ODE

Boundary Value Problem

$$\text{Solve } y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta$$

It is not very easy to solve such problems because we don't have a simple marching scheme unlike Initial Value Problems

One of the commonest method is to solve using *Shooting Method*

Shooting Method

Second-Order Differential Equation

The simplest two-point boundary value problem is a second-order differential equation with one condition specified at $x = a$ and another one at $x = b$. Here is a typical second-order boundary value problem:

$$\boxed{y'' = f(x, y, y'), \qquad y(a) = \alpha, \qquad y(b) = \beta} \tag{8.1}$$

Let us now attempt to turn Eqs. (8.1) into the initial value problem

$$\boxed{y'' = f(x, y, y'), \qquad y(a) = \alpha, \qquad y'(a) = u} \tag{8.2}$$

The key to success is finding the correct value of u.

Solve initial value problem by marching from x = a to x = b

If the solution matches with y(b) = β, then OK.

Or else more iteration on *u*

This can be converted into a root finding problem
Any technique we use (ex. Runge Kutta) will be some kind of "function θ" which takes "*u*" as the input and should give us y(b)

$$\theta(u) = \beta$$

Thus shooting method involves using some initial guess "*uo*" and solve iteratively till we get:

$$\mathbf{r}(u) = \theta(u) - \beta \qquad (8.3)$$

To iteratively solve for *u* we need to use any of the methods that we have learnt earlier.
Newton's method is not all that good because it involves finding the derivative $d\theta(u)/du$

Solution strategy

1. Specify the starting values u_1 and u_2 which *must bracket the root u* of Eq. (8.3).
2. Apply Ridder's method to solve Eq. (8.3) for u. Note that each iteration requires evaluation of $\theta(u)$ by solving the differential equation as an initial value problem.
3. Having determined the value of u, solve the differential equations once more and record the results.

When the differential equation is linear we need only two points to find the complete solution. So a simple secant method is preferable over general ridder's method

EXAMPLE 8.1

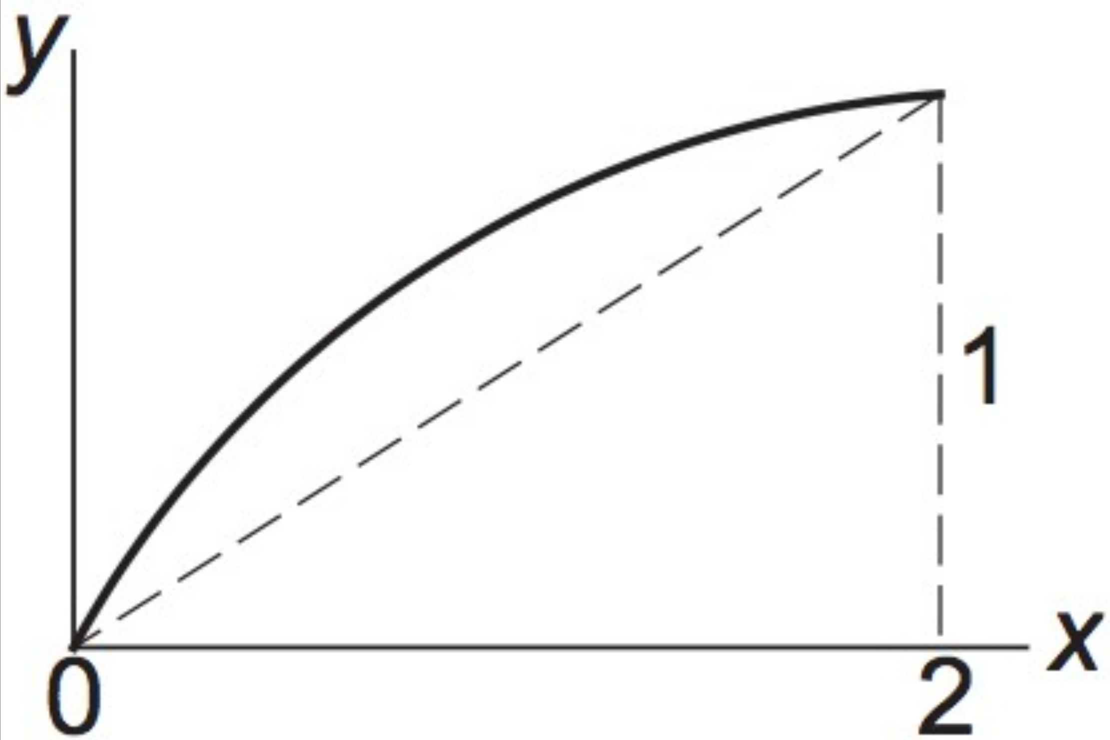
Solve the nonlinear boundary value problem

$$y'' + 3yy' = 0 \qquad y(0) = 0 \qquad y(2) = 1$$

Solution Letting $y = y_1$ and $y' = y_2$, the equivalent first-order equations are

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -3y_1 y_2 \end{bmatrix}$$

with the boundary conditions



$$y_1(0) = 0 \qquad y_1(2) = 1$$

Looking at the sketch it is clear that $y'(0) > 0.5$, so that $y'(0) = 1$ and 2 appear to be reasonable estimates for the brackets of $y'(0)$; if they are not, Ridder's method will display an error message.

EXAMPLE 8.3

Solve the third-order boundary value problem

$$y''' = 2y'' + 6xy \qquad y(0) = 2 \qquad y(5) = y'(5) = 0$$

and plot y versus x.

EXAMPLE 8.2

Numerical integration of the initial value problem

$$y'' + 4y = 4x \qquad y(0) = 0 \qquad y'(0) = 0$$

yielded $y'(2) = 1.653\,64$. Use this information to determine the value of $y'(0)$ that would result in $y'(2) = 0$.

Solution We use linear interpolation – see Eq. (4.2)

$$u = u_2 - \theta(u_2)\frac{u_2 - u_1}{\theta(u_2) - \theta(u_1)}$$

where in our case $u = y'(0)$ and $\theta(u) = y'(2)$. So far we are given $u_1 = 0$ and $\theta(u_1) = 1.653\,64$. To obtain the second point, we need another solution of the initial value problem. An obvious solution is $y = x$, which gives us $y(0) = 0$ and $y'(0) = y'(2) = 1$. Thus the second point is $u_2 = 1$ and $\theta(u_2) = 1$. Linear interpolation now yields

$$y'(0) = u = 1 - (1)\frac{1 - 0}{1 - 1.653\,64} = 2.529\,89$$

Since the problem is linear, no further iterations are needed.

EXAMPLE 8.3

Solve the third-order boundary value problem

$$y''' = 2y'' + 6xy \qquad y(0) = 2 \qquad y(5) = y'(5) = 0$$

and plot y versus x.

Solution The first-order equations and the boundary conditions are

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ 2y_3 + 6xy_1 \end{bmatrix}$$

$$y_1(0) = 2 \qquad y_1(5) = y_2(5) = 0$$

The program listed below is based on `shoot2` in Example 8.1. Because two of the three boundary conditions are specified at the right end, we start the integration at $x = 5$ and proceed with negative h toward $x = 0$. Two of the three initial conditions are prescribed as $y_1(5) = y_2(5) = 0$, whereas the third condition $y_3(5)$ is unknown. Because the differential equation is linear, the two guesses for $y_3(5)$ (u_1 and u_2) are not important; we left them as they were in Example 8.1. The adaptive Runge–Kutta method (`runKut5`) was chosen for the integration.