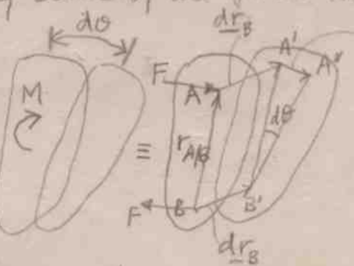
## VIRTUAL WORK & ENERGY METHODS

... No FBD reqd & we deal with only some of the forces acting on the body to get equil eqns.

$dU = \underline{F} \cdot d\underline{r} = F\,dr\cos\alpha$


$\boxed{U = \int \underline{F} \cdot d\underline{r}}$




both $\perp^{ar}$ to $\underline{r}_{A/B}$

$dU = \cancel{\underline{F} \cdot d\underline{r}_B} - \cancel{\underline{F} \cdot d\underline{r}_B} + \underline{F} \cdot d\underline{r}_{A/B}$

$= |\underline{F}|\,|d\underline{r}_{A/B}| = \underbrace{F|\underline{r}_{A/B}|}_{M}\,d\theta = M\,d\theta$

$\boxed{U = \int M\,d\theta}$

<u>Definition</u>: $\delta\underline{r}$ is an infinitesimal hypothetical displ <u>consistent with constraints</u>. Thus $\delta\underline{r}$ may be different from $d\underline{r}$ which is the actual infinitesimal displ that could occur during $dt$. Thus $d\underline{r}$ can be integrated but $\delta\underline{r}$ cannot. Similar definition holds for $\delta\theta$.

### <u>Principle of Virtual Work for Particle</u>

Consider particle constrained to move on frictionless surface under action of applied forces $\underline{F}_1 \cdots \underline{F}_n$. $\underline{N}$ is normal reaction from surface

Equil $\Rightarrow \sum \underline{F} = \underbrace{\sum_{i=1}^{n} \underline{F}_i}_{= \underline{F}_R} + \underline{N} = 0 \Rightarrow (\underline{F}_R + \underline{N}) \cdot \delta\underline{r} = 0 \Rightarrow \underline{F}_R \cdot \delta\underline{r} = 0 \;(\because \underline{N} \cdot \delta\underline{r} \equiv 0)$

Note: keep forces/moments const during infinitesimal virtual displ (else HOT's will arise which will be dropped later anyway)

$\therefore \boxed{\delta U = \underline{F}_R \cdot \delta\underline{r} = 0}$ → ① → Necessary & sufficient (shown later) condt for equil.

Sufficiency proof:
For particle initially @ rest (when $F_R$ applied), it is also sufficient condt for equil. We prove this by contradiction. Assume that ① holds but particle not in equil. This means particle initially @ rest moves $d\underline{r}$ (in dir of $\underline{N} + \underline{F}_R$) in $dt$.

$\therefore dU = (\underline{N} + \underline{F}_R) \cdot d\underline{r} = \underline{F}_R \cdot d\underline{r} > 0$

Now choose $\delta r = dr$ (always a possible choice for $\delta r$) & hence we get a contradiction of ①.

Another way to prove sufficiency is relax constraints (i.e. take them as active forces) & then let $\delta\underline{r} = \delta x\,\underline{i}, \delta y\,\underline{j}, \delta z\,\underline{k}$ resply in order to get $\sum F_x = \sum F_y = \sum F_z = 0$

### <u>Principle of Virtual Work for a Rigid Body</u> (Assume no friction, i.e, Ideal RB)

RB in equil ⇒ each particle in equil. ⇒ We sum ① over all particles.

Virtual work due to constraint forces is zero

Virtual work due to internal forces is zero ∵ they cancel out in pairs (equal opp forces undergoing identical [∵ R virtual displ)

$\boxed{\therefore \delta U = \sum_{i=1}^{n} (\underline{F}_R)_i \cdot \delta\underline{r}_i = 0}$, n = nos of particles having active (applied) forces = nos of (resultant) active forces

↳ Necessary & sufficient condt for equil. Proof of sufficiency is same as that for single particle.

<u>Note</u>: Proof of Shames that int forces do no work for RB's (p. 372) breaks down when considering system of RB's.

### <u>Principle of Virtual Work for Ideal System of Interconnected Rigid Bodies</u> (Assume no friction @ supports & connections)

The only difference is internal forces @ connections. Since they occur in pairs (equal & opp forces) and undergo identical displ, their contribution to $\delta U$ is zero.

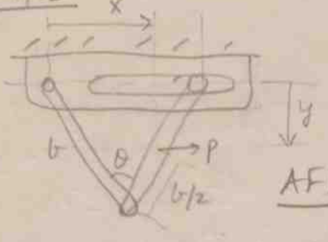

Active forces : $F_1, F_2$

Reactive forces : reactions/constraints forces

Internal forces : forces @ connections (plus int. forces on constituent particles).

MK 7/12

√




Links: each has mass m, length b ; Find θ given P.

1 DoF $\quad x = b\sin\frac{\theta}{2} + \frac{b}{2}\sin\frac{\theta}{2} = \frac{3b}{2}\sin\frac{\theta}{2}$

$y = \frac{b}{2}\cos\frac{\theta}{2}$

AFD

$\therefore\ \delta x = \frac{3b}{2}\cdot\frac{1}{2}\cos\frac{\theta}{2}\ , \quad \delta y = -\frac{b}{2}\cdot\frac{1}{2}\sin\frac{\theta}{2}$

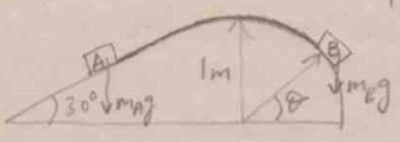
$\delta U = P\delta x + 2(mg)\delta y = P\frac{3\cancel{b}}{\cancel{2}}\cdot\frac{1}{\cancel{2}}\cdot\cos\frac{\theta}{2} - 2mg\frac{\cancel{b}}{2}\cdot\frac{1}{\cancel{2}}\sin\frac{\theta}{2} = 0 \Rightarrow \tan\frac{\theta}{2} = \frac{3P}{2mg}$ ◀

Shames 10.25

√

No friction, $m_A g = 200\,N$, $m_B g = 150\,N$

Find θ at equil ($=\theta_e$)




AFD

$\delta U = 0 = m_B g\cos\theta_e[r\delta\theta] - m_A g\sin 30°[r\delta\theta] = 0$

$\theta_e = \cos^{-1}\left(\frac{200\times 0.5}{150}\right) = 48.2°$ ◀

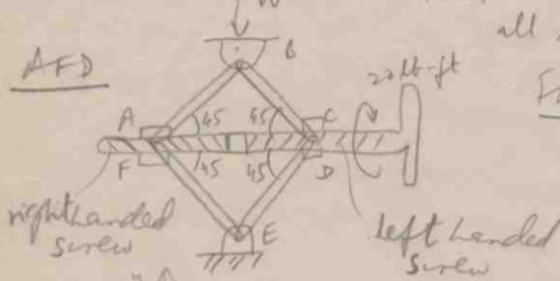
Shames 10.12

√

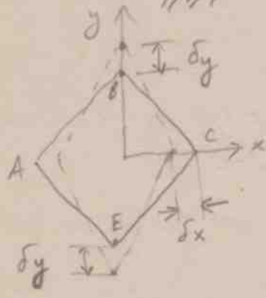
No friction, 20 lb-ft torque applied, pitch = 0.3" in opp sense, all links of equal length = 1ft

Find: W that can be maintained in equil.

AFD




The mechanism is as follows. Under the action of the torque in dir. shown, the left handed [right handed] half tends to move towards the right [left]. Since that is not possible, the collars AF and CD move inwards thus raising the load.




∵ pt E cannot move down the screw, (line AC) rises by δy and the load (pt B) by 2δy.

$\delta U = 0 = M\delta\theta - W(2\delta y) = 0$

Now $\frac{\delta x}{\delta\theta} = \frac{-\text{pitch}}{2\pi} = \frac{-0.3}{2\pi}$

Also $y^2 + x^2 = \text{const} \Rightarrow \quad \delta y = -\frac{x}{y}\delta x = \frac{0.3}{2\pi}\frac{x}{y}\delta\theta$

$\therefore\ \delta U = 0 = M\delta\theta - W * 2 * \left(\frac{0.3}{2\pi}\frac{x}{y}\right)\delta\theta = 0$

$\Rightarrow W = 2513\ lb$ ◀

## Advantages of Virtual Work Principle

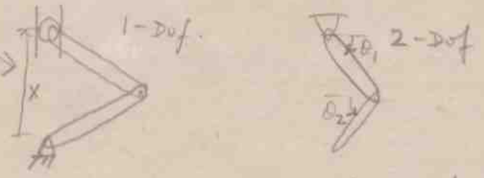
- Main advantage only for frictionless (ideal) system of interconnected RB's
- No need to dismember system, i.e. to draw FBD's revealing reactions/constraint forces or internal forces @ connection. So we can get relation between active forces w/o bothering about these other forces that would appear in FBD reqd in equilibrium approach.
- These advantages make VW method attractive when we want to find equil. config for known ext loading. This contrasts with the problem of finding reactions given ext loading + equil config.
- When internal friction present we need to include it as active force in $\delta U$ expression so VW principle becomes less attractive.

## Degrees of Freedom

- Nos. of indep coords (dist or angle) that are reqd to completely specify config of system are number of d.o.f's.
- Their nos. is reqd to decide how many indep eqns we can get by VW principle. Let $x_1 .. x_m$ be d.o.f's. Then write $\delta U = 0$ m times, each time taking $[(\delta x_j \neq 0,\ \delta x_i = 0,\ i \neq j,\ i = 1, m),\ j = 1, m]$.

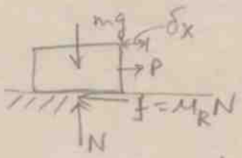
## Active Force Diag (AFD)

Unlike FBD this only shows forces that have non-zero contribution for $\delta U$. Only AFD reqd for VW principle.

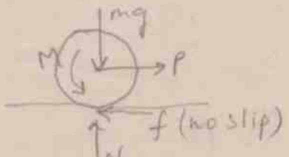



## Systems with friction (Real system). Mechanical efficiency

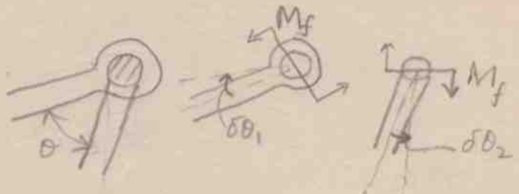
- Friction does -ve work since it always opposes motion.
- (-ve) work done by friction generates heat. Thus some of the +ve work done by ext AF's is dissipated as heat. This dissipated work is unrecoverable.




(a) Dissipative system f does -ve work



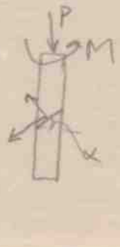

(b) Non-Dissipative ∴ f does no work




(c) $\delta U_f = -M_f\,\delta\theta_1 - M_f\,\delta\theta_2 = -M_f\,\delta\theta$

mech efficiency = $e = \dfrac{\text{output work}}{\text{input work}}$


$$e = \frac{mg\,\delta s \sin\theta}{mg(\sin\theta + \mu_k \cos\theta)\,\delta s}$$




$$M = Pr\,\frac{(\mu\cos\alpha + \sin\alpha)}{(\cos\alpha - \mu\sin\alpha)}$$

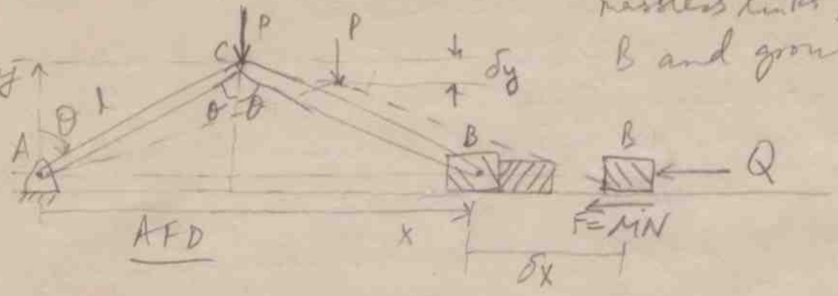
input work = $M\,\delta\theta$

output work = $P\left(\underbrace{\frac{L}{2\pi r}}_{\tan\alpha} \times r\,\delta\theta\right)$

(over)

eg. Toggle vise from BPJ p. 406

massless links, $\mu$ is coeff of friction between block B and ground.




AFD

$\delta U = -Q\,\delta x - P\,\delta y - F\,\delta x$

Now, $\delta x = 2l\cos\theta\,\delta\theta$, $\delta y = -l\sin\theta\,\delta\theta$

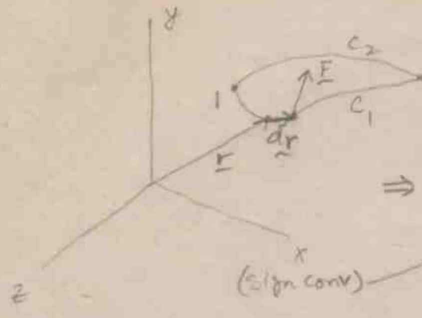
We must draw FBD & do $\Sigma M_A = 0$ to get $N = P/2$ (then put $F = \mu N$)

Thus, from $\delta U = 0$ we get $Q = \frac{1}{2}P(\tan\theta - \mu)$ ◀

Thus $Q = 0$ for $\tan\theta = \mu$, $Q < 0$ for $\tan\theta < \mu$ so vise is ineffective when when $\tan\theta < \mu$.

$\eta$ = efficiency = $\dfrac{\text{output work}}{\text{input work}} = \dfrac{2Ql\cos\theta\,\delta\theta}{Pl\sin\theta\,\delta\theta} = 1 - \mu\cot\theta$ ◀

## Potential Energy Method.




$\underset{\sim}{F} = \underset{\sim}{F}(x,y,z)$.

In general $U_{1-2} = \int_c \underset{\sim}{F}\cdot d\underset{\sim}{r}$ is path dependent.

Suppose $\underset{\sim}{F}\cdot d\underset{\sim}{r}$ is an exact differential, $-dV$, of a scalar f^n^ $V(x,y,z)$.

$\Rightarrow U_{1-2} = \int \underset{\sim}{F}\cdot d\underset{\sim}{r} \equiv \int_1^2 -dV = V_1 - V_2 = -\Delta V =$ path indep, hence $\underset{\sim}{F}$ is consv field

Now $-dV = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = \underset{\sim}{F}\cdot d\underset{\sim}{r} = F_x dx + F_y dy + F_z dz$

Thus $\boxed{F = -\nabla V}$ → i.e, for a conservative force field $\underset{\sim}{F}$.

Note: If $\underset{\sim}{F}$ is consv we also have $\oint \underset{\sim}{F}\cdot d\underset{\sim}{r} = 0$

Note: in general for any $\underset{\sim}{F} \nexists$ a $V$, ie, only for consv $\underset{\sim}{F}$ $\exists$ $V$.

Thus consv $F$ is always derivable from a pot. f^n^ $V$.

Note: For consv. $\underset{\sim}{F}$, $\underset{\sim}{F}$ (& hence $V$) cannot (explicitly) depend on time.

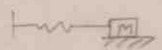
(i) Grav. pot. energy ($V_g$) Let $\underset{\sim}{F} = -mg\underset{\sim}{j} \Rightarrow \frac{\partial V_g}{\partial y} = mg,\ \frac{\partial V_g}{\partial x} = \frac{\partial V_g}{\partial z} = 0 \Rightarrow V_g = mgy$

So work done by wt $= U_{1-2} = \int \underset{\sim}{F}\cdot d\underset{\sim}{r} = \underbrace{V_{g_1} - V_{g_2}}_{-\Delta V_g} = mg(y_1 - y_2)$

Thus grav pot energy is −ve of work done by wt. in slowly raising/lowering body from a datum through a ht.
" " " " +ve " " " on wt " " " " " " " " "

(ii) Elastic Pot energy – springs → $V_e$

Let $\underset{\sim}{F} = -kx\underset{\sim}{i} \Rightarrow V_e = \frac{1}{2}kx^2 =$ −ve [+ve] work done by [on] spring in deforming it from mean position to x



So work done by spring $= U_{1-2} = \frac{1}{2}k(x_1^2 - x_2^2) = -\Delta V_e$, so during tension/compr spring does −ve work
" relaxation " " +ve "

Similarly for torsional spring, $V_e = \frac{1}{2}k\theta^2$

### Equilibrium

※ Thus $(\delta U)_{\text{consv forces}} = -(\delta V)_{\text{consv forces}}$. Thus we get $\boxed{\delta U = \delta V}$ where now $\delta U$ is virtual work done by non-consv active forces. So now in AFD we do not isolate system from springs (i.e., dont show spring forces in AFD) & we dont show wt's in AFD.

※ Suppose $F_i$ $(i=1,n)$ are consv forces acting on system

Then $\delta U = \sum_{i=1}^{n} \underset{\sim}{F}_i\cdot\delta\underset{\sim}{r}_i = -\sum_{i=1}^{n}\delta V_i = -\delta\sum_{i=1}^{n} V_i = -\delta V$

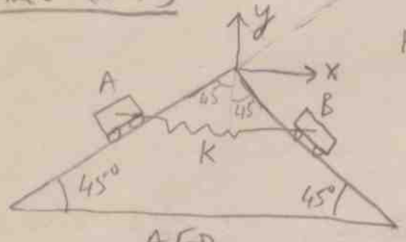
but $\delta U = 0 \Rightarrow \boxed{\delta V = 0}$ → i.e, stationarity condt. for V which ensures equil.
i.e, zero first order change in V @ equil. (maxima, minima or inflexion pt.).

For n. d.o.f system $V = V(q_1 \ldots q_n)$

$\delta V = \sum_{i=1}^{n}\frac{\partial V}{\partial q_i}\delta q_i = 0 \Rightarrow$ for $\delta q_i$ arbitrary, $\boxed{\frac{\partial V}{\partial q_i} = 0,\quad i = 1, n}$ → equil. eqns.

Shames (10-49)

✓




AFD.

$K = 3\ N/mm$ ; unstretched spring length $= 450mm$

$W_A = 60N$, $W_B = 90N$,

Find: stretched length of spring @ equil.

2 d.o.f's $x_A, x_B$.

Datum is x-axis.

$$V = V_g + V_s = W_A \cancel{y_A}^{x_A} + W_B \cancel{y_B}_{-x_B} + \frac{1}{2}K\left[\sqrt{(x_B-x_A)^2+(y_B-y_A)^2} - 450\right]^2$$

$$= W_A x_A - W_B x_B + \frac{1}{2}K\left[\sqrt{(x_B-x_A)^2+(-x_B-x_A)^2} - 450\right]^2 = W_A x_A - W_B x_B + \frac{1}{2}K\left[\sqrt{2}\sqrt{x_A^2+x_B^2} - 450\right]$$

$\delta U = \delta V$, $\because$ no n/c ext forces $\Rightarrow \delta V = 0$ for equil $\Rightarrow \frac{\partial V}{\partial x_A} = 0$, $\frac{\partial V}{\partial x_B} = 0$

$$\therefore \frac{\partial V}{\partial x_A} = W_A + K\left[\sqrt{2}\sqrt{x_A^2+x_B^2} - 450\right]\left[\sqrt{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x_A^2+x_B^2}}\cdot 2x_A\right] = 0 \rightarrow ①$$

$$\frac{\partial V}{\partial x_B} = -W_B + K\left[\sqrt{2}\sqrt{x_A^2+x_B^2} - 450\right]\left[\sqrt{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x_A^2+x_B^2}}\cdot 2x_B\right] = 0 \rightarrow ②$$

from $\frac{①}{②} \longrightarrow \frac{x_A}{x_B} = -\frac{W_A}{W_B} = -\frac{2}{3}$

Subst this result in ② (or ①) & get,

$$K\left[\sqrt{2}\,x_A(-\sqrt{1+9/4}) - 450\right]\left[\frac{1}{\sqrt{2}}\frac{1}{\cancel{x_A}(-\sqrt{1+9/4})}\cdot\cancel{2}\left(\frac{-3}{\cancel{2}}\right)\cancel{x_A}\right] = W_B$$

$$\therefore x_A = \left\{\frac{W_B}{K\left[\frac{-3}{\sqrt{2}\left(\frac{-\sqrt{13}}{2}\right)}\right]} + 450\right\} \Big/ \left\{\sqrt{2}\left(\frac{-\sqrt{13}}{2}\right)\right\} = -186.50\,mm$$

Note: $\because$ we subst $x_B$ in terms of $x_A$, & $x_A < 0$ is implied, then $\because \sqrt{\ }$ term represents a length (hence it must be +ve) we have to take −ve $\sqrt{2}$. If we did everything in terms of $x_B$ then $\because x_B > 0$ implied, we would take +ve $\sqrt{2}$

$x_B = 279.75\,mm$

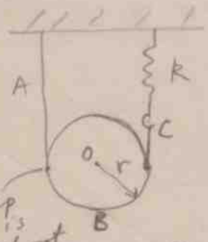
$L = \sqrt{2} * \sqrt{x_A^2 + x_B^2} = 475.49\ mm$ ◄

(Stretched length)

The other method would be to take $\overset{x_A}{\longleftrightarrow} x_B$, $\downarrow y_B, y_A$ coord system. Then $x_A > 0$, $x_B > 0$ so we always take +ve $\sqrt{\ }$. The change will be that $V_g = -W_A y_A - W_B y_B$, $y_B = x_B$, $y_A = x_A$, $L = \left[(x_B+x_A)^2 + (y_B - y_A)^2\right]^{1/2}$ & $x_A/x_B = +2/3$

MK-7.61

✓




no slip so this is inst center of rot.

uniform wheel mass m supported by light band ABC (in vertical plane) & spring. Released initially from position where spring has no force. Find clockwise angle $\theta$ thru which wheel rotates from initial position to equil.

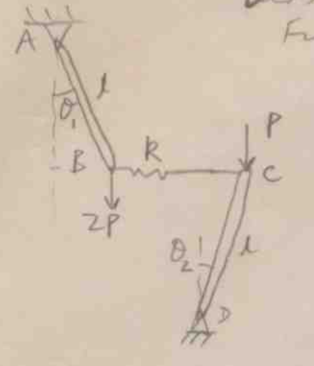
Assume no slip during downward motion $\Rightarrow$ consv system.

$\Rightarrow$ @ every instant, $\frac{\text{dist moved by } O}{\text{dist moved by spring, ie. pt } C} = \frac{2r}{r} = 2$.

dist moved by $O = r\theta$ ($\because$ no slip). Now fix datum at horz line thru O when spring unstretched

$\therefore V = V_g + V_s = -mgr\theta + \frac{1}{2}k(2r\theta)^2 \Rightarrow \delta V = 0$ gives, $-mgr + k(2r\theta)(2r) = 0 \Rightarrow \theta_e = mg/4kr$ ◄

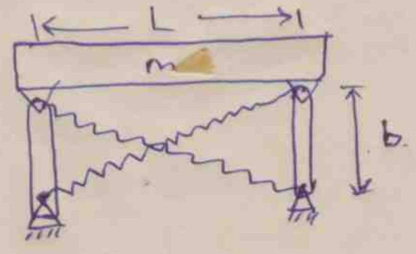
.10.78, BJ




Bars are massless. Springs undef when $\theta_i = 0$, $i = 1, 2$, & $\theta_i = 0$ is equil pos.
Find range of $P$ for which equil is stable. Assume small $\theta_i$ i.e. spring stays horz.

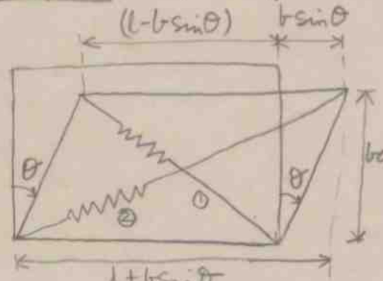
$V_g = 0$

$V_S =$




Platform mass $m$, length $L$, vertical rods length $b$, springs stiffness $k$, prestress $\Delta$. Find minimum $k$ for stability.

MK 7/72 = 7/67 of 5th ed.




1 d.o.f system.

$V_g = mgb\cos\theta$

$l_s =$ unstretched length $= \sqrt{l^2+b^2} - \Delta$

$(\Delta x)_1 = \sqrt{(l-b\sin\theta)^2 + (b\cos\theta)^2} - l_s = \sqrt{(l-b\sin\theta)^2+(b\cos\theta)^2} - \sqrt{l^2+b^2} + \Delta$

$(\Delta x)_2 = \sqrt{(l+b\sin\theta)^2 + (b\cos\theta)^2} - \sqrt{l^2+b^2} + \Delta$

$V_S = \frac{1}{2}k\left[(\Delta x_1)^2 + (\Delta x_2)^2\right]$

$$\therefore V_S = \tfrac{1}{2}k\Big[(l-b\sin\theta)^2+(b\cos\theta)^2+(l+b\sin\theta)^2+(b\cos\theta)^2+2\Delta^2+2(l^2+b^2)-4\Delta\sqrt{l^2+b^2}$$
$$+2\Delta\left\{\sqrt{(l-b\sin\theta)^2+(b\cos\theta)^2}+\sqrt{(l+b\sin\theta)^2+(b\cos\theta)^2}\right\}-2\sqrt{l^2+b^2}\left\{\sqrt{(l-b\sin\theta)^2+(b\cos\theta)^2}+\sqrt{(l+b\sin\theta)^2+(b\cos\theta)^2}\right\}\Big]$$

$$= \tfrac{1}{2}k\left[2l^2+2b^2+2\Delta^2+2l^2+2b^2-4\Delta\sqrt{l^2+b^2}+2\left(\sqrt{l^2+b^2-2lb\sin\theta}+\sqrt{l^2+b^2+2lb\sin\theta}\right)\left(\Delta-\sqrt{l^2+b^2}\right)\right]$$

$$\frac{\partial V}{\partial\theta} = -mgb\sin\theta + k\left(\Delta-\sqrt{l^2+b^2}\right)\left[\frac{1(-2lb\cos\theta)}{2\sqrt{l^2+b^2-2lb\sin\theta}}+\frac{(2lb\cos\theta)}{2\sqrt{l^2+b^2+2lb\sin\theta}}\right] \to \text{so } \theta=0^\circ \text{ satisfies equil eqn } \left(\left.\frac{\partial V}{\partial\theta}\right|_{\theta=}=0\right)$$

$$\frac{\partial^2 V}{\partial\theta^2} = -mgb\cos\theta + k\left(\Delta-\sqrt{l^2+b^2}\right)\left\{\left[\frac{-1}{\sqrt{l^2+b^2-2lb\sin\theta}}+\frac{1}{\sqrt{l^2+b^2+2lb\sin\theta}}\right](-lb\sin\theta)+(lb\cos\theta)\left[\frac{1(-2lb\cos\theta)}{2\left(\sqrt{l^2+b^2-2lb\sin\theta}\right)^3}-\frac{1(2lb\cos\theta)}{2\left(\sqrt{l^2+b^2+2lb\sin\theta}\right)^3}\right.\right.$$

Stability $\Leftrightarrow \left.\frac{\partial^2 V}{\partial\theta^2}\right|_{\theta=\theta_e=0^\circ} > 0$

$$\Rightarrow k > \frac{mgb}{\left(\Delta-\sqrt{l^2+b^2}\right)\left\{l^2b^2\left[-\frac{1}{(\sqrt{l^2+b^2})^3}-\frac{1}{(\sqrt{l^2+b^2})^3}\right]\right\}}.$$

For $\Delta = 0$, $k > \frac{mg}{2b}\left(1+\frac{b^2}{l^2}\right)$

## Stability

cyl w/o friction ⇒ consv Force field (only gravity).

stable $V = \min$ | unstable $V = \max$ | neutral $V = \text{const.}$ | unstable

Let $V = V(x)$

$$V(\Delta x + x_{eq}) = V_{eq} + \underbrace{\left(\frac{dV}{dx}\right)_{eq}}_{=0 \text{ for equil.}} \Delta x + \frac{1}{2!}\left(\frac{d^2V}{dx^2}\right)_{eq} \Delta x^2 + \cdots$$

$$\therefore \Delta V = V - V_{eq} = \frac{1}{2!}\left(\frac{d^2V}{dx^2}\right)_{eq} + \cdots$$