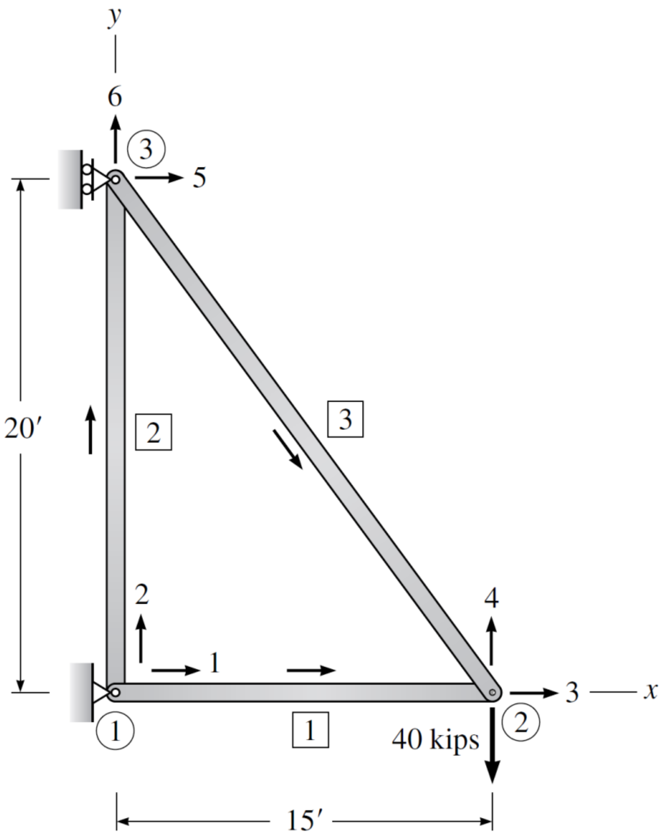
$$\begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} \Delta_5 = 0 \\ \Delta_6 = 0 \\ \Delta_1 = -0.16 \\ \Delta_2 = -0.547 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.20 \end{bmatrix}$$

- Substituting into Equation 17.53 yields

$$F_{23} = 250[0 \;\; 0.20]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 50 \text{ kips (tension)} \qquad \textbf{Ans.}$$

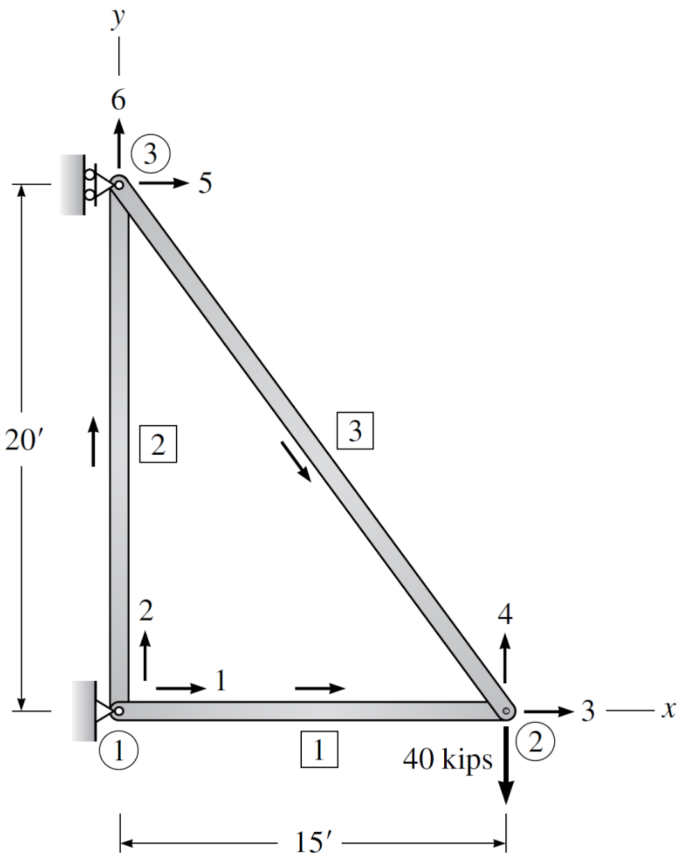
Example 17.3

Analyze the truss in Figure 17.8 by the direct stiffness method. Construct the structure stiffness matrix without considering if joints are restrained or unrestrained against displacement. Then, rearrange the terms and partition the matrix so that the unknown joint displacements Δ_f can be determined by Equation 17.30. Use $k_1 = k_2 = AE/L = 250$ kips/in and $k_3 = 2AE/L = 500$ kips/in.



Example 17.3 Solution

- Number the joints arbitrarily. Arrows are shown along the axis of each truss bar to indicate the direction from the near end to the far end of the member. Superimpose on the truss a global coordinate system with origin at joint 1. Form the member stiffness matrices using Equation 17.48. For bar 1, *i* = joint 1 and *j* = joint 2. Using Equation 17.37,



$$\cos\phi = \frac{x_j - x_i}{L} = \frac{15 - 0}{15} = 1$$

$$\sin\phi = \frac{y_j - y_i}{L} = \frac{0 - 0}{15} = 0$$

$$\mathbf{k}_1 = 250 \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Example 17.3 Solution (continued)

- For bar 2, *i* = joint 1 and *j* = joint 3.

$$\cos\phi = \frac{0 - 0}{20} = 0 \qquad \sin\phi = \frac{20 - 0}{20} = 1$$

$$\mathbf{k}_2 = 250 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

- For bar 3, *i* = joint 3 and *j* = joint 2.

$$\cos\phi = \frac{15 - 0}{25} = 0.6 \qquad \sin\phi = \frac{0 - 20}{25} = -0.8$$

$$\mathbf{k}_3 = 500 \begin{array}{c} \begin{array}{cccc} 5 & 6 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ -0.36 & 0.48 & 0.36 & -0.48 \\ 0.48 & -0.64 & -0.48 & 0.64 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 3 \\ 4 \end{array}$$

Example 17.3 Solution (continued)

- Add $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ by inserting the elements of the member stiffness matrices into the structure stiffness matrix at the appropriate locations. Multiply the elements of $\mathbf{k}_3$ by 2 so that all matrices are multiplied by the same scalar *AE/L*, i.e. 250.

$$\mathbf{K} = 250 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.72 & -0.96 & -0.72 & 0.96 \\ 0 & 0 & -0.96 & 1.28 & 0.96 & -1.28 \\ 0 & 0 & -0.72 & 0.96 & 0.72 & -0.96 \\ 0 & -1 & 0.96 & -1.28 & -0.96 & 2.28 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Example 17.3 Solution (continued)

- Establish the force-displacement matrices of Equation 17.30 by shifting the rows and columns of the structure stiffness matrix so that elements associated with the joints that displace (i.e., direction components 3, 4, and 6) are located in the upper left corner.

$$\left[\begin{array}{l} Q_3=0 \\ Q_4=-40 \\ Q_6=0 \\ \hdashline Q_1 \\ Q_2 \\ Q_5 \end{array}\right] = 250 \begin{array}{c} \begin{array}{cccccc} \;\;3\;\; & \;\;4\;\; & \;\;6\;\; & \;\;1\;\; & \;\;2\;\; & \;\;5\;\; \end{array} \\ \left[\begin{array}{rrr:rrr} 1.72 & -0.96 & 0.96 & -1 & 0 & -0.72 \\ -0.96 & 1.28 & -1.28 & 0 & 0 & 0.96 \\ 0.96 & -1.28 & 2.28 & 0 & -1 & -0.96 \\ \hdashline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -0.72 & 0.96 & -0.96 & 0 & 0 & 0.72 \end{array}\right] \end{array} \left[\begin{array}{l} \Delta_3 \\ \Delta_4 \\ \Delta_6 \\ \hdashline \Delta_1=0 \\ \Delta_2=0 \\ \Delta_5=0 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 6 \\ 1 \\ 2 \\ 5 \end{array}$$

Example 17.3 Solution (continued)

- Partition the matrix and solve for the unknown joint displacements, using Equation 17.33.

$$\mathbf{Q}_f = \mathbf{K}_{11}\boldsymbol{\Delta}_f$$

$$\begin{bmatrix} 0 \\ -40 \\ 0 \end{bmatrix} = 250 \begin{bmatrix} 1.72 & -0.96 & 0.96 \\ -0.96 & 1.28 & -1.28 \\ 0.96 & -1.28 & 2.28 \end{bmatrix} \begin{bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_6 \end{bmatrix}$$

- Solving the set of equations above gives

$$\begin{bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_6 \end{bmatrix} = \begin{bmatrix} -0.12 \\ -0.375 \\ -0.16 \end{bmatrix} \qquad \textbf{Ans.}$$

Example 17.3 Solution (continued)

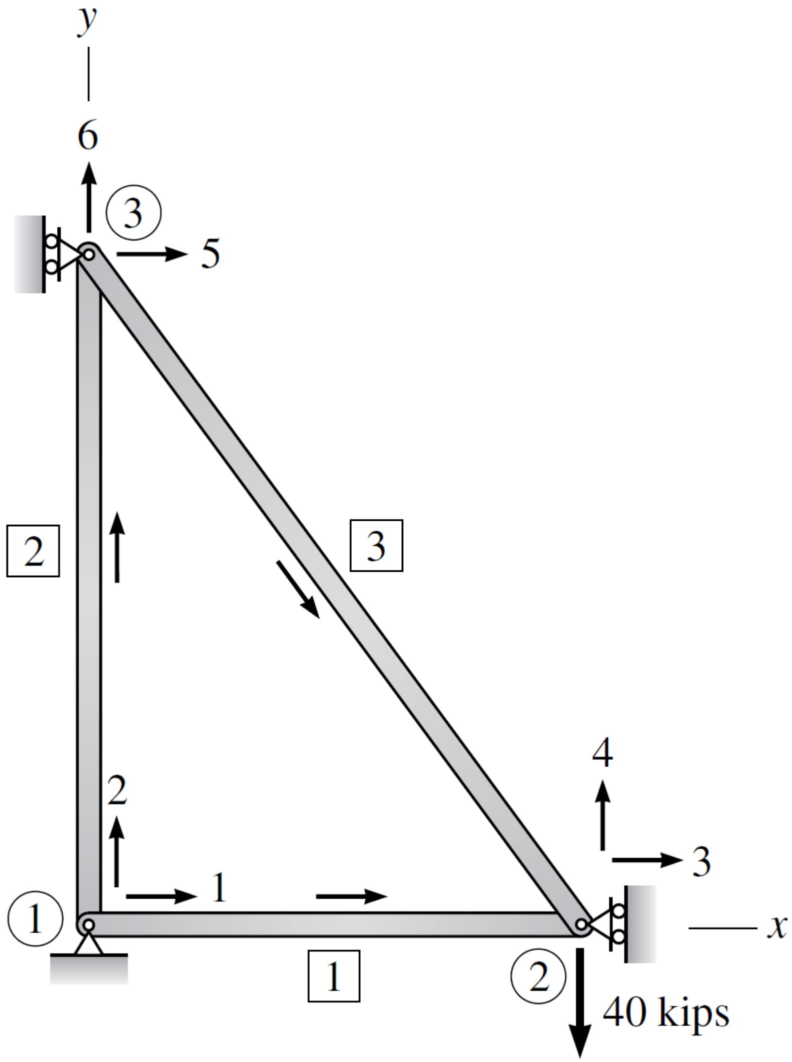
- Solve for the support reactions, using Equation 17.34.

$$\mathbf{Q}_s = \mathbf{K}_{21}\boldsymbol{\Delta}_f$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_5 \end{bmatrix} = 250\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ -0.72 & 0.96 & -0.96 \end{bmatrix}\begin{bmatrix} -0.12 \\ -0.375 \\ -0.16 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \\ -30 \end{bmatrix} \qquad \textbf{Ans.}$$

Example 17.4

If the horizontal displacement of joint 2 of the truss in Example 17.3 is restrained by the addition of a roller (see Figure 17.9), determine the reactions.



Example 17.4 Solution

- Although the addition of an extra support creates an indeterminate structure, the solution is carried out in the same manner. The rows and columns associated with the degrees of freedom that are free to displace are shifted to the upper left corner of the structure stiffness matrix.

$$\left[\begin{array}{l} Q_4=-40 \\ Q_6=0 \\ \hdashline Q_1 \\ Q_2 \\ Q_3 \\ Q_5 \end{array}\right] = 250 \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 1 & 2 & 3 & 5 \end{array} \\ \left[\begin{array}{cc:cccc} 1.28 & -1.28 & 0 & 0 & -0.96 & 0.96 \\ -1.28 & 2.28 & 0 & 0 & 0.96 & -0.96 \\ \hdashline 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.96 & 0.96 & -1 & 0 & 1.72 & -0.72 \\ 0.96 & -0.96 & 0 & 0 & -0.72 & 0.72 \end{array}\right] \end{array} \left[\begin{array}{l} \Delta_4 \\ \Delta_6 \\ \hdashline \Delta_1=0 \\ \Delta_2=0 \\ \Delta_3=0 \\ \Delta_5=0 \end{array}\right] \begin{array}{c} 4 \\ 6 \\ 1 \\ 2 \\ 3 \\ 5 \end{array}$$

Example 17.4 Solution (continued)

- Partition the matrix and solve for the unknown joint displacements

$$\mathbf{Q}_f = \mathbf{K}_{11}\boldsymbol{\Delta}_f$$

$$\begin{bmatrix} -40 \\ 0 \end{bmatrix} = 250 \begin{bmatrix} 1.28 & -1.28 \\ -1.28 & 2.28 \end{bmatrix} \begin{bmatrix} \Delta_4 \\ \Delta_6 \end{bmatrix}$$

- Solution of the set of equations above gives

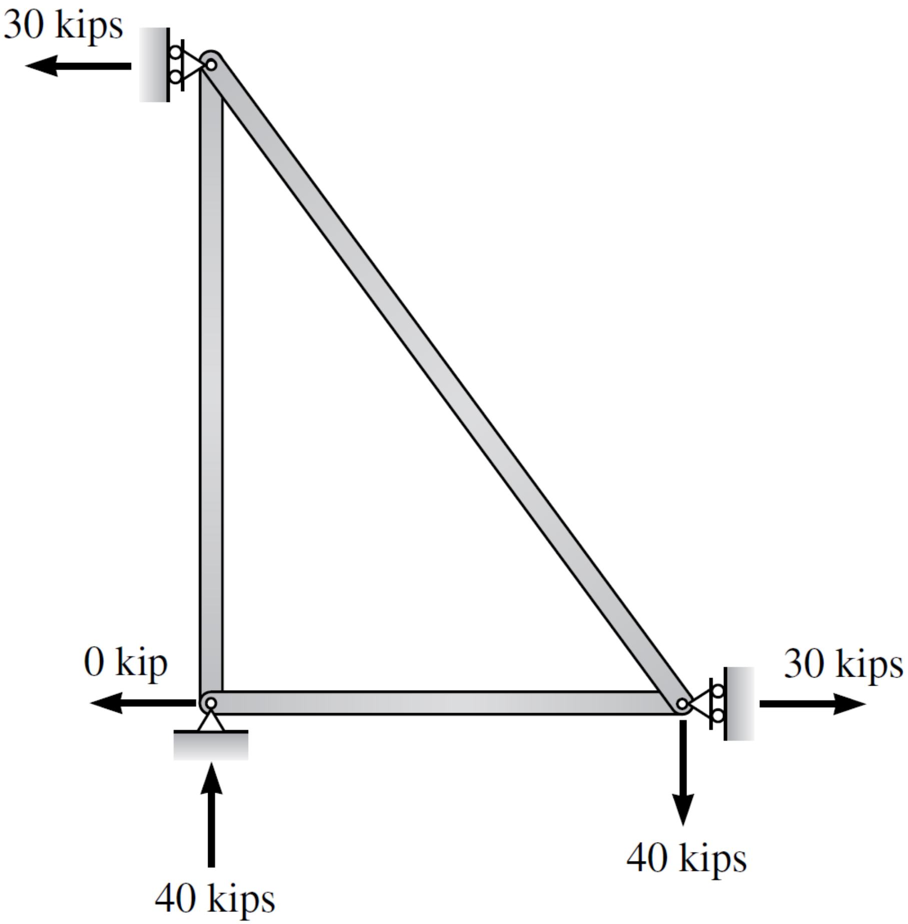
$$\begin{bmatrix} \Delta_4 \\ \Delta_6 \end{bmatrix} = \begin{bmatrix} -0.285 \\ -0.160 \end{bmatrix}$$

- Solve for the reactions using Equation 17.34.

$$\mathbf{Q}_s = \mathbf{K}_{21}\boldsymbol{\Delta}_f$$

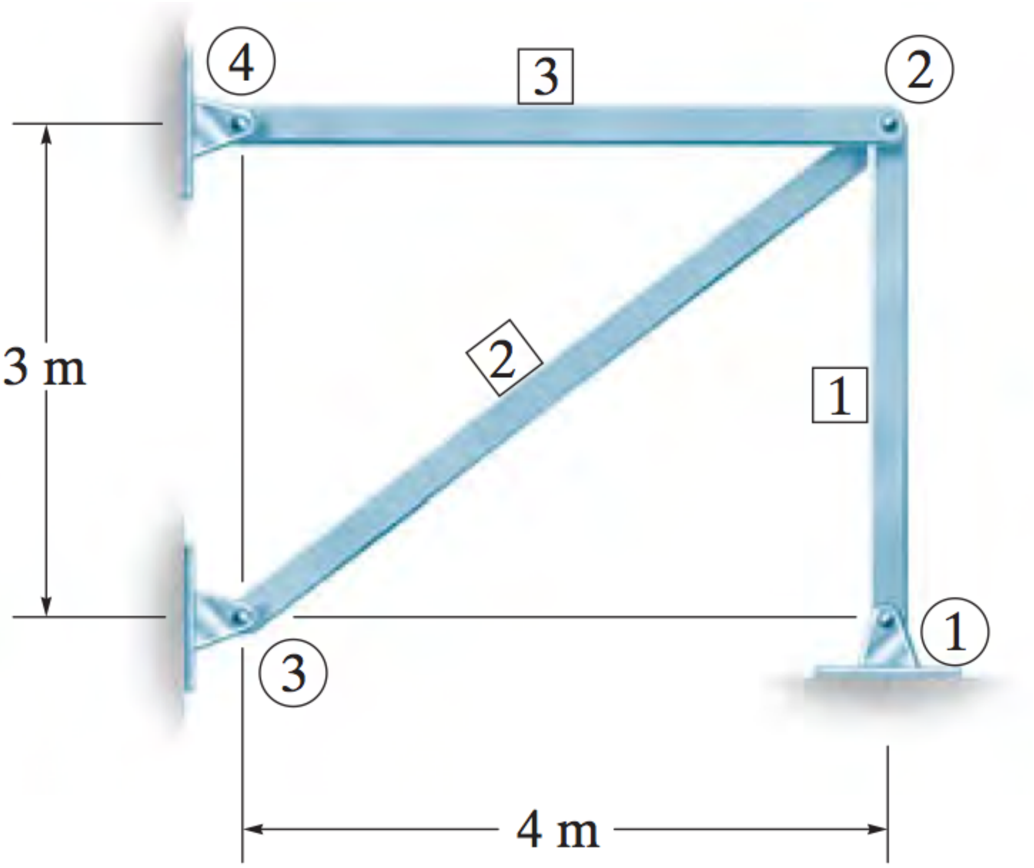
$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_5 \end{bmatrix} = 250 \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ -0.96 & 0.96 \\ 0.96 & -0.96 \end{bmatrix} \begin{bmatrix} -0.285 \\ -0.160 \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \\ 30 \\ -30 \end{bmatrix} \qquad \textbf{Ans.}$$

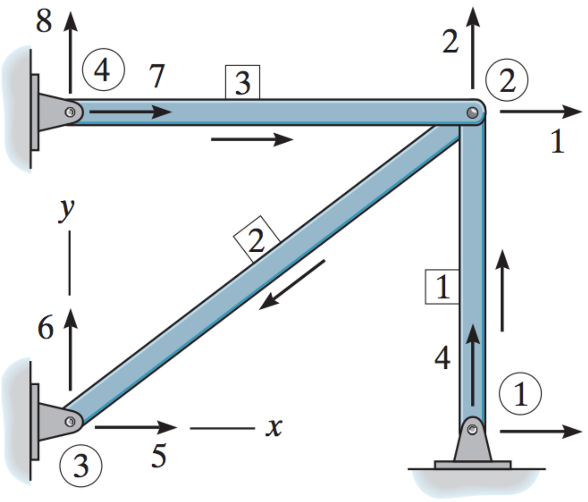
Example 17.4 Solution (continued)



Results of analysis

Determine the force in member 2 of the assembly in Fig. 14–11*a* if the support at joint ① settles *downward* 25 mm. Take $AE = 8(10^3)$ kN.





$$\mathbf{D}_k = \begin{array}{cc} \begin{bmatrix} 0 \\ -0.025 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \end{array} \qquad \mathbf{Q}_k = \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

1: $\lambda_x = 0, \lambda_y = 1, L = 3$ m, so that

$$\mathbf{k}_1 = AE \begin{array}{cc} \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.333 & 0 & -0.333 \\ 0 & 0 & 0 & 0 \\ 0 & -0.333 & 0 & 0.333 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} \end{array}$$

2: $\lambda_x = -0.8, \lambda_y = -0.6, L = 5$ m, so that

$$\mathbf{k}_2 = AE \begin{array}{cc} \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 0.128 & 0.096 & -0.128 & -0.096 \\ 0.096 & 0.072 & -0.096 & -0.072 \\ -0.128 & -0.096 & 0.128 & 0.096 \\ -0.096 & -0.072 & 0.096 & 0.072 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} \end{array}$$

3: $\lambda_x = 1, \lambda_y = 0, L = 4$ m, so that

$$\mathbf{k}_3 = AE \begin{array}{cc} \begin{matrix} 7 & 8 & 1 & 2 \end{matrix} & \\ \begin{bmatrix} 0.25 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \\ -0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix} \end{array}$$

Displacements and Loads. Here $\mathbf{Q} = \mathbf{K}\mathbf{D}$ yields

$$\begin{bmatrix} 0 \\ 0 \\ \hline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix} = AE \left[\begin{array}{cc|cccccc} 0.378 & 0.096 & 0 & 0 & -0.128 & -0.096 & -0.25 & 0 \\ 0.096 & 0.405 & 0 & -0.333 & -0.096 & -0.072 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.333 & 0 & 0.333 & 0 & 0 & 0 & 0 \\ -0.128 & -0.096 & 0 & 0 & 0.128 & 0.096 & 0 & 0 \\ -0.096 & -0.072 & 0 & 0 & 0.096 & 0.072 & 0 & 0 \\ -0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} D_1 \\ D_2 \\ \hline 0 \\ -0.025 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = AE \begin{bmatrix} 0.378 & 0.096 \\ 0.096 & 0.405 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + AE \begin{bmatrix} 0 & 0 & -0.128 & -0.096 & -0.25 & 0 \\ 0 & -0.333 & -0.096 & -0.072 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -0.025 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 = AE[(0.378D_1 + 0.096D_2) + 0]$$

$$0 = AE[(0.096D_1 + 0.405D_2) + 0.00833]$$

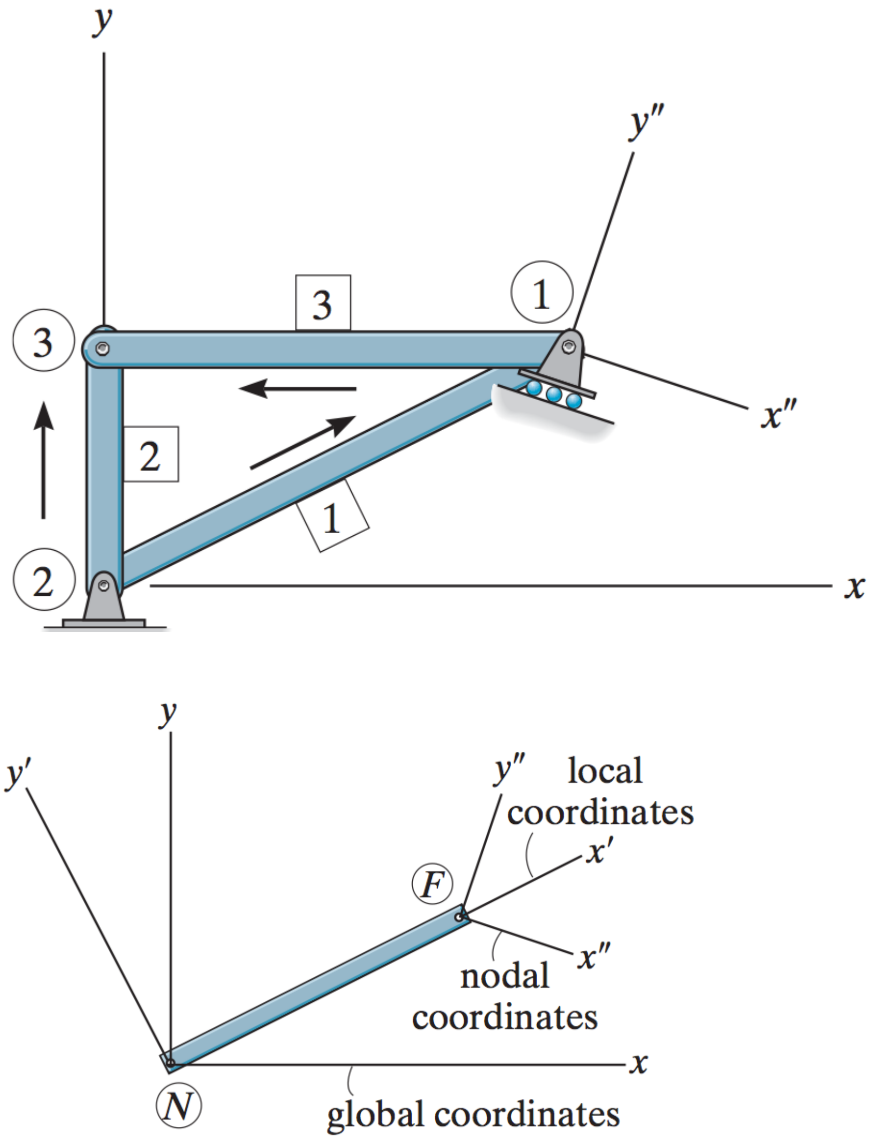
Member 2: $\lambda_x = -0.8$, $\lambda_y = -0.6$, $L = 5$ m, $AE = 8(10^3)$ kN, so that

$$q_2 = \frac{8(10^3)}{5}[0.8 \quad 0.6 \quad -0.8 \quad -0.6]\begin{bmatrix} 0.00556 \\ -0.021875 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{8(10^3)}{5}(0.00444 - 0.0131) = -13.9 \text{ kN}$$ *Ans.*

-

Inclined Supports Using Nodal Coordinates

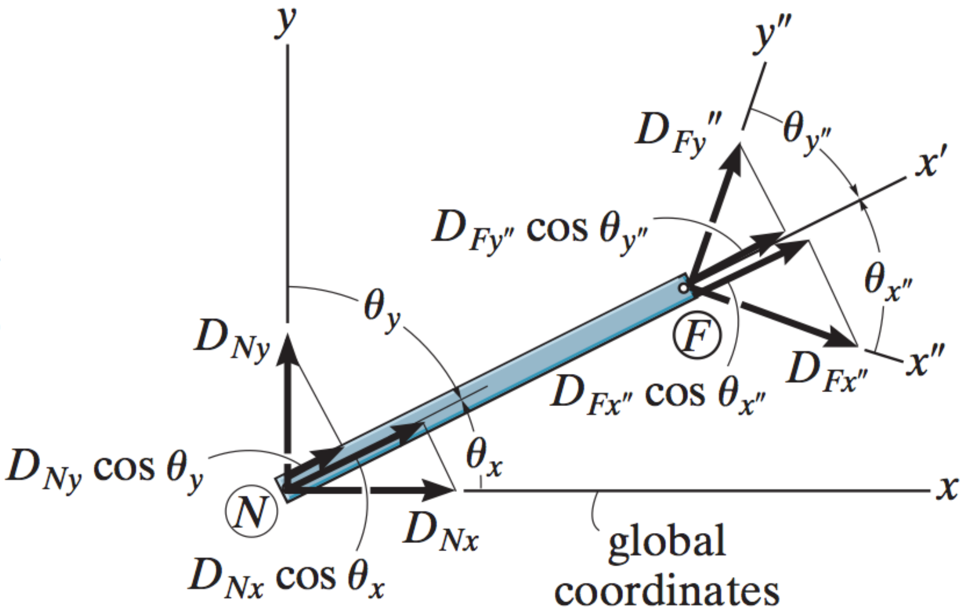


(b)

Inclined Supports Using Nodal Coordinates

$$d_N = D_{Nx}\cos\theta_x + D_{Ny}\cos\theta_y$$

$$= D_{Fx''}\cos\theta_{x''} + D_{Fy''}\cos\theta_{y''}$$

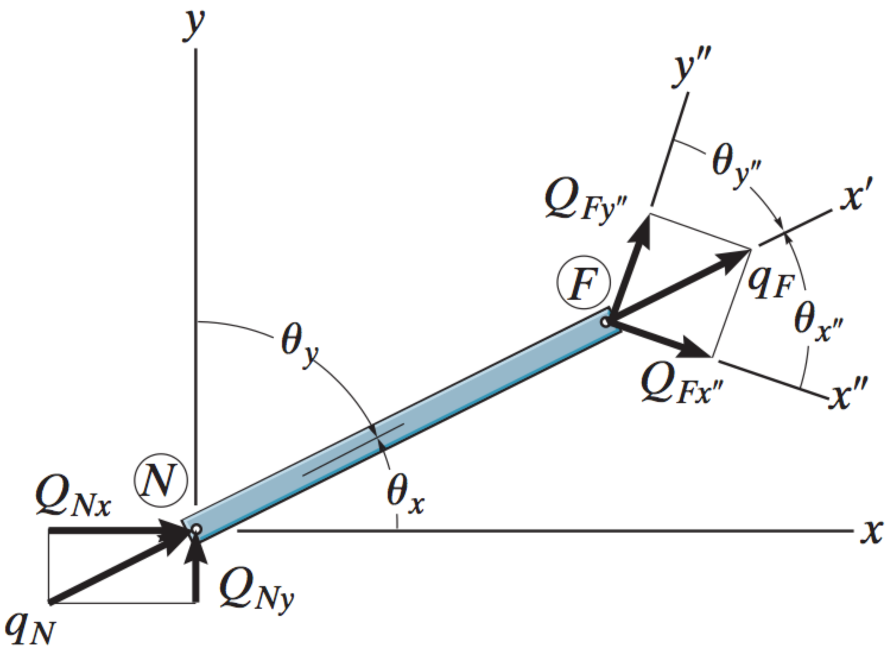


$$\begin{bmatrix} d_N \\ d_F \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_{x''} & \lambda_{y''} \end{bmatrix} \begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx''} \\ D_{Fy''} \end{bmatrix}$$

Inclined Supports Using Nodal Coordinates: Force Transformation

$$Q_{Nx} = q_N \cos \theta_x \qquad Q_{Ny} = q_N \cos \theta_y$$

$$Q_{Fx''} = q_F \cos \theta_{x''} \qquad Q_{Fy''} = q_F \cos \theta_{y''}$$



$$\begin{bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx''} \\ Q_{Fy''} \end{bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_{x''} \\ 0 & \lambda_{y''} \end{bmatrix} \begin{bmatrix} q_N \\ q_F \end{bmatrix}$$

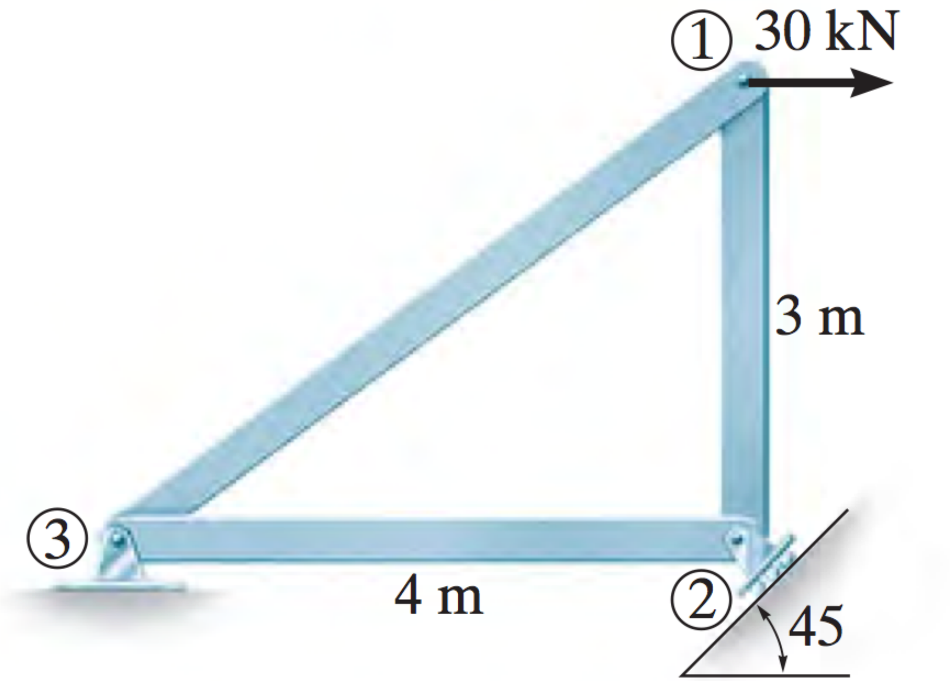
Resulting Stiffness Matrix

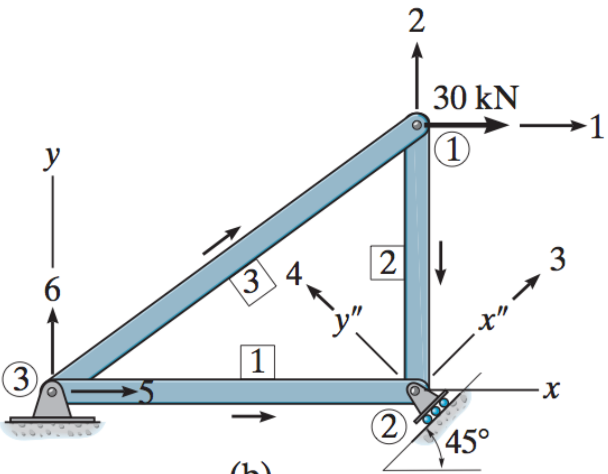
$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

$$\mathbf{k} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_{x''} \\ 0 & \lambda_{y''} \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_{x''} & \lambda_{y''} \end{bmatrix}$$

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}$$

Determine the support reactions for the truss

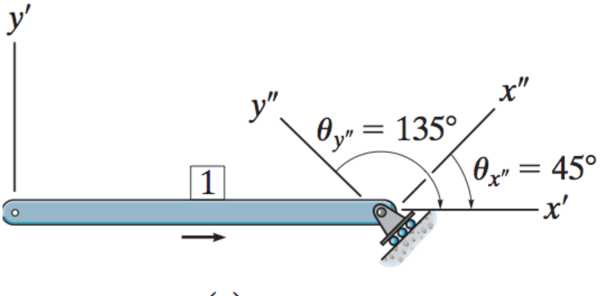




(b)

r 1. Fig. 14–13c, $\lambda_x = 1$, $\lambda_y = 0$, $\lambda_{x''} = 0.707$, $\lambda_{y''} = -0.707$

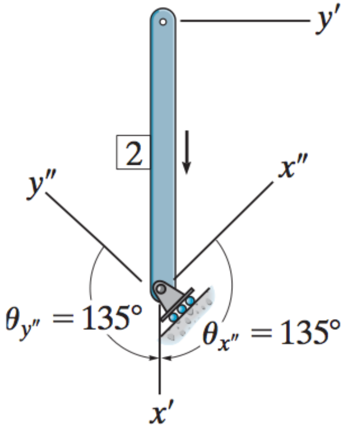
$$\mathbf{k}_1 = AE\begin{array}{c} \\ \left[\begin{array}{cccc} 5 & 6 & 3 & 4 \\ 0.25 & 0 & -0.17675 & 0.17675 \\ 0 & 0 & 0 & 0 \\ -0.17675 & 0 & 0.125 & -0.125 \\ 0.17675 & 0 & -0.125 & 0.125 \end{array}\right] \end{array}\begin{array}{c} \\ 5 \\ 6 \\ 3 \\ 4 \end{array}$$



(c)

2. Fig. 14–13d, $\lambda_x = 0$, $\lambda_y = -1$, $\lambda_{x''} = -0.707$, $\lambda_{y''} = -0.707$

$$\mathbf{k}_2 = AE\begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0.3333 & -0.2357 & -0.2357 \\ 0 & -0.2357 & 0.1667 & 0.1667 \\ 0 & -0.2357 & 0.1667 & 0.1667 \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$



Member 3. $\lambda_x = 0.8$, $\lambda_y = 0.6$

$$\mathbf{k}_3 = AE\begin{array}{c} \\ \left[\begin{array}{cccc} 5 & 6 & 1 & 2 \\ 0.128 & 0.096 & -0.128 & -0.096 \\ 0.096 & 0.072 & -0.096 & -0.072 \\ -0.128 & -0.096 & 0.128 & 0.096 \\ -0.096 & -0.072 & 0.096 & 0.072 \end{array}\right] \end{array}\begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array}$$

Global K matrix

$$\left[\begin{array}{c}30\\0\\0\\\hdashline Q_4\\Q_5\\Q_6\end{array}\right] = AE\left[\begin{array}{ccc:ccc}0.128 & 0.096 & 0 & 0 & -0.128 & -0.096\\0.096 & 0.4053 & -0.2357 & -0.2357 & -0.096 & -0.072\\0 & -0.2357 & 0.2917 & 0.0417 & -0.17675 & 0\\\hdashline 0 & -0.2357 & 0.0417 & 0.2917 & 0.17675 & 0\\-0.128 & -0.096 & -0.17675 & 0.17675 & 0.378 & 0.096\\-0.096 & -0.072 & 0 & 0 & 0.096 & 0.072\end{array}\right]\left[\begin{array}{c}D_1\\D_2\\D_3\\\hdashline 0\\0\\0\end{array}\right]$$

$$D_1 = \frac{352.5}{AE}$$

$$D_2 = \frac{-157.5}{AE}$$

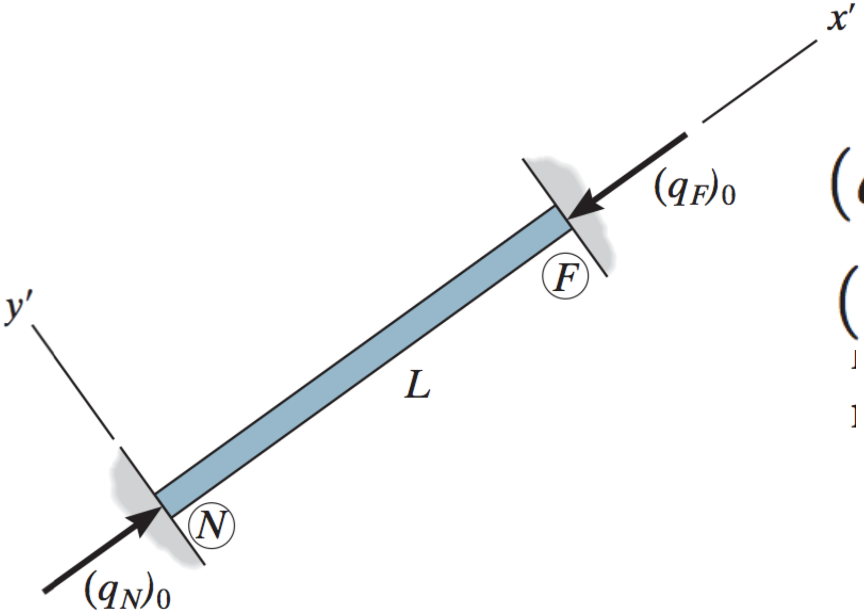
$$D_3 = \frac{-127.3}{AE}$$

$$\begin{aligned}Q_4 &= 0(352.5) - 0.2357(-157.5) + 0.0417(-127.3)\\&= 31.8\ \text{kN}\end{aligned}$$

$$\begin{aligned}Q_5 &= -0.128(352.5) - 0.096(-157.5) - 0.17675\ (-127.3)\\&= -7.5\ \text{kN}\end{aligned}$$

$$\begin{aligned}Q_6 &= -0.096(352.5) - 0.072(-157.5) + 0(-127.3)\\&= -22.5\ \text{kN}\end{aligned}$$

Temperature effects fabrication errors



$$(q_N)_0 = AE\alpha\Delta T$$

$$(q_F)_0 = -AE\alpha\Delta T$$

$$\begin{bmatrix} (Q_{Nx})_0 \\ (Q_{Ny})_0 \\ (Q_{Fx})_0 \\ (Q_{Fy})_0 \end{bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} AE\alpha\Delta T \begin{bmatrix} 1 \\ -1 \end{bmatrix} = AE\alpha\Delta T \begin{bmatrix} \lambda_x \\ \lambda_y \\ -\lambda_x \\ -\lambda_y \end{bmatrix}$$

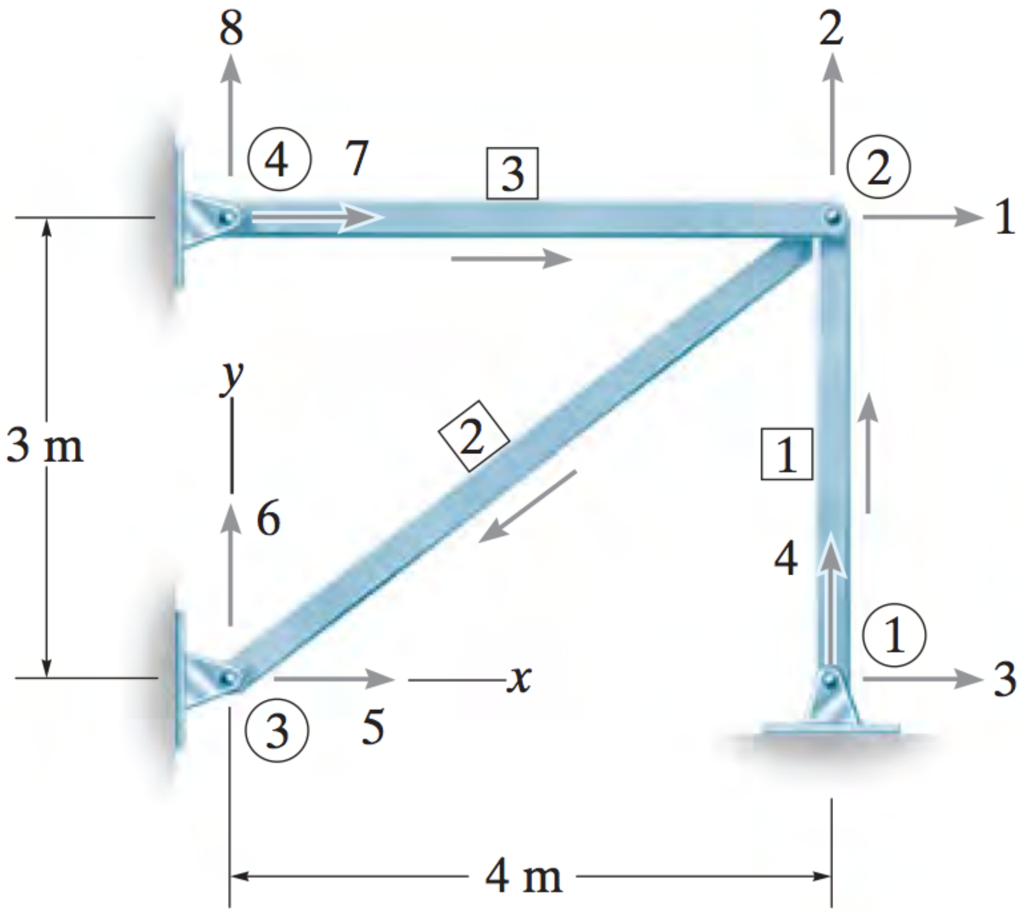
$$\left[\begin{array}{c} \mathbf{Q}_k \\ \hdashline \mathbf{Q}_u \end{array}\right] = \left[\begin{array}{c:c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{D}_u \\ \hdashline \mathbf{D}_k \end{array}\right] + \left[\begin{array}{c} (\mathbf{Q}_k)_0 \\ \hdashline (\mathbf{Q}_u)_0 \end{array}\right]$$

Force in the member

$$\mathbf{q} = \mathbf{k}'\mathbf{T}\mathbf{D} + \mathbf{q}_0$$

$$q_F = \frac{AE}{L}\begin{bmatrix} -\lambda_x & -\lambda_y & \lambda_x & \lambda_y \end{bmatrix}\begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{bmatrix} + (q_F)_0$$

Determine the force in members 1 and 2 of the pin-connected assembly of Fig. 14–15 if member 2 was made 0.01 m too short before it was fitted into place. Take $AE = 8(10^3)$ kN.



Since the member is short, then $\Delta L = -0.01$ m, and therefore applying Eq. 14–26 to member 2, with $\lambda_x = -0.8$, $\lambda_y = -0.6$, we have

$$\begin{bmatrix} (Q_1)_0 \\ (Q_2)_0 \\ (Q_5)_0 \\ (Q_6)_0 \end{bmatrix} = \frac{AE(-0.01)}{5}\begin{bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{bmatrix} = AE \begin{array}{c} \begin{bmatrix} 0.0016 \\ 0.0012 \\ -0.0016 \\ -0.0012 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\left[\begin{array}{c} 0 \\ 0 \\ \hline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{array}\right] = AE \left[\begin{array}{cc|cccccc} 0.378 & 0.096 & 0 & 0 & -0.128 & -0.096 & -0.25 & 0 \\ 0.096 & 0.405 & 0 & -0.333 & -0.096 & -0.072 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.333 & 0 & 0.333 & 0 & 0 & 0 & 0 \\ -0.128 & -0.096 & 0 & 0 & 0.128 & 0.096 & 0 & 0 \\ -0.096 & -0.072 & 0 & 0 & 0.096 & 0.072 & 0 & 0 \\ -0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} D_1 \\ D_2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] + AE \left[\begin{array}{r} 0.0016 \\ 0.0012 \\ \hline 0 \\ 0 \\ -0.0016 \\ -0.0012 \\ 0 \\ 0 \end{array}\right]$$

$$0 = AE[0.378D_1 + 0.096D_2] + AE[0] + AE[0.0016]$$

$$0 = AE[0.096D_1 + 0.405D_2] + AE[0] + AE[0.0012]$$

$$D_1 = -0.003704 \text{ m}$$

$$D_2 = -0.002084 \text{ m}$$

Member 1. $\lambda_x = 0, \lambda_y = 1, L = 3\text{ m}, AE = 8(10^3)\text{ kN}$, so that

$$q_1 = \frac{8(10^3)}{3}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}0 \\ 0 \\ -0.003704 \\ -0.002084\end{bmatrix} + [0]$$

$$q_1 = -5.56\text{ kN}$$ *Ans.*

Member 2. $\lambda_x = -0.8, \lambda_y = -0.6, L = 5\text{ m}, AE = 8(10^3)\text{ kN}$, so

$$q_2 = \frac{8(10^3)}{5}\begin{bmatrix}0.8 & 0.6 & -0.8 & -0.6\end{bmatrix}\begin{bmatrix}-0.003704 \\ -0.002084 \\ 0 \\ 0\end{bmatrix} - \frac{8(10^3)\,(-0.01)}{5}$$

$$q_2 = 9.26\text{ kN}$$ *Ans.*

§17.7 Coordinate Transformation of a Member Stiffness Matrix

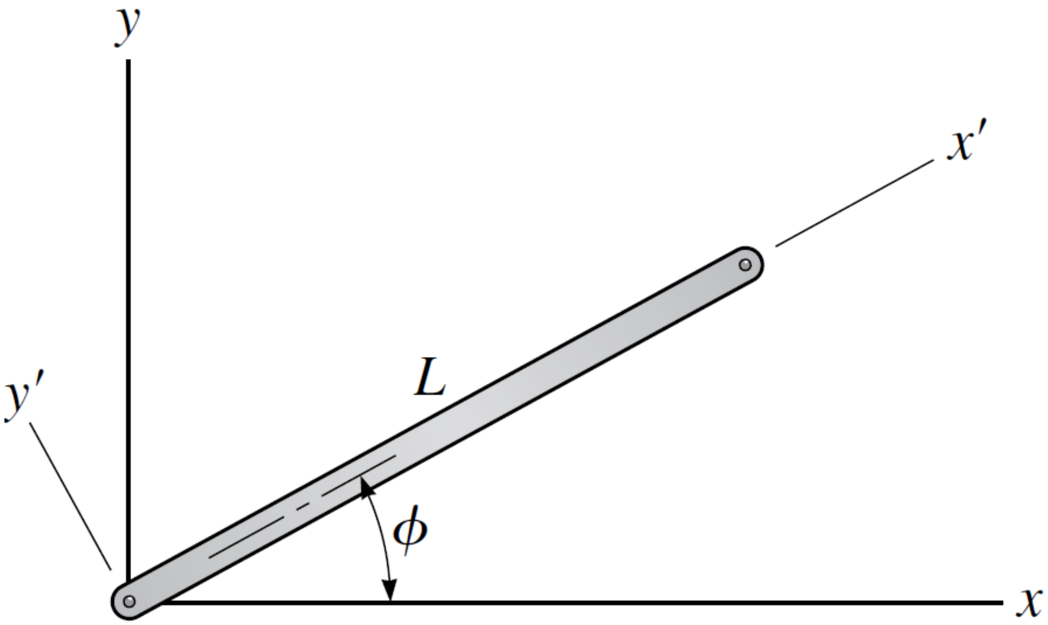


Figure 17.10 Global coordinates shown by *xy* system; member or local coordinates shown by *x′ y′* system

-

§17.7 Coordinate Transformation of a Member Stiffness Matrix

$$\mathbf{k} = \mathbf{T}^{\mathrm{T}}\mathbf{k}'\,\mathbf{T} \tag{17.54}$$

where $\mathbf{k}$ = 4 × 4 member stiffness matrix referenced to global coordinates

$\mathbf{k}'$ = 2 × 2 member stiffness matrix referenced to local coordinate system

$\mathbf{T}$ = transformation matrix, that is, matrix that converts 4 × 1 displacement vector $\boldsymbol{\Delta}$ in global coordinates to the 2 × 1 axial displacement vector $\boldsymbol{\delta}$ in the direction of bar's longitudinal axis

Equation 17.54 Stiffness matrix