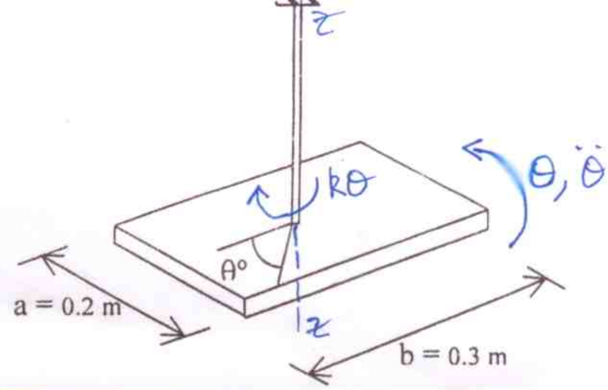
P.61

P.61. A 10 kg rectangular plate, shown in the figure, is suspended at its center from a rod having a torsional stiffness $k = 1.5$ *N.m/rad*. Develop the equation of motion for small angular rotation θ (in the plane of the plate), and determine the natural period of vibration of the plate.



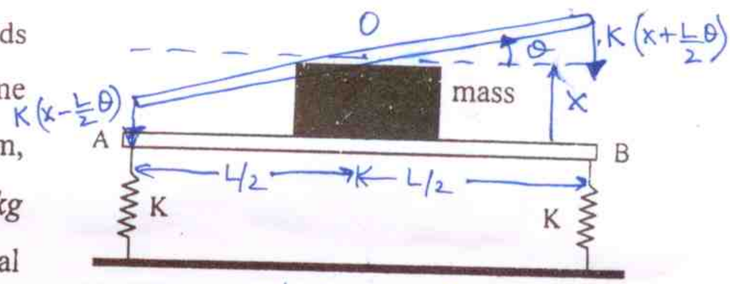
$$I_z\ddot{\theta} = -k\theta \Rightarrow \boxed{m\frac{(a^2+b^2)}{12}\ddot{\theta} + k\theta = 0} \rightarrow \text{EOM (Equation of motion).}$$

$$\Rightarrow \frac{10(0.04+0.09)}{12}\ddot{\theta} + 1.5\theta = 0 \Rightarrow \ddot{\theta} + \omega_n^2\theta = 0$$

$$\omega_n = \sqrt{\frac{1.5 * 12}{1.3}} = 3.721 \text{ rad/sec}, \quad T_n = \frac{2\pi}{\omega_n} = 1.689 \text{ secs.}$$

P.62

P.62. The uniform beam is supported at its ends by two springs *A* and *B*, each having the same stiffness *k*. When nothing is supported on a beam, it has a period of vibration of 0.83 *s*. If a 50 *kg* mass is placed at its center, the period of vertical vibration is 1.52 *s*. Compute the stiffness of each spring and the mass of the beam.



x measured from static equilibrium position (δ_{st}) so no need to show $(M+m)g$ in FBD.

$$\Sigma F_x = (M+m)\ddot{x} \Rightarrow (M+m)\ddot{x} = -K\left(\left[x - \frac{L}{2}\theta\right] + \left[x + \frac{L}{2}\theta\right]\right)$$

$$\Rightarrow \boxed{(M+m)\ddot{x} + 2Kx = 0} \longrightarrow 1 \longrightarrow \text{(EOM for } x\text{).}$$

$$\Sigma M_z = I\ddot{\theta} \Rightarrow \frac{ML^2}{12}\ddot{\theta} = K\left(\left(x - \frac{L}{2}\theta\right)\frac{L}{2} - \left(x + \frac{L}{2}\right)\frac{L}{2}\right)$$

$$\Rightarrow \boxed{\frac{ML^2}{12}\ddot{\theta} + K\frac{L^2}{4}\theta = 0} \longrightarrow 2 \longrightarrow \text{(EOM for } \theta\text{).}$$

1, 2 are the equations of motion for the free-vibration of this 2-dof system. Notice that they are uncoupled (i.e, x, θ appear in separate equations, hence they dont depend on each other). So, for vertical vibrations we need 1 only.

Given, $m = 0$, $\sqrt{\frac{2K}{M}} = (0.83)^{-1} * 2\pi$

$m = 50$, $\sqrt{\frac{2K}{M+m}} = \sqrt{\frac{2K}{M+50}} = (1.52)^{-1}\, 2\pi$

$$\Rightarrow \frac{M+50}{M} = \left(\frac{1.52}{0.83}\right)^2 \Rightarrow M = 21.24 \text{ kg}, \Rightarrow K = \frac{M}{2}\left(\frac{2\pi}{0.83}\right)^2 = 608.7 \text{ N/m}^2$$

Note: If we measure vertical displacement from position of undeformed spring, and call it y, then in place of ① we have,

$$(M+m)\ddot{y} + 2Ky + (M+m)g = 0 \rightarrow ③$$

Put $\ddot{y} = 0$, for static equilibrium, get $y_{st} = \delta_{st} = -\frac{(M+m)g}{2K}$ (static equilibrium position). → ④

Now let $y = x + \delta_{st}$ → insert in ③,

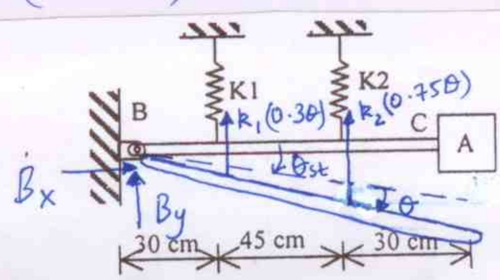
$$(M+m)\ddot{x} + 2K(x+\cancel{\delta_{st}}) + \cancel{(M+m)g} = 0 \rightarrow ①$$ (so we get back ①).

by virtue of definition of static equilibrium position, δ_{st}, i.e., ④ above.

At the static equilibrium position shown in the figure, we have additional spring forces directed upward, equal to $K\delta_{st}$ on each spring. We have not shown these, since they cancel $(M+m)g$ (also not shown) in the summation of vertical forces (ΣF_y), and their moments about O cancel each other in the sum of moments (ΣM_z).

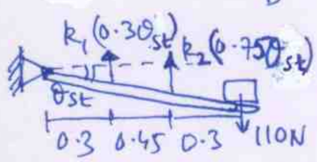
P.63.

P.63. What is the natural frequency of motion for block *A* for small oscillation? Consider *BC* to have negligible mass and body *A* to be a particle. When body *A* is attached to the rod, the static defection is 25 mm. The spring constant K_1 = *1.75 N/mm*. Body *A* weighs 110 *N*. What is K_2?



Static deflection:

$$\Sigma M_B = 0 \rightarrow \underset{1.75}{k_1}\left(25 * \frac{30}{105}\right)(30) + k_2\left(25 * \frac{75}{105}\right)(75) = 110\,(105) \quad ①$$



$$\Rightarrow k_2 = 8.344 \text{ N/mm} = 8344 \text{ N/m}$$

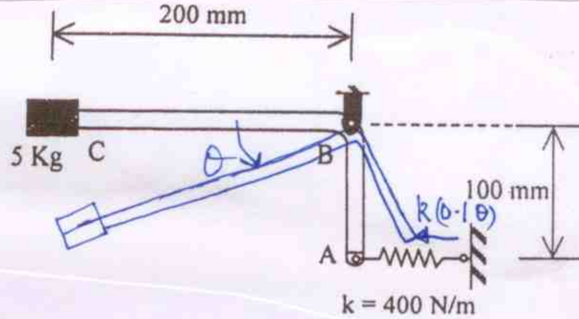
Now if we measure θ from static equilibrium position, then due to static equilibrium equation ① we can omit the contribution of 110N force in moment equation, since it is cancelled out by the LHS terms in ①. Thus the dynamic equilibrium equation, for small θ, is

EOM

$$\Sigma M_z = I\ddot{\theta} \Rightarrow \boxed{-\left[k_1(0.3\theta)(0.3) + k_2(0.75)(0.75\theta)\right] = \left(\frac{110}{9.81}\right)(1.05)^2\ddot{\theta}}$$

$$\omega_n = \sqrt{\frac{0.09k_1 + 0.5625k_2}{\left(\frac{110}{9.81}\right)(1.05)^2}} = 19.809 \text{ rad/s.}$$

P.64 P.64.The bent rod shown in the figure has negligible mass and supports a 5 kg collar at its end. Develop the equation of a motion and determine the natural period of vibration for the system.



The static equilibrium position is shown, i.e, spring is sufficiently compressed in the position shown so that its force provides a moment about B to balance moment due to collar weight.
So, for motion about static equilibrium position, the moment due to 5kg and spring force at static equil cancel each other, hence not shown.

$$\Sigma M_z = I\ddot{\theta} \Rightarrow \boxed{\underset{5}{m}\underset{(0.2)^2}{r^2}\ddot{\theta} = -\underset{400\,N/m}{k}(0.1\theta)(0.1)}$$

So θ measured from static equilibrium position.

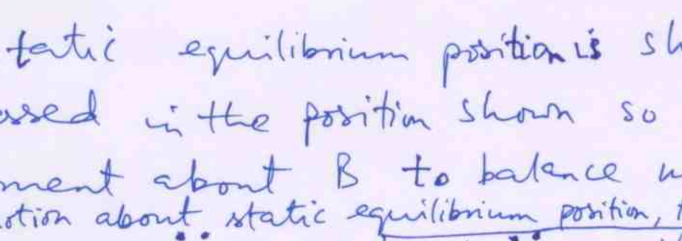
$$\Rightarrow \boxed{0.2\ddot{\theta} + 4\theta = 0} \rightarrow \text{EOM.}$$

$$\omega_n = \sqrt{\frac{4}{0.2}} = 4.47 \text{ rad/sec.}$$

$$T_n = 2\pi/\omega_n = 1.405 \text{ sec}$$

P.65 Consider rod ABC (shown in P64) to have uniformly distributed mass of 3 kg. Develop the equation of motion and determine the natural period of vibration for the system

Assume that $\theta = 0°$ is the static equilibrium position.



Here 2g and 1g are the weights of horizontal & vertical legs, respectively. $F_{st} + k(0.1\theta)$ is static equil spring force + incremental spring force due to small motion θ about static equil ($\theta = 0°$) position.

$$\Sigma M_B = I_B\ddot{\theta}$$

$$\Rightarrow \left(\frac{m_1 l_1^2}{3} + \frac{m_2 l_2^2}{3}\right)\ddot{\theta} = m_1 g\frac{l_1}{2} - m_2 g\frac{l_2}{2}\theta - \left(F_{st} + k(0.1\theta)\right)l_2 \rightarrow (1)$$

Static equilibrium equation is,

$$\Sigma M_B = 0 \rightarrow m_1 g\frac{l_1}{2} - F_{st}l_2 = 0 \rightarrow (2)$$

← can also get this by putting $\ddot{\theta} = 0$ for static equilibrium and $\theta = 0°$, ie the static equil position in (1)

$$(2) \text{ in } (1) \rightarrow \boxed{\left(\frac{m_1 l_1^2 + m_2 l_2^2}{3}\right)\ddot{\theta} + \left(m_2 g\frac{l_2}{2} + 0.1k\right)\theta = 0}$$

EOM.

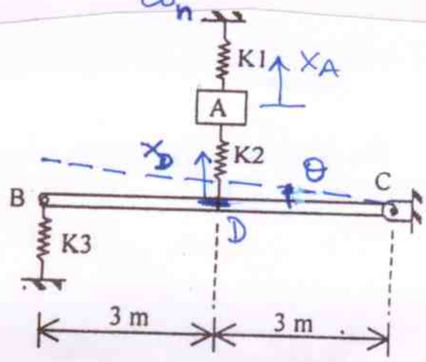
$$\left(\frac{2*0.2^2+1*0.1^2}{3}\right)\ddot{\theta} + \left(1*9.81*\frac{0.1}{2} + 0.1*400\right)\theta = 0$$

$$\omega_n = \sqrt{\left(\frac{0.981}{2}+40\right)\Big/\left(\frac{2*0.04+0.01}{3}\right)} = 36.74 \text{ rad/sec.}$$

$$\Rightarrow T_n = \frac{2\pi}{\omega_n} = 0.171\,s$$

P66.

P.66. What are the differential equation of motion about the static equilibrium configuration and the natural frequency of motion of body *A* for small motion of *BC*? Neglect inertial effects from *BC*. Assume $K_1 = 15$ N/m, $K_2 = 20$ N/m, $K_3 = 30$ N/m and $W_A = 30$N.



Can assume, without loss of generality, that configuration shown is the static equilibrium configuration. Given that BC is massless (i.e., neglect inertial effects of BC), and for motions about static equilibrium i.e., dropping the weight of A from FBD, we have

(I) $\leftarrow m_A \ddot{x}_A = -k_1 x_A - k_2(x_A - x_D) \xrightarrow{\text{EOM}}$ from vertical dynamic equil of A.

(II) $\leftarrow \underbrace{I_D^{BC}}_{=0}\ddot{\theta} = -k_3(6\theta)(6) + k_2(x_A - x_D)(3) \xrightarrow{\text{EOM}}$ from angular dynamic equil of BC

$=0$ ($\because$ BC is massless, i.e. has no inertia).

(III) $\leftarrow \theta = x_D/3 \rightarrow$ from kinematics (i.e. geometry).

(III) in (II) $\rightarrow \quad -36k_3\frac{x_D}{3} + 3k_2(x_A - x_D) = 0 \Rightarrow x_D = \left(\frac{\cancel{3}k_2}{\cancel{3}k_2 + \cancel{12}^{4}k_3}\right)x_A$ (IV)

Thus due to inertialess rod BC the apparent 2-dof problem actually reduces to 1-dof problem. (Since x_D depends purely on x_A).

(IV) in (I) $\rightarrow$

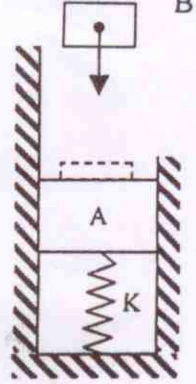
$$\boxed{m_A\ddot{x}_A + \left[k_1 + k_2\left(\frac{4k_3}{k_2+4k_3}\right)\right]x_A = 0}$$

$\rightarrow$ EOM for x_A

$$\omega_n = \sqrt{\frac{k_1k_2 + 4k_3(k_1+k_2)}{m_A(k_2+4k_3)}}$$

$$= 3.242 \text{ rad/sec}$$

P.67. A rigid body *A* rests on a spring with stiffness *K = 8.80 N/mm*. A lead pad *B* falls on to the block *A* with a speed on impact of 7 m/sec. If the impact is perfectly plastic, what are the frequency and amplitude of the motion of the system, provided that the lead pad sticks to A at all times? Take $W_A = 134\ N$ and $W_B = 22\ N$. What is the distance moved by *A* in 0.02 *s*?



Perfectly plastic impact, $e=0$ (coeff of restitution), ie bodies stick to each other just after impact. During impact momentum is conserved, i.e.,

$$m_B v_B = (m_A + m_B) v \Rightarrow v = \frac{m_B}{(m_A+m_B)} v_B = \text{combined (A+B) velocity just after impact.}$$

Eqn of motion (EOM) of spring-block system is

$$(m_A + m_B)\ddot{x} + kx = 0$$

, x measured about static equilibrium, upward (+).

So initial conditions (IC's) are:

$$x_0 \triangleq x(0) = +m_B g/k = 22/8800 \text{ m}$$

$$\dot{x}_0 \triangleq \dot{x}(0) = -v = -\frac{m_B}{(m_A+m_B)} v_B = -\frac{154}{156}$$

$$x = C\sin(\omega_n t + \psi), \quad C = \left(x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}\right)^{1/2}, \quad \tan\psi = \frac{x_0 \omega_n}{\dot{x}_0}$$

$$\omega_n = \sqrt{\frac{k}{(m_A+m_B)}} = \sqrt{\frac{8800}{(134+22)/9.81}} = 23.52 \text{ rad/s}$$

$$C = -\sqrt{\frac{(154/156)^2 * 156}{8800 * 9.81} + \left(\frac{22}{8800}\right)^2} = -0.042039 \text{m}, \quad \psi = -0.0594933 \text{ rad}$$

Amplitude $= |C| = 0.042039$m

$$x\Big|_{t=0.02s} = -0.016792 \text{ m}$$

Now $\omega_n t + \psi = 0.4109$ rad for $t = 0.02$, ie, $< \pi/2$.

So, dist moved by A in 0.02S $= \dfrac{22}{8800} - (-0.016792)$

$= 0.019292$ m

NOTE: We fudge the $-$ sign to make it match initial conditions, ie $x_0 = C\sin\psi$, $\dot{x}_0 = C\omega_n\cos\psi$

$\because \sin\psi < 0$, $\cos\psi > 0$, $x_0 > 0$, $\dot{x}_0 < 0$ we must have $C < 0$.

Alternative is to use

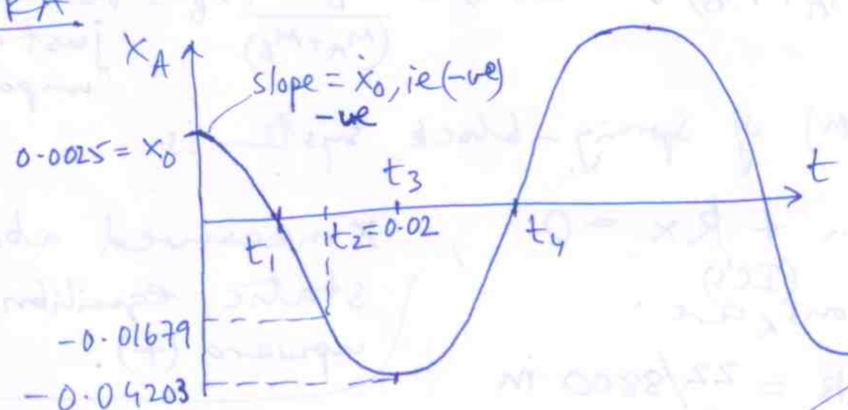
$$x = \underbrace{A}_{x_0}\cos\omega_n t + \underbrace{B}_{\dot{x}_0/\omega_n}\sin\omega_n t \quad \text{and get}$$

$$x\Big|_{t=0.02} = \frac{22}{8800}\cos(23.52*0.02) + \frac{(-154/156)}{23.52}\sin(23.52*0.02)$$

$= -0.016795\,m$ (same as ealier)

$\Rightarrow$ dist moved by A is same as by ealier way.

EXTRA



$T_n = \frac{2\pi}{\omega_n} = 0.2671\,s$

First zero crossing when

$\omega_n t_1 + \psi = 0$

$\Rightarrow t_1 = \frac{-\psi}{\omega_n} = 0.00253$ secs.

First peak when

$\omega_n t_3 + \psi = \pi/2$

$t_3 = 0.06932\,s$.

Second zero crossing,

$\omega_n t_4 + \psi = \pi$

$\Rightarrow t_4 = 0.1361\,s$.

So period $= 2(t_4 - t_1)$

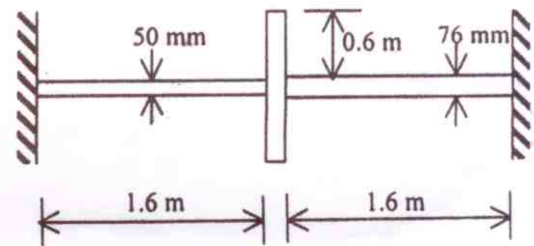
$= 0.2671$ ✓

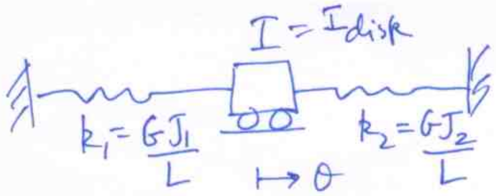
checks out.

$x(t_3) = C\sin(\omega_n t_3 + \psi) = -0.04203\,m$

P68

P.68. What is the equivalent torsional spring constant on the disc from the shafts? The modulus of rigidity for the shaft is 10×10^{10} N/ m^2. What is the natural frequency of the system? If the disc is twisted 10° and then released, what will its angular position be in 1 *sec*? Neglect the mass of shaft. The disc weighs 143N





Equivalent system

EOM: $I\ddot{\theta} = -(k_1+k_2)\theta = -\frac{G}{L}(J_1+J_2)\theta$

$$I\ddot{\theta} + \frac{G}{L}(J_1+J_2)\theta = 0$$

$$k_{eq} = \frac{G}{L}(J_1+J_2) = \frac{10^{11}}{1.6} * \frac{\pi}{32}\left(\left[\frac{50}{1000}\right]^4 + \left[\frac{76}{1000}\right]^4\right) = 243057.3 \text{ N.m/rad}$$

$$\omega_n = \sqrt{\frac{G(J_1+J_2)}{L\,I}} = \sqrt{\frac{k_{eq}}{\left(\frac{143}{9.81}\right)\frac{0.6^2}{2}}} = 304.36 \text{ rad/sec}$$

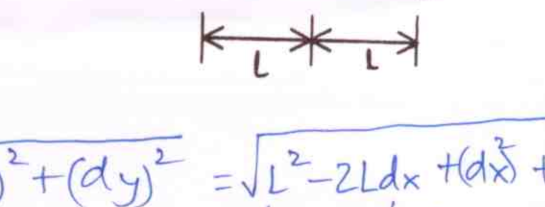
IC's: $\theta_0 \triangleq \theta(0) = 10^\circ \frac{\pi}{180^\circ} = 0.1745$, $\dot{\theta}_0 \triangleq \dot{\theta}(0) = 0$ rad/sec

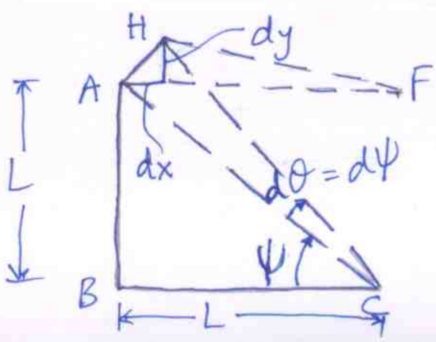
$$C = \left[\theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2\right]^{1/2} = \frac{\pi}{18}, \quad \psi = \overbrace{\tan^{-1}\left(\frac{\theta_0 \omega_n}{\dot{\theta}_0}\right)}^{} = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\theta\Big|_{t=1sec} = C\sin\left[\omega_n(1) + \frac{\pi}{2}\right] = -0.1623 \text{ rad} = -9.3^\circ$$

P.69.

P.69. a) If bar *ABCDE* is of negligible mass, what is the natural frequency of free oscillation of the system for small amplitude of motion? Take $K = 25$ *N/m*, $m = 2$ *kg*, $l = 1$ *m*. b) If mass of bar *ABCDE* is 0.4 kg/m, what is the natural frequency of free oscillation of the system for small amplitude of motion? Take $K = 25$ N/m, $m = 2$ kg, $l = 1$ m.





$AF = L$

$$HF = \sqrt{(L-dx)^2 + (dy)^2} = \sqrt{L^2 - 2Ldx + (dx)^2 + (dy)^2}$$

$$= L\left(1 - \frac{dx}{L} + \frac{1}{2}\left\{\left(\frac{dx}{L}\right)^2 + \left(\frac{dy}{L}\right)^2\right\} + HOT's\right)$$

neglect

higher order terms.

$$= L - dx.$$

So compression of spring AF is $(AF - HF) = dx$

Likewise, extension of " AG is $(GH - GA) = dx$.

We could have guessed this directly, i.e, for small motions the change in length of a spring equals (nearly) its component along the original direction of the spring.

Now for small motion, B essentially moves vertically and A horizontally. Thus $dx = L d\theta$.

Alternatively, a cleaner (non graphical) way to get this is as follows. Let x be horizontal coordinate of A measured from C.

$$x = -AC\cos\psi \Rightarrow dx = AC\sin\psi \, d\psi = d\theta = L d\theta$$

Now replace dx, $d\theta$ with x, θ throughout (i.e. x, θ are small).

Also note that position shown can be assumed as static equilibrium position, so in this position the moments about C, due to spring forces balance that due to self-weight hence self-wt is not accounted in EOM about static equilibrium (as usual).

(a) $I_C \ddot{\theta} = \sum M_c \Rightarrow m(\sqrt{2}L)^2 \ddot{\theta} = -2K(L\theta)(L)$

$$\cancel{2} m \cancel{L^2} \ddot{\theta} + \cancel{2} K \cancel{L^2} \theta = 0$$

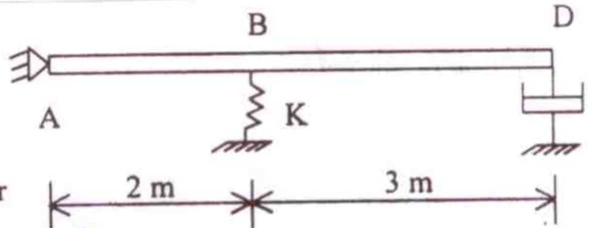
$$\omega_n = \sqrt{K/m} = \sqrt{25/2} = 3.5355 \text{ rad/sec}$$

(b) Only I_c changes here. (Compare with P. 65 where part of self-wt had to be accounted for in EOM).

$$I_c = \underbrace{\left[(2\rho L)\frac{(2L)^2}{12}\right]}_{I_{BC}} + \underbrace{\left[2\left\{(\rho L)\frac{L^2}{12} + (\rho L)\left(\frac{L^2}{4} + L^2\right)\right]\right.}_{I_{AB} + I_{DE}} + m(2L^2)$$

$$\Rightarrow I_c = 5.33\bar{3} \text{ kg m}^2 \Rightarrow \omega_n = \sqrt{\frac{2KL^2}{I_c}} = \sqrt{\frac{50}{5.33}} = 3.062 \text{ rad/sec.}$$

P.70 P.70. Derive equation of motion for small rotation the rigid uniform beam of mass 12kg. Take K=2500 N/m, C=8 Kg/sec. What is the damping ratio? If initial rotation of 0.1 rad at A is applied and the bar is released, compute the displacement at D at the end of 3sec. (8)



$$I_A \ddot{\theta} = \sum M_A \Rightarrow \frac{\overset{12}{m}\,\overset{25}{L^2}}{3}\ddot{\theta} = -\underset{2500}{K}(2\theta)(2) - \underset{8}{c}(5\dot{\theta})(5)$$

$$\Rightarrow 100\ddot{\theta} + 200\dot{\theta} + 10000\theta = 0$$

$$\underset{m}{1}\ddot{\theta} + \underset{c}{2}\dot{\theta} + \underset{k}{100}\theta = 0$$

$$\zeta = \frac{c}{2m\underset{\sqrt{k/m}}{\omega_n}} = \frac{2}{2*1*\sqrt{100/1}} = 0.1$$

$\theta_0 = 0.1$ rad, $\dot{\theta}_0 = 0$

$$\theta = (A\cos\omega_d t + B\sin\omega_d t)e^{-\zeta\omega_n t}$$

$$A = \theta_0 = 0.1, \quad B\omega_n\sqrt{1-\zeta^2} - \omega_n\zeta A = \dot{\theta}_0 = 0$$

$$\Rightarrow B = \frac{\zeta}{\sqrt{1-\zeta^2}}A = \frac{0.1}{\sqrt{1-0.01}}*0.1 = 0.0100503$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{0.99}$$

$$\theta\Big|_{3\,sec} = \left\{0.1\cos[10\sqrt{0.99}*3] + \frac{0.01}{\sqrt{0.99}}\sin[10\sqrt{0.99}*3]\right\}e^{-0.1\sqrt{100}*3}$$

$$\theta[3] = -0.000478005 \text{ rad} = -0.02739^\circ$$

Or by alternate approach,

$$\theta = C\sin(\omega_d t + \psi)e^{-\zeta\omega_n t}$$

$$C = \oplus\left[\theta_0^2 + \left(\frac{\dot{\theta}_0 + \zeta\omega_n\theta_0}{\omega_d}\right)^2\right]^{1/2} = \left[0.1^2 + \left(\frac{0.1*10*0.1}{10\sqrt{0.99}}\right)^2\right]^{1/2}$$

$\because \sin\psi > 0$, $\theta_0 > 0$ $\Rightarrow C > 0$

$$= 0.100503$$

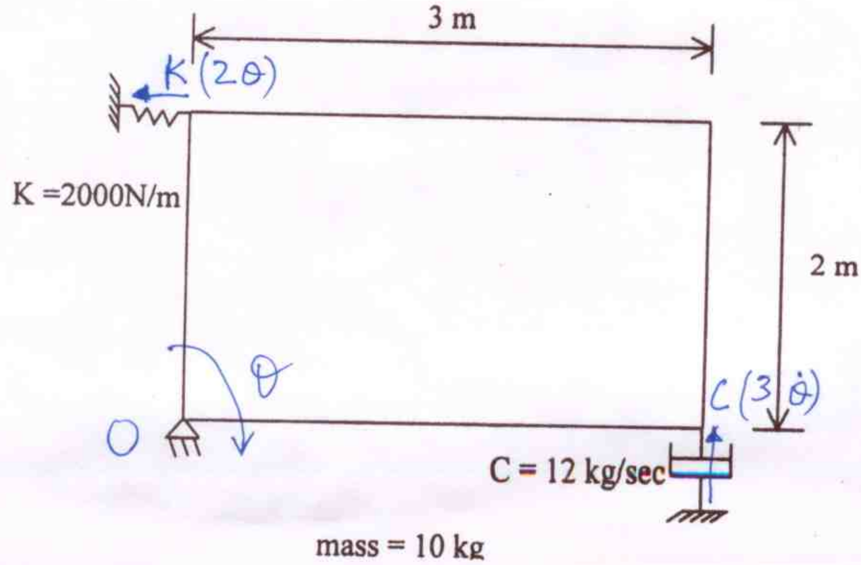
$$\psi = \tan^{-1}\left(\frac{\theta_0\omega_d}{\dot{\theta}_0 + \zeta\omega_n\theta_0}\right) = \tan^{-1}\left(\frac{0.1*10*\sqrt{0.99}}{0.1*10*0.1}\right) = 84.26^\circ = 1.4706289$$

$$\theta[3] = 0.100503\sin(10\sqrt{0.99}*3 + 1.4706289)e^{-0.1\sqrt{100}*3}$$

$$= -0.000478005 \text{ rad} = -0.02739^\circ$$

P.71

P.71. For the rigid, uniform plate shown in the diagram, derive equation of motion for small oscillation.



Assume static equil position shown. (so no mg term in ΣM_o).

$$I_o \ddot{\theta} = \Sigma M_o$$

$$\left[m\left(\frac{a^2+b^2}{12}\right) + m\left(\frac{a^2}{4}+\frac{b^2}{4}\right)\right]\ddot{\theta} = -K(2\theta)(2) - c(5\dot{\theta})(5)$$

$$10\left(\frac{3^2}{3}+\frac{2^2}{3}\right)\ddot{\theta} + (2000*4)\theta + (12*25)\dot{\theta} = 0$$

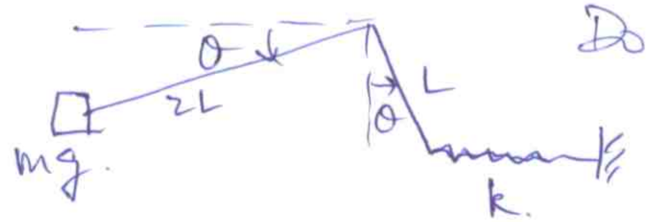
$$\Rightarrow \frac{130}{3}\ddot{\theta} + 300\dot{\theta} + 8000\theta = 0$$

P64 Modified

Assume given position is undeformed springs.

Assume (as before) that spring is long so essentially it stays horizontal.

Do moment (dynamic) equilibrium.



Do $I\ddot{\theta} = \sum M$.

$$m(2L)^2\ddot{\theta} = mg\,2L\cos\theta - kL\sin\theta\, L\cos\theta \longrightarrow (1)$$

Static equilibrium: Put $\frac{d()}{dt} = \frac{d^2()}{dt^2} = 0$.

$$mg\,2L\cos\theta_{st} - kL^2\sin\theta_{st}\cos\theta_{st} = 0 \longrightarrow (2)$$

$$\theta_{st} = \frac{\pi}{2},\ \sin^{-1}\left(\frac{mg\,2L}{kL^2}\right) \longrightarrow (3)$$

STABLITY OF EQUILIBRIUM.

From energy methods, to find stability of equilibrium

$$V = -mg2L\sin\theta + \frac{1}{2}k(L\sin\theta)^2$$

$$\frac{dV}{d\theta} = \left[-mg2L\cos\theta + kL^2\sin\theta\cos\theta\right] = 0.$$

gives same θ_{st} as (3) above

$$\frac{d^2V}{d\theta^2} = \left[mg2L\sin\theta + \frac{kL^2}{\cancel{2}}\cancel{2}\cos 2\theta\right]$$

For $\theta_{st_1} = \frac{\pi}{2}$, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta_{st_1}=\frac{\pi}{2}} = 2mgL - kL^2$

So $\theta_{st_1} = \frac{\pi}{2}$ is stable for $2mgL > kL^2$

So when θ_{st1} is stable, θ_{st2} is not real (ie does not exist)

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta_{st2}} = 2mgL\frac{mg2L}{kL^2} + kL^2\left[1-2\left(\frac{mg2L}{kL^2}\right)^2\right]$$

$$= kL^2 + \frac{2mgL}{kL^2}\left[2mgL - 2\cancel{kL^2}\left(\frac{2mgL}{\cancel{kL^2}}\right)\right]$$

$$= kL^2 - \frac{(2mgL)(2mgL)}{kL^2}$$

$$> 0 \quad \text{if} \quad 2mgL < kL^2$$

So θ_{st2} is stable when it exists (ie when it is real).

SMALL MOTIONS (α) ABOUT θ_{st} (NON-TRIVIAL)

So we want to write the linearized equations for small motion (α) about a non-trivial (ie non-zero) θ_{st}.

Put $\theta = \theta_{st} + \alpha$ in (1) & linearize (ie retain only first order terms in α in the Taylor series expansions of $\sin\theta$ and $\cos\theta$, where α is small).

So,
$$\left.\begin{aligned}\ddot\theta &= \ddot\theta_{st} + \ddot\alpha = \ddot\alpha\\ \sin\theta &= \sin\theta_{st} + \cos\theta_{st}\,\alpha\\ \cos\theta &= \cos\theta_{st} - \sin\theta_{st}\,\alpha\end{aligned}\right\} \rightarrow \text{substitute in (1).}$$

So we get,

$$m(2L)^2\ddot\alpha = mg2L(\underline{\cos\theta_{st}} - \sin\theta_{st}\,\alpha) - \frac{kL^2}{2}(\underline{\sin 2\theta_{st}} + 2\cos 2\theta_{st}\,\alpha) \longrightarrow (4)$$

Now use 2 in 4, & see that the double-underlined terms cancel out (since they represent 2, ie static equil equation). Thus 4 becomes,

$$m(2L)^2\ddot{\alpha} = -\left(2mgL\sin\theta_{st} + \frac{kL^2}{2}2\cos 2\theta_{st}\right)\alpha$$

$$m(2L)^2\ddot{\alpha} + \left(2mgL\sin\theta_{st} + \frac{kL^2}{\cancel{2}}\cancel{2}\cos 2\theta_{st}\right)\alpha = 0. \quad (5)$$

↳ simple harmonic oscillator.

Specializations of Modified problem

(i) $\theta_{st_1} = 90° \rightarrow$ small motions about this.

$$m(2L)^2\ddot{\alpha} + \underbrace{(2mgL - kL^2)}\alpha = 0.$$

this coefficient is >0, so stable vibrations.

(ii) $\theta_{st_2} \rightarrow$ small motions about this.
(assume θ_{st_2} is real, ie exists).
(ie, $mg2L < kL^2$).

$$m(2L)^2\ddot{\theta} + \left(kL^2 - \frac{(2mgL)^2}{kL^2}\right)\alpha = 0.$$

positive

x

Recall solution of original problem (64):
$\theta_{st} = 0°$, and

$$(2L)^2\ddot{\theta} + kL^2\theta = 0 \quad \text{for small } \theta,$$ see p3 of notes.

ie, inconsistent, $\because \theta_{st} = 0°$ violates 2

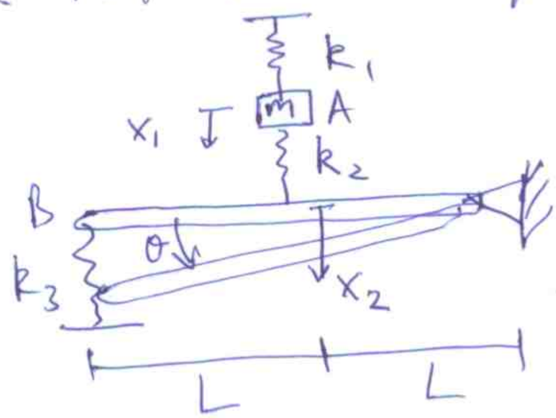
↳ Note: If you put $\theta_{st} = 0°$, $\theta = \theta_{st} + \alpha$, α small in 5 and linearize, you get same result — coincidence.

P 66 Modified.

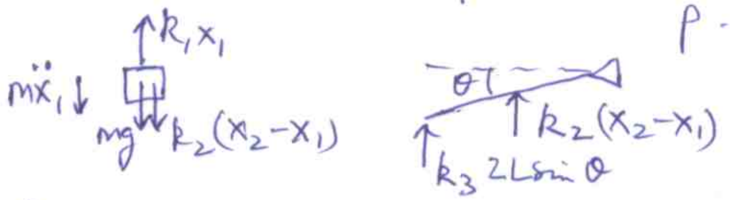
Assume given position is undeformed springs.

Assume (as before) that all springs are long so essentially they remain vertical.

Do translational equilibrium for mass A & rotational equilibrium for rod.



(Note: $x_1 = -x_A$, $x_2 = -x_D$ in the notation of original problem — see p. 4)



$$m\ddot{x}_1 = mg - k_1 x_1 + k_2(x_2 - x_1) \longrightarrow (1)$$

$$I_{rod}\,\ddot{\theta} = -k_3(2L\sin\theta)(2L\cos\theta) - k_2(x_2 - x_1)L\cos\theta \longrightarrow (2)$$

$I_{rod} = 0$ ($\because$ massless rod).

$$x_2 = L\sin\theta \longrightarrow (3) \quad \text{(from kinematics).}$$

Substitute (3) in (1):

$$m\ddot{x}_1 = mg - k_1 x_1 + k_2(L\sin\theta - x_1) \longrightarrow (*)$$

From (3) in (2), $-L\cos\theta\left[4k_3 L\sin\theta + k_2(L\sin\theta - x_1)\right] = 0.$

$\because \cos\theta \neq 0$ in general (note we are not finding θ_{st} as yet),

we get, $$\sin\theta = \frac{k_2}{L(4k_3 + k_2)}\, x_1 \longrightarrow (4).$$

Relation (3) is a purely kinematical relation.

Relation (4) is a relation arising from (3) & dynamics (2), with $I_{rod} = 0$.

Both ③, ④ valid for all 't'.

Substitute ④ in ⊛, get,

$$m\ddot{x}_1 + \left[k_1 + k_2 - \frac{k_2^2}{4k_3+k_2}\right]x_1 - mg = 0.$$

$$m\ddot{x}_1 + \left[k_1 + \frac{4k_2k_3}{4k_3+k_2}\right]x_1 - mg = 0 \rightarrow ⑤$$

STATIC EQUILIBRIUM.

In ⑤ put $\frac{d()}{dt} = \frac{d^2()}{dt^2} = 0$, get,

$$(x_1)_{st} = mg \Big/ \left[k_1 + \frac{4k_2k_3}{4k_3+k_2}\right] = \frac{30}{\left(15 + \frac{4*20*30}{4*30+20}\right)}$$

$$= \frac{30}{32.14} = 0.9333$$

$$(\theta)_{st} = \sin^{-1}\left[\frac{k_2}{L(4k_3+k_2)}(x_1)_{st}\right] = \sin^{-1}\left[\frac{20*0.9333}{3*(4*30+20)}\right]$$

$$\sin^{-1}(0.04444)$$

SMALL MOTIONS ABOUT EQUILIBRIUM.

put $x_1 = (x_1)_{st} + x$, where x is small,

in ⑤ and use ⑥. You get,

$$m\ddot{x} + \left[k_1 + \frac{4k_2k_3}{4k_3+k_2}\right]\left[\underline{\underline{(x_1)_{st}}} + x\right] - \underline{\underline{mg}} = 0$$

underlined terms cancel due to ⑥, so,

$$m\ddot{x} + \left[k_1 + \frac{4k_2k_3}{4k_3+k_2}\right]x = 0$$

↳ Simple harmonic oscillator, same result as on page ④.

So vibration of mass A unaffected by static equil in this case.