Lesson 39

Chp 8 VIBRATIONS & TIME RESPONSE

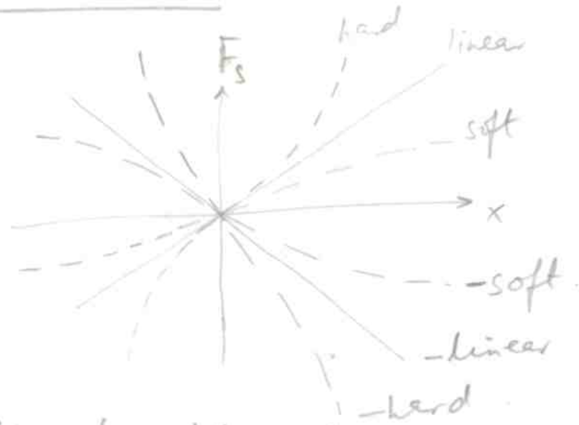
Sec 8/1 — Introduction

- Continuous or distributed parameter v/s discrete or lumped-parameter
  eg. plucked strings, membranes, flutter of wings, roll pitch of ships v/s shafts with masses at ends (ships propeller)
- Need to make mathematical model.
- Degrees of freedom < single / multi arising in math model.
- Forced v/s Free

Sec 8/2 — Free Vibrations.

Springs.

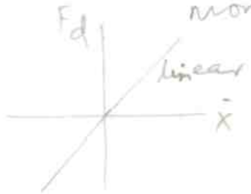
$F_s = kx$



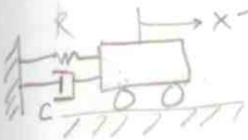
Damping. Dashpot → viscous damper

$F_d = c\dot{x}$ (linear)

→ Frictional damping more complex.



(b) Damped



→ define x from static equil position

$\Sigma F_x = ma_x$

$$\boxed{\begin{aligned}&\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0\\ &\omega_n = \sqrt{\frac{k}{m}}\ ,\quad \zeta = \frac{c}{2m\omega_n}\end{aligned}}$$

Assume $x = Ae^{\lambda t} \Longrightarrow \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

$\lambda_1 = \omega_n\left[-\zeta + \sqrt{\zeta^2-1}\right]$, $\lambda_2 = \omega_n\left[-\zeta - \sqrt{\zeta^2-1}\right]$

$\therefore\ x = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} = A_1e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$

Case I $\zeta > 1$ (Overdamped), $\lambda_1, \lambda_2 \rightarrow$ real, distinct, < 0.

Motion decays, no periodic oscillations

Case II $\zeta = 1$ (critically)

$\lambda_1 = \lambda_2 = -\omega_n \Rightarrow x = (A_1 + A_2 t)e^{-\omega_n t}$

Motion decays, no periodic oss.

Note :- Can show that for same IC's x_0, v_0, motion for Case II decays faster than Case I

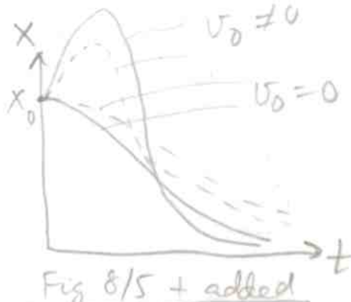


Fig 8/5 + added

Lesson 39

$$x = \left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]e^{-\zeta\omega_n t}$$

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Case III $\zeta < 1$ (Underdamped)

$$x = \left[A_1 e^{i\sqrt{1-\zeta^2}\,\omega_n t} + A_2 e^{-i\sqrt{1-\zeta^2}\,\omega_n t}\right]e^{-\zeta\omega_n t}$$

Let $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

$$\therefore x = \left\{(A_1+A_2)\cos\omega_d t + i(A_1-A_2)\sin\omega_d t\right\}e^{-\zeta\omega_n t}$$

$$x = \left\{A_3\cos\omega_d t + A_4\sin\omega_d t\right\}e^{-\zeta\omega_n t}$$
$$= C\sin(\omega_d t + \psi)\,e^{-\zeta\omega_n t}$$

$$\dot{x} = \left[(-A_3\omega_d - A_4\zeta\omega_n)\sin\omega_d t + (A_4\omega_d - \zeta\omega_n A_3)\cos\omega_d t\right] e^{-\zeta\omega_n t}$$

$x_0 = A_3$, $\dot{x}_0 = A_4\omega_d - \zeta\omega_n A_3$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ rad/s — damped nat. freq

$T_d = \frac{2\pi}{\omega_d}$, $f_d = \frac{\omega_d}{2\pi}$ cyc/s = Hertz

$$\dot{x} = C\left[\omega_d\cos(\omega_d t+\psi) - \zeta\omega_n\sin(\omega_d t + \psi)\right]e^{-\zeta\omega_n t}$$

$$\left.\begin{array}{l} x_0 = C\sin\psi \\ v_0 = \dot{x}_0 = C\left[\omega_d\cos\psi - \zeta\omega_n\sin\psi\right]\end{array}\right\} \Rightarrow$$

$$\tan\psi = \left(\frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}\right)$$

$$C = \left[x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2\right]^{1/2}$$

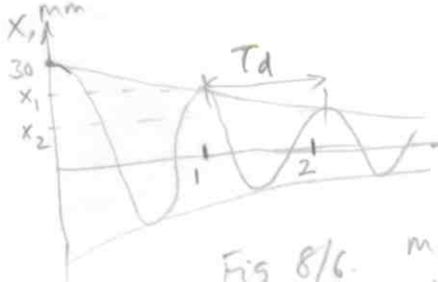


Fig 8/6. $m = 1$ kg, $k = 36$ N/m, $c = 1$ N-s/m ($\zeta = 0.083$)
$x_0 = 30$ mm, $\dot{x}_0 = 0$

Log decrement. → For underdamped systems → ref. Fig above.

amplitudes / cycles apart.

$$\frac{x_1}{x_{j+1}} = \frac{C e^{-\zeta\omega_n t_1}}{C e^{-\zeta\omega_n (t_1 + jT_d)}} = e^{j\zeta\omega_n T_d}$$

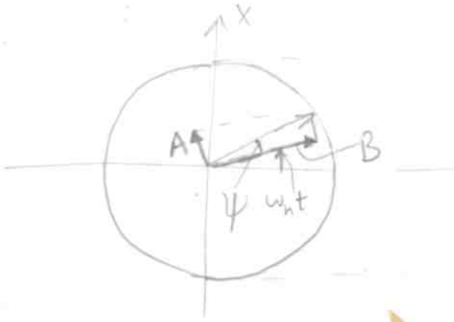
$$\delta = \ln\left(\frac{x_1}{x_{j+1}}\right) = j\zeta\omega_n T_d = \frac{j2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \frac{\delta}{\sqrt{(2\pi j)^2 + \delta^2}}$$

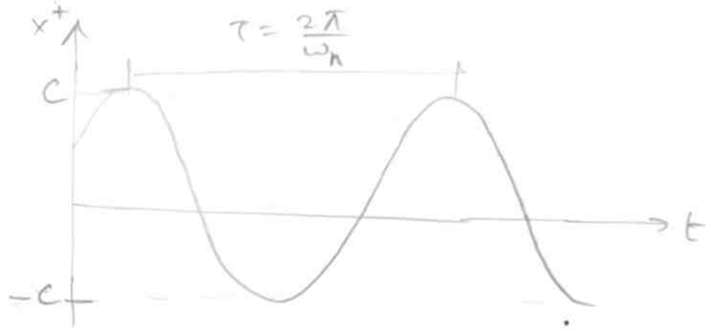
if j is small (say $j=1$), and $x_1 \simeq x_2$ ie. δ is small
ie. ζ is small, $\zeta \simeq \frac{\delta}{2\pi}$

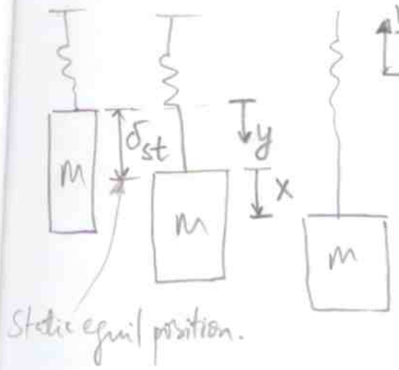
(a) Undamped.
Put $\zeta = 0$ in Case III (Underdamped) results → get SHM

$x = A\cos\omega_n t + B\sin\omega_n t$ $= C\sin(\omega_n t + \psi)$	$A = x_0$ $B = \frac{\dot{x}_0}{\omega_n}$	or	$C = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2}$ $\tan\psi = \frac{x_0\omega_n}{\dot{x}_0}$

$\omega_d = \omega_n$ rad/s
$T_n = \frac{2\pi}{\omega_n}$
$f_n = \frac{\omega_n}{2\pi}$ cyc/s = Hz







$\underset{\sim}{i}$, $\underset{\sim}{j}$

$\sum F_i = m a_i$

$m\ddot{y} - mg + ky = 0 \longrightarrow$ set $\frac{d}{dt} = 0 \Rightarrow k\delta_{st} = mg$.

let $y = x + \delta_{st}$

$m\ddot{x} - \cancel{mg} + kx + \cancel{k\delta_{st}} = 0 \longrightarrow$ by definition $k\delta_{st} = mg$

$\therefore \boxed{m\ddot{x} + kx = 0}$

- So better to define disp variable about static equil.
- Wont work for nonlinear systems so we have to deal with y itself.

do whole page

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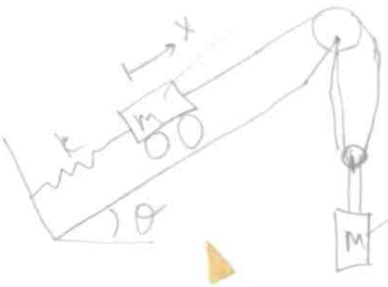
LmE101

Find ω_n



for m, $\sum F_x = T - kx \overset{-mg\sin\theta}{} = m\ddot{x}$

for M, $\sum F_y = Mg - 2T = M\ddot{y}$

Kinematics :- $x = 2y$.

$\therefore\ Mg \overset{-2mg\sin\theta}{} - 4ky = m5\ddot{y}$

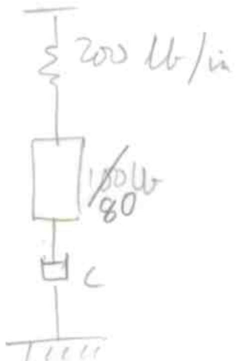
$\delta_{st} = mg(1 - 2\sin\theta)/4k$

let $z = y - \delta_{st}$

$\therefore\ 5m\ddot{z} + 4kz = 0$

$$\omega_n = \sqrt{\frac{4k}{5m}}$$

8/~~30~~ 31 (slightly modified)



Find c so that system is critically damped

$$\zeta = \frac{c}{2m\omega_n} = 1$$

$$\therefore\ c = 2m\omega_n = 2\,\frac{\cancel{100}\ 80}{32.2}\sqrt{\frac{200\times 12}{\cancel{100}\ 80/32.2}}$$

$$= 154.4\ \frac{\text{lb-sec}}{\text{ft}}$$

Do also 8/26 (see next pg.) (undamped free)
8/38 L.41 (damped free)

HWP Answers

8/26 $C = 43.1\ \frac{Ns}{m}$

8/~~9~~ 12 (don't do. It's HWP)

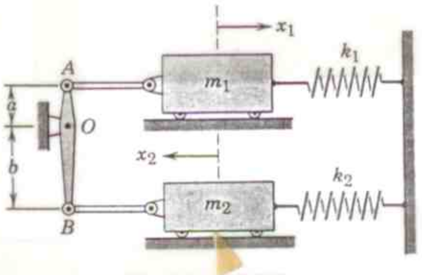
Given :- both springs unstretched.
Find δ_{st}
ω_n

$\sum F_x = 3kx - mg\sin\theta = m\ddot{x}$

$\delta_{st} = mg\sin\theta/3$

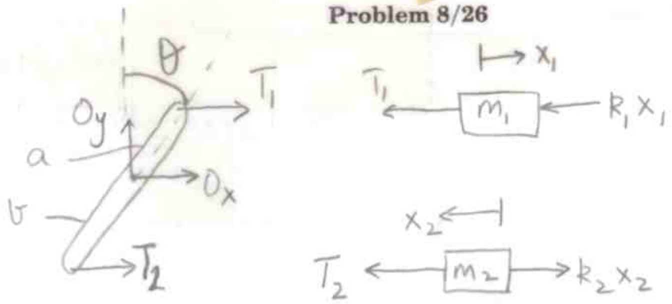
$\omega_n = \sqrt{\frac{3k}{m}}$ $T = 2\pi\sqrt{\frac{m}{3k}}$

8/26 (new book 3rd ed.)



Problem 8/26

8/26 Derive the differential equation of motion for the system shown in terms of the variable x_1. The mass of the linkage is negligible. State the natural frequency ω_n' in rad/s for the case $k_1 = k_2 = k$ and $m_1 = m_2 = m$. Assume small oscillations throughout.



$$\frac{x_1}{a} = \frac{x_2}{b} \longrightarrow ① \text{ (from similar triangles)}$$

FBD Link

$$\sum M_O = \cancel{I_O \alpha} = 0 \text{ (mass negligible)}$$

$$T_2 b - T_1 a = 0 \longrightarrow ② \quad (\because \text{small } \theta)$$

FBD m_1, m_2

$$\sum F_{x_1}: \quad -T_1 - k_1 x_1 = m_1 \ddot{x}_1 \longrightarrow ③$$

$$\sum F_{x_2}: \quad T_2 - k_2 x_2 = m_2 \ddot{x}_2 \longrightarrow ④$$

From ①–④,

$$-T_2 \frac{b}{a} - k_1 x_1 = m_1 \ddot{x}_1$$

$$-(k_2 x_2 + m_2 \ddot{x}_2)\frac{b}{a} - k_1 x_1 = m_1 \ddot{x}_1$$

$$-\left(k_2 \frac{b}{a} x_1 + m_2 \frac{b}{a} \ddot{x}_1\right)\frac{b}{a} - k_1 x_1 = m_1 \ddot{x}_1$$

$$\left(m_1 + m_2 \frac{b^2}{a^2}\right)\ddot{x}_1 + \left(k_1 + k_2 \frac{b^2}{a^2}\right) x_1 = 0 \longrightarrow ⑤$$

Put $k_1 = k_2 = k$, $m_1 = m_2 = m$ in ⑤

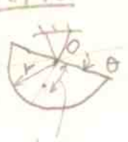
$$\therefore \; m\ddot{x}_1 + kx_1 = 0$$

$$\therefore \; \omega_n = \sqrt{\frac{k}{m}}$$

Rigid Body Vibrations

- Analogy between $F - x$ & $M - \theta$ (i.e., $M = k_\theta \theta$ for spring, $c_\theta \dot{\theta}$ for damper).
- no need to consider static forces as they cancel. So write EOM about static equil & drop wt. terms.
- apply moment equil eqns.

MK – 8/76



$4r/3\pi$

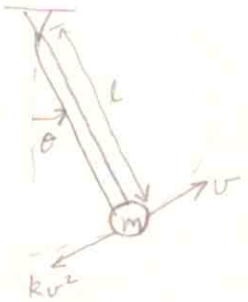
Find f_n for small oscillations.

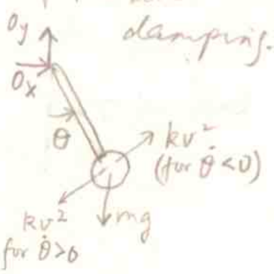
$$\Sigma M_0 = -mg\frac{4r}{3\pi}\sin\theta = \underbrace{I_0}_{\frac{mr^2}{2}}\ddot{\theta} \Rightarrow \ddot{\theta} + \frac{8g}{3r\pi}\theta = 0 \quad (\text{for small } \theta)$$

$$\Rightarrow \omega_n = \sqrt{\frac{8g}{3r\pi}}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{T} = \frac{1}{\pi}\sqrt{\frac{2g}{3r\pi}}$$

MK 8/87





Pendulum bob oscillates in fluid. Drag is kv^2. Neglect mass & drag of rod and bearing friction. Find EOM & comment on form of damping.

$$\Sigma M_0 : -(\pm k v^2 + mg\sin\theta)l = I_0\alpha = ml^2\ddot{\theta}, \quad \boxed{\begin{array}{l} + \text{ for } \dot{\theta} > 0 \\ - \text{ for } \dot{\theta} < 0 \end{array}}$$

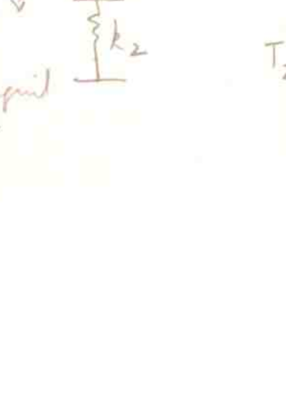
$v = l\dot{\theta} \rightarrow$ Kinematics

$$\therefore -(\pm k l^2\dot{\theta}^2 + mg\theta)l = ml^2\ddot{\theta} \quad (\text{for small } \theta)$$

$$\therefore \ddot{\theta} + \frac{g}{l}\theta \pm \frac{kl}{m}\dot{\theta}^2 = 0$$

or $$\ddot{\theta} + \frac{g}{l}\theta + \frac{kl}{m}|\dot{\theta}|\dot{\theta} = 0$$ → Coulomb damping.

MK 8/91



Pulley is homogenous & solid cylindrical, m_1, r_1
Find EOM in terms of x.
Assume no slip of cord wrt pulley.

$$\Sigma M_0 : (T_2 - \underbrace{T_1}_{k_1(x-x_B)})r_1 = \frac{m_1 r_1^2}{2}\alpha = \frac{m_1 r_1^2}{2}\ddot{\theta} \rightarrow ①$$

$$\Sigma F_x : -k_2 x - T_2 = m_2\ddot{x} \rightarrow ②$$

Kinematics: pt A on pulley / cord → $r_1\ddot{\theta} = \ddot{x}$ → ③

$$\therefore [-k_2 x - m_2\ddot{x} - k_1(x - b\cos\omega t)] = \frac{m_1}{2}\ddot{x}$$

$$\therefore (m_2 + m_1/2)\ddot{x} + (k_1 + k_2)x = k_1 b\cos\omega t$$

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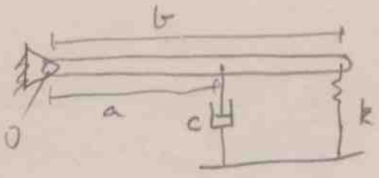
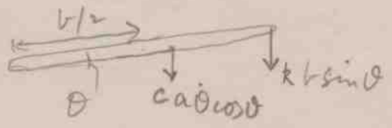
ME 101



uniform rod of mass m freely pivoted at O.
For small oscillations find ζ.
For what value of c_{cr} is $\zeta = 1$.

* Assume equil pos. shown (i.e., spring compressed)



$$\Sigma M_O: \; -kb\sin\theta(b\cos\theta) - ca\dot{\theta}\cos\theta(a\cos\theta) = \frac{mb^2}{3}\ddot{\theta}$$

for small θ, $\ddot{\theta} + \dfrac{3ca^2}{mb^2}\dot{\theta} + \dfrac{3k}{m}\theta = 0$

$$\omega_n = \sqrt{\frac{3k}{m}}, \quad 2\zeta\omega_n = 3\frac{ca^2}{mb^2} \Rightarrow \zeta = \frac{1}{2}\frac{a^2}{b^2}c\sqrt{\frac{3}{km}}$$

$$\zeta = 1 \text{ for } c = c_{cr} = \frac{2b^2}{a^2}\sqrt{\frac{km}{3}}$$