CHAPTER

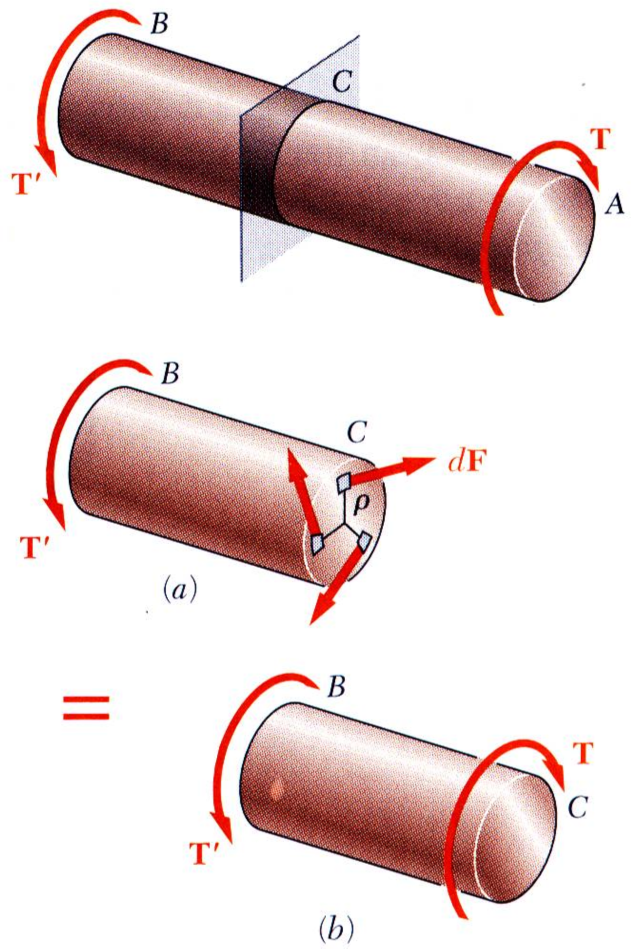
3

MECHANICS OF MATERIALS

Torsion

Contents

Net Torque Due to Internal Stresses



- Resultant of internal shearing stresses is an internal torque, equal and opposite to applied torque,

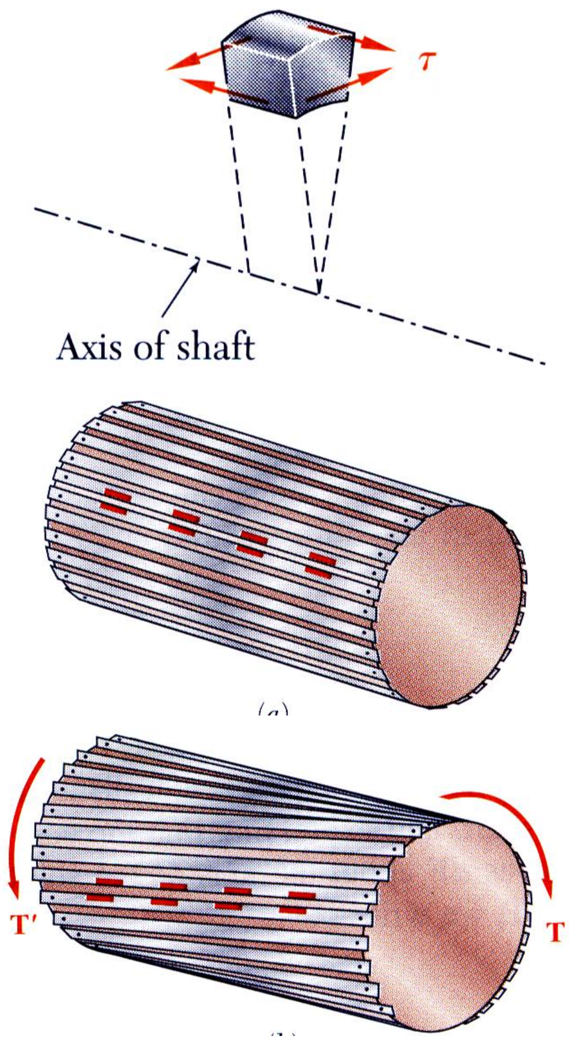
$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

- Although net torque due to shearing stresses is known, the distribution of the stresses is not

- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations

- Unlike normal stress due to axial loads, the distribution of shearing stresses due to torsional loads cannot be assumed uniform.

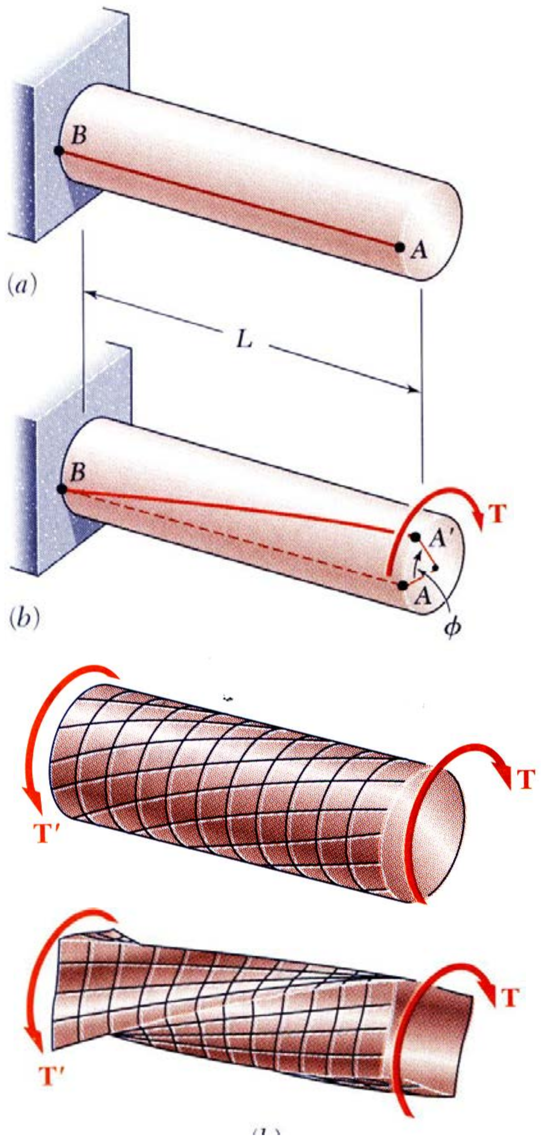
Axial Shear Components



- Torque applied to shaft produces shearing stresses on planes perpendicular to shaft axis.
- Moment equilibrium requires existence of equal shear stresses on planes containing the shaft axis, i.e., “axial shear stresses”.
- Existence of axial shear stresses is demonstrated by considering a shaft made up of axial slats.

Slats slide with respect to each other when equal and opposite torques applied to shaft ends.

Shaft Deformations



- Will see that angle of twist of shaft is proportional to applied torque and to shaft length.

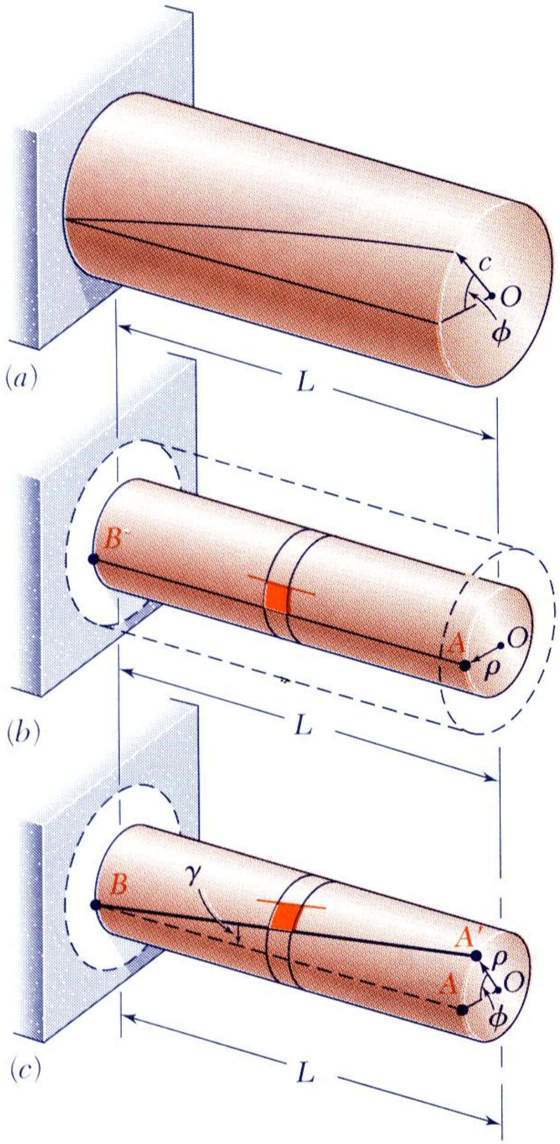
$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of circular (solid or hollow) shaft remains plane and undistorted. This is due to axisymmetry of cross section.

- Cross-sections of noncircular (hence non-axisymmetric) shafts are distorted when subjected to torsion – since no axisymm.

Shearing Strain



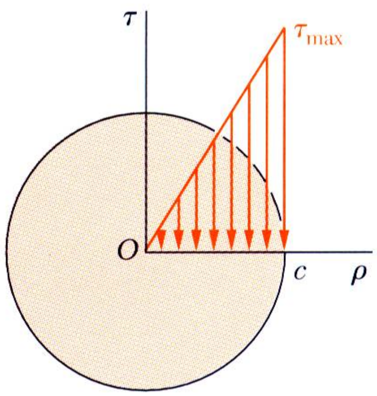
- Consider interior section of shaft. When torsional load applied, a rectangular element on the interior cylinder deforms into rhombus.
- So shear strain equals angle between *BA* and *BA'*
- Thus,

$$L\gamma = \rho\phi \quad \Rightarrow \quad \gamma = \frac{\rho\phi}{L}$$

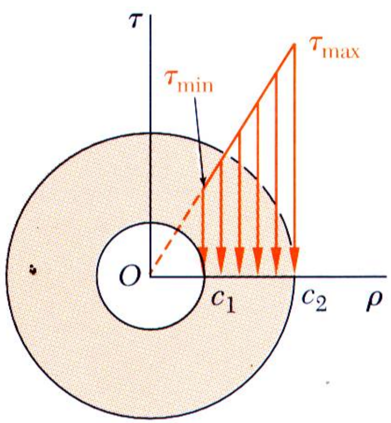
so shear strain proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Torsion Formulae in elastic range (shear stress, angle of twist



$J = \frac{1}{2}\pi c^4$



$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$

J= Polar moment of inertia

- Hooke's law,

$$\tau = G\gamma = G\frac{\rho\phi}{L} = G\frac{\rho}{c}\gamma_{\max} = \frac{\rho}{c}\tau_{\max}$$

So shearing stress also varies linearly with radial position in the section.

- Recall: sum of moments from internal stress distribution equals internal torque at the section,

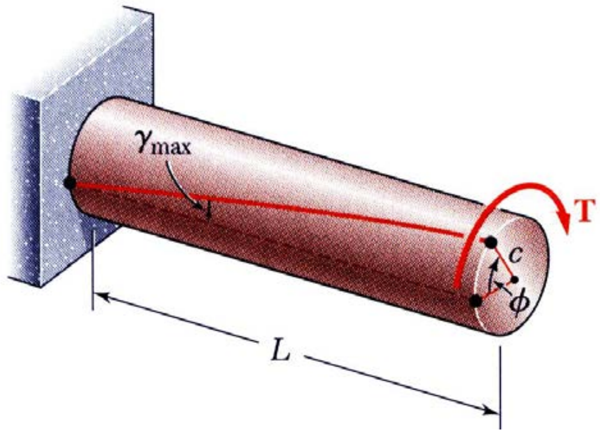
$$T = \int \rho\tau \, dA = \int \rho G \frac{\rho\phi}{L} dA = G\frac{\phi}{L}\int \rho^2 \, dA$$

$$= GJ\frac{\phi}{L} = \tau\frac{J}{\rho}$$

- Thus, elastic torsion formulas are

$$\tau = \frac{T\rho}{J} \quad ; \quad \phi = \frac{TL}{GJ}$$

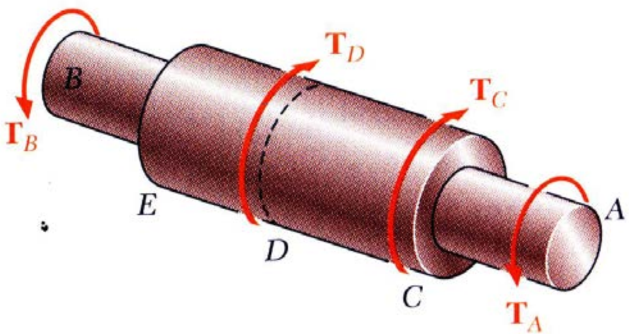
Torsion formulae in Elastic Range



$$\tau = \frac{T\rho}{J} \quad ; \quad \phi = \frac{TL}{GJ}$$

$$\tau_{max} = \frac{Tc}{J}$$

$$\text{so } \frac{\phi}{L} = \alpha = \frac{T}{GJ} \text{ , a constant if } T \text{ constant}$$



- If torsional loading or shaft cross-section changes (discretely) along length, the angle of rotation is found as sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Comparison of Axial and Torsion formulae.

AE =Axial rigidity

Axial Stiffness

$$k_A = \frac{AE}{L}$$

Axial Flexibilty: $f_A = k_A^{-1}$

Axial displacement

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Axial stress

$$\sigma = \frac{P}{A}$$

GJ =Torsional rigidity

Torsional Stiffness

$$k_T = \frac{GJ}{L}$$

Torsional Flexibilty: $f_T = k_T^{-1}$

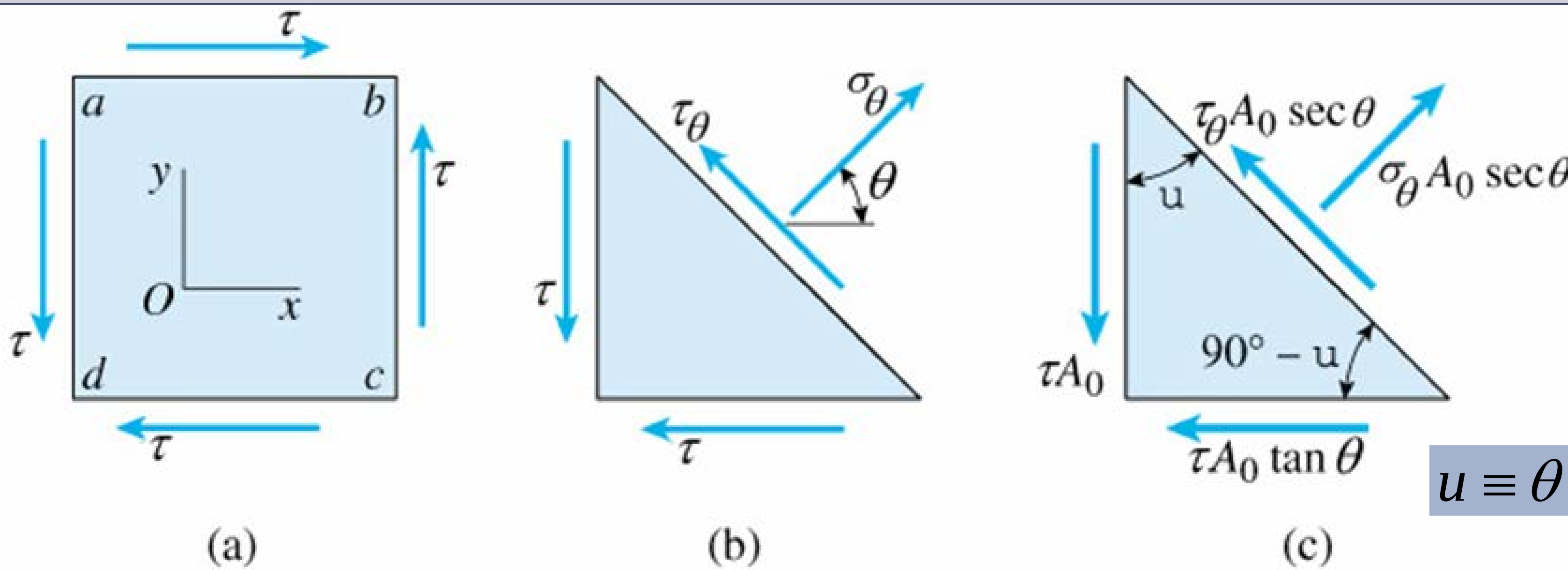
Torsional displacement

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Torsional stress

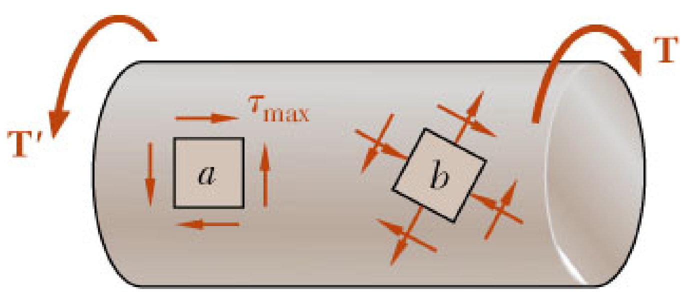
$$\tau = \frac{T\rho}{J}$$

Stressed on Inclined Plane



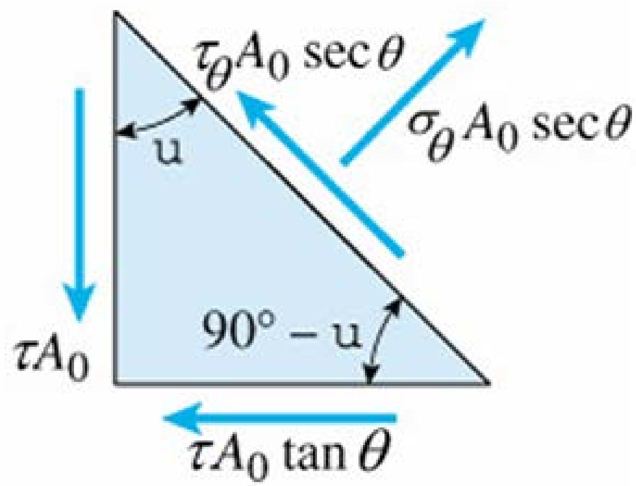
$u \equiv \theta$

(a) element in pure shear generated due to applied torque,
(b) stresses acting on inclined plane of a triangular stress element,
(c) forces acting on the triangular stress element (FBD).



Sign convention for stresses on inclined plane (Normal stresses tensile positive, shear stresses producing counterclockwise rotation positive.)

Stressed on Inclined Plane



$u \equiv \theta$

(c)

Equilibrium normal to plane,

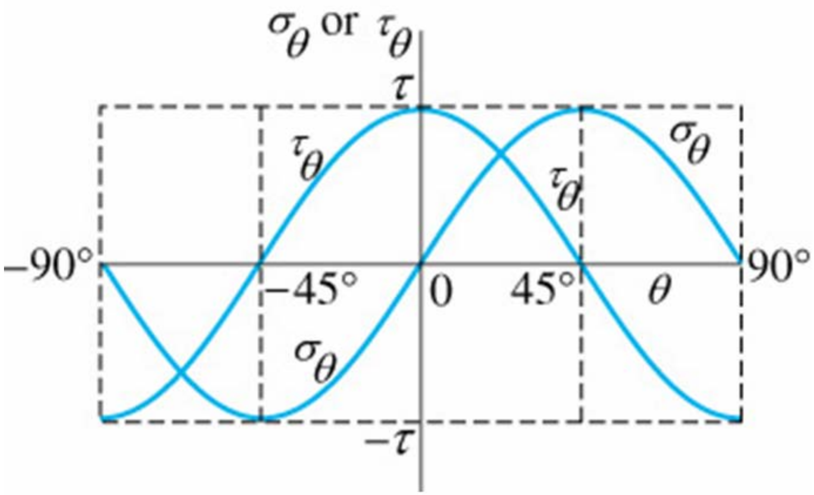
$$\sigma_\theta A_o \sec\theta = \tau A_o \sin\theta + \tau A_o \tan\theta \cos\theta \qquad (1)$$

$$\sigma_\theta = \tau \sin 2\theta$$

Equilibrium along plane,

$$\tau_\theta A_o \sec\theta = \tau A_o \cos\theta - \tau A_o \tan\theta \sin\theta \qquad (2)$$

$$\tau_\theta = \tau \cos 2\theta$$

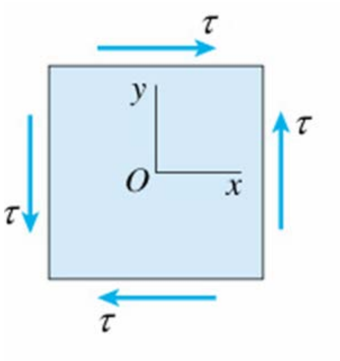


Graph of σ_θ and τ_θ versus θ.

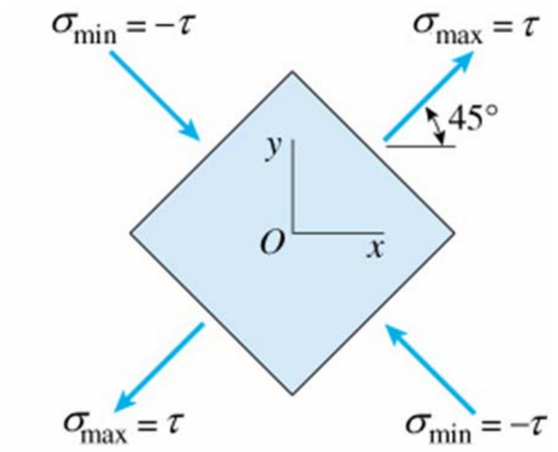
$$\sigma_\theta = \tau \sin 2\theta$$

$$\tau_\theta = \tau \cos 2\theta$$

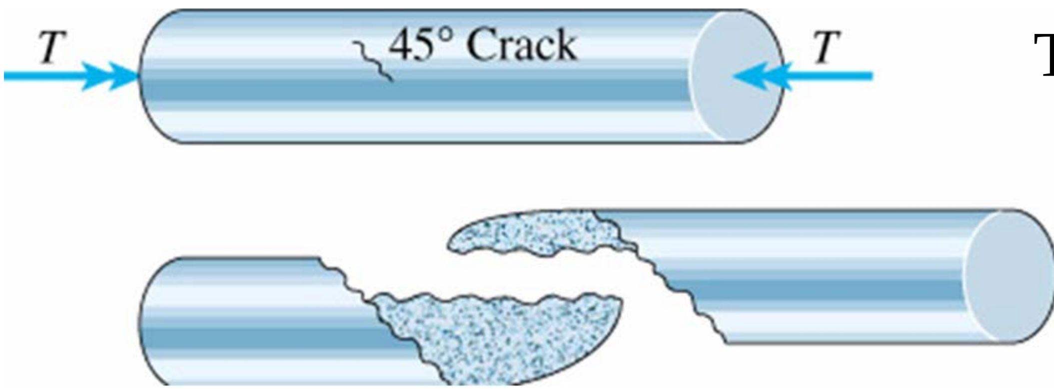
Maximum/Minimum shear stress occurs at $\theta = 0^o$ or 90^o plane $\tau_{max} = +\tau;\ \tau_{min} = -\tau$



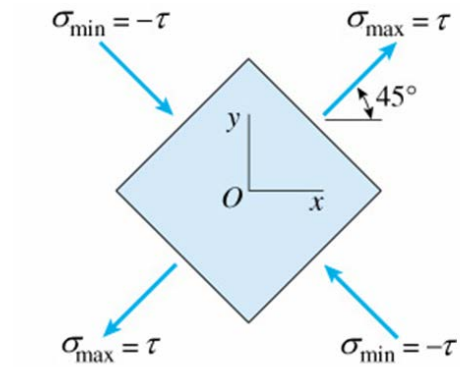
Maximum/minimum normal stress occurs at θ=+45 or -45^o plane $\sigma_{max} = +\tau;\ \sigma_{min} = -\tau$

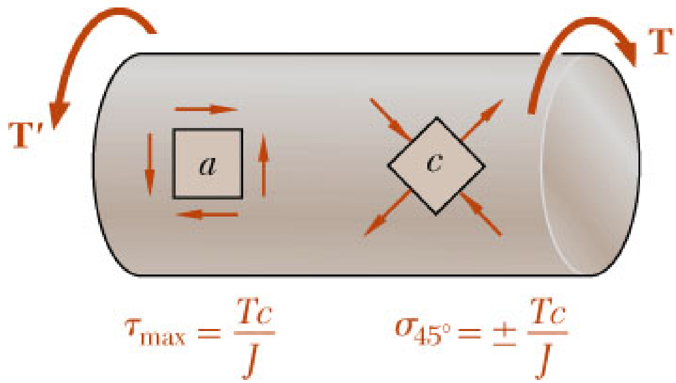


Failure of Brittle material



Try on a piece of chalk!

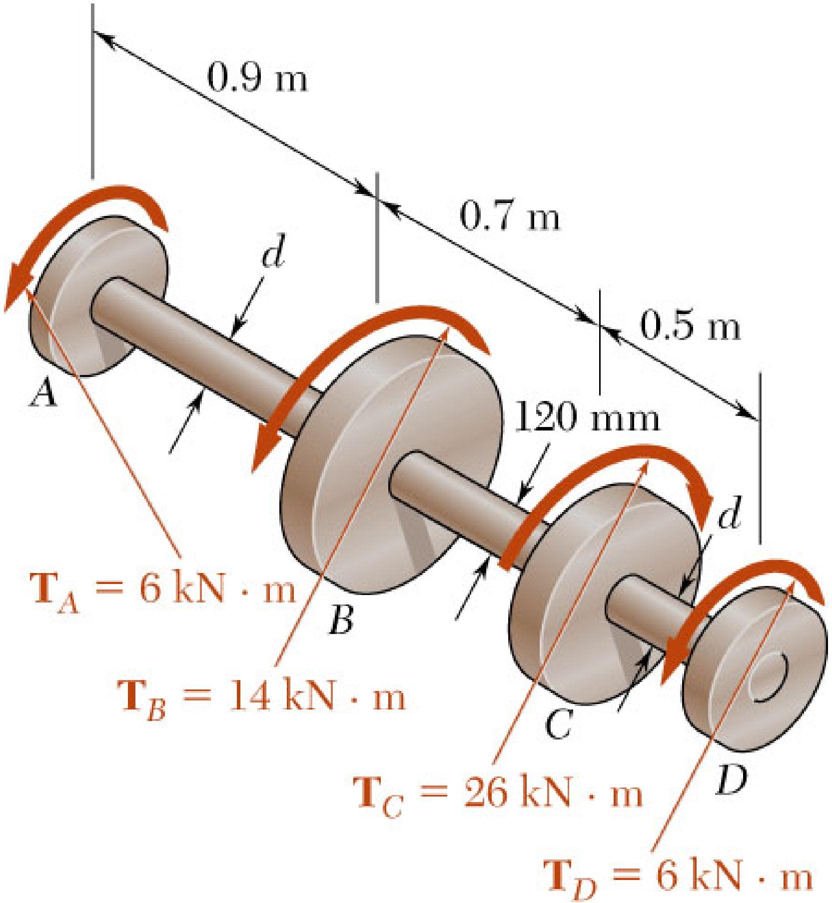




Reason: Brittle materials are weak in tension and maximum normal stress (tensile) plane in this case is 45°

Remember: Ductile materials are weak in shear and brittle materials are weak in tension. Thus, for ductile material failure occurs on maximum shear stress plane, and for brittle material failure occurs on maximum normal (tensile) stress plane.

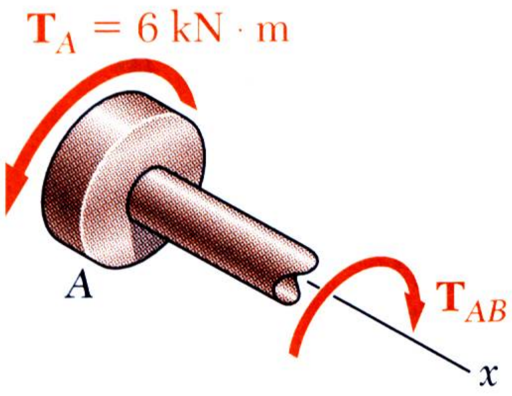
Example 3.1

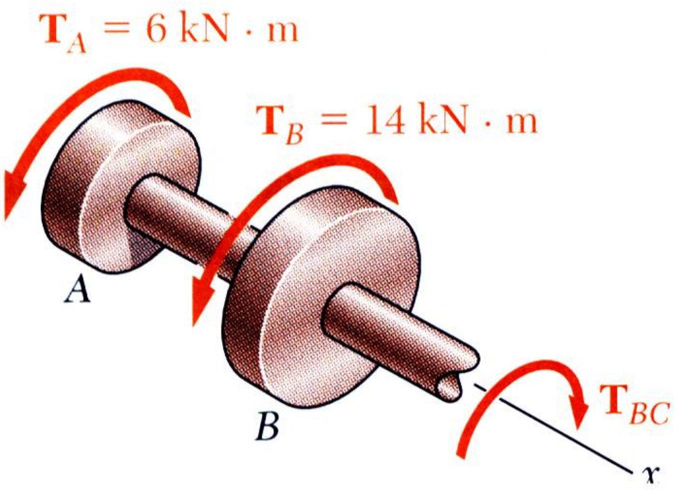


Shaft *BC* hollow, inner dia. 90 mm, outer dia. 120mm. Shafts *AB* and *CD* solid, dia. *d*. For loading shown, find (*a*) min. and max. shearing stress in *BC*, (*b*) required dia. *d* of *AB* and *CD* if allowable shearing stress in them is 65 MPa.

Example 3.1

- Cut sections, use equilibrium to find internal torque.





$$\sum M_x = 0 = (6\,\mathrm{kN\cdot m}) - T_{AB}$$

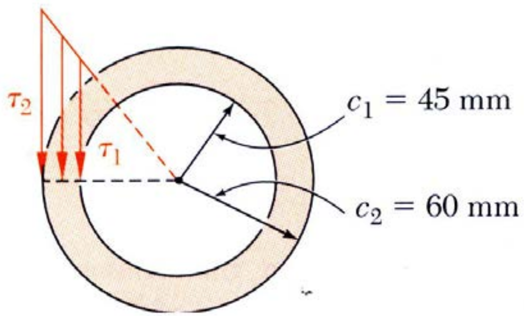
$$T_{AB} = 6\,\mathrm{kN\cdot m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\mathrm{kN\cdot m}) + (14\,\mathrm{kN\cdot m}) - T_{BC}$$

$$T_{BC} = 20\,\mathrm{kN\cdot m}$$

Example 3.1

- Apply elastic torsion formulae to find min. and max. stress in *BC*



$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92 \times 10^{-6}\,\mathrm{m}^4$$

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\mathrm{kN\cdot m})(0.060\,\mathrm{m})}{13.92\times10^{-6}\,\mathrm{m}^4}$$

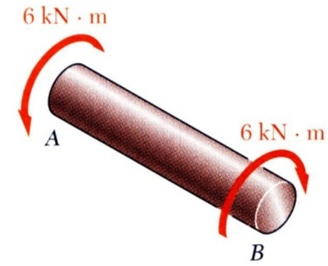
$$= 86.2\,\mathrm{MPa}$$

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{\min}}{86.2\,\mathrm{MPa}} = \frac{45\,\mathrm{mm}}{60\,\mathrm{mm}}$$

$$\tau_{\min} = 64.7\,\mathrm{MPa}$$

$$\boxed{\begin{aligned}\tau_{\max} &= 86.2\,\mathrm{MPa}\\ \tau_{\min} &= 64.7\,\mathrm{MPa}\end{aligned}}$$

- Given allowable shearing stress and applied torque, find required dia. of *AB* and *CD*

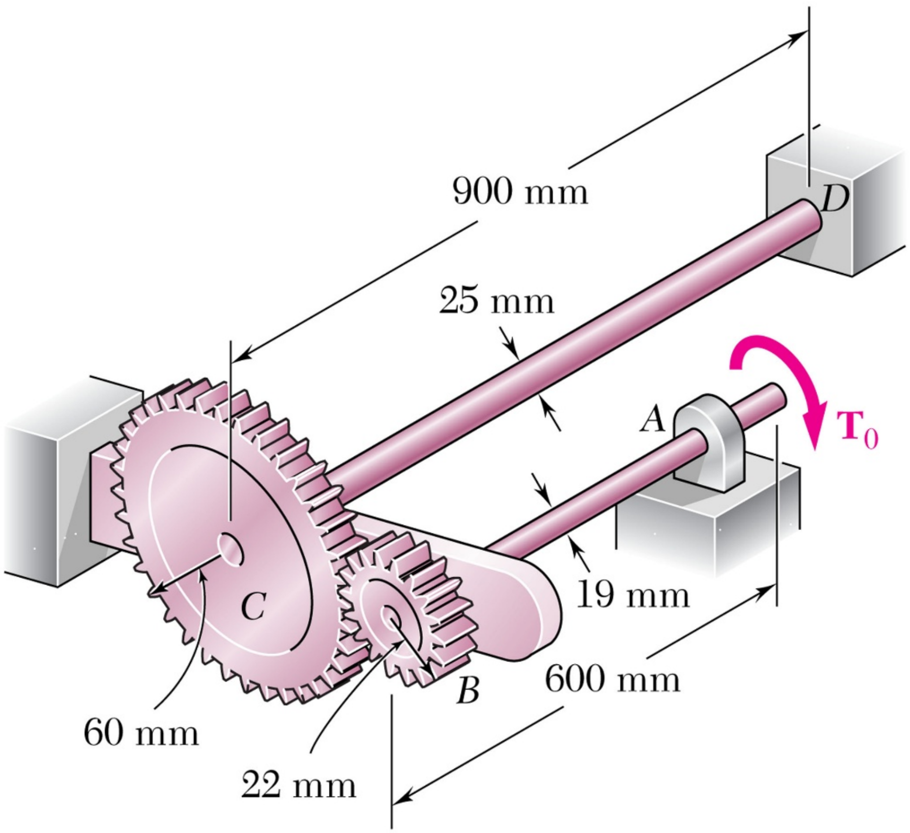


$$\tau_{\max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\mathrm{kN\cdot m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times10^{-3}\,\mathrm{m}$$

$$\boxed{d = 2c = 77.8\,\mathrm{mm}}$$

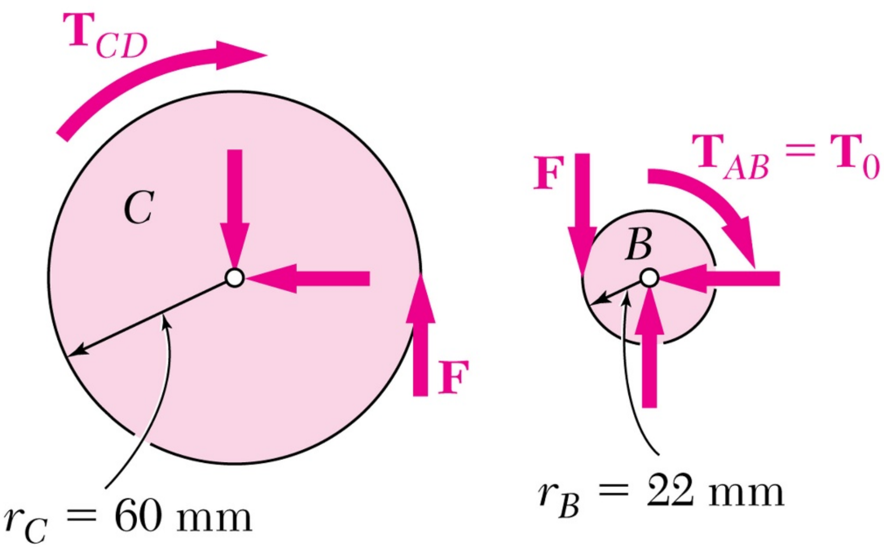
Example 3.2



Two solid steel shafts connected by gears. For each shaft G = 77 GPa and allowable shearing stress 55 Mpa. Find (*a*) largest torque T_0 that can be applied to end of *AB*, (*b*) corresponding angle through which end *A* rotates.

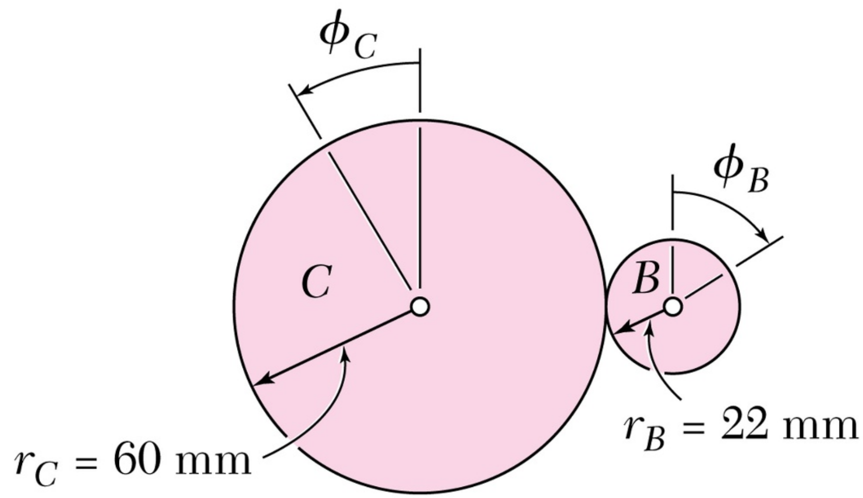
Example 3.2

- Equilibrium



$$\sum M_B = 0 = F(22\,\mathrm{mm}) - T_0$$
$$\sum M_C = 0 = F(60\,\mathrm{mm}) - T_{CD}$$
$$T_{CD} = 2.73 T_0$$

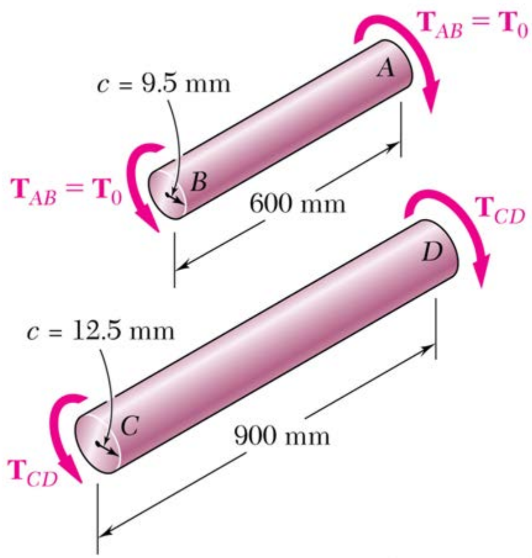
- Kinematic constraint of no slipping between gears (to relate rotations)



$$r_B \phi_B = r_C \phi_C$$
$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{60\,\mathrm{mm}}{20\,\mathrm{mm}}\phi_C$$
$$\phi_B = 2.73\phi_C$$

Example 3.2

- Find T_0 for max allowable torque on each shaft – choose smallest



$$\tau_{\max} = \frac{T_{AB}c}{J_{AB}} \quad 55\times10^6\,\text{Pa} = \frac{T_0\left(9.5\times10^{-3}\,\text{m}\right)}{\frac{\pi}{2}\left(9.5\times10^{-3}\,\text{m}\right)^4}$$

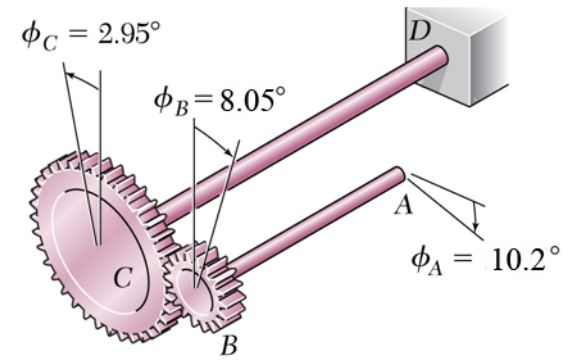
$$T_0 = 74.1\,\text{Nm}$$

$$\tau_{\max} = \frac{T_{CD}c}{J_{CD}} \quad 55\times10^6\,\text{Pa} = \frac{2.8T_0\left(12.5\times10^{-3}\,\text{m}\right)}{\frac{\pi}{2}\left(12.5\times10^{-3}\,\text{m}\right)^4}$$

$$T_0 = 61.8\,\text{Nm}$$

$$\boxed{T_0 = 61.8\,\text{Nm}}$$

- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*



$$\phi_{A/B} = \frac{T_{AB}L}{J_{AB}G} = \frac{(61.8\,\text{Nm})(0.6\,\text{m})}{\frac{\pi}{2}(0.0095\,\text{m})^4\left(77\times10^9\,\text{Pa}\right)}$$

$$= 0.0376\,\text{rad} = 2.15^\circ$$

$$\phi_{C/D} = \frac{T_{CD}L}{J_{CD}G} = \frac{2.73(61.8\,\text{Nm})(0.6\,\text{m})}{\frac{\pi}{2}(0.0125\,\text{m})^4\left(77\times10^9\,\text{psi}\right)}$$

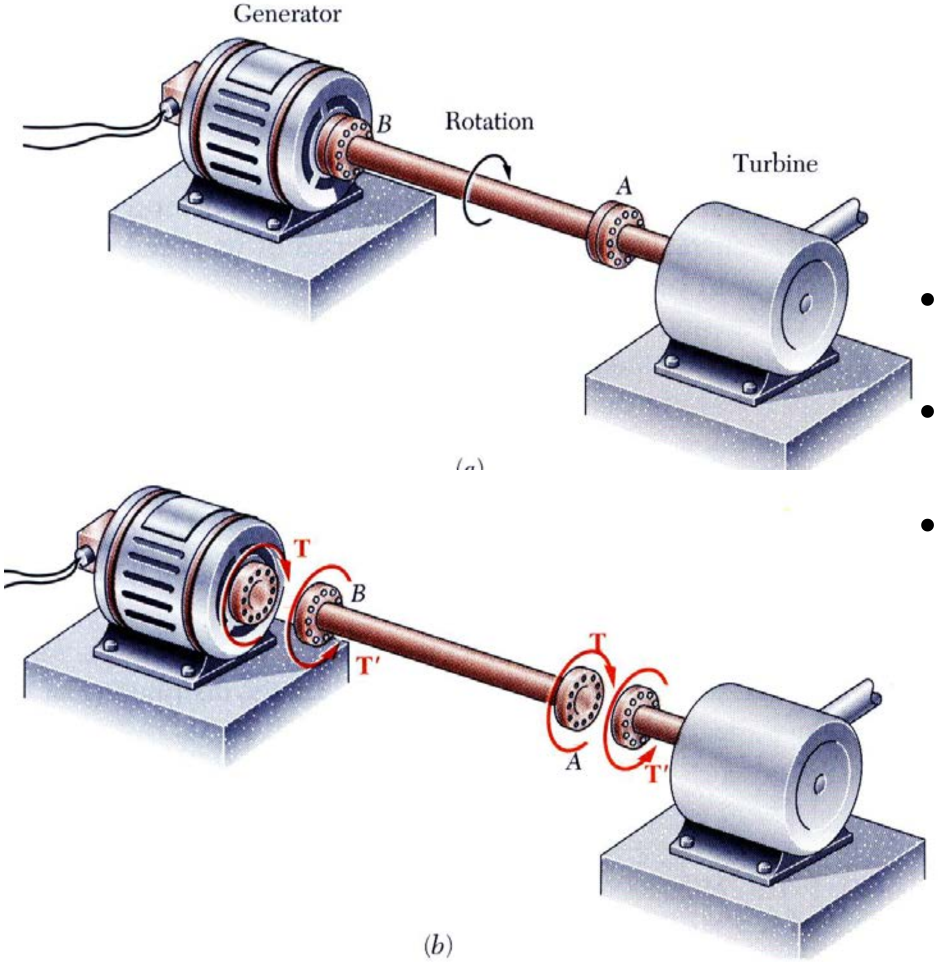
$$= 0.0514\,\text{rad} = 2.95^\circ$$

$$\phi_B = 2.73\phi_C = 2.73\left(2.95^\circ\right) = 8.05^\circ$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.05^\circ + 2.15^\circ$$

$$\boxed{\phi_A = 10.2^\circ}$$

Design of Transmission Shafts



- Turbine exerts torque *T* on shaft
- Shaft transmits torque to generator
- Generator applies equal and opposite torque *T*’ on shaft.

Design of Transmission Shafts

- Transmission shaft performance specifications are:
  - power
  - speed

- Designer must select shaft material and cross-section to meet performance specs. without exceeding allowable shearing stress.

- Determine torque applied to shaft at specified power and speed,

$$\mathbf{P} = \mathbf{T}\boldsymbol{\omega} = 2\pi\mathbf{NT}$$

$$\mathbf{T} = \frac{\mathbf{P}}{\boldsymbol{\omega}} = \frac{\mathbf{P}}{2\pi\mathbf{N}}$$

P= Power (Watt)
T= torque (N-m)
ω= angular speed (rad/s)
N= revolution per sec

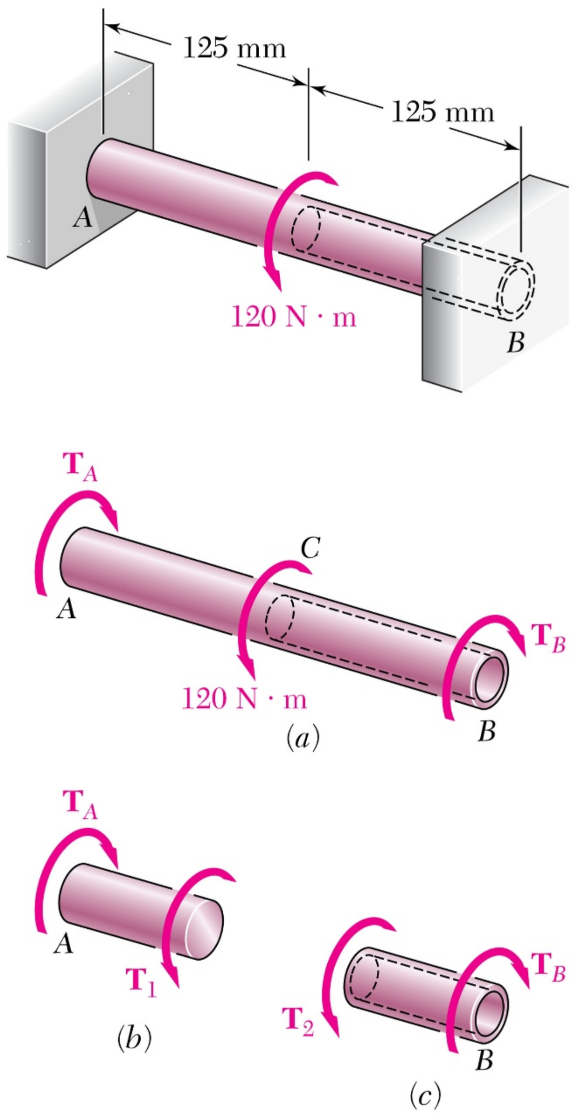
- Find cross-section so that max allowable shearing stress not exceeded,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2} c^3 = \frac{T}{\tau_{\max}} \quad (\text{solid shafts})$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad (\text{hollow shafts})$$

Statically Indeterminate Shafts



- Given applied torque, find torque reactions at *A* and *B*.

- Equilibrium,

$$T_A + T_B = 120 \ \text{N.m}$$

  So problem is SID.

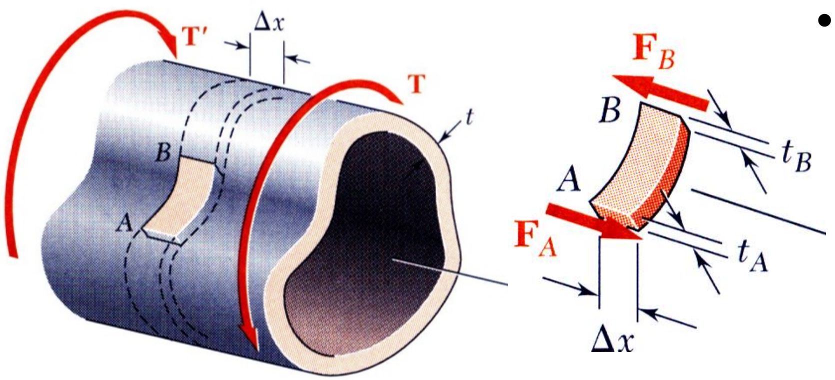
- Compatibility,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0$$

- Solve equil. and compat,

$$T_A \left(1 + \frac{L_1 J_2}{L_2 J_1}\right) = 120 \, \text{N} \cdot \text{m}$$

Thin-Walled Hollow Shafts



- Since wall is thin, assume shear stress constant thru wall thickness. For *AB*, summing forces in *x*(shaft-axis)-direction,

$$\sum F_x = 0 = \tau_A\left(t_A \Delta x\right) - \tau_B\left(t_B \Delta x\right)$$

$$\tau_A t_A = \tau_B t_B = \tau t = q = \text{shear flow}$$

So shear flow constant and shear stress at section varies inversely with thickness

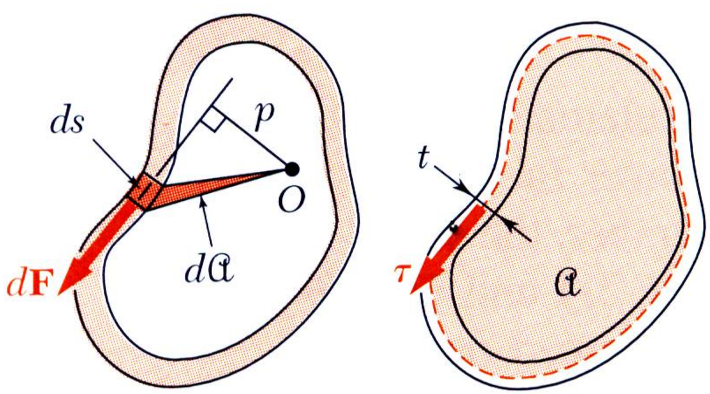
- Compute shaft torque from integral of the moments due to shear stress

$$dM_0 = p\,dF = p\left(\tau\,t\,ds\right) = q\left(p ds\right)$$

$$= q(r\sin\theta\,ds) = q(\mathbf{r}\times\mathbf{ds}) = 2q\,dA$$

$$T = \oint dM_0 = \oint 2q\,dA = 2qA$$

$$\tau = \frac{T}{2tA} \Rightarrow q = \frac{T}{2A}$$

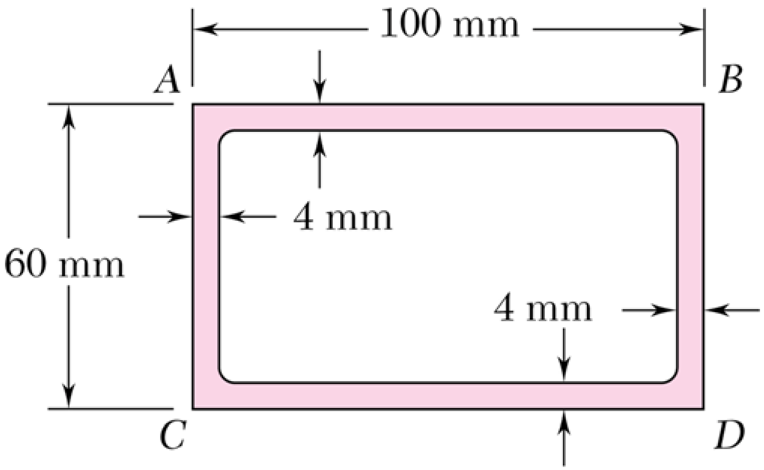


$$\tau t = q = \text{constant}; \quad q = \frac{T}{2A}$$

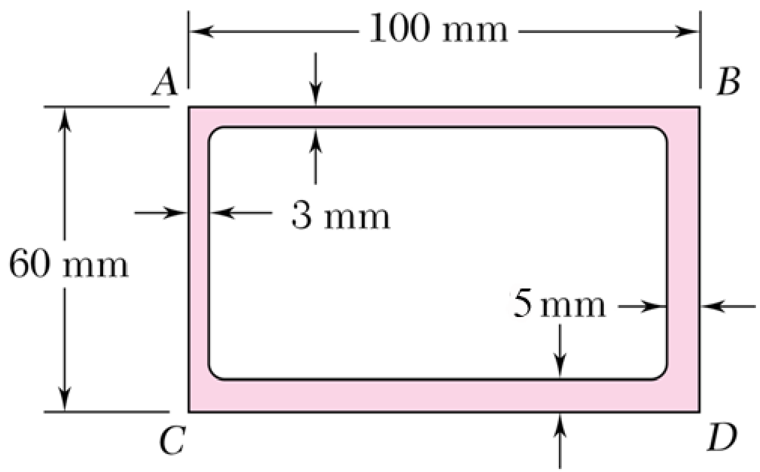
- Angle of twist (from Chapt 11)

$$\phi = \frac{TL}{4A^2G}\oint\frac{ds}{t}$$

Example 3.3



(a)

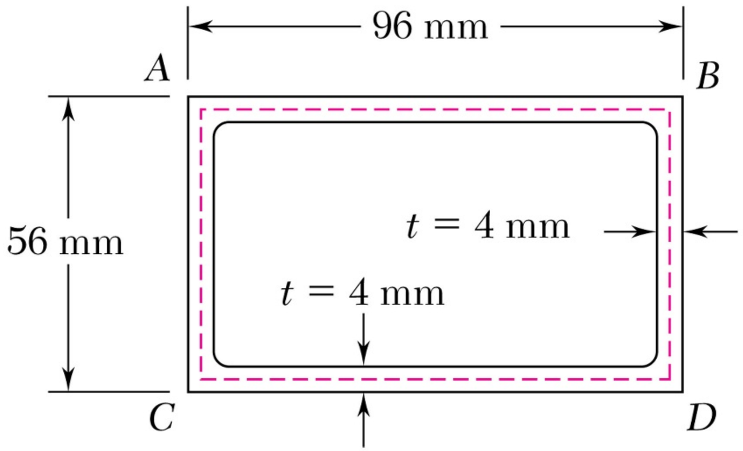


(b)

Extruded aluminum tubing with rectangular cross-section has torque loading 2.7 kNm. Find shearing stress in each of four walls considering (a) uniform wall thickness of 4 mm and (b) wall thicknesses of 3 mm on *AB* and *AC* and 5 mm on *CD* and *BD*.

Example 3.3

Find shear flow q.



$$A = (96\,\text{mm})(56\,\text{mm}) = 5376\,\text{mm}^2$$

$$q = \frac{T}{2A} = \frac{2700\,\text{Nm}}{2\left(5376\times10^{-6}\,\text{mm}^2\right)} = 251.12\times10^3\,\frac{\text{N}}{\text{m}}$$

Find corresponding shearing stress for each wall thickness.

With uniform wall thickness,

$$\tau = \frac{q}{t} = \frac{251.12\times10^3\,\text{N/m}}{0.004\,\text{m}}$$

$$\boxed{\tau = 62.8\,\text{MPa}}$$

With variable wall thickness

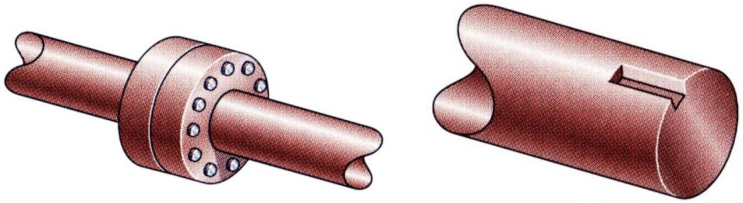
$$\tau_{AB} = \tau_{AC} = \frac{251.12\times10^3\,\text{N/m}}{0.003\,\text{m}}$$

$$\boxed{\tau_{AB} = \tau_{BC} = 83.7\,\text{MPa}}$$

$$\tau_{BD} = \tau_{CD} = \frac{251.12\times10^3\,\text{N/m}}{0.005\,\text{m}}$$

$$\boxed{\tau_{BC} = \tau_{CD} = 50.2\,\text{MPa}}$$

Stress Concentrations



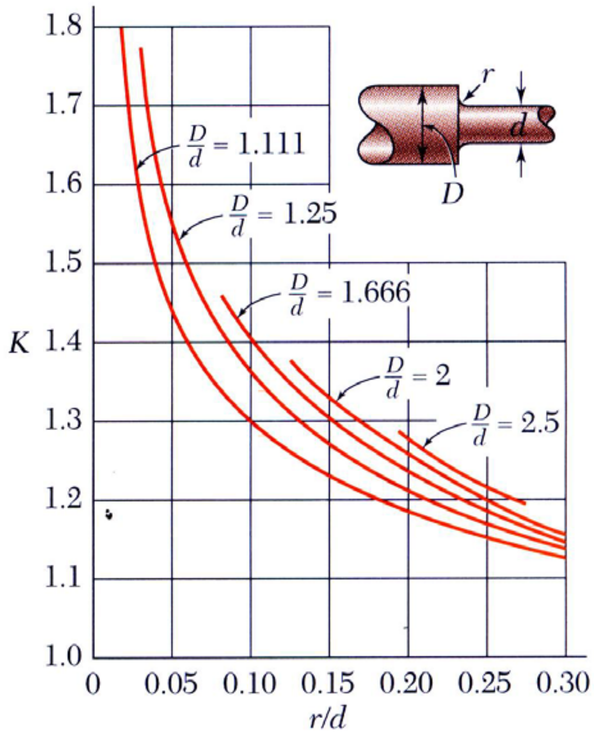


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

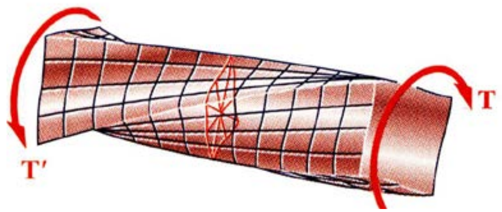
  assumed a circular shaft with uniform cross-section loaded through rigid end plates.

- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$

Torsion of Noncircular Members



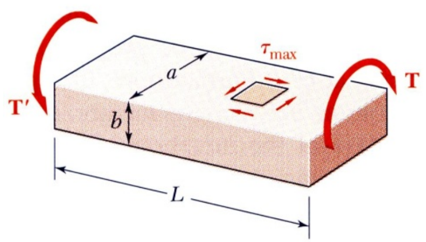
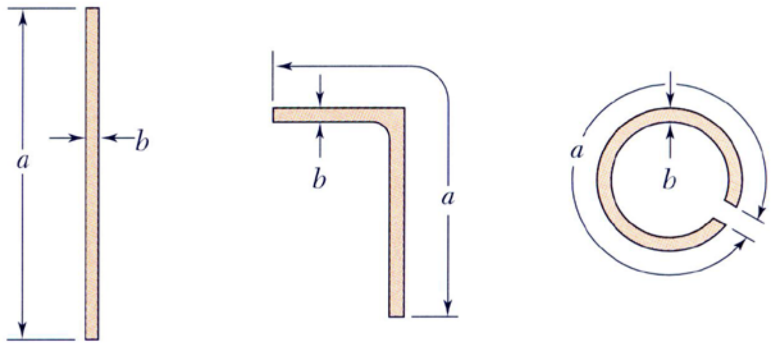


TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

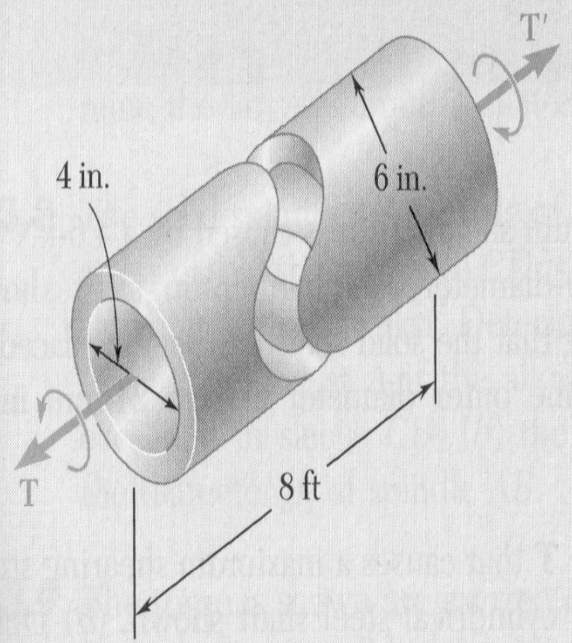


- Previous torsion formulas are valid for axisymmetric or circular shafts
- Planar cross-sections of noncircular shafts do not remain planar and stress and strain distribution do not vary linearly
- For uniform rectangular cross-sections,

$$\tau_{\max} = \frac{T}{c_1 a b^2} \qquad \phi = \frac{TL}{c_2 a b^3 G}$$

- At large values of a/b, the maximum shear stress and angle of twist for other open sections are the same as a rectangular bar.

Example 3.4



The preliminary design of a large shaft connecting a motor to a generator calls for the use of a hollow shaft with inner and outer diameters of 4 in. and 6 in., respectively. Knowing that the allowable shearing stress is 12 ksi, determine the maximum torque that can be transmitted (*a*) by the shaft as designed, (*b*) by a solid shaft of the same weight, (*c*) by a hollow shaft of the same weight and of 8-in. outer diameter.

Example 3.4

Ex 3-4

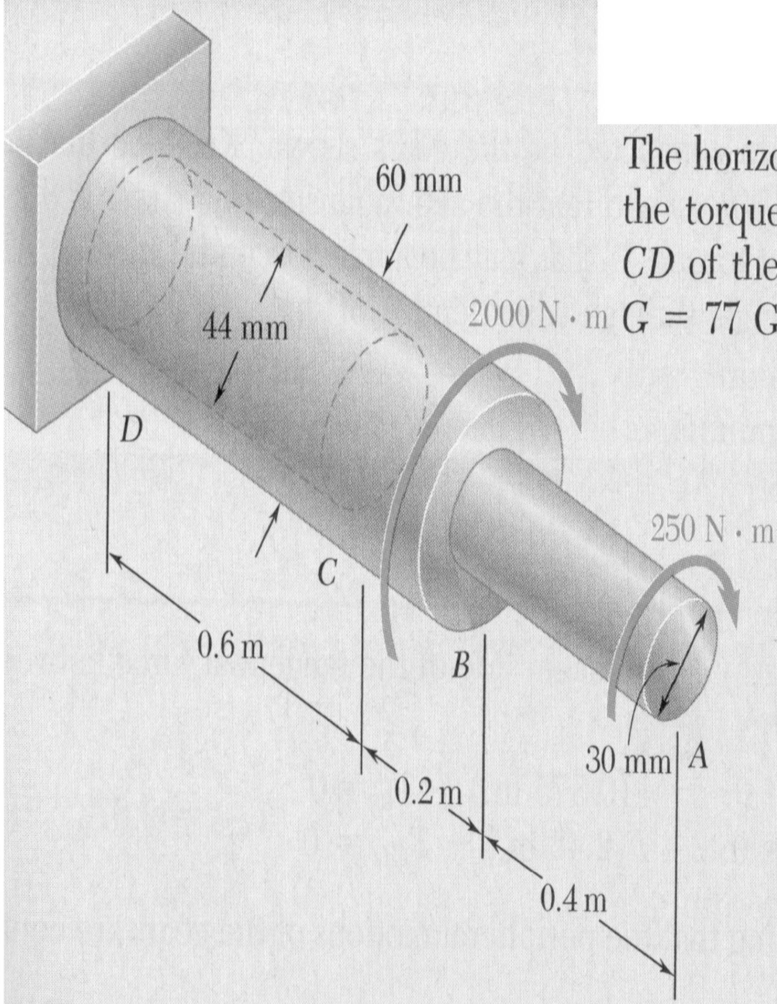
1

(a) $12 = \dfrac{T(6/2)}{\frac{\pi}{32}(6^4 - 4^4)} \Rightarrow T = 408.4$ kip·in

(b) $d = \sqrt{6^2 - 4^2} \Rightarrow 12 = \dfrac{T(d/2)}{\frac{\pi}{32} d^4} \Rightarrow T = 210.7$ kip·in

(c) $d_i^2 = 8^2 - (6^2 - 4^2) \Rightarrow 12 = \dfrac{T(8/2)}{\frac{\pi}{32}(8^4 - d_i^4)} \Rightarrow T = 636.2$ kip·in

Example 3.5



The horizontal shaft *AD* is attached to a fixed base at *D* and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion *CD* of the shaft. Knowing that the entire shaft is made of steel for which $G = 77$ GPa, determine the angle of twist at end *A*.

Example 3.5

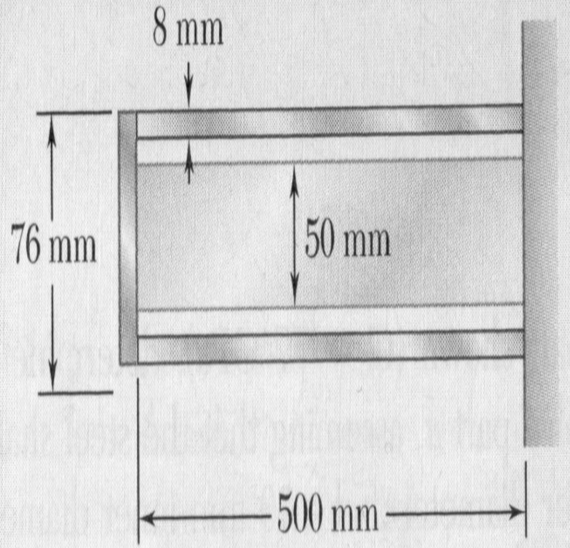
Ex 3.5 —— X ——

$$\theta_A = \theta_{A/B} + \theta_{B/C} + \theta_{C/D} + \cancel{\theta_D} \to 0$$

$$= \frac{(250)(0.4)}{(77E9)\frac{\pi}{32}\left(\frac{30}{1000}\right)^4} + \frac{(2250)(0.2)}{(77E9)\frac{\pi}{32}\left(\frac{60}{1000}\right)^4} + \frac{(2250)(0.6)}{(77E9)\frac{\pi}{32}\left[\left(\frac{60}{1000}\right)^4 - \left(\frac{44}{1000}\right)^4\right]}$$

$$= 0.0403 \text{ rad.}$$

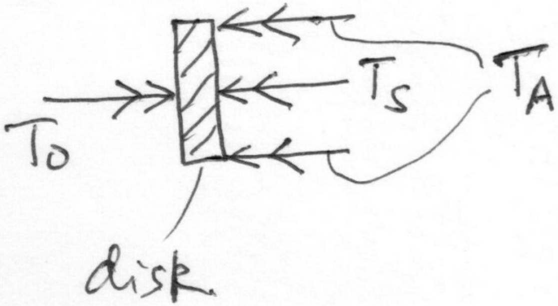
Example 3.6



A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the initial stresses are zero, determine the maximum torque $\mathbf{T}_0$ that can be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use $G = 77$ GPa for steel and $G = 27$ GPa for aluminum.

Example 3.6

Ex 3.6



Equilibrium: $T_0 = T_A + T_S \longrightarrow (1)$

Compatibility: $\theta_S = \theta_A$

$$\frac{T_A(.5)}{(27E9)\frac{\pi}{32}\left(\frac{76^4-60^4}{1000^4}\right)} = \frac{T_S(.5)}{(77E9)\frac{\pi}{32}\left(\frac{50^4}{1000^4}\right)} \longrightarrow (2)$$

We see that $T_A/J_A = \frac{27}{77} T_S/J_S$.

$$\Rightarrow (\tau_{max})_A = \frac{27}{77}\underbrace{\left(\frac{T_S(25)}{J_S}\right)}_{(\tau_{max})_S}\frac{1}{25}\cdot(76/2) = 0.53\,(\tau_{max})_S$$

and $\frac{(\tau_{all})_A}{(\tau_{all})_S} = \frac{70}{120} = 0.58$

*So failure in steel first.

$\Rightarrow T_S = (\tau_{all})_S J_S/(25mm) = 2945.2$ N.m

Example 3.6

(2)

From (2) $\rightarrow T_A = 3371.2$

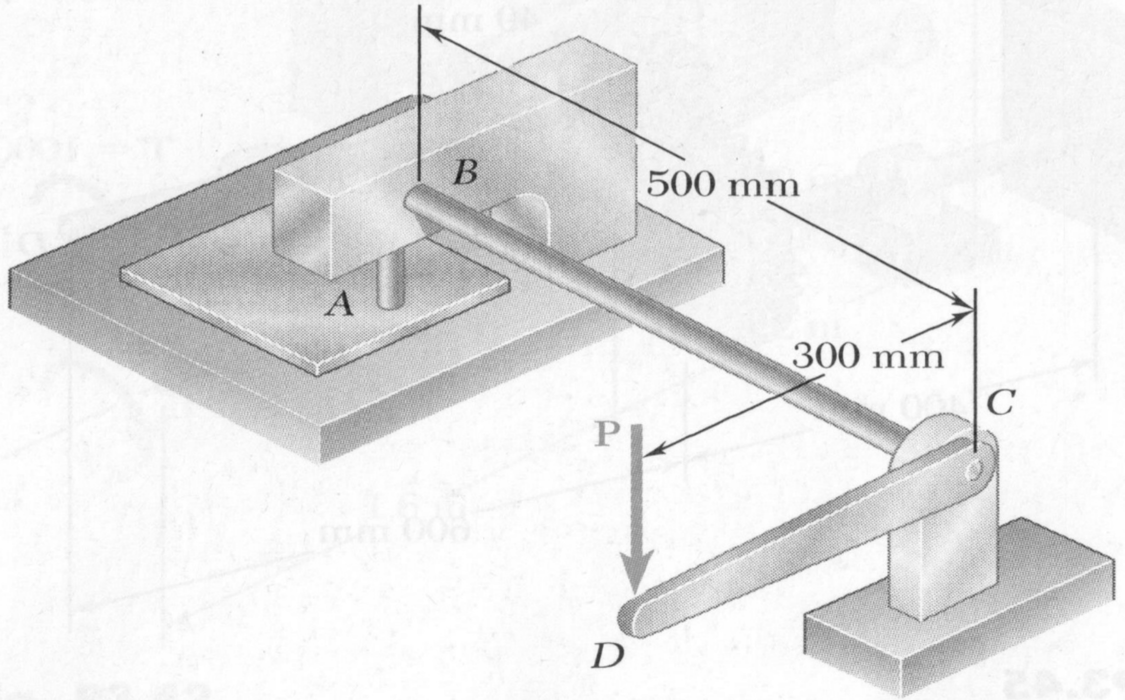
From (1) $\rightarrow T_0 = 6316.4$ N.m.

Note: we concluded failure in steel is critical $\because$

$$\frac{(\tau_{max})_A}{(\tau_{max})_s} < \frac{(\tau_{all})_A}{(\tau_{all})_s}$$

Example 3.7

A hole is punched at A in a plastic sheet by applying a 600-N force **P** to end D of lever CD, which is rigidly attached to the solid cylindrical shaft BC. Design specifications require that the displacement of D should not exceed 15 mm from the time the punch first touches the plastic sheet to the time it actually penetrates it. Determine the required diameter of shaft BC if the shaft is made of a steel with $G = 77$ GPa and $\tau_{all} = 80$ MPa.



Example 3.7

Ex 3.7 ———— x ————

So, For shaft BC, end B is fixed.

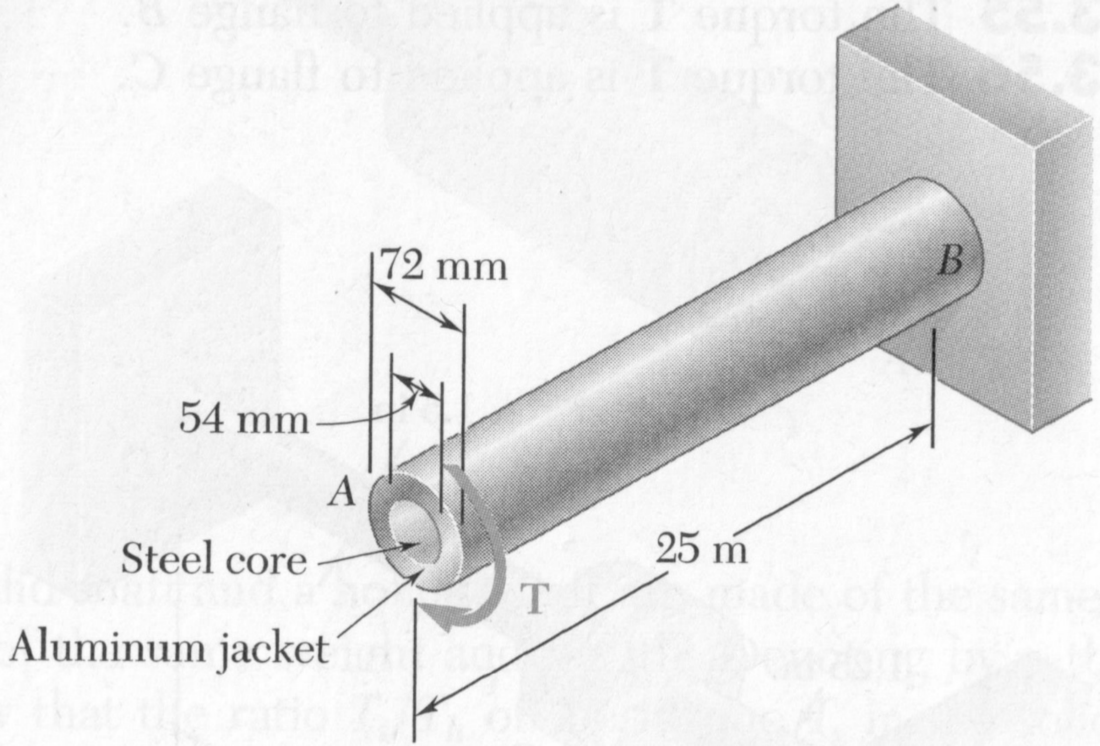
$$(\theta_C)_{max} = \frac{15}{300} = \frac{1}{20} \text{ rad.}$$

(ie, angle during deformation stage, this excludes rigid body rot.)

$$= \frac{(600)(300/1000)(0.5)}{(77E9)\frac{\pi}{32}d^4} \Rightarrow d = 22.09 \text{ mm.}$$

Example 3.8

A torque of magnitude $T = 4\ \mathrm{kN \cdot m}$ is applied at end A of the composite shaft shown. Knowing that the modulus of rigidity is 77 GPa for the steel and 27 GPa for the aluminum, determine (*a*) the maximum shearing stress in the steel core, (*b*) the maximum shearing stress in the aluminum jacket, (*c*) the angle of twist at A.



Example 3.8

Ex 3.8

Applied Torque distributes itself to act partly on steel & partly on aluminium.

Equilibrium $\longrightarrow$ $4000 = T_S + T_A \longrightarrow$ (1)

Compatibility $\longrightarrow$ $\theta_S = \theta_A \Rightarrow \dfrac{T_S(25)}{(77E9)\frac{\pi}{32}\left(\frac{54}{1000}\right)^4} = \dfrac{T_A(25)}{\left\{27E9 * \frac{\pi}{32} * \left(\frac{72^4-54^4}{1000^4}\right)\right\}}$ (2)

(1), (2) $\longrightarrow T_S = 2275.9\ N.m$

$T_A = 1724.1\ N.m \longleftarrow T_A = \left(\frac{27}{77}\right)\left(\frac{72^4-54^4}{54^4}\right)T_S \Longleftarrow$ (2)

$$(\tau_{max})_S = \frac{(2275.9)\left(\frac{54}{2}\cdot\frac{1}{1000}\right)}{\left(\frac{\pi}{32}\right)\left(\frac{54}{1000}\right)^4} = 73.61\ MPa$$

$$(\tau_{max})_A = \frac{(1724.1)\left(\frac{72}{2}\cdot\frac{1}{1000}\right)}{\left(\frac{\pi}{32}\right)\left(\frac{72^4-54^4}{1000^4}\right)} = 34.41\ MPa$$

$\theta = \theta_S = \theta_A$
$= 0.885\ rad.$

Example 3.8

(3)

$$(\tau_{max})_A = \frac{T_A}{J_A}\left(\frac{72}{2}\right) = \left(\underbrace{\frac{T_S}{J_S}\cdot\frac{54}{2}}_{(\tau_{max})_S}\cdot\frac{2}{54}\right)\cdot\frac{27}{77}\cdot\left(\frac{72}{2}\right)$$

$$= \frac{72}{54}\cdot\frac{27}{77}(\tau_{max})_S = 0.468(\tau_{max})_S.$$

$$\frac{(\tau_{all})_A}{(\tau_{all})_S} = \frac{45}{60} = 0.75.$$

$\because \dfrac{(\tau_{max})_A}{(\tau_{max})_S} < \dfrac{(\tau_{all})_A}{(\tau_{all})_S}$, steel will reach critical state first.

$$\Rightarrow T_S = \frac{(\tau_{all})_S\, J_S}{\left(\frac{54}{2}\cdot\frac{1}{1000}\right)} = 1855.1\ \text{N.m.}$$

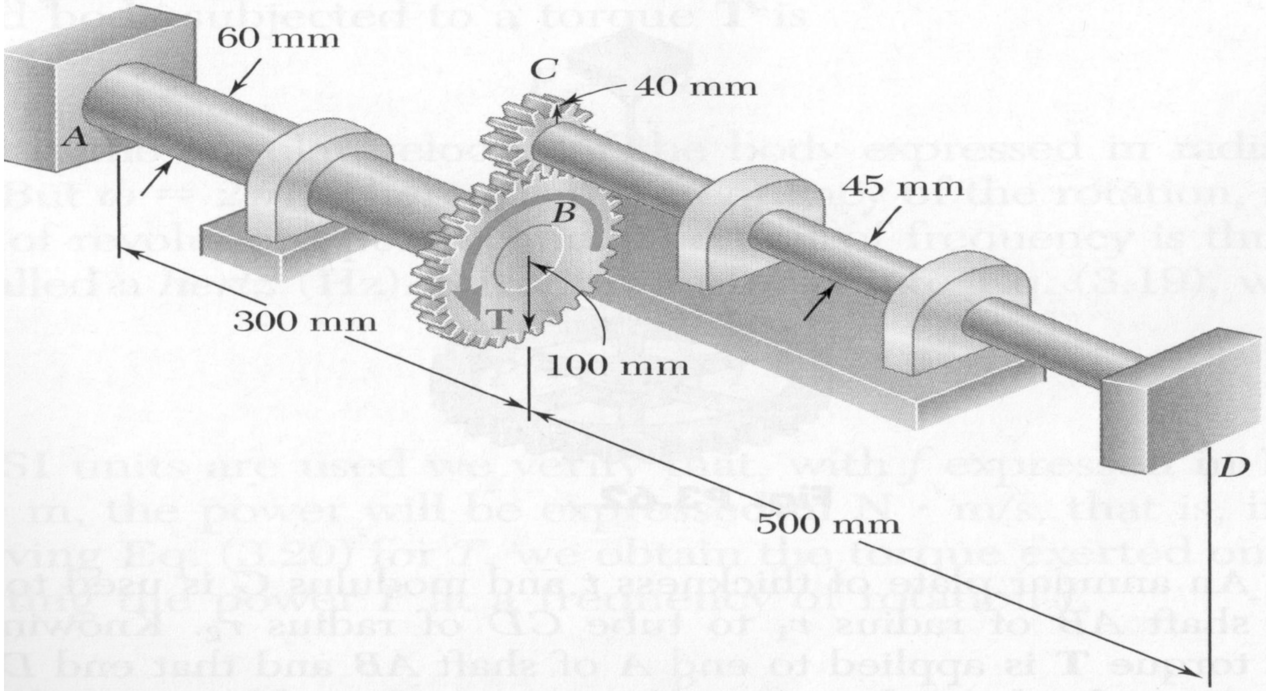
from (2), $\mathbf{T}_A = T_S\,\dfrac{J_A}{J_S}\cdot\dfrac{27}{77} = 1405.4$

$$T = T_A + T_S = 3260.5\ \text{N.m}$$

$$\theta = \theta_S = \theta_A = \frac{(1855.1)(25)}{(77E9)\frac{\pi}{32}\left(\frac{54}{1000}\right)^4} = 0.7215\ \text{rad}$$

Example 3.9

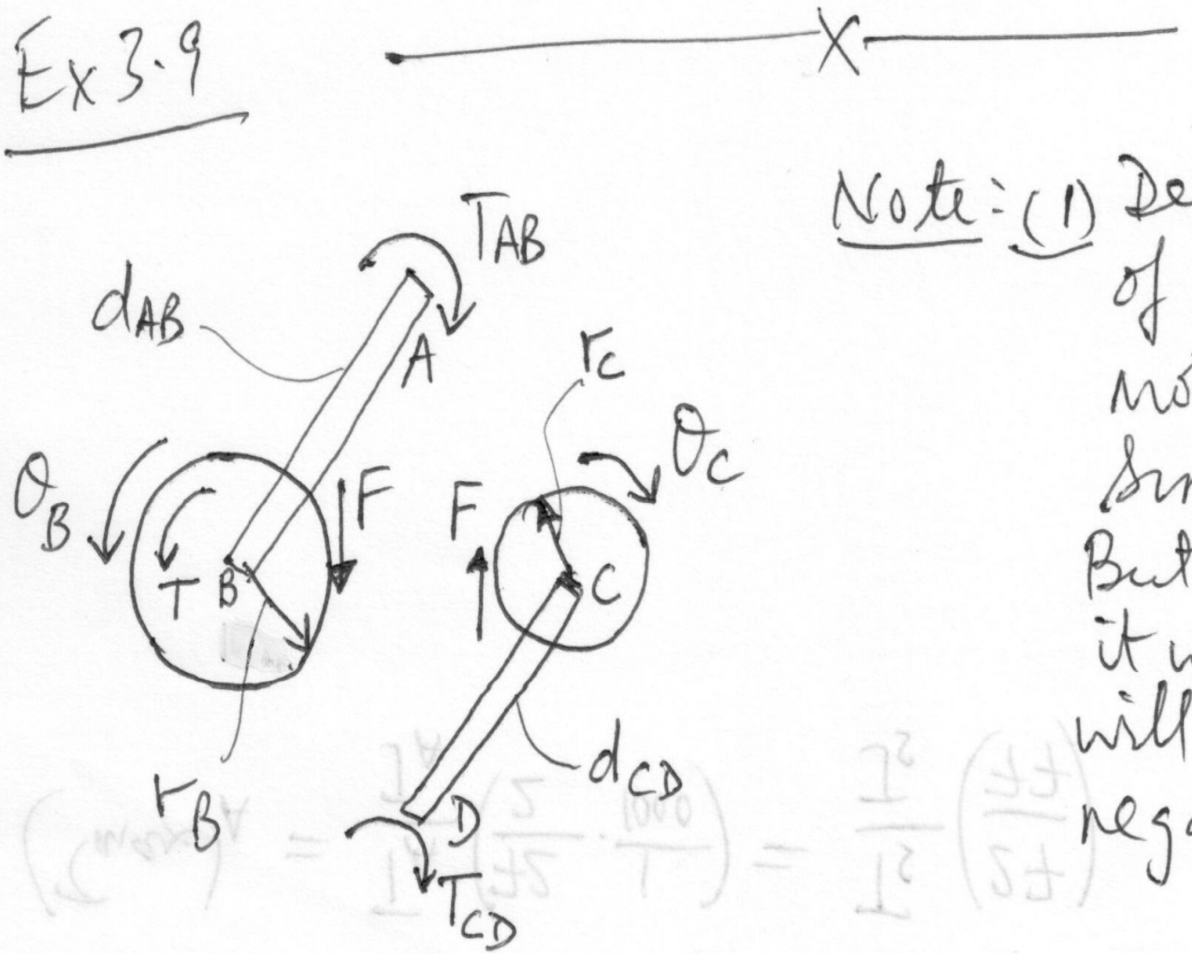
Ends A and D of the two solid steel shafts AB and CD are fixed, while ends B and C are connected to gears as shown. Knowing that a 4-kN · m torque **T** is applied to gear B, determine the maximum shearing stress (a) in shaft AB, (b) in shaft CD.



Ends A and D of the two solid steel shafts AB and CD are fixed, while ends B and C are connected to gears as shown. Knowing that the allowable shearing stress is 50 MPa in each shaft, determine the largest torque **T** that can be applied to gear B.

Example 3.9

Ex 3.9



Note: (1) Determine direction of F from impending motion concept, similar to friction. But even if its wrong, it wont matter, you will just get a negative number for F.

Example 3.9

4

Note (2): Assume directions of internal torques T_{AB}, T_{CD}, and remain consistent when writing equilibrium equations.

Note (3): θ_B, θ_C should be consistent with each other and with T_{AB}, T_{CD}.

Equilibrium:

$$T = T_{AB} + F \cdot r_B \quad (1)$$

$$0 = F \cdot r_C + T_{CD} \quad (2)$$

3 unknowns, F, T_{AB}, T_{CD}.

Compatibility: $r_B \theta_B = r_C \theta_C$

$$\theta_B = \frac{T_{AB} L_{AB}}{G \frac{\pi}{32} (d_{AB})^4}, \quad \theta_C = (-) \frac{T_{CD} L_{CD}}{G \frac{\pi}{32} (d_{CD})^4}$$

imp!! for consistency

Example 3.9

$$\Rightarrow \frac{T_{AB} L_{AB} r_B}{G \frac{\pi}{32} (d_{AB})^4} = - \frac{T_{CD} L_{CD} r_C}{G \frac{\pi}{32} (d_{CD})^4} \longrightarrow (3).$$

Use $r_B = 100\,mm$, $r_C = 40\,mm$, $d_{AB} = 60\,mm$, $d_{CD} = 45\,mm$,
$L_{AB} = 300\,mm$, $L_{CD} = 500\,mm$, $T = 4000\,N.m$,

$$(1),(2),(3) \rightarrow \frac{T}{r_B} = \frac{T_{AB}}{r_B} - \frac{T_{CD}}{r_C} = T_{AB}\left(\frac{1}{r_B} + \frac{L_{AB}}{L_{CD}}\left(\frac{d_{CD}}{d_{AB}}\right)^4 \cdot \frac{1}{r_C} \frac{r_B}{r_C}\right)$$

$$\Rightarrow T_{AB} = 1829.39\ N.m, \quad T_{CD} = (-)\ 868.25\ N.m.$$

expected ∵ F, T_{CD} chosen positive in same direction.

$$\tau_{AB} = \frac{(1829.39)(30/1000)}{\frac{\pi}{32}\left(\frac{60}{1000}\right)^4} = 43.13\ \text{MPa}$$

$$\tau_{CD} = \frac{(868.25)\left(\frac{22.5}{1000}\right)}{\frac{\pi}{32}\left(\frac{45}{1000}\right)^4} = 48.53\ \text{MPa}$$

Example 3.9

(5)

From previous part, for given torque T applied,

$(\tau_{AB})_{max} < (\tau_{CD})_{max}$.

$$\Rightarrow T_{CD} = \frac{(\tau_{All}) \cdot J_{CD}}{(d_{CD}/2)} = \frac{(50E6)\frac{\pi}{32}\left(\frac{45}{1000}\right)^4}{\left(\frac{45}{2}\cdot\frac{1}{1000}\right)} = -894.62 \text{ N.m.}$$

Also from (3), $T_{AB} = 1884.96$ N.m

From (1), (2), eliminating F, $T = r_B\left(\frac{T_{AB}}{r_B} - \frac{T_{CD}}{r_C}\right) = 4121.50$ N-m

Example 3.10

A solid shaft and a hollow shaft are made of the same material and are of the same weight and length. Denoting by n the ratio c_1/c_2, show that the ratio T_s/T_h of the torque T_s in the solid shaft to the torque T_h in the hollow shaft is (*a*) $\sqrt{(1-n^2)/(1+n^2)}$ if the maximum shearing stress is the same in each shaft, (*b*) $(1-n^2)/(1+n^2)$ if the angle of twist is the same for each shaft.

Example 3.10

Ex 3.10

Solid shaft radius = r

Hollow shaft radius = r_o (outer) & r_i (inner)

Given $n = \frac{r_i}{r_o}$. Also ∵ lengths & weights are same,

$\Rightarrow$ areas same $\Rightarrow r^2 = r_o^2 - r_i^2$

For max shear stresses same,

$$\tau_s = \frac{T_s r}{G \frac{\pi}{2} r^4} = \tau_h = \frac{T_h r_o}{G \frac{\pi}{2} (r_o^4 - r_i^4)}$$

$$\frac{T_s}{T_h} = \frac{r^3 r_o}{(r_o^2 - r_i^2)(r_o^2 + r_i^2)} = \frac{\left(\sqrt{r_o^2 - r_i^2}\right) r_o}{(r_o^2 + r_i^2)} = \frac{\sqrt{1-n^2}}{1+n^2}$$

For angle of twist same,

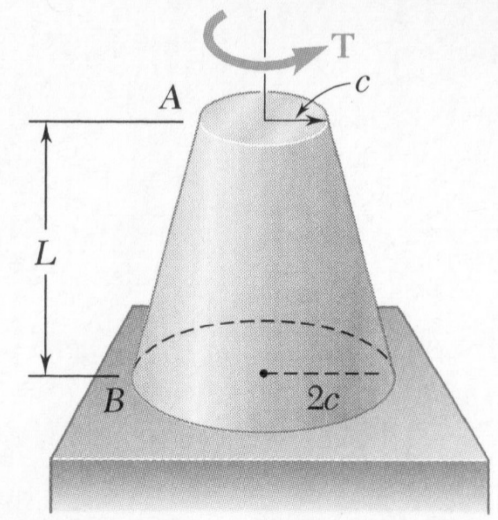
$$\frac{T_s \cancel{L_s}}{\cancel{G} J_s} = \frac{T_h \cancel{L_h}}{\cancel{G} J_h} \Rightarrow \frac{T_s}{T_h} = \frac{J_s}{J_h} = \frac{r^4}{r_o^4 - r_i^4} = \frac{r_o^2 - r_i^2}{r_o^2 + r_i^2}$$

$$= \frac{1-n^2}{1+n^2}$$

Example 3.11

A torque **T** is applied as shown to a solid tapered shaft AB. Show by integration that the angle of twist at A is

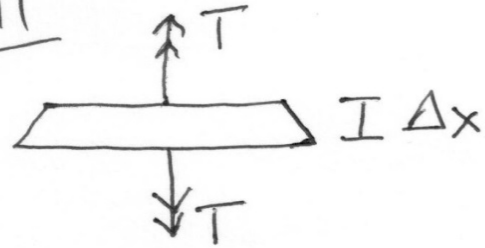
$$\phi = \frac{7TL}{12\pi G c^4}$$



Example 3.11

Ex 3-11

(6)



Equilibrium of element shown.

$$\Delta\theta = \frac{T\,\Delta x}{G\,J(x)} \Rightarrow \int_0^{\theta} d\theta = \int_0^{L} \frac{T\,dx}{G\,J(x)}$$

where $J(x) = \frac{\pi}{2} r^4 = \frac{\pi}{2}\left(2c - \frac{c}{L}x\right)^4$

$$\Rightarrow \theta = \frac{T}{G}\int_0^L \frac{2}{\pi c^4}\,\frac{1}{\left(2-\frac{x}{L}\right)^4}\,dx = \frac{2T}{G\pi c^4}\int_2^1 \frac{1}{y^4}\left(-L\,dy\right)$$

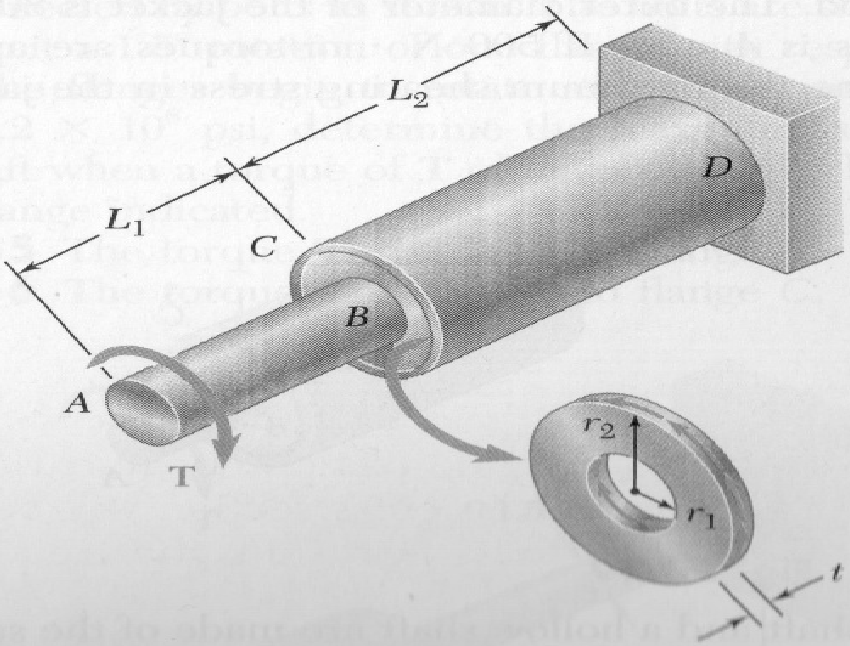
$$= \frac{2T}{G\pi c^4}\left[-\frac{1}{y^3}\,\frac{1}{3}\right]_2^1 \cdot (-L) = \frac{2TL}{3G\pi c^4}\left(1-\frac{1}{8}\right)$$

$$= \frac{7}{12}\,\frac{TL}{G\pi c^4}$$

Example 3.12

An annular plate of thickness t and modulus G is used to connect shaft AB of radius r_1 to tube CD of radius r_2. Knowing that a torque $\mathbf{T}$ is applied to end A of shaft AB and that end D of tube CD is fixed, (a) determine the magnitude and location of the maximum shearing stress in the annular plate, (b) show that the angle through which end B of the shaft rotates with respect to end C of the tube is

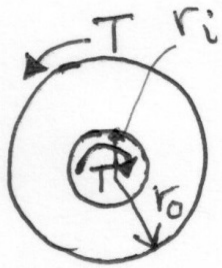
$$\phi_{BC} = \frac{T}{4\pi G t}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$



Example 3.12

Ex 3.12

Consider annular plate as rigid.



Torque balance on annular plate is shown. These torques are resultants of shear stress distribution on inner & outer radii.

$$T = \int (\tau_i \, dA) r_i = \int (\tau_o \, dA) r_o = \tau_i (2\pi r_i^2 t) = \tau_o (2\pi r_o^2 t)$$

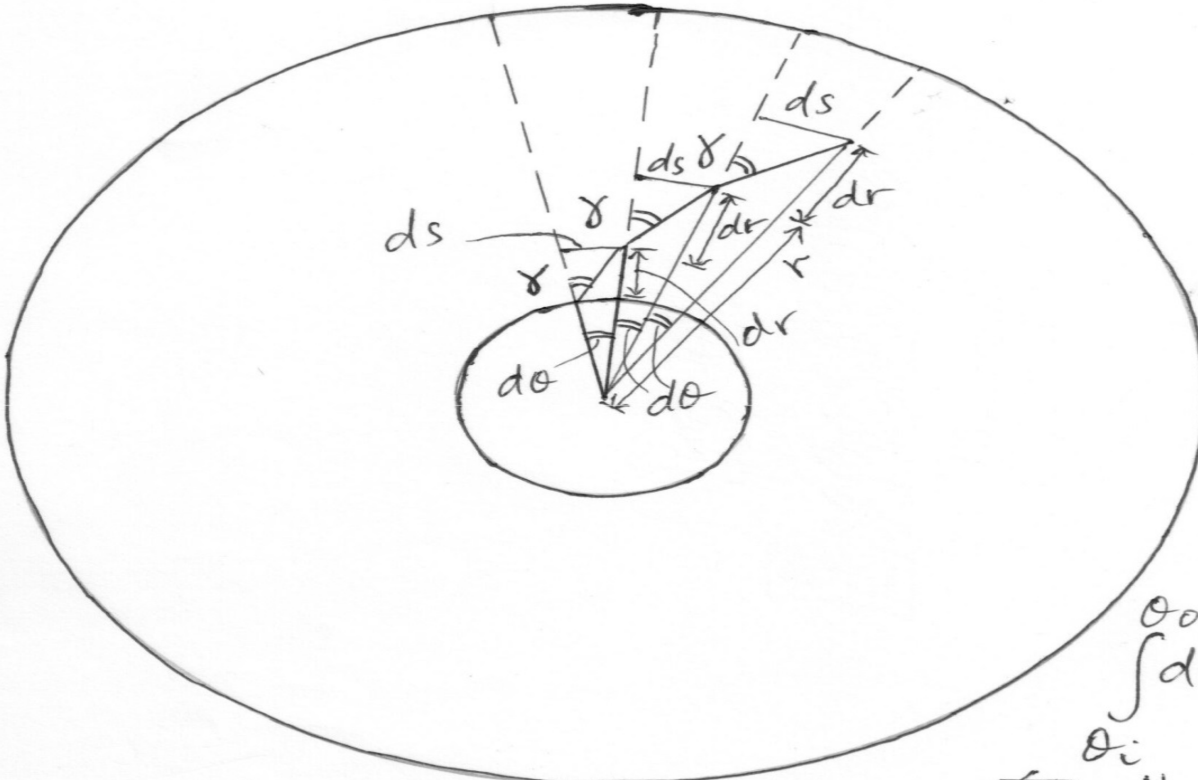
$$\Rightarrow \tau_{max} = \tau_i = T / (2\pi r_1^2 t) \qquad \begin{pmatrix} r_i = r_1 \\ r_o = r_2 \end{pmatrix}$$

Thus for any r,

$$T = \int (\tau \, dA) r = \tau (2\pi r^2 t)$$

(Contd)

Example 3.12



$$\gamma = \text{shear strain}$$

$$= \frac{\tau}{G} = \frac{T}{2\pi G t}\frac{1}{r^2}$$

$$ds = \gamma\, dr$$

$$= r\, d\theta$$

$$\Rightarrow d\theta = \frac{\gamma}{r}\, dr$$

$$= \frac{T}{2\pi G t}\frac{1}{r^3}\, dr$$

$$\int_{\theta_i}^{\theta_o} d\theta = \int_{r_i}^{r_o} \frac{T}{2\pi G t}\frac{1}{r^3}\, dr$$

$$\theta_o - \theta_i = \frac{T}{2\pi G t}\left[-\frac{1}{2}\frac{1}{r^2}\right]_{r_i}^{r_o}$$

$$\theta_i - \theta_o = \theta_{BC} = \frac{T}{4\pi G t}\left(\frac{1}{r_i^2} - \frac{1}{r_o^2}\right),$$ Note $r_i = r_1$, $r_o = r_2$