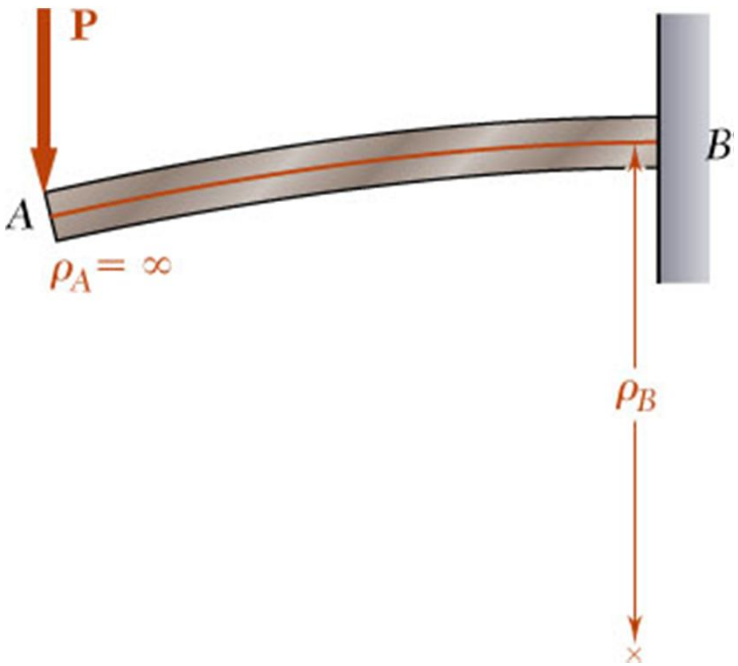
CHAPTER

9

MECHANICS OF MATERIALS

Deflection of Beams

Deformation of a Beam Under Transverse Loading



Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

Cantilever beam subjected to tip load,

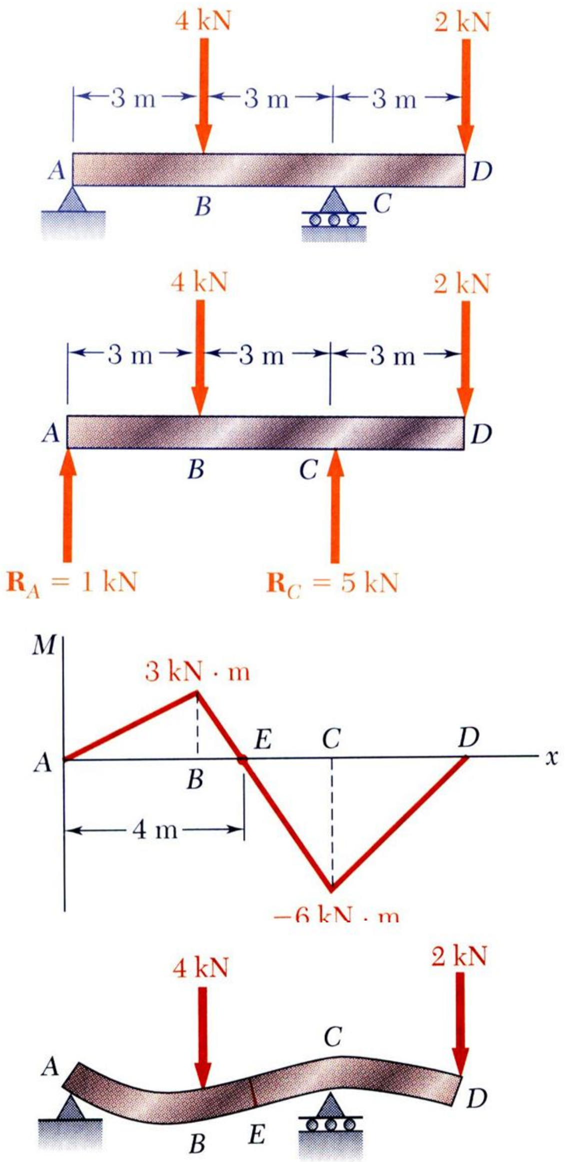
$$\frac{1}{\rho} = -\frac{Px}{EI}$$

Curvature varies linearly with *x*

At free end *A*, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$

At support *B*, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Deformation of a Beam Under Transverse Loading

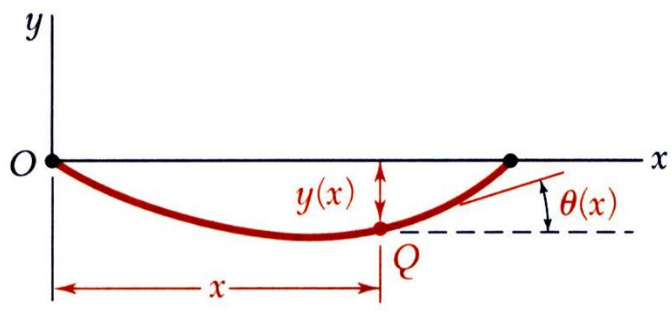


- Overhanging beam
- Reactions at *A* and *C*
- Bending moment diagram
- Curvature is zero at points where bending moment is zero, i.e., at each end and at *E*.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Beam is concave upwards where bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is maximum.
- An equation for the beam shape, i.e., *elastic curve*, is required to determine maximum deflection and slope.

Equation of the Elastic Curve



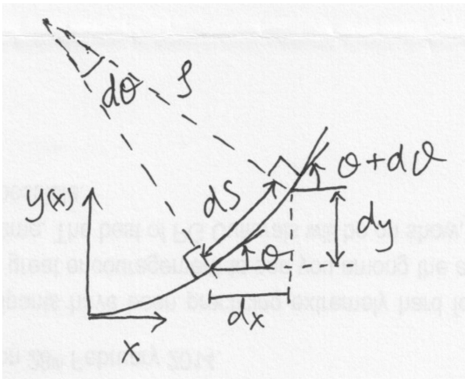
$$\frac{dy}{dx} = \tan\theta$$

$$\frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\frac{d(\tan\theta)}{dx} = \frac{d^2y}{dx^2} = \frac{d(\tan\theta)}{d\theta}\frac{d\theta}{dx} = \left(1+\tan^2\theta\right)\frac{d\theta}{dx} = \left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$



- Thus,

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

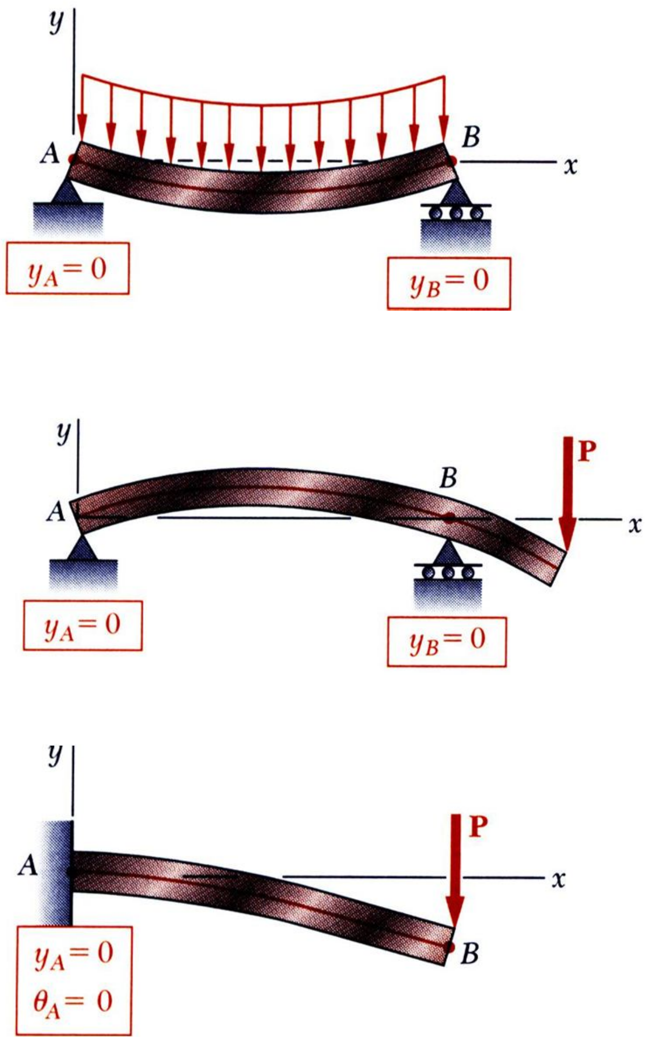
- Substituting and integrating,

$$EI\frac{1}{\rho} = EI\frac{d^2y}{dx^2} = M(x)$$

$$EI\,\theta \approx EI\frac{dy}{dx} = \int M(x)dx + C_1$$

$$EI\,y = \int dx \int M(x)dx + C_1x + C_2$$

Equation of the Elastic Curve



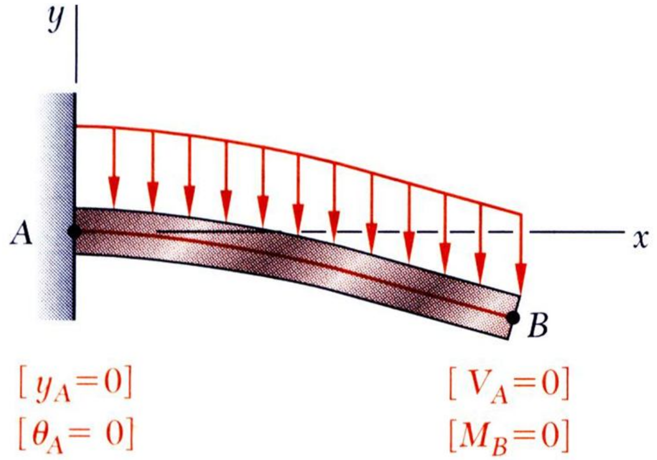
- Constants are determined from boundary conditions

$$EI\,y = \int dx \int M(x)\,dx + C_1 x + C_2$$

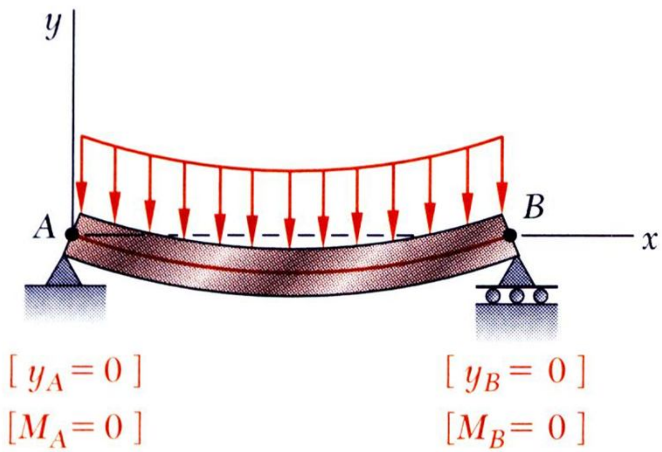
- Three cases for statically determinate beams,
  - Simply supported beam
    $y_A = 0, \quad y_B = 0$
  - Overhanging beam
    $y_A = 0, \quad y_B = 0$
  - Cantilever beam
    $y_A = 0, \quad \theta_A = 0$

- More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

Direct Determination of Elastic Curve from Load Distribution



(*a*) Cantilever beam



(*b*) Simply supported beam

- For a beam subjected to a distributed load,

$$\frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x)$$

- Equation for beam displacement becomes

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = -w(x)$$

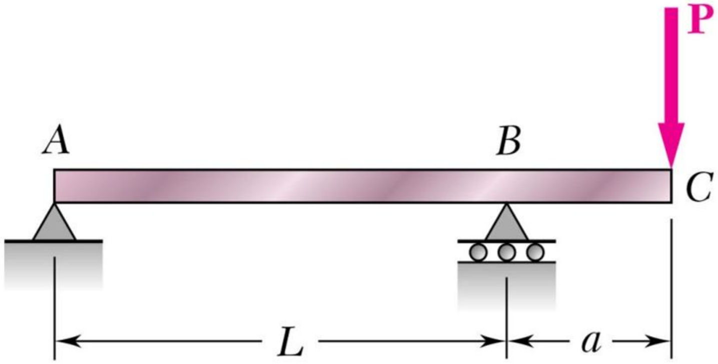
- Integrating four times yields

$$EI\,y(x) = -\int dx \int dx \int dx \int w(x)\,dx$$

$$+\tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from boundary conditions.

Example 1



W360 × 101 $\quad I = 300 \times 10^6 \text{ mm}^4$

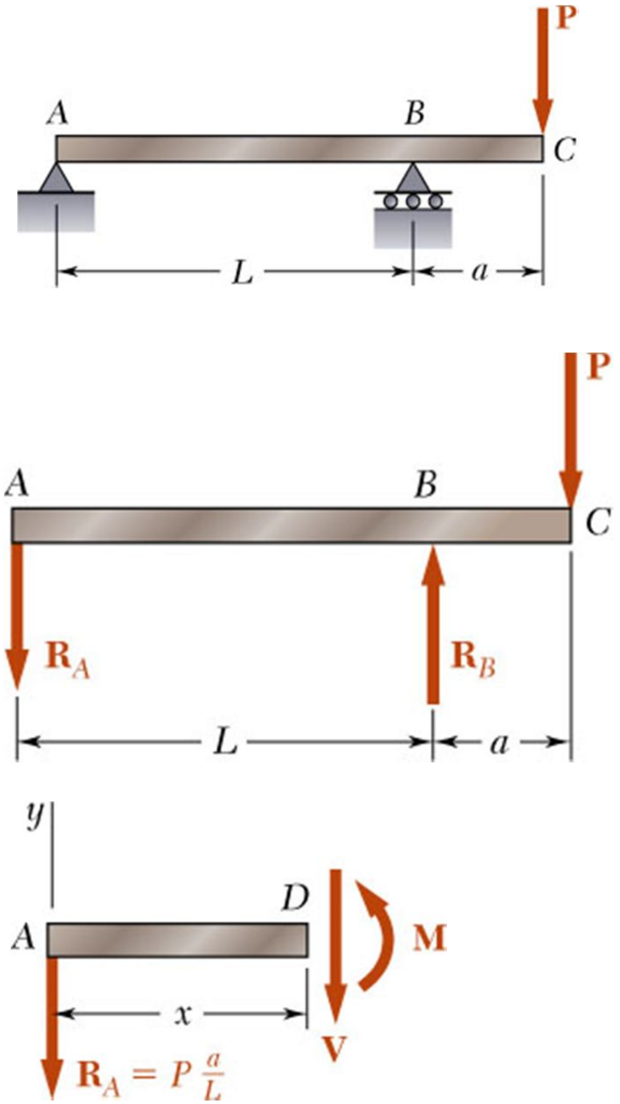
$E = 200$ GPa

$P = 200$ kN $\quad L = 4.5$ m

$a = 1.2$ m

For portion *AB* of the overhanging beam, (*a*) derive equation for the elastic curve, (*b*) find maximum deflection, (*c*) evaluate y_{max}.

Example 1



SOLUTION:

Develop expression for M(x) and derive differential equation for elastic curve.

- Reactions:

$$R_A = \frac{Pa}{L}\downarrow \quad R_B = P\left(1+\frac{a}{L}\right)\uparrow$$

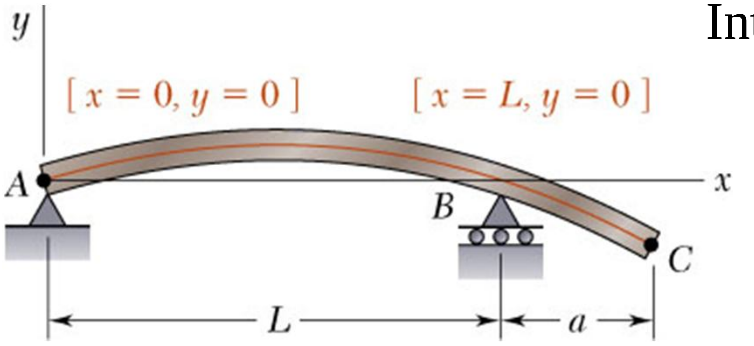
- From FBD for section *AD*,

$$M = -P\frac{a}{L}x \quad (0 < x < L)$$

- Differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Example 1



$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2$$

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ y = 0:\quad 0 = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \quad C_1 = \frac{1}{6}PaL$

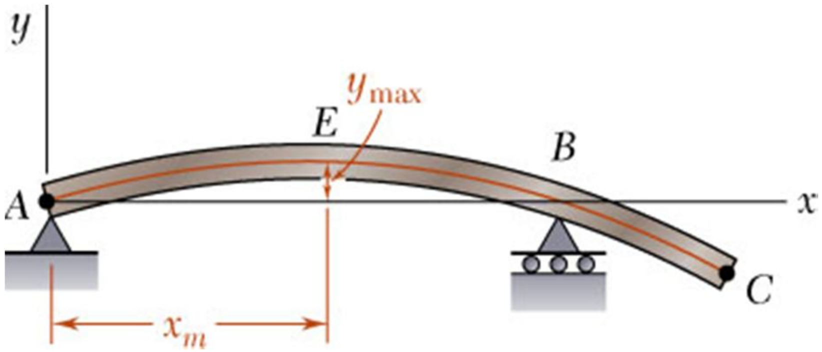
Substituting,

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx$$

$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

Example 1



$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

Locate point of zero slope i.e., point of maximum deflection.

$$\frac{dy}{dx} = 0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

Evaluate maximum deflection.

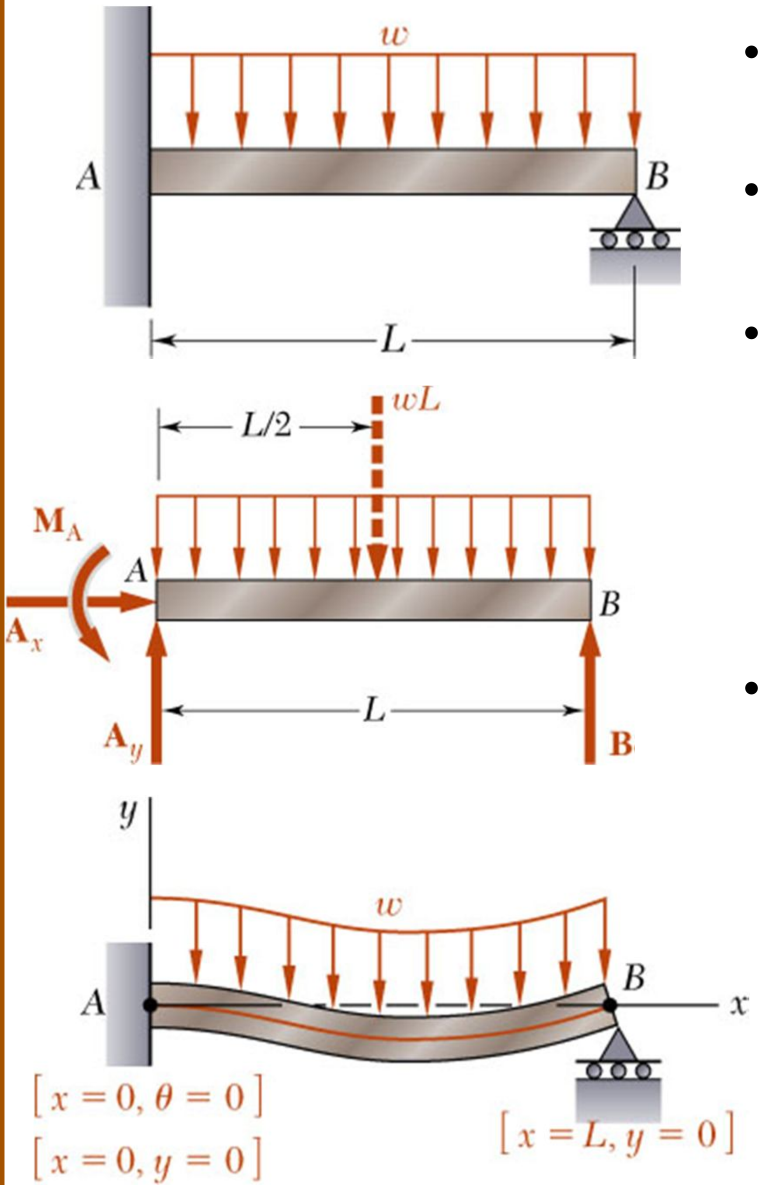
$$y_{max} = \frac{PaL^2}{6EI}\left[0.577 - (0.577)^3\right]$$

$$y_{max} = 0.0642\frac{PaL^2}{EI}$$

$$y_{max} = 0.0642\frac{\left(200\times10^3\ \text{N}\right)\left(1.2\,\text{m}\right)\left(4.5\,\text{m}\right)^2}{\left(200\times10^9\ \text{Pa}\right)\left(300\times10^{-6}\ \text{m}^4\right)}$$

$$y_{max} = 5.2\,\text{mm}$$

Statically Indeterminate Beams



- Consider beam fixed at *A* and roller support at *B*.
- There are four unknown reaction components.
- Conditions for static equilibrium are

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

So beam statically indeterminate to degree one.
Say R_B is redundant.

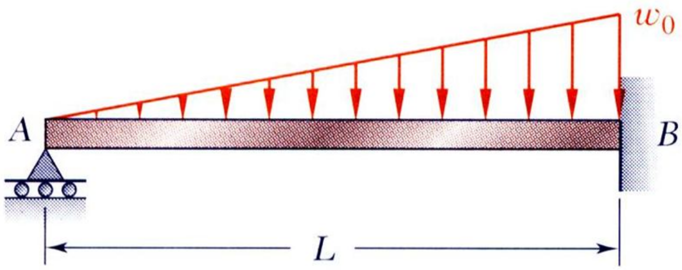
- We also have beam deflection equation,

$$EI\, y = \int dx \int M(x)\,dx + C_1 x + C_2$$

which introduces two unknowns but provides three additional equations from the boundary conditions (used to solve for C_1 , C_2 , R_B):

At $x = 0,\ \theta = 0\ y = 0$ At $x = L,\ y = 0$

Example 2



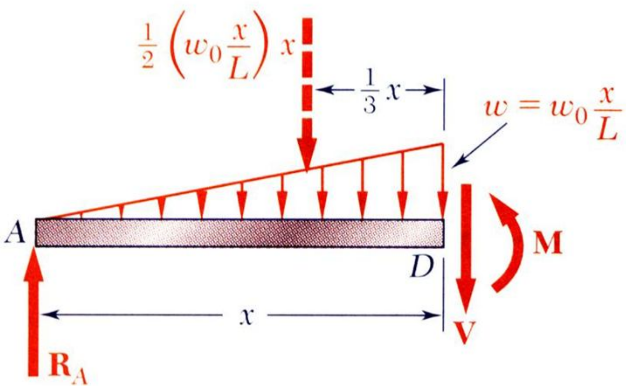
For the uniform beam, find reaction at *A*, derive equation for elastic curve, and find slope at *A*.

Beam is statically indeterminate to one degree (i.e., one excess reaction which static equilibrium alone cannot solve for).

SOLUTION:

- Develop differential equation for elastic curve (will be functionally dependent on reaction at *A*).
- Integrate twice, apply boundary conditions, solve for reaction at *A* and obtain elastic curve.

Example 2



- Consider moment acting at section *D*,

$$\sum M_D = 0$$

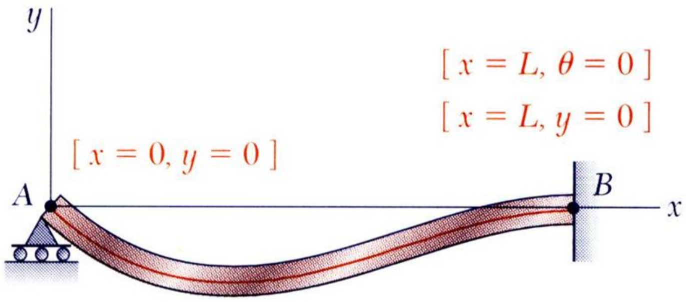
$$R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0$$

$$M = R_A x - \frac{w_0 x^3}{6L}$$

- The differential equation for the elastic curve,

$$EI\frac{d^2 y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

Example 2



$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

- Integrate twice

$$EI\frac{dy}{dx} = EI\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

- Apply boundary conditions:

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ \theta = 0:\quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0$

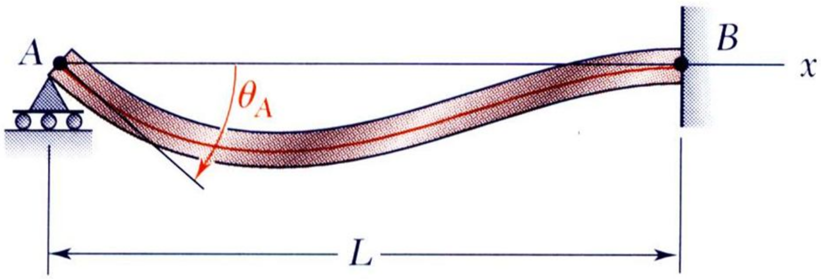
at $x = L,\ y = 0:\quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0$

- Solve for reaction at A

$$\frac{1}{3}R_A L^3 - \frac{1}{30}w_0 L^4 = 0$$

$$R_A = \frac{1}{10}w_0 L \uparrow$$

Example 2



- Substitute for C_1, C_2, and R_A in the elastic curve equation,

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

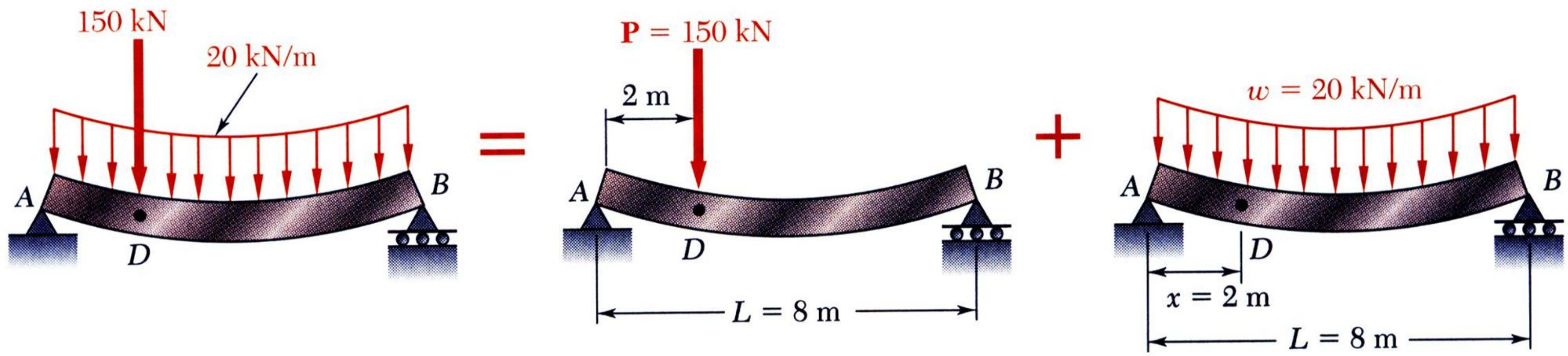
$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

- Differentiate once to find the slope,

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right)$$

at $x = 0$, $$\theta_A = \frac{w_0 L^3}{120EI}$$

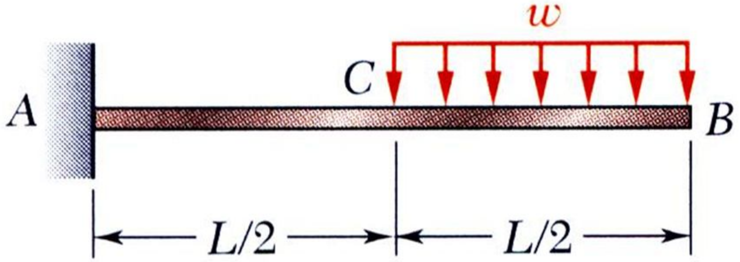
Method of Superposition



Principle of Superposition:

- Deformations of beams subjected to combinations of loadings may be obtained as a linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

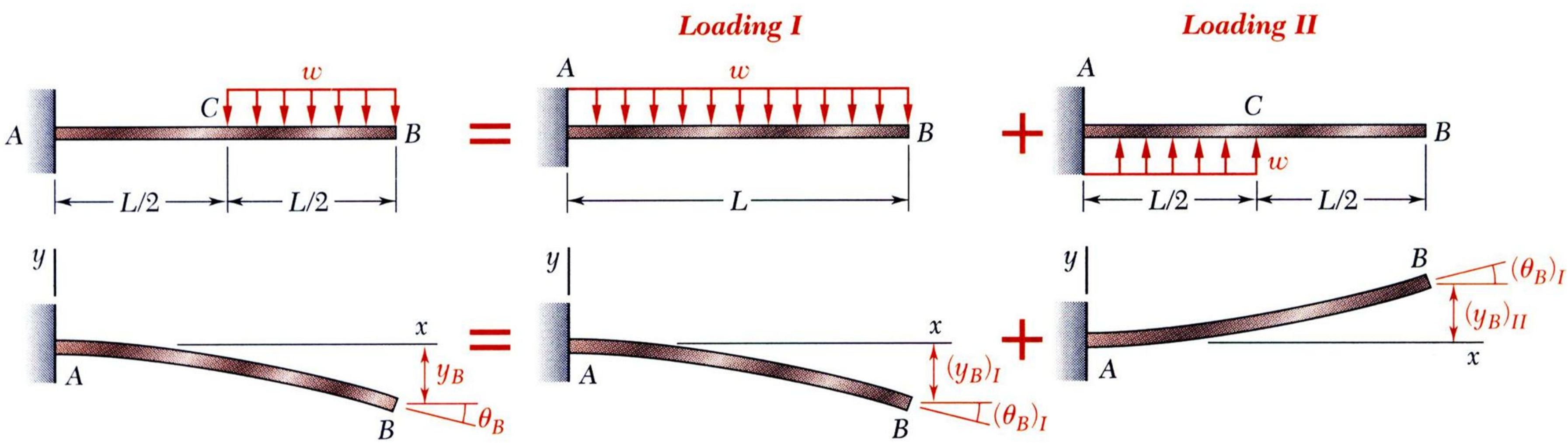
Example 3



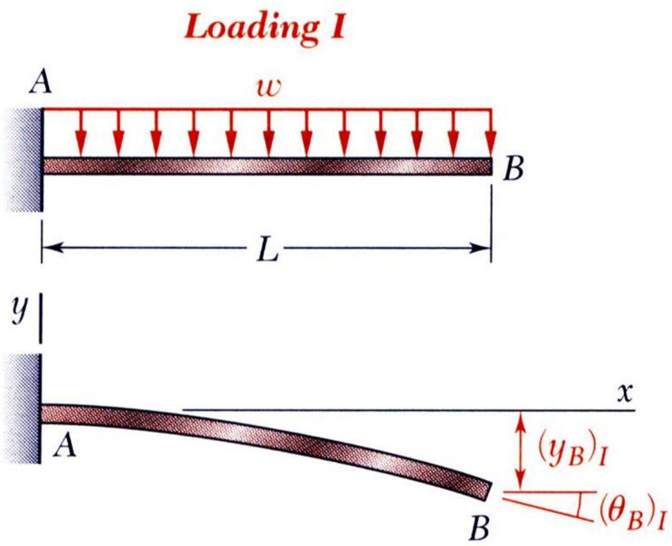
For the beam and loading shown, find slope and deflection at point *B*.

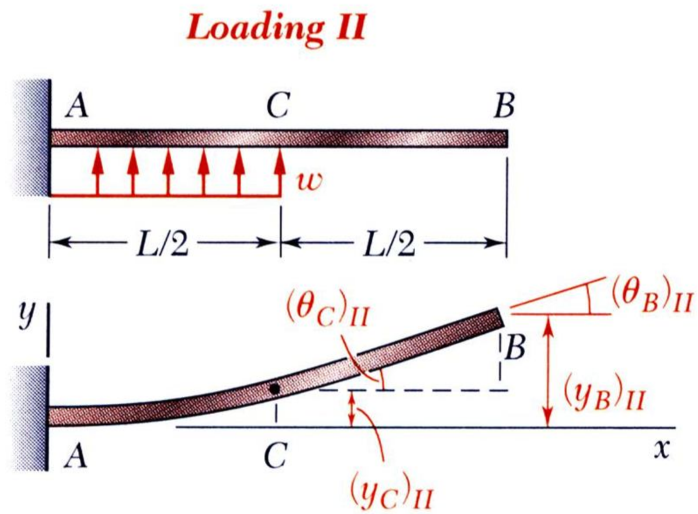
SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.



Example 3





Loading I

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

Loading II

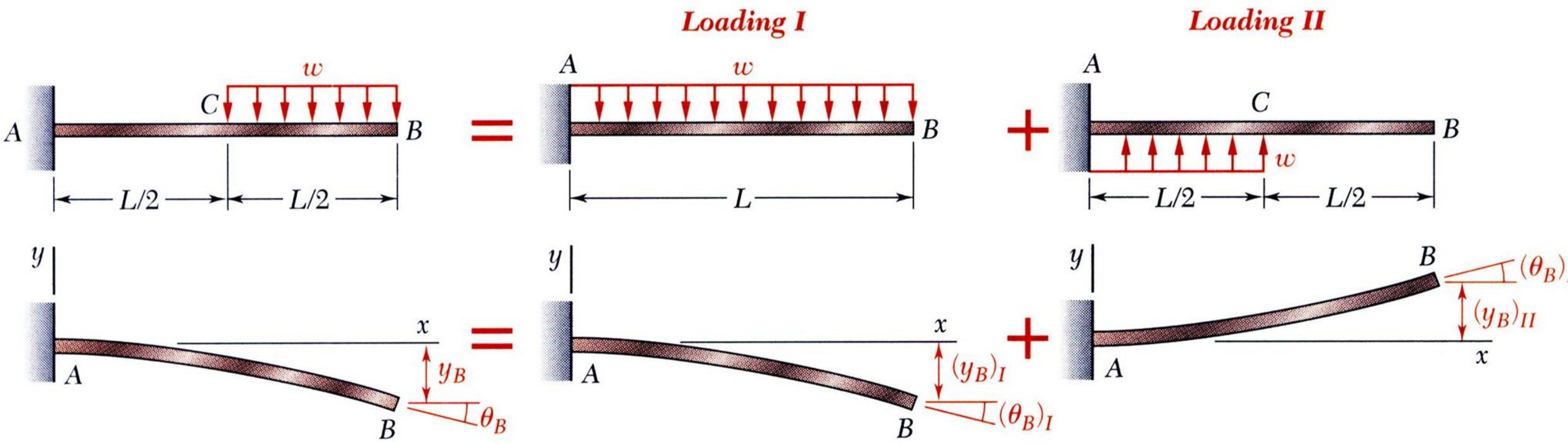
$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$

In beam segment CB, bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$

Example 3



Combine the two solutions,

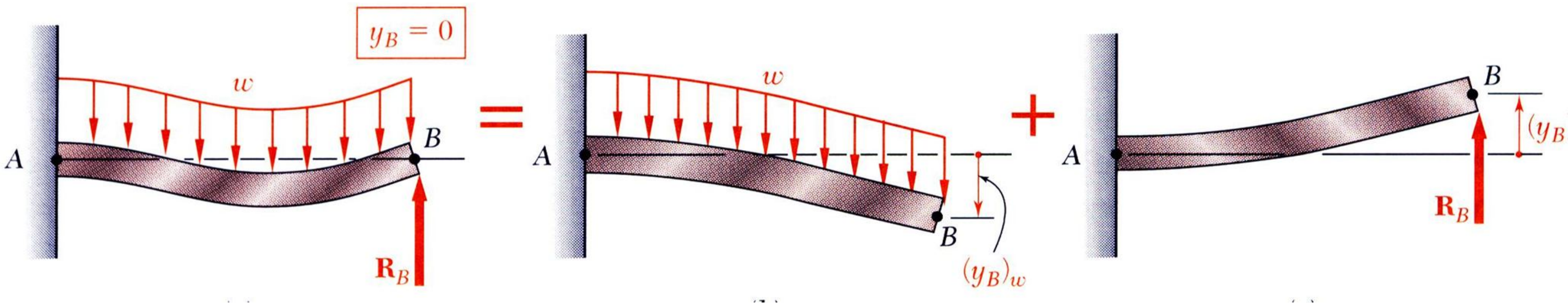
$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI}$$

$$\boxed{\theta_B = \frac{7wL^3}{48EI}}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI}$$

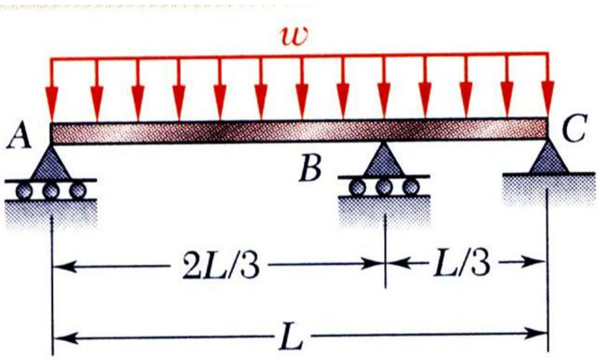
$$\boxed{y_B = \frac{41wL^4}{384EI}}$$

Application of Superposition to Statically Indeterminate Beams



- Method of superposition can be applied to find reactions of statically indeterminate beams.
- Designate one of the reactions as the redundant and eliminate or modify the support.
- Note that you must ensure that redundant chosen does not make structure unstable
- Determine beam deformation without redundant reaction.
- Treat redundant reaction as an unknown load which, together with the other (i.e., applied) loads, must produce deformations compatible with the original supports.

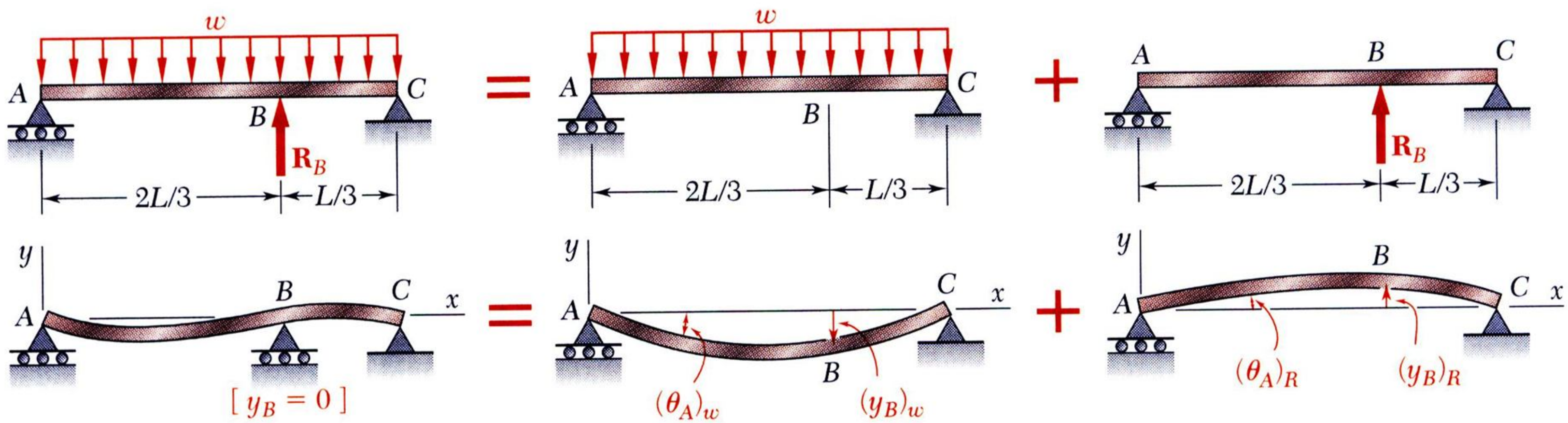
Example 4



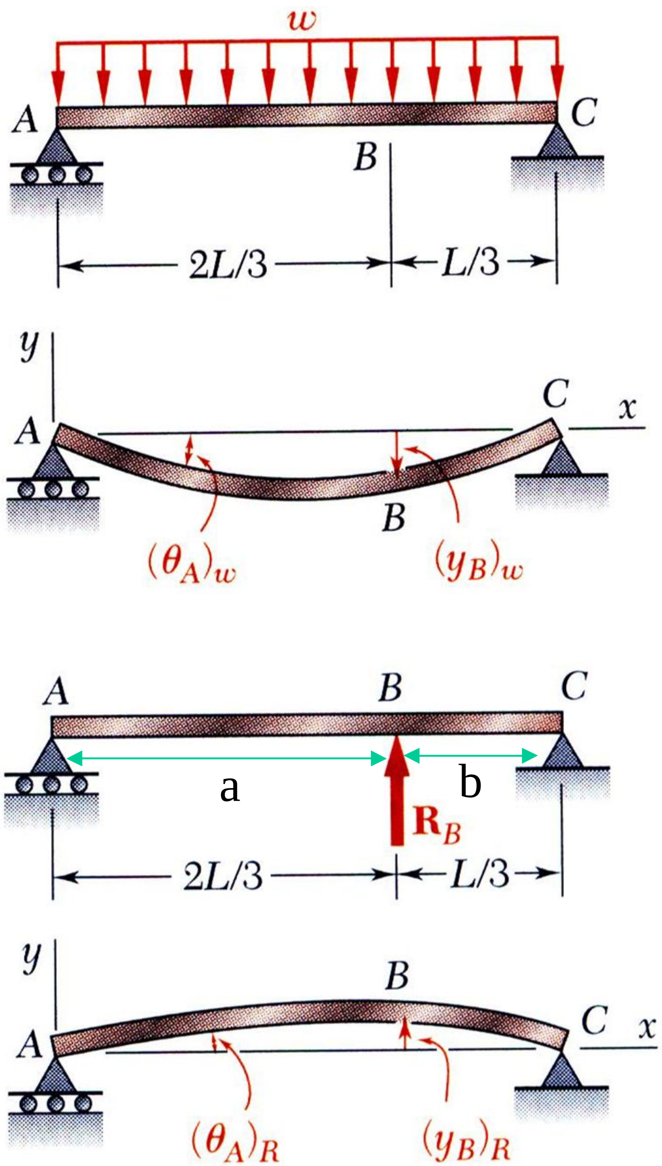
For the uniform beam and loading shown, find reaction at each support and slope at *A*.

SOLUTION:

- Release "redundant" support/reaction at B, and find deformation.
- Apply reaction at *B* as an unknown load to ensure zero displacement at *B*.



Example 4



- Distributed Load: $(y)_w = -\dfrac{w}{24EI}\left[x^4 - 2Lx^3 + L^3x\right]$

$$(y_B)_w = -\frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3}L\right)^3 + L^3\left(\frac{2}{3}L\right)\right]$$

$$= -0.01132\frac{wL^4}{EI}$$

- Redundant Reaction Load: At $x = a, \quad y = -\dfrac{Pa^2b^2}{3EIL}$

$$(y_B)_R = \frac{R_B}{3EIL}\left(\frac{2}{3}L\right)^2\left(\frac{L}{3}\right)^2 = 0.01646\frac{R_BL^3}{EI}$$

- For compatibility with original supports, $y_B = 0$

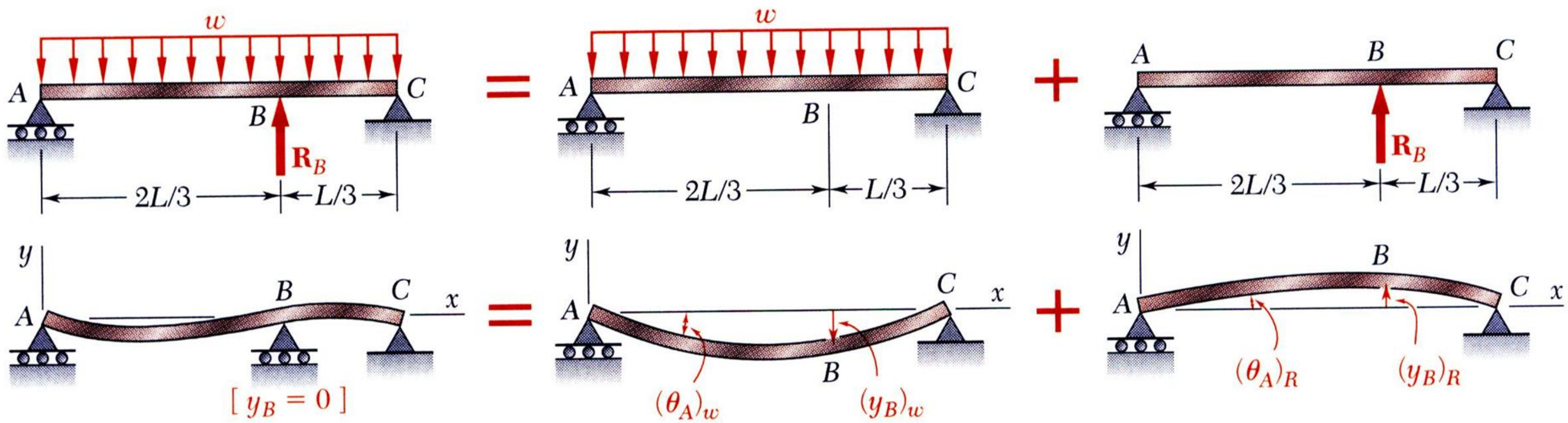
$$0 = (y_B)_w + (y_B)_R = -0.01132\frac{wL^4}{EI} + 0.01646\frac{R_BL^3}{EI}$$

$$R_B = 0.688wL \uparrow$$

- From statics,

$$R_A = 0.271wL \uparrow \qquad R_C = 0.0413wL \uparrow$$

Example 4



Slope at A,

$$(\theta_A)_w = -\frac{wL^3}{24EI} = -0.04167\frac{wL^3}{EI}$$

$$(\theta_A)_R = \frac{0.0688wL}{6EIL}\left(\frac{L}{3}\right)\left[L^2 - \left(\frac{L}{3}\right)^2\right] = 0.03398\frac{wL^3}{EI}$$

$$\theta_A = (\theta_A)_w + (\theta_A)_R = -0.04167\frac{wL^3}{EI} + 0.03398\frac{wL^3}{EI}$$

$$\theta_A = -0.00769\frac{wL^3}{EI}$$