CHAPTER

9

# MECHANICS OF MATERIALS

## Columns

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# Columns

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Sample Problem 10.2

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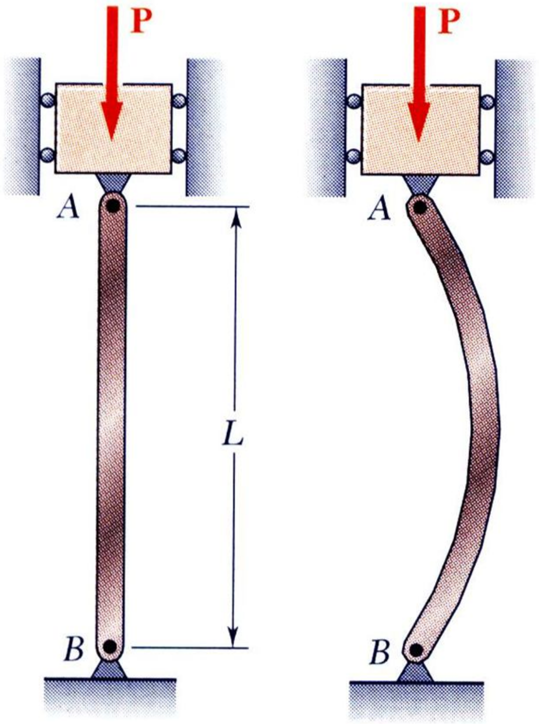
Sample Problem 10.4

Design of Columns Under an Eccentric Load

# Stability of Structures




- When designing columns, cross-sectional area is selected such that
  - allowable stress not exceeded

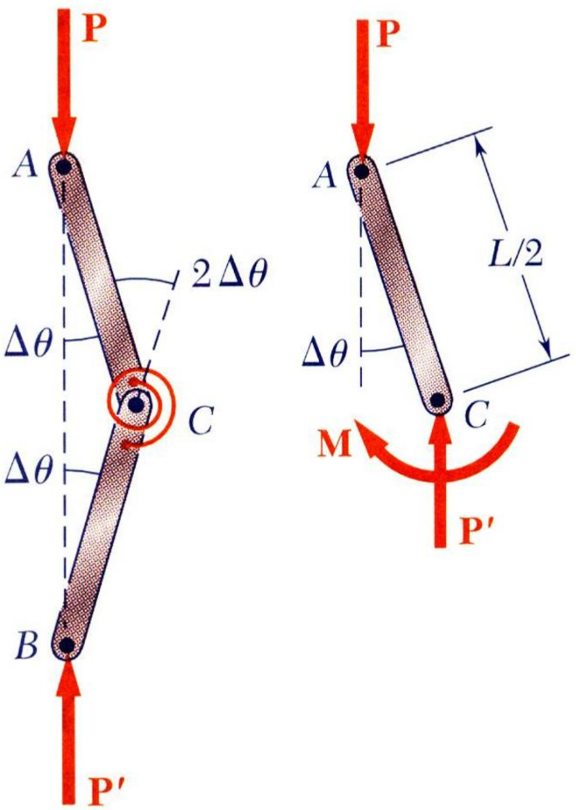
$$\sigma = \frac{P}{A} \le \sigma_{all}$$

  - deformation falls within specifications

$$\delta = \frac{PL}{AE} \le \delta_{spec}$$

- After such design, may still find that column is unstable under loading, i.e., that it suddenly becomes sharply curved or buckles.

# Stability of Structures




- Consider model with two rods and torsional spring. After small perturbation,

$$K(2\Delta\theta) = \text{restoring moment}$$

$$P\frac{L}{2}\sin\Delta\theta = P\frac{L}{2}\Delta\theta = \text{destabilizing moment}$$

- Column is stable (tends to return to straight orientation) if

$$P\frac{L}{2}\Delta\theta < K(2\Delta\theta)$$

$$P < P_{cr} = \frac{4K}{L}$$

## Stability of Structures

- A better way is to write equilibrium equation in the slightly perturbed (i.e., deformed) configuration, and then seek conditions for a non-trivial equilibrium solution (i.e., buckled state) to occur.

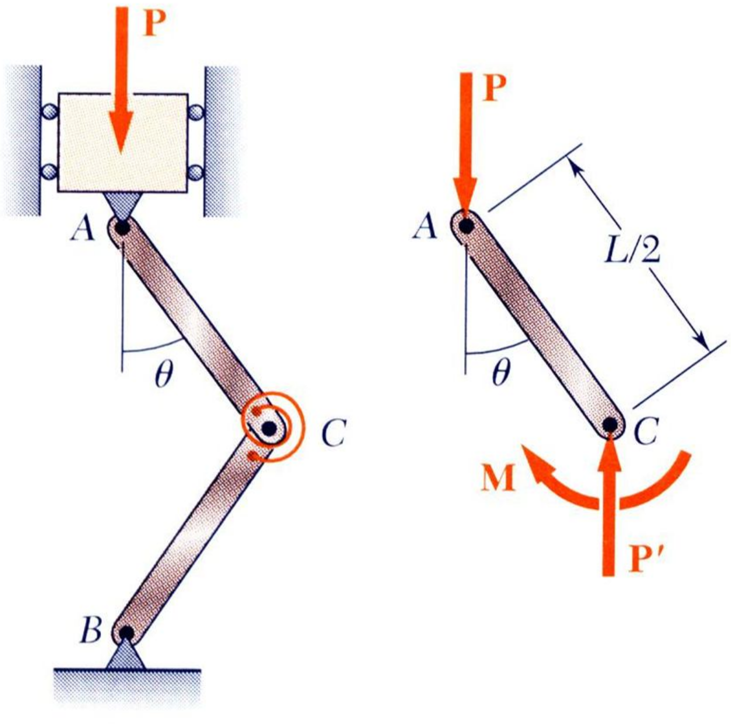
$$P\frac{L}{2}\Delta\theta - M = P\frac{L}{2}\Delta\theta - K\left(2\Delta\theta\right) = 0$$

Solution : either trivial equilibrium $\Delta\theta = 0$ or $P = \dfrac{4k}{L} = P_{cr}$.

So for $P = P_{cr}$ non – trivial, i.e., buckled equilibrium possible.

In fact in practice it will occur if any small perturbation or imperfection exists in the system.

# Stability of Structures – Post-buckling behavior



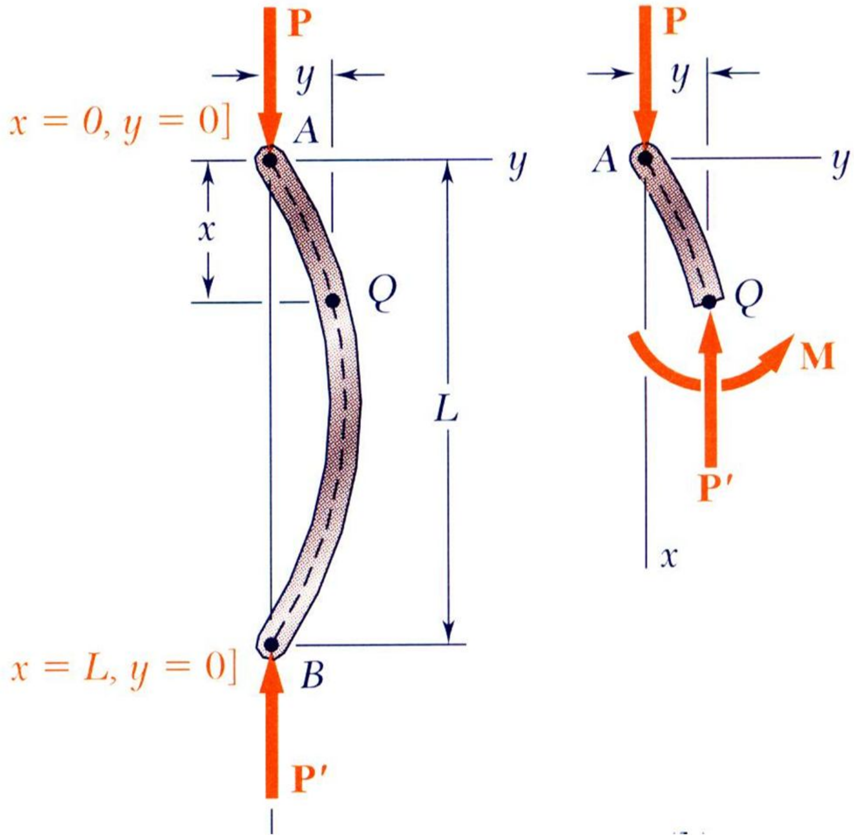

- Assume that load $P$ applied and after a perturbation the system settles to a new equilibrium configuration at a finite $\theta$.

$$P\frac{L}{2}\sin\theta = K(2\theta)$$

$$\frac{PL}{4K} = \frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

- Noting that $\sin\theta < \theta$, the assumed configuration is only possible if $P > P_{cr}$.
- Plot $\theta / \sin\theta$ versus $\theta$, which is same as plot of $P/P_c$ ie., $PL/4K$ versus $\theta$. Thus we get post-buckling behavior.

# Euler's Formula for Pin-Ended Beams




- Consider axially loaded beam. If after small perturbation, the system reaches a non-trivial equilibrium configuration, then equilibrium requires that,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y \qquad (\text{we used } M + Py = 0)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

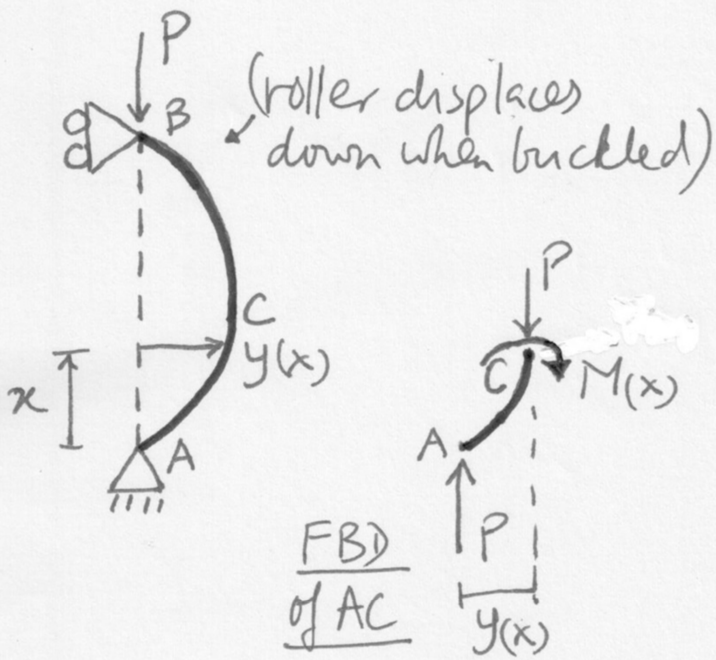
- Solution with assumed configuration can only be obtained if (see following slides for Case I-IV details)

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2}$$

# Case I: Simply Supported

Case I Simply supported.




$$\sum M_C = 0 \Rightarrow Py + M(x) = 0$$

$$Py + EIy'' = 0$$

$$\boxed{k^2 y + y'' = 0}, \quad \boxed{k^2 = \frac{P}{EI}}$$

put $y = Ae^{sx}$,

$$A(k^2 + s^2)e^{sx} = 0 \Rightarrow s = \pm ik$$

$$\Rightarrow y = A_1 e^{s_1 x} + A_2 e^{s_2 x} = A_1 e^{ikx} + A_2 e^{-ikx}$$

$\because y = \text{real}$, $A_1 = \bar{A}_2$ (complex conjugates)

## Case I: Simply Supported

$y = 2\,\mathrm{Re}\,(A_1 e^{ikx}) = \boxed{C_1 \cos kx + C_2 \sin kx = y = y_h}$

where $A_1 = \frac{C_1}{2} - i\frac{C_2}{2}$

general soln. — well known result. — homogenous soln.

<u>BC's</u> (boundary conditions)

$x=0,\ y=0 \Rightarrow C_1 = 0$

$x=L,\ y=0 \Rightarrow C_2 \sin kL = 0$ → This is an (evp) eigenvalue problem

So either $C_2 = 0 \Rightarrow y = 0$ (trivial soln., no buckling)

or $\sin kL = 0$ (ie $C_2 \neq 0$, non-trivial $y = C_2 \sin kx$, ie buckling).

↳ CHARACTERISTIC EQN (CE)

So for buckling,

$\sin kL = 0 \Rightarrow kL = n\pi, \quad n = 1, 2, 3, \ldots$

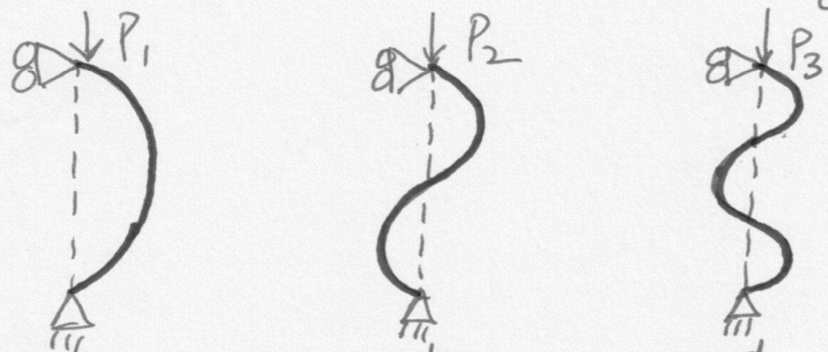
$$P = P_n = \frac{n^2 \pi^2 EI}{L^2}$$

$P_1 = \boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$ for $n=1$ → EULER BUCKLING LOAD.

or $\boxed{P_{cr} = \frac{\pi^2 EI}{L_e^2}, \; L_e = L}$

Buckling mode shapes:

$y(x) = C_2 \sin kx = C_2 \sin \frac{n\pi x}{L}$




1st mode 2nd mode 3rd mode

(Note: roller displaces down in deformed, i.e. buckled, configuration).

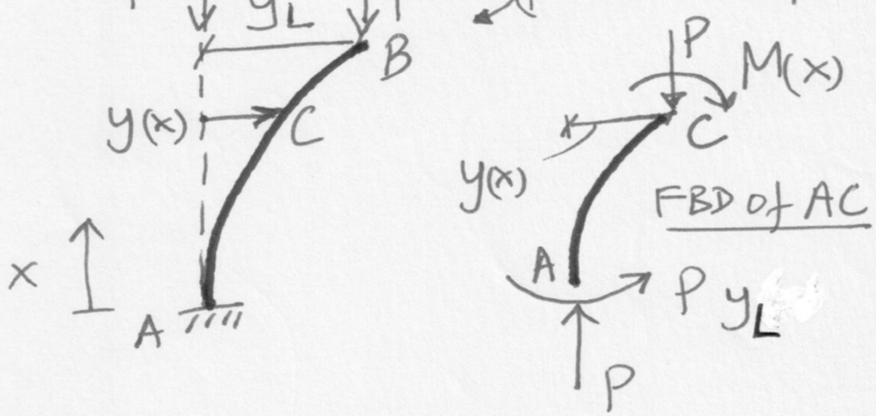
Lowest buckling load is $P_1 = P_{cr}$ → physically realized buckling load.

If you have stoppers (to arrest y-displacement) at: $x = \frac{L}{2}$ you can physically realize 2nd mode,

$x = \frac{L}{3}, \frac{2L}{3}$, " " " " 3rd ", etc.

## Case II — Fixed – Free

(point B displaces down when buckled)




$\Sigma M_C = 0 \Rightarrow Py + M(x) - P y_L = 0$

$y_h$ = homogenous sol$^{\underline{n}}$ $= C_1 \cos kx + C_2 \sin kx$

$y_p$ = particular sol$^{\underline{n}}$ $= y_L$

$$\Rightarrow y = y_h + y_p = C_1 \cos kx + C_2 \sin kx + y_L$$

BC's : $x = 0,\ y = 0 \Rightarrow C_1 = -y_L$

$x = 0,\ y' = 0 \Rightarrow C_2 = 0$

$x = L,\ y = y_L \Rightarrow y_L = y(L)$

$y_L = y_L(1 - \cos kL) \rightarrow$ EVP

$\Rightarrow \cos kL = 0 \rightarrow$ CE

## Case II: Fixed-Free

$$kL = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

$$P_n = (2n-1)^2 \frac{\pi^2 EI}{4L^2} \Rightarrow P_{cr} = \text{lowest } P_n = P_1 = \frac{\pi^2 EI}{4L^2}$$

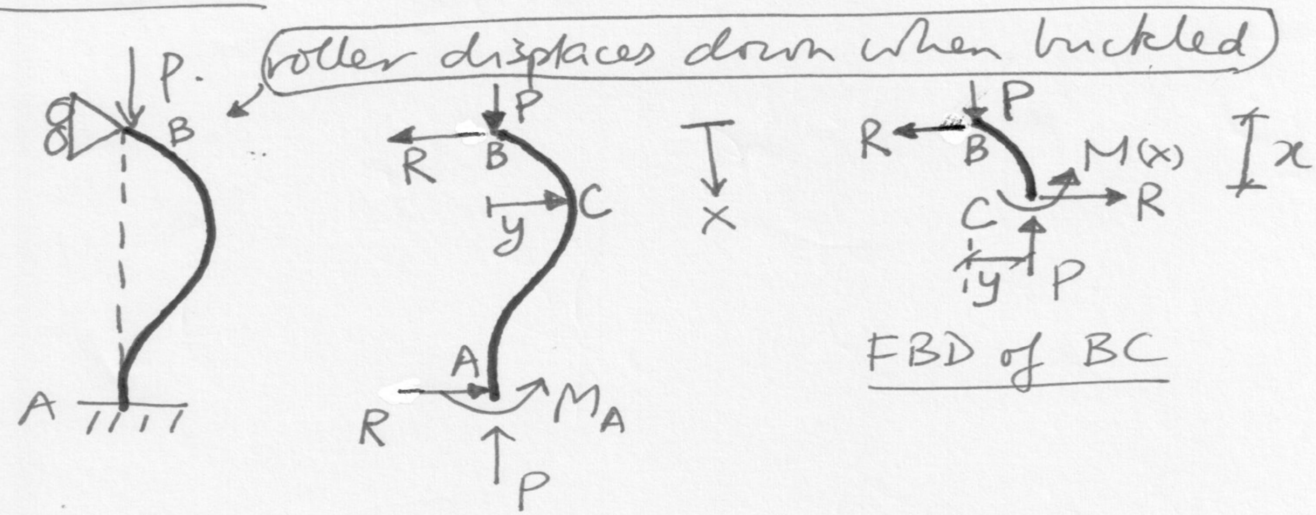
$$\boxed{P_{cr} = \frac{\pi^2 EI}{L_e^2}, \quad L_e = 2L}$$

Mode shapes:

$$y(x) = y_L \left\{1-\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{L}\right]\right\}$$

1st mode shape is $\boxed{y(x) = y_L\left(1-\cos\frac{\pi}{2}\frac{x}{L}\right)}$

Case III  Fixed – Roller.




$$\Sigma M_C = 0 \Rightarrow Py + M(x) + Rx = 0$$

(note: we conveniently chose coordinate system to avoid $M_A$ appearing in the moment equilibrium equation – but that is not necessary, only convenient).

$$y = y_h + y_P = \underbrace{C_1 \cos kx + C_2 \sin kx}_{y_h} \underbrace{- \frac{R}{P} x}_{y_P}$$

## Case III: Fixed-Roller

BC's: $x=0,\ y=0 \Rightarrow C_1=0$

$x=L,\ y=0 \Rightarrow C_2 \sin kL - \frac{R}{P} L = 0$

$x=L,\ y'=0 \Rightarrow C_2 k \cos kL - \frac{R}{P} = 0$

} write in matrix form

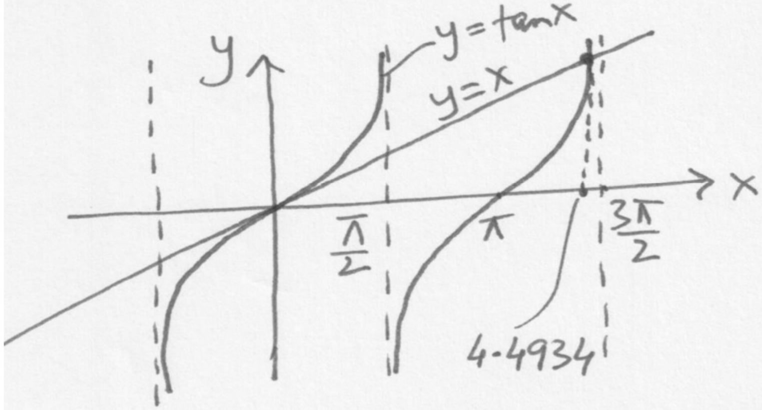
$$\begin{bmatrix} \sin kL & -L \\ k\cos kL & -1 \end{bmatrix} \begin{Bmatrix} C_2 \\ R/P \end{Bmatrix} = 0 \rightarrow \underline{\text{evp.}}$$

→ if this is zero, $y=0$ ie no buckling.

So for buckling, this should be non-trivial, ie det $[\ \ ]=0$

$$\det\begin{bmatrix} \sin kL & -L \\ k\cos kL & -1 \end{bmatrix} = 0 \Rightarrow -\sin kL + kL\cos kL = 0$$

$$\tan kL = kL \rightarrow \underline{CE}$$




transcedental equation. Needs to be solved numerically. Has infinite roots. First positive root is

$$kL = 4.4934$$

$$\Rightarrow P_{cr} = 20.19\frac{EI}{L^2} = \frac{\pi^2 EI}{\left(L/\sqrt{2.0456}\right)^2} = \frac{\pi^2 EI}{L_e^2}$$

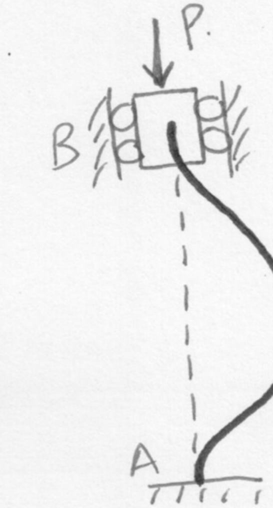
$$L_e = 0.7L$$

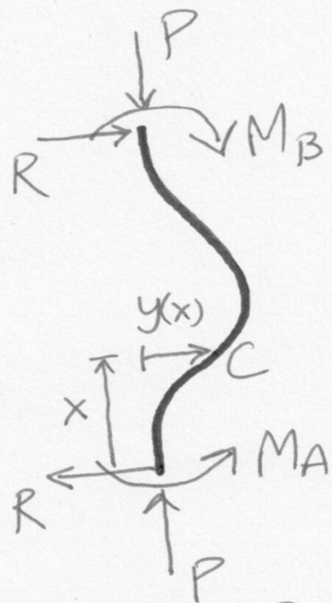
Mode shapes: $y(x) = C_2\left\{\sin kx - (k\cos kL)\,x\right\}$

1st mode shape $y(x) = C_2\left\{\sin\left(4.4934\,\frac{x}{L}\right) - \left(4.4934\cos 4.4934\right)\frac{x}{L}\right\}$

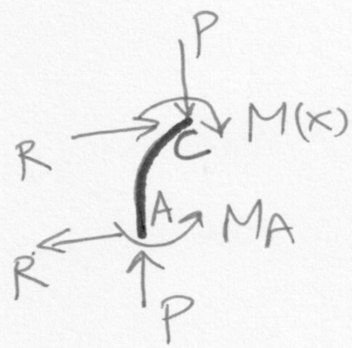
## Case IV Fixed – Fixed

Note: one of the fixed ends must move in roller guide, for load to be transmitted.




FBD of AB




FBD of AC

Note: Due to symmetry of load, BC's, buckled shape, we know that $M_A = M_B$, $R = 0$. This simplifies sol$^{\underline{n}}$. But we won't use it

$$\Sigma M_C = 0 = Rx - M_A + Py + M(x) = Rx - M_A + Py + EIy''$$

$$y = y_h + y_p = \underbrace{A_1 \cos kx + A_2 \sin kx}_{y_h} \underbrace{- \frac{R}{P}x + \frac{M_A}{P}}_{y_p}$$

# Case IV: Fixed-Fixed

BC's : $x=0,\ y=0 \Rightarrow A_1 + \frac{M_A}{P} = 0$

$x=0,\ y'=0 \Rightarrow A_2 k - \frac{R}{P} = 0$

$x=L,\ y=0 \Rightarrow A_1 \cos kL + A_2 \sin kL - \frac{R}{P} L + \frac{M_A}{P} = 0$

$x=L,\ y'=0 \Rightarrow -A_1 k \sin kL + A_2 k \cos kL - \frac{R}{P} = 0$

In matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & k & -1 & 0 \\ \cos kL & \sin kL & -L & 1 \\ -k\sin kL & k\cos kL & -1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ R/P \\ M_A \end{Bmatrix} = 0 \rightarrow \underline{\text{EVP}}$$

→ if zero then $y=0$, no buckling.

For buckled equilibrium, ie $y \neq 0$, det[ ] = 0. → CE

$$\Rightarrow 1\left[(k)(1)-(-1)(-k\cos kL)\right]-1\left[(-k)(-\cos kL-kL\sin kL)+(-1)(k)\right]=0$$

$$\Rightarrow 2k(1-\cos kL)-k^2 L\sin kL=0$$

$$\Rightarrow \sin\frac{kL}{2}\left(-2\sin\frac{kL}{2}+kL\cos\frac{kL}{2}\right)=0 \rightarrow \underline{CE}$$

solutions → $\sin\frac{kL}{2}=0$ or $\frac{kL}{2}=\tan\frac{kL}{2}$

$\frac{kL}{2}=n\pi$ $(n=1,2,\ldots)$ or $\frac{kL}{2}=4.493,\ldots\ldots$ (numerically evaluated roots)

↓

For $n=1$, $\frac{kL}{2}=\pi$ gives critical (lowest) buckling load

## Case IV: Fixed-Fixed

$$\Rightarrow \boxed{P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{L_e^2}, \quad L_e = \frac{L}{2}}$$

Buckling modes: solve for $A_1$, $A_2$, $\frac{R}{P}$, $\frac{M_A}{P}$, using $\sin\frac{kL}{2} = 0$.

$$\Rightarrow \left.\begin{aligned} A_1 + M_A &= 0 \\ A_2 - R/P &= 0 \\ A_1 - L\frac{R}{P} + M_A &= 0 \end{aligned}\right\} \Rightarrow \frac{R}{P} = A_2 = 0$$

$\frac{R}{P}$ as expected.

$$\boxed{y(x) = A_1(\cos kx - 1)}$$

1st mode: $\boxed{y(x) = A_1\left(\cos 2\pi\frac{x}{L} - 1\right)}$

Shorter way is to take $k=0$ from begining.

$$\Rightarrow \quad -M_A + Py + EIy'' = 0$$

$$y = A_1 \cos kx + A_2 \sin kx + \frac{M_A}{P}$$

BC's

$$\begin{cases} x=0,\ y=0 \Rightarrow A_1 + \frac{M_A}{P} = 0 \\ x=0,\ y'=0 \Rightarrow A_2 k = 0 \Rightarrow A_2 = 0 \\ x=L,\ y=0 \Rightarrow A_1 \cos kL + \cancel{A_2}^{0} \sin kL + \frac{M_A}{P} = 0 \\ x=L,\ y'=0 \Rightarrow -A_1 k \sin kL + \cancel{A_2}^{0} k \cos kL = 0 \end{cases}$$

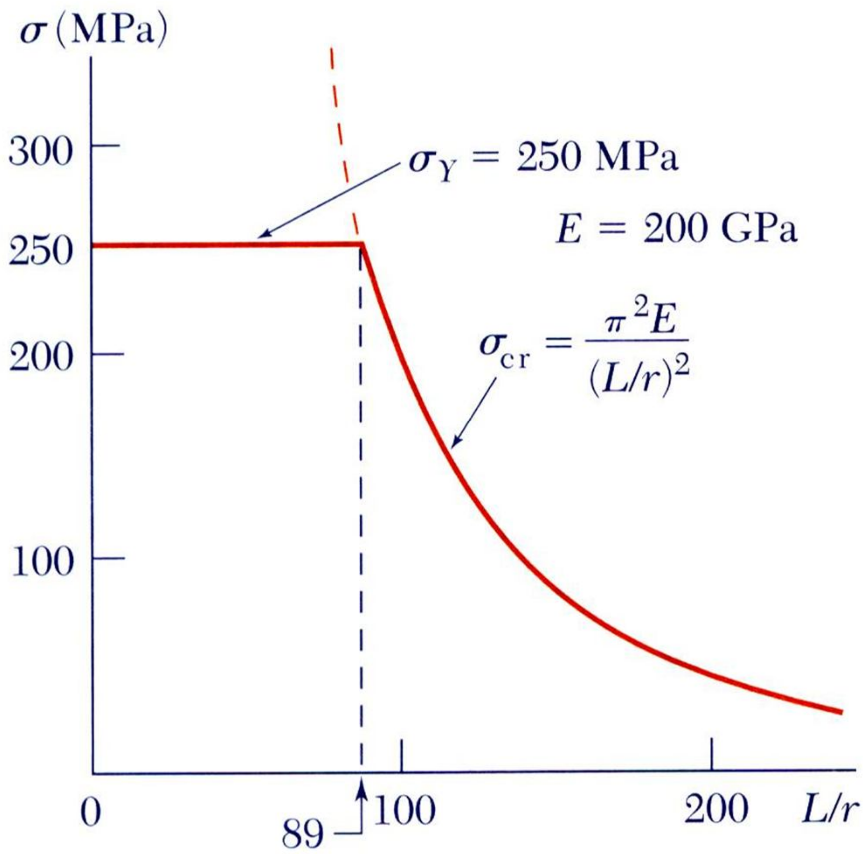
$$\Rightarrow \sin kL = 0 \quad \underline{\underline{\text{AND}}} \quad 1 - \cos kL = 0$$

$$kL = n\pi \quad \underline{\underline{\text{AND}}} \quad kL = 2n\pi$$

common solution is $kL = 2n\pi$ (as before)

Mode shape $y = A_1(\cos kx - 1)$ (as before).

# Euler's Formula for Pin-Ended Beams




- The value of stress corresponding to the critical load,

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{P_{cr}}{A}$$

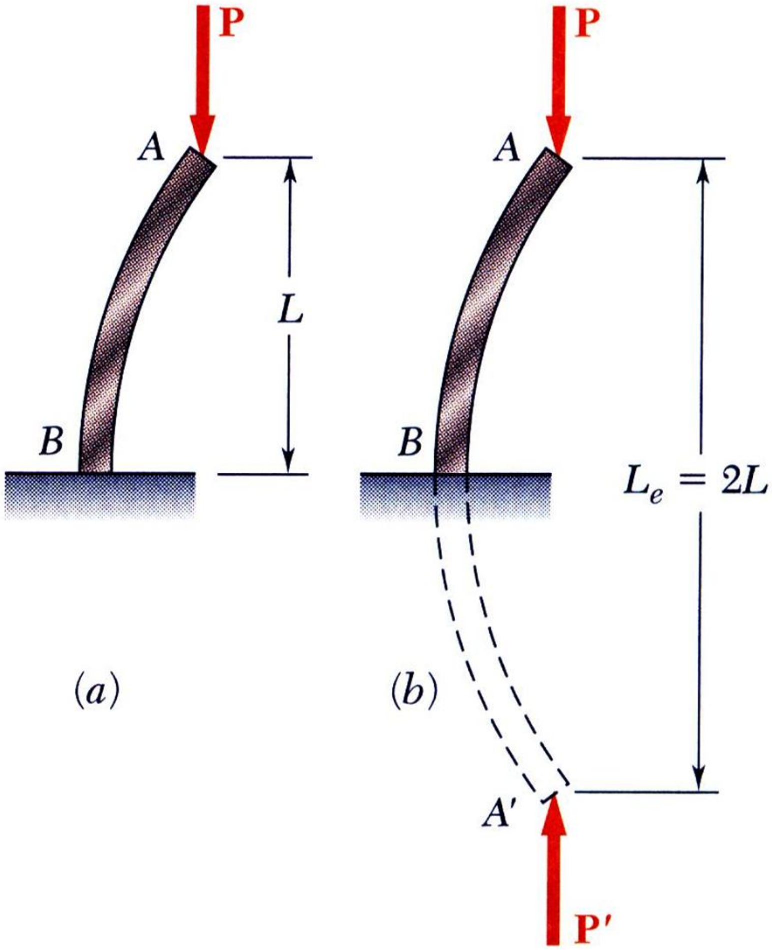
$$\sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A}$$

$$= \frac{\pi^2 E}{\left(L/r\right)^2} = \textit{critical stress}$$

$$\frac{L}{r} = \textit{slenderness ratio}$$

- Preceding analysis is limited to centric loadings.

# Extension of Euler's Formula




- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.

- The critical loading is calculated from Euler's formula,

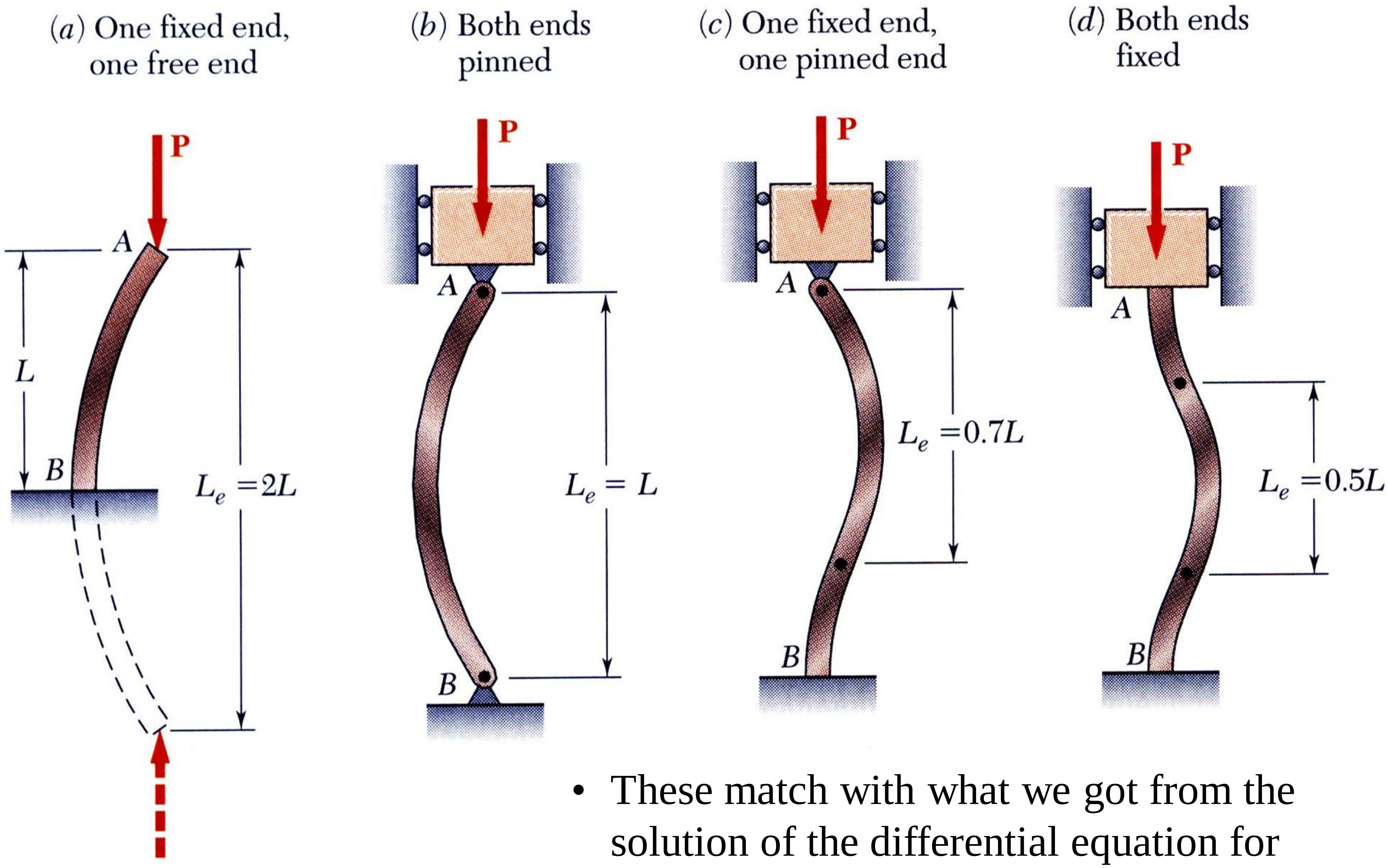
$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{\left(L_e/r\right)^2}$$

$$L_e = 2L = \text{equivalent length}$$

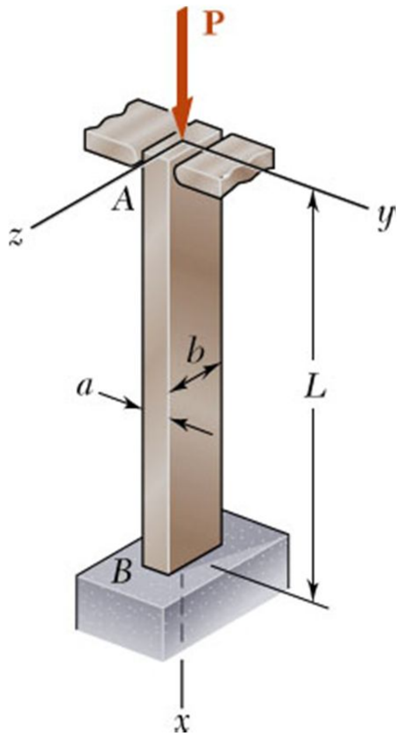
- This matches with what we got from the solution of the differential equation for Case II (Fixed-Free)

# Extension of Euler's Formula




- These match with what we got from the solution of the differential equation for Cases I-IV

# Example 10.1




$L = 0.5$ m
$E = 70$ GPa
$P = 20$ kN
$FS = 2.5$

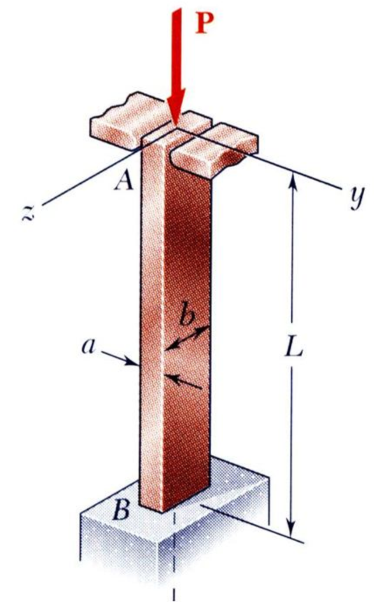
Aluminum column, length L, rectangular cross-section, has fixed end at B, supports centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

a) Determine ratio a/b of the two sides of the cross-section corresponding to the most efficient design against buckling.

b) Design the most efficient cross-section for the column.

# Example 9.1

Most efficient design is when the resistance to buckling in *xy* and *xz* planes are equal, i.e., buckling load for buckling in *xy* and *xz* planes are equal, i.e., slenderness ratios in *xy* and *xz* planes are equal.




- Buckling in *xy* Plane:

$$P_{\mathrm{cr},z} = \frac{\pi^2 E I_z}{L_{e,z}^2} = \frac{\pi^2 E \frac{a^3 b}{12}}{(0.7L)^2}$$

- Buckling in *xz* Plane:

$$P_{\mathrm{cr},y} = \frac{\pi^2 E I_y}{L_{e,y}^2} = \frac{\pi^2 E \frac{a b^3}{12}}{(2L)^2}$$

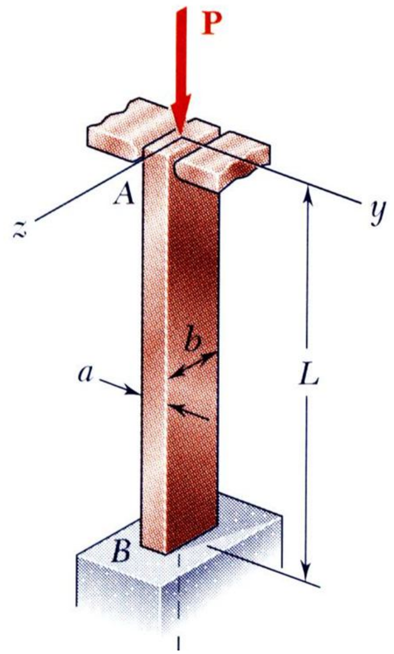
- Most efficient design:

$$P_{\mathrm{cr},z} = P_{\mathrm{cr},y}$$

$$\frac{a^2}{0.7^2} = \frac{b^2}{2^2}$$

$$\frac{a}{b} = \frac{0.7}{2} = 0.35$$

# Example 9.1




$L = 0.5\ \text{m}$
$E = 70\ \text{GPa}$
$P = 20\ \text{kN}$
$FS = 2.5$
$a/b = 0.35$

Design:

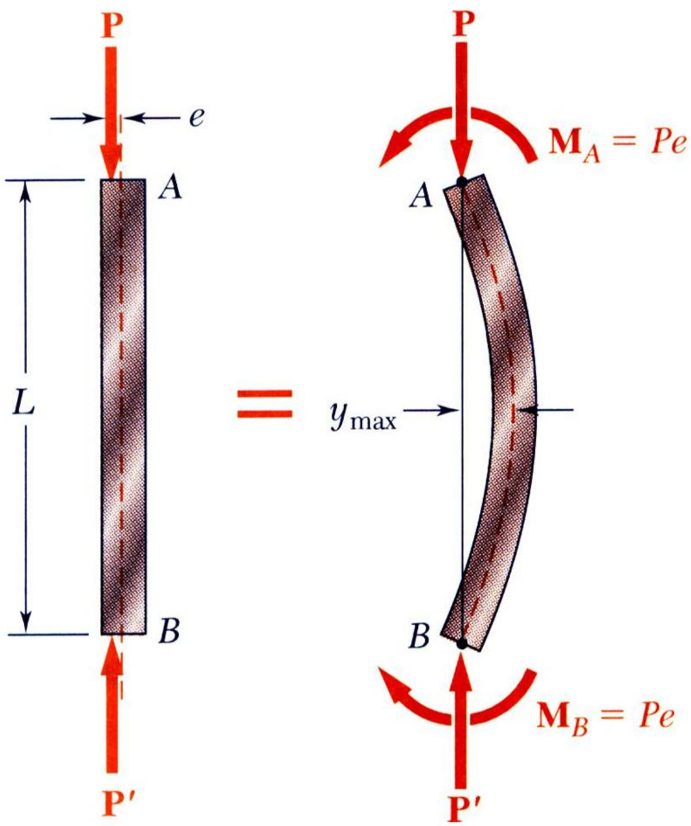
$$P_{cr} = (FS)P = (2.5)(20\ \text{kN}) = 50\ \text{kN}$$

$$= \frac{\pi^2 EI_y}{L_{e,y}^2} = \frac{\pi^2 E \frac{ab^3}{12}}{(2L)^2} = \frac{\pi^2 E \frac{0.35b^4}{12}}{(2L)^2}$$

Use given data

$$b = 39.7\ \text{mm}$$
$$a = 0.35b = 13.9\ \text{mm}$$

# Eccentric Loading: The Secant Formula




- Eccentric loading is equivalent to a centric load and a couple.
- Bending occurs for any nonzero eccentricity. Question of buckling becomes whether the resulting deflection is excessive (infinite).
- The deflection become infinite when $P = P_{cr}$

$$\frac{d^2y}{dx^2} = \frac{-Py - Pe}{EI}$$

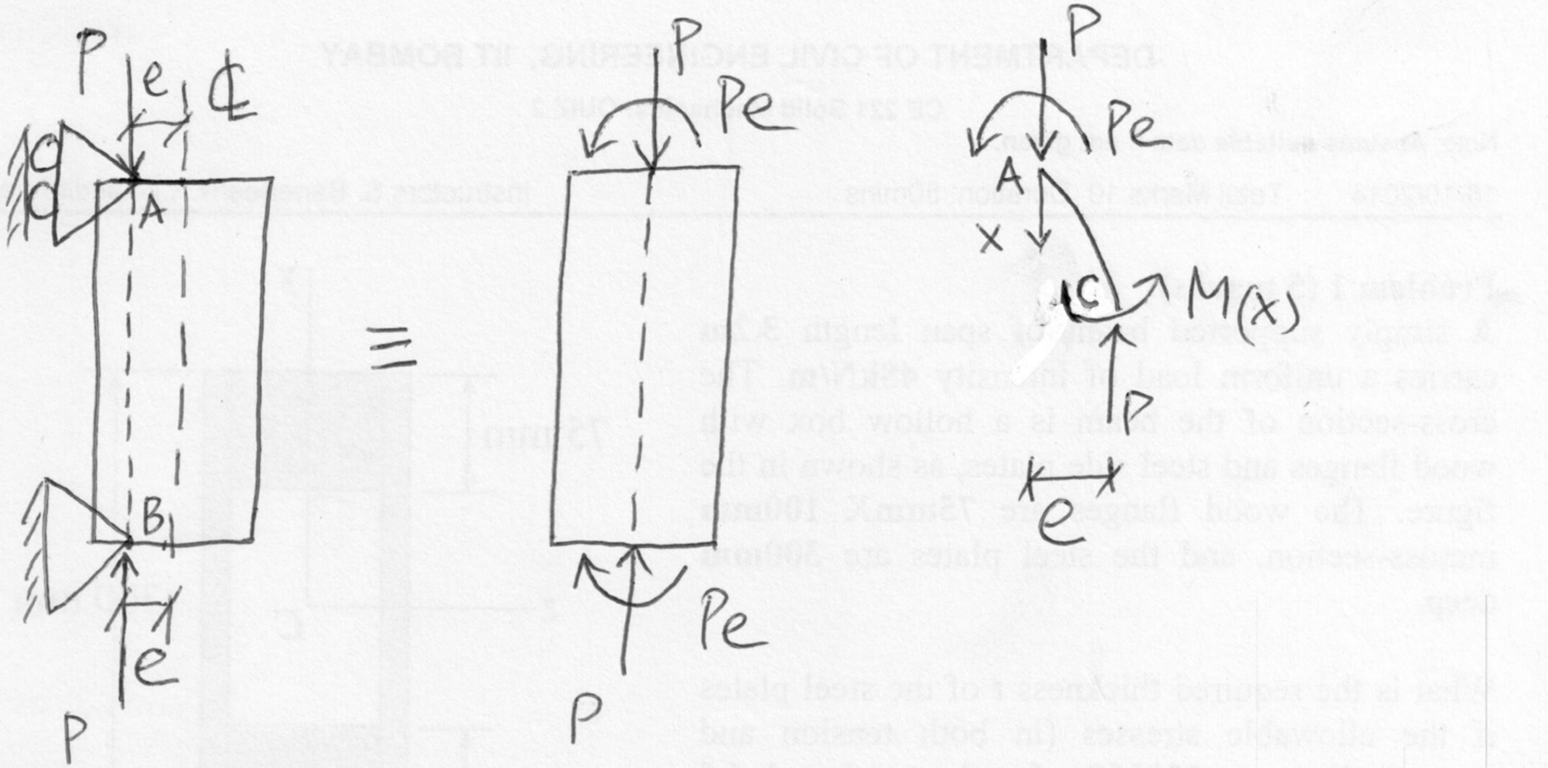
$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

- Maximum stress

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{(y_{max} + e)c}{r^2}\right]$$

$$= \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L_e}{r}\right)\right]$$

## Derivation of Secant formula




$$\Sigma M_c = 0 = Py + M(x) + Pe = Py + EIy'' + Pe$$

$$k^2 = P/EI$$

## Derivation of Secant formula

$$y'' + k^2 y + k^2 e = 0$$

$$y = C_1 \cos kx + C_2 \sin kx - e$$

$$x = 0,\ y = 0,\ \Rightarrow\ C_1 = e$$

$$x = L,\ y = 0,\ \Rightarrow\ e(\cos kL - 1) + C_2 \sin kL = 0$$

$$y = e\left[\cos kx + \left(\frac{1 - \cos kL}{\sin kL}\right)\sin kx - 1\right]$$

$$= e\left[\cos kx + \tan\frac{kL}{2}\sin kx - 1\right]$$

This is bent (not buckled) shape, since for the smallest non-zero $k\ (\equiv P)$ and $e$, we get non-zero $y$.

## Derivation of Secant formula

In this case buckling defined as $y \to \infty$

$$\Rightarrow \tan\frac{kL}{2} \to \infty, \Rightarrow \frac{kL}{2} = (2n-1)\frac{\pi}{2} = \frac{k_{cr}L}{2}$$

So $P_{cr}$ (for $n=1$) $= \pi^2 \dfrac{EI}{L^2}$ (just as in S.S. Euler column).

Can apply this to Fixed-Free and Fixed-Fixed column by using appropriate $L_e$. Thus $P_{cr} = \dfrac{\pi^2 EI}{L_e^2}$ in general.

Max deflection occurs at $x = \frac{L}{2}$. Thus

$$y\Big|_{x=\frac{L}{2}} = y_{max} = e\left(\cos\frac{kL}{2} + \tan\frac{kL}{2}\sin\frac{kL}{2} - 1\right)$$

$$= e\left(\sec\frac{kL}{2} - 1\right)$$

$$y_{max} = e\left(\sec\frac{kL}{2} - 1\right) = e\left(\sec\left[\frac{\pi}{k_{cr}L}\,\frac{kL}{2}\right] - 1\right)$$

$$= e\left(\sec\left[\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right] - 1\right)$$

$$M\Big|_{x=\frac{L}{2}} = M_{max} = P y_{max} + Pe = P\left(y_{max} + e\right)$$

$$\sigma_{max} = \frac{P}{A} \pm \frac{P\left(y_{max} + e\right)c}{I \rightarrow Ar^2} \quad \left(\begin{array}{c}\text{compression} \\ +\text{ve}\end{array}\right).$$

$$\left(\sigma_{max}\right)_{compr} = \frac{P}{A}\left(1 + \frac{\left(e + y_{max}\right)c}{r^2}\right)$$

$$= \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\left[\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right]\right)$$

use $P_{cr} = \frac{\pi^2 EI}{L_e^2}$ here

$$= \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\left[\frac{1}{2}\sqrt{\frac{P}{AE}}\,\frac{L_e}{r}\right]\right) = \left(\sigma_{max}\right)_{compr} = \sigma_y$$

# Eccentric Loading: The Secant Formula

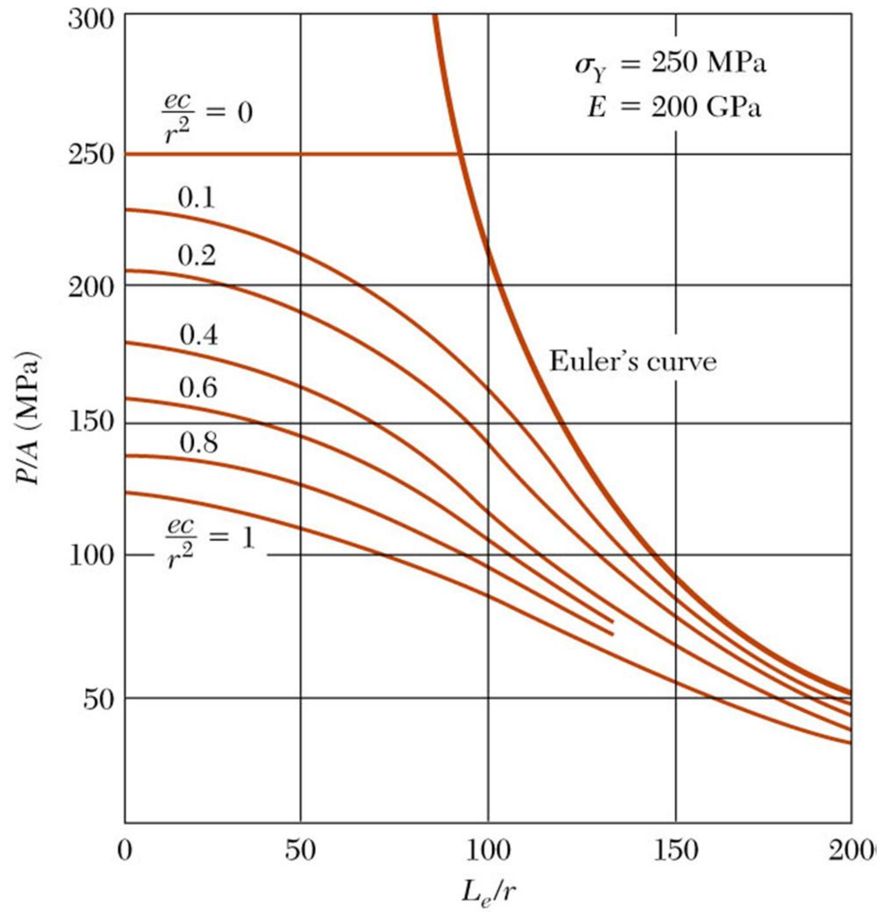
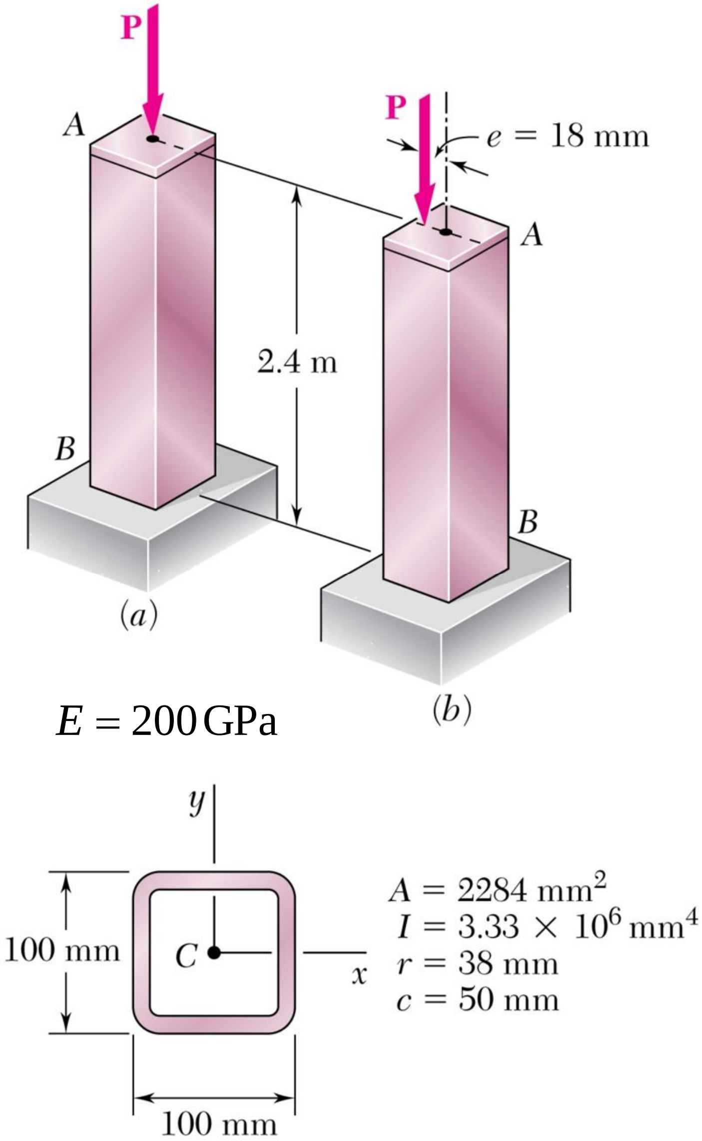



**Fig. Load per unit area, P / A, causing yield in column.**

$$\sigma_{max} = \sigma_Y = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L_e}{r}\right)\right]$$

# Example 9.2



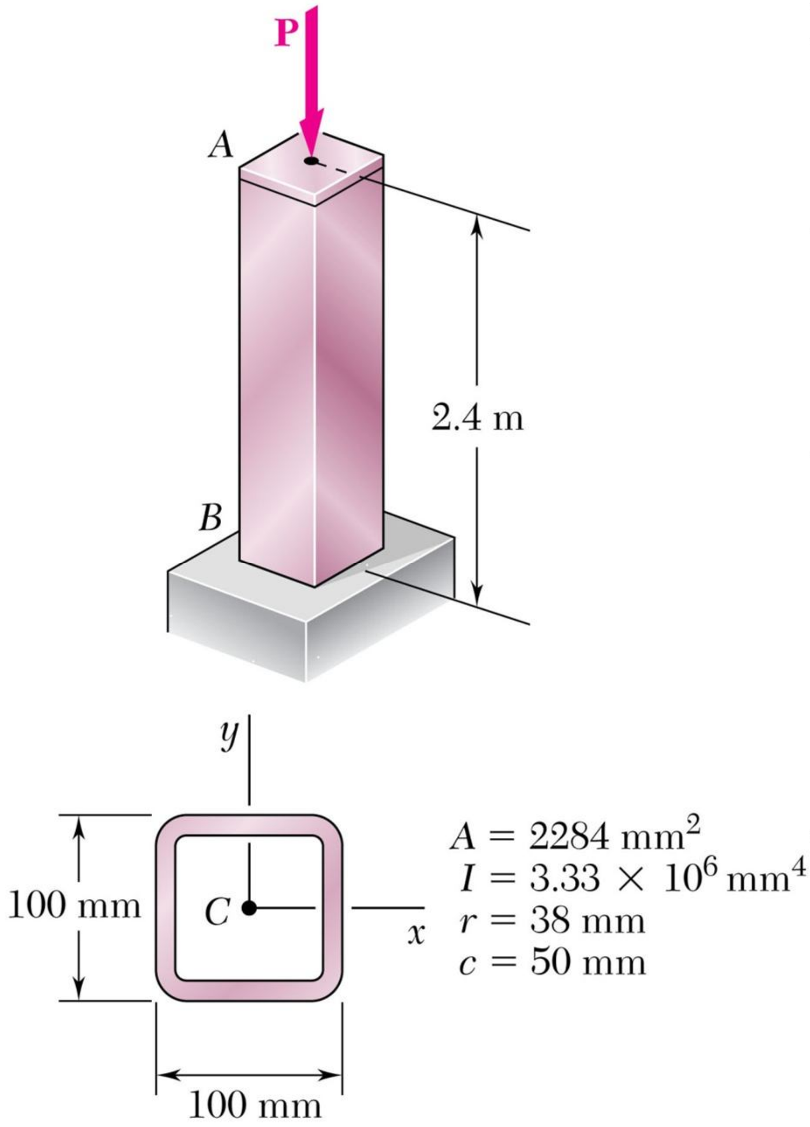

$E = 200\,\text{GPa}$

Uniform column consists of a 2.4 m section of structural tubing having the cross-section shown.

a) Using Euler's formula and a FS=2, determine allowable centric load for the column and corresponding normal stress.

b) Assuming that allowable load found in part *(a)* is applied at a point 18 mm from the geometric axis of the column. Find horizontal deflection of the top of the column and the maximum normal stress in the column.

# Example 9.2




Maximum allowable centric load:

- Effective length,

$$L_e = 2(2.4\,\text{m}) = 4.8\,\text{m}$$

- Critical load,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (200\times10^9\ \text{Pa})(3.33\times10^{-6}\ \text{m}^4)}{(4.8\,\text{m})^2}$$

$$= 285.3\,\text{kN}$$

- Allowable load,

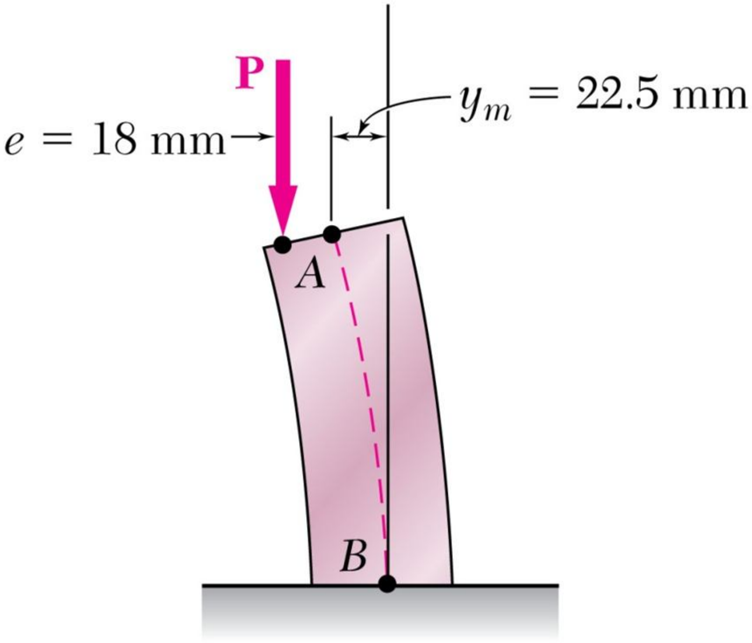
$$P_{all} = \frac{P_{cr}}{FS} = \frac{285.3\,\text{kN}}{2}$$

$$P_{all} = 142.7\ \text{kN}$$

$$\sigma = \frac{P_{all}}{A} = \frac{142700\ \text{N}}{2284\times10^{-6}\ \text{m}^2}$$

$$\sigma = 62.5\ \text{MPa}$$

# Example 9.2




Eccentric load:

- End deflection,

$$y_m = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]$$

$$= (18\text{ mm})\left[\sec\left(\frac{\pi}{2\sqrt{2}}\right) - 1\right]$$

$$y_m = 22.5\text{ mm}$$

- Maximum normal stress,

$$\sigma_m = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$= \frac{142700\text{ N}}{2284\times10^{-6}\text{ m}^2}\left[1 + \frac{(0.018\text{ m})(0.05\text{ m})}{(0.038\text{ m})^2}\sec\left(\frac{\pi}{2\sqrt{2}}\right)\right]$$

$$\sigma_m = 150.2\text{ MPa}$$