CHAPTER

4

MECHANICS OF MATERIALS

Pure Bending

Pure Bending

Pure Bending

Other Loading Types

Symmetric Member in Pure Bending

Bending Deformations

Strain Due to Bending

Stress Due to Bending

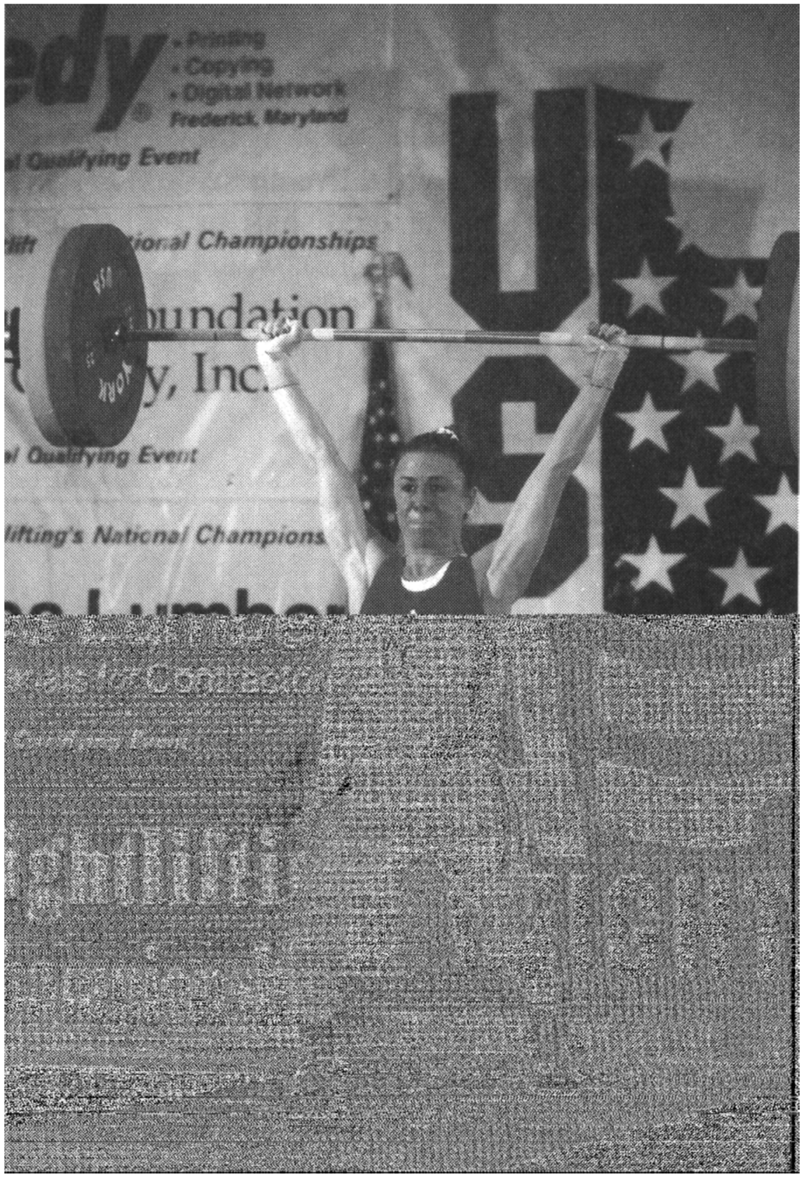
Beam Section Properties

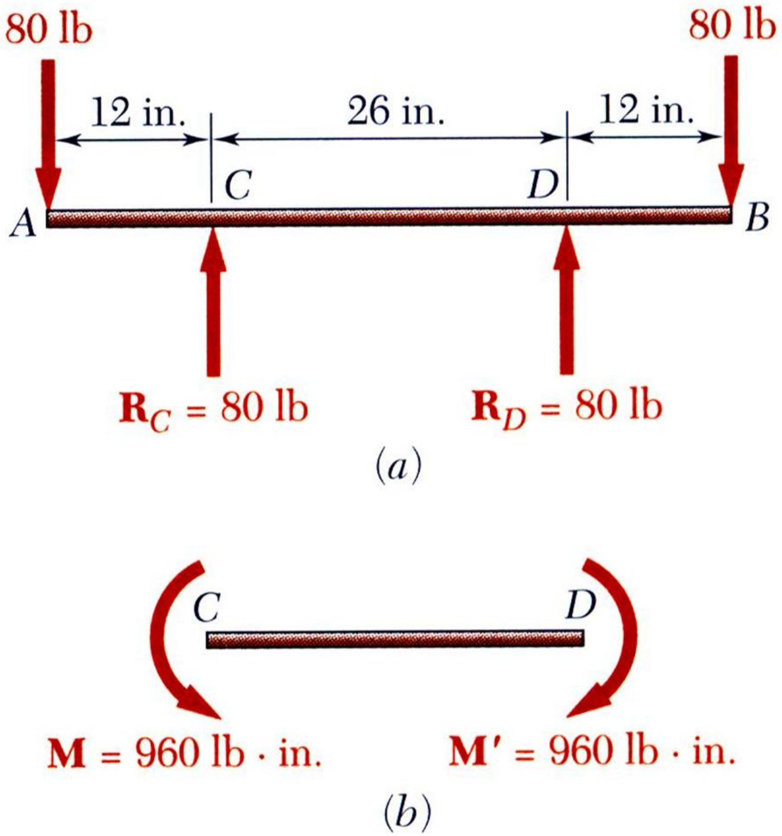
Bending of Members Made of Several Materials

Reinforced Concrete Beams

Unsymmetric Bending

Pure Bending

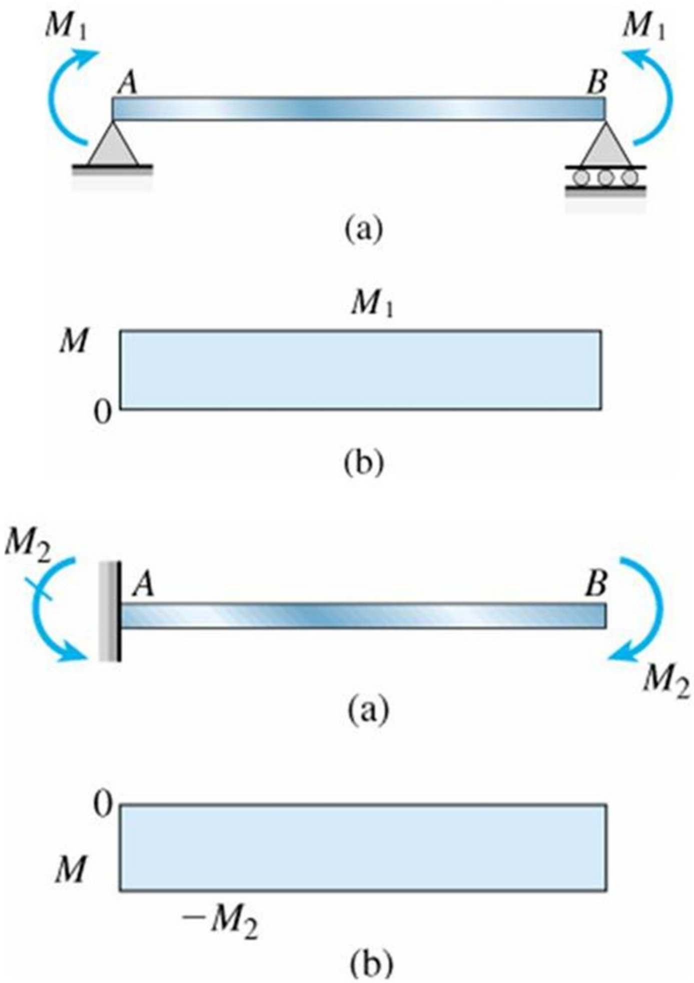




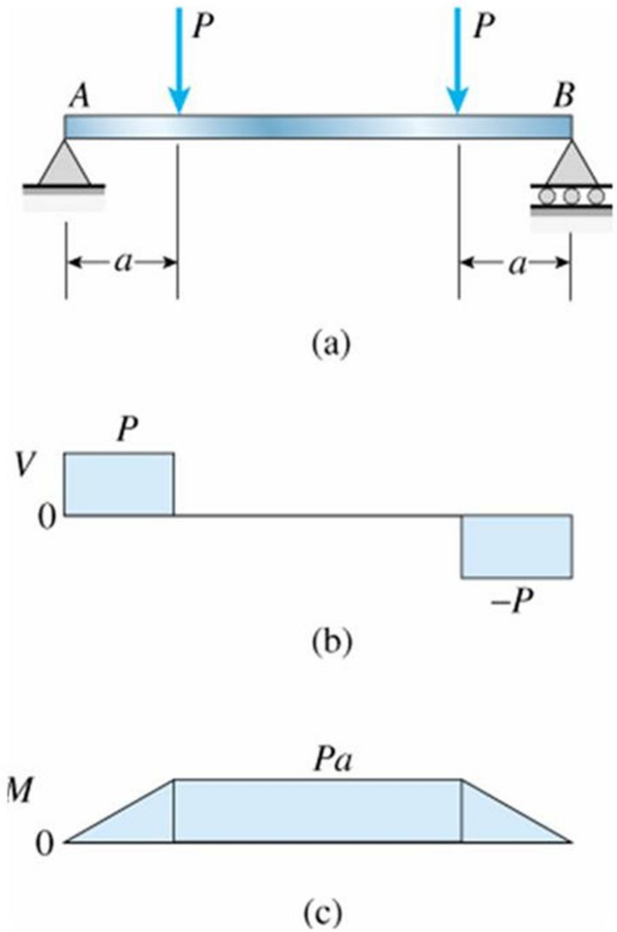
Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Pure/Non-Pure Bending

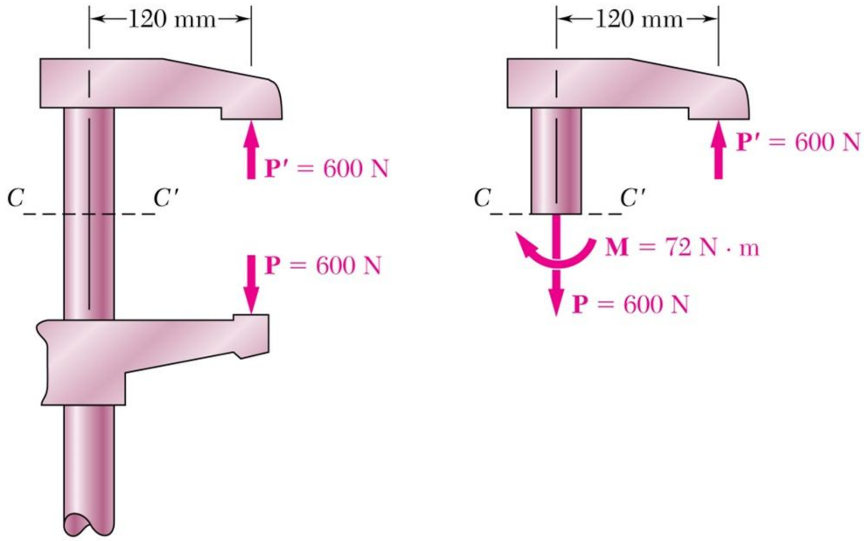
Pure Bending

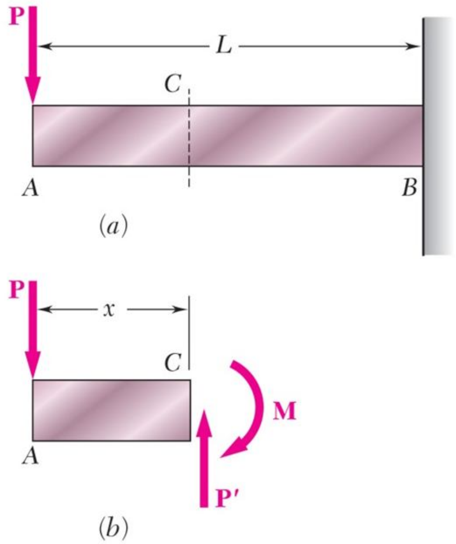


Non-Uniform Bending



Other Loading Types



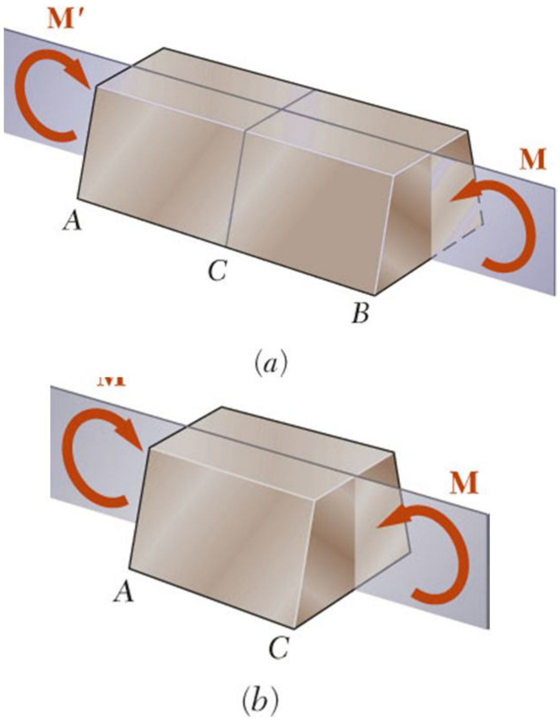


Eccentric Loading: Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple

Transverse Loading: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple

Principle of Superposition: The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.

Symmetric Member in Pure Bending



Internal forces in a cross section are equivalent to a couple moment, i.e., the section *bending moment*.

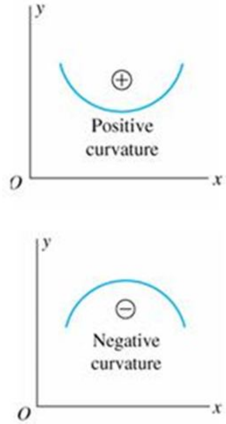
Couple moment is same about any axis perpendicular to the plane of the couple.

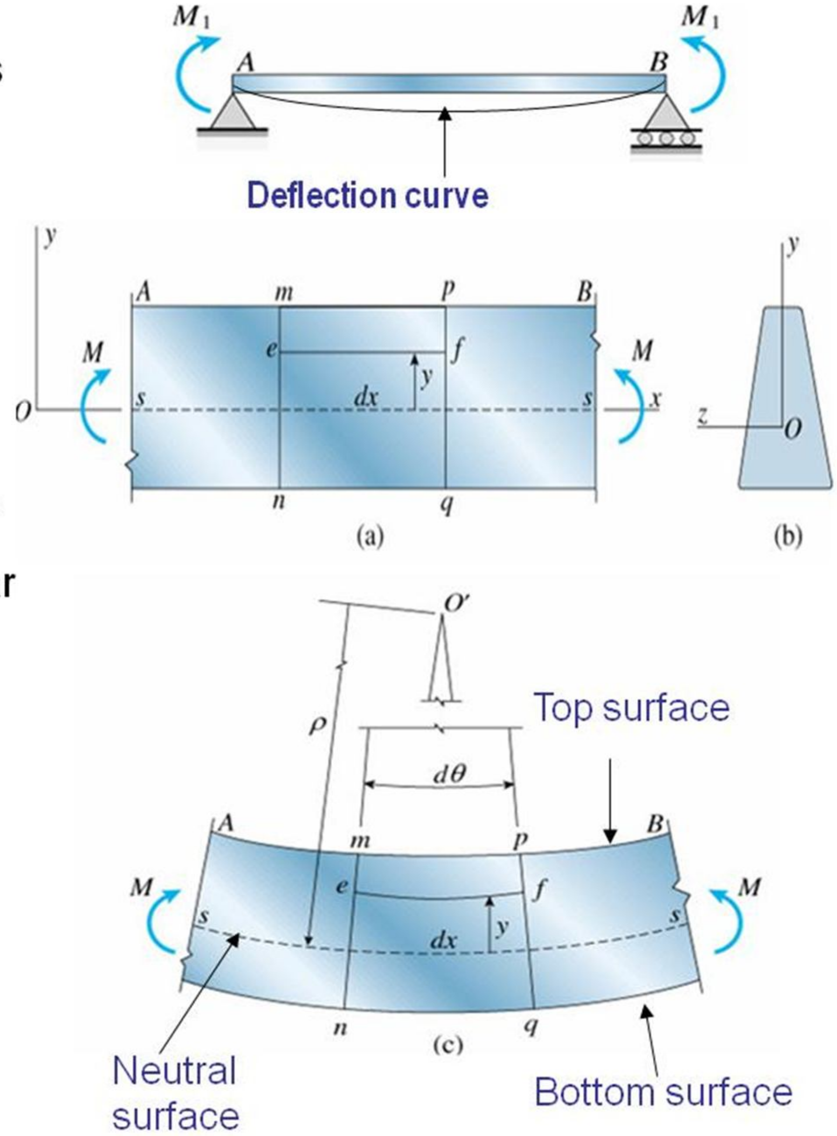
Due to symmetry of loading and symmetry of cross-section, beam bends uniformly into arc of circle and plane sections remain plane.

Terminologies/Assumptions

1. Longitudinal Plane of Symmetry (LPS) – Y axis
2. Top surface under compression
3. Bottom surface under tension
4. Neutral surface (NS) preserves original length (no strain)
5. Beam rotates about the neutral axis (z axis)
6. Intersection of NS and LPS – Deflection Curve
7. Plane sections remain plane and perpendicular to the deflection curve after deformation
8. Radius of curvature (ρ) or curvature (κ) of the deflection curve

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$





Physical meaning of assumptions.

- Assumption that plane sections remain plane and perpendicular to deflection curve after deformation implies shear strain $\gamma_{xy} = 0$. Thus Hooke's law gives shear stress $\tau_{xy} = 0$. **This is exactly true in pure bending (since no Shear force).**
- If plane sections remain plane but not perpendicular to deflection curve after deformation, γ_{xy} (hence τ_{xy}) is nonzero but constant through depth (y-direction).
- If plane sections don't remain plane or perpendicular to deflection curve after deformation, γ_{xy} (hence τ_{xy}) is nonzero and varies through depth (y-direction). This is what we will obtain later when we find shear stresses.

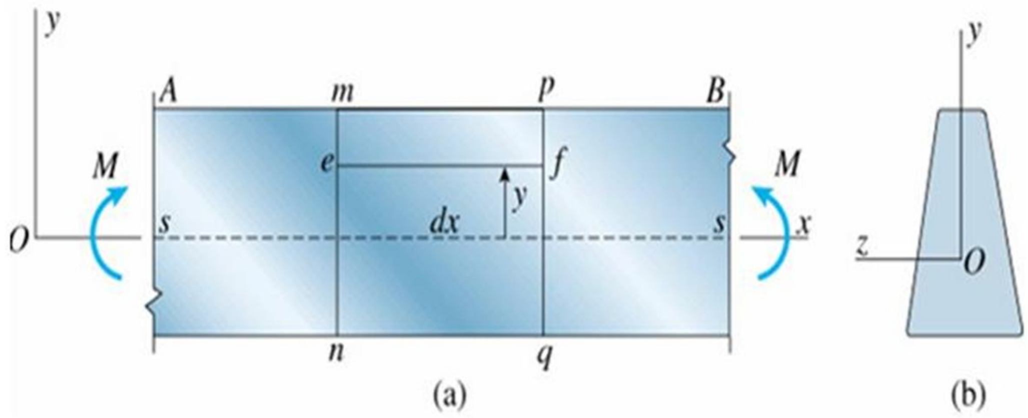
Longitudinal Strain

Initial length of the line ef

$= dx = \rho \, d\theta$

Deformed length of the line ef

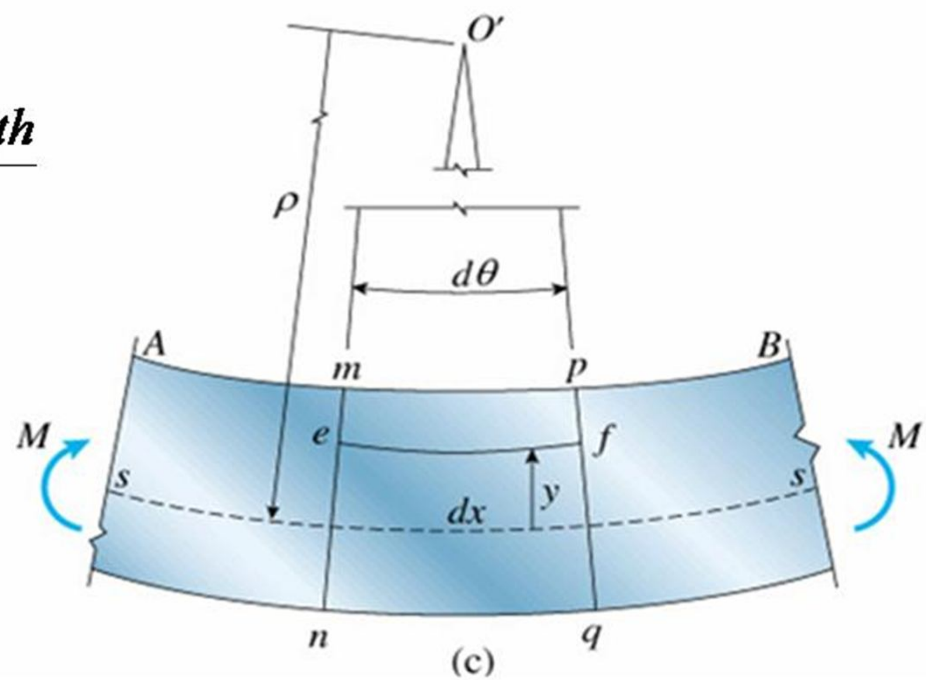
$= (\rho - y) \, d\theta$



Longitudinal strain =

$$\frac{\textit{Deformed length} - \textit{undeformed length}}{\textit{Undeformed length}}$$

$$\varepsilon_x = -\frac{y}{\rho} = -\kappa y$$

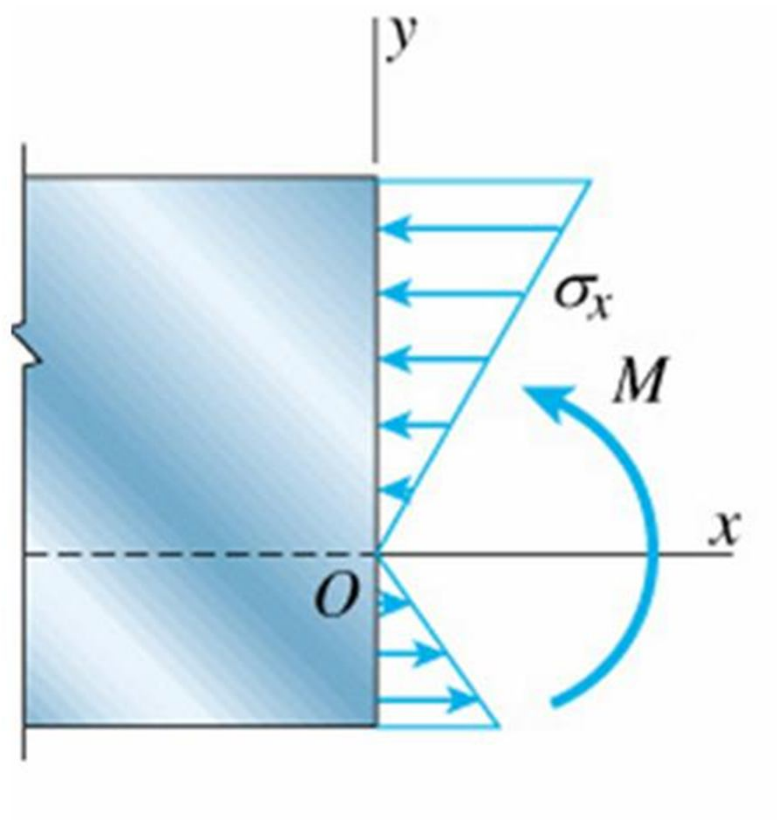


Longitudinal strain varies linearly with the distance from the neutral axis

Normal Stress

Normal Stress-Strain relationship (Hooke's law in tension/compression)

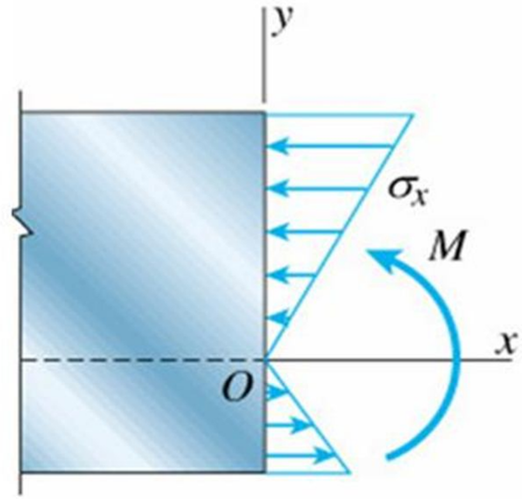
$$\sigma_x = E\varepsilon_x = -E\frac{y}{\rho} = -E\kappa y$$



Resultant of Normal Stress Distribution

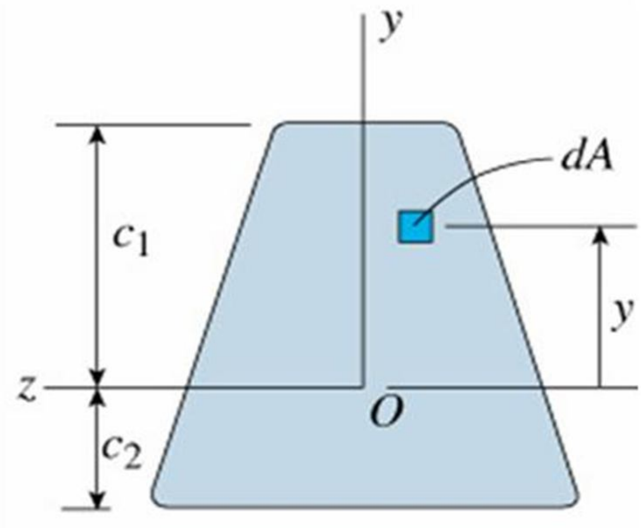
(1) The resultant force in the x-direction is zero

$$\int_A \sigma_x dA = 0$$



(2) The resultant moment is equal to the bending moment M induced at the cross-section

$$-\int_A \sigma_x y dA = M$$

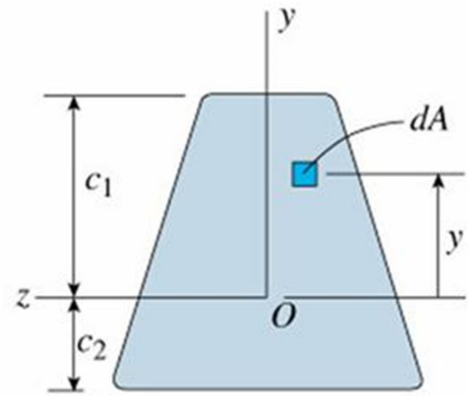


Resultant Force

(1) The resultant force in the x-direction is zero – **establishes location for Neutral axis**

$$\int_A \sigma_x dA = 0 = \int_A -E\kappa y dA$$

Since $\sigma_x = -E\kappa y$



$$\Rightarrow \int_A y dA = 0$$

This equation states that the first moment of the area of the cross section evaluated with respect to z axis is zero, i.e z axis must pass through the centroidal axis. Since z axis is also the neutral axis we arrived at the conclusion that the neutral axis passes through the centroid of the cross-section

Resultant Moment

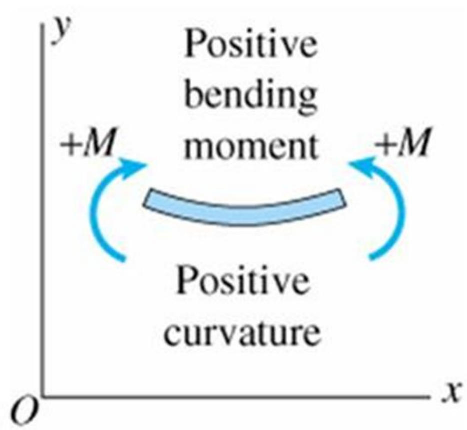
(2) The resultant moment is equal to the bending moment M induced at the cross-section- **establishes moment curvature relationship**

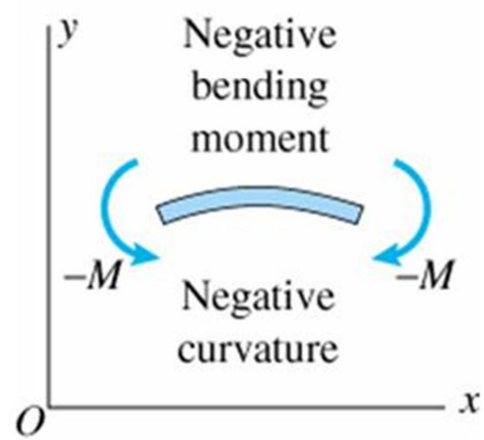
$$-\int_A \sigma_x y\,dA = M$$

$$\Rightarrow \int_A E\kappa y^2 dA = E\kappa \int_A y^2 dA = M \qquad Since\ \ \sigma_x = -E\kappa y$$

But $\int y^2 dA = I =$ Moment of Inertia (second moment) of the cross-sectional area with respect to the *z* axis. [Unit m^4 or in^4]

$$\boxed{\kappa = \frac{1}{\rho} = \frac{M}{EI}}$$

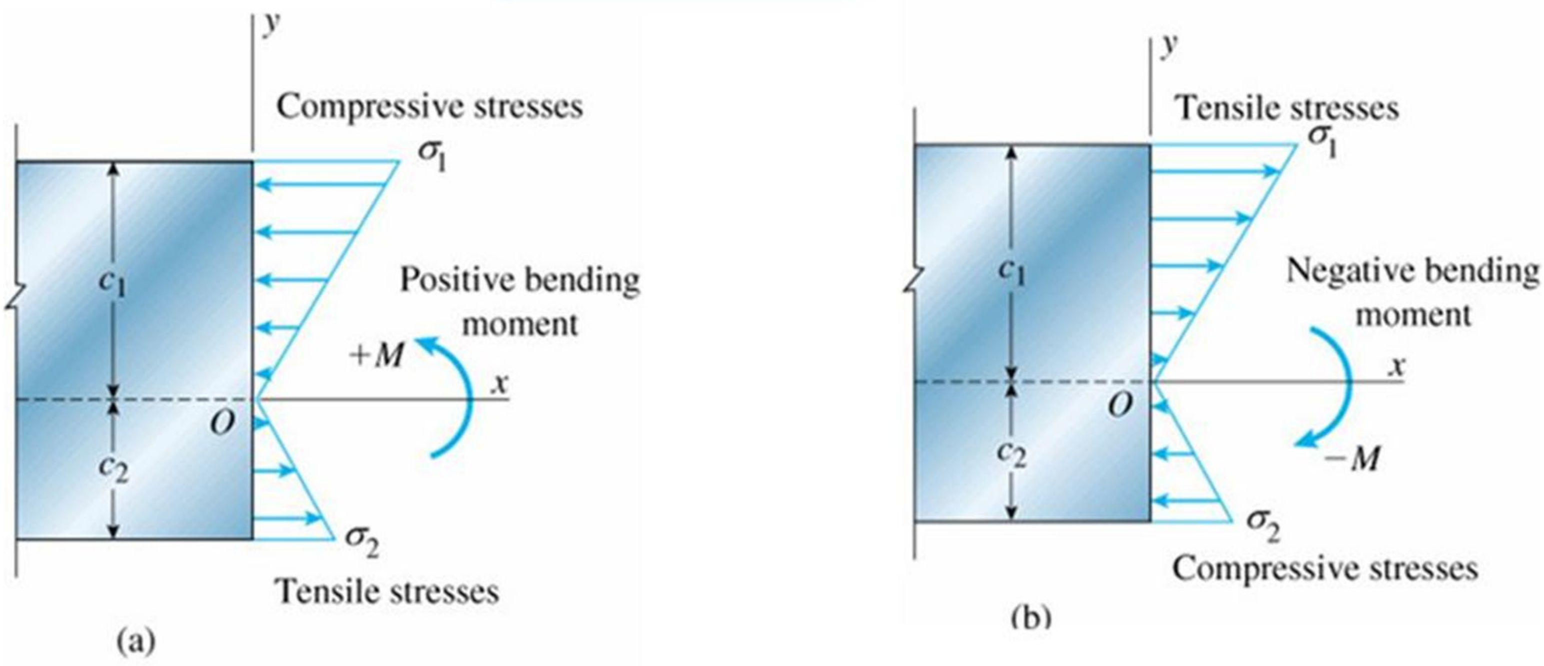




Relationship between Bending Stress and Moment

$$\sigma_x = -E\kappa y$$ and $$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

Hence $$\sigma_x = -\frac{My}{I}$$



Area Moments of Inertia

$$I_{zz} = \int_A y^2 \mathrm{d}A; \quad I_{yy} = \int_A z^2 \mathrm{d}A; \quad I_{zy} = I_{yz} = \int_A yz\,\mathrm{d}A;$$

Here y, z, are centroidal axes, so $\int y\,\mathrm{d}A = \int z\,\mathrm{d}A = 0$

Translation of axes:

$$I_{z'z'} = \int y'^2 \mathrm{d}A = \int (y-d_y)^2 \mathrm{d}A, \text{ expand, use } \int y\,\mathrm{d}A = 0$$

$$I_{z'z'} = I_{zz} + Ad_y^2. \quad \text{Similarly, } I_{y'y'} = I_{yy} + Ad_z^2$$

$$I_{y'z'} = \int (y-d_y)(z-d_z)\,\mathrm{d}A = I_{yz} + d_y d_z A$$

Rotation of axes:

$$z' = z\cos\theta + y\sin\theta; \quad y' = -z\sin\theta + y\cos\theta;$$

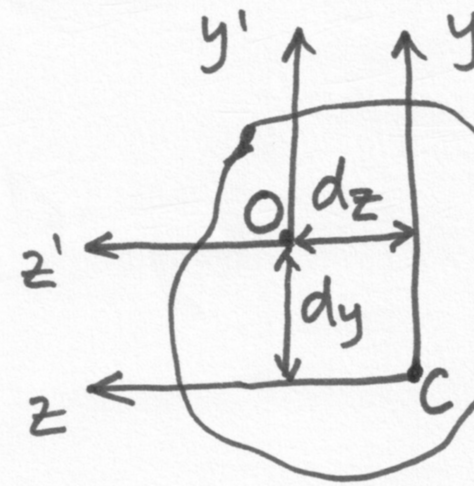
$$I_{y'z'} = \int y'z'\,\mathrm{d}A = \left(I_{zz} - I_{yy}\right)\frac{\sin 2\theta}{2} + I_{yz}\cos 2\theta$$

$$I_{z'z'} = I_{yy}\sin^2\theta + I_{zz}\cos^2\theta - I_{yz}\frac{\sin 2\theta}{2}$$

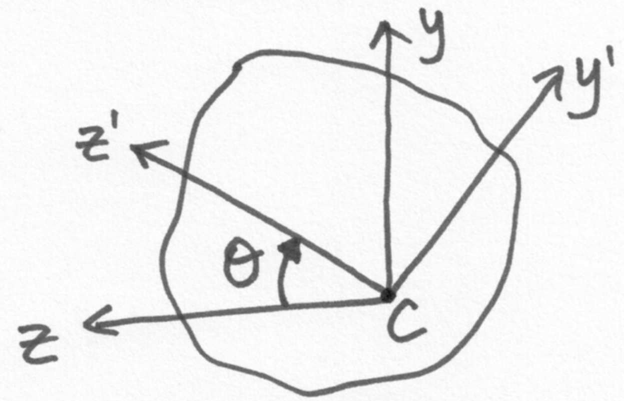
$$I_{y'y'} = I_{yy}\cos^2\theta + I_{zz}\sin^2\theta + I_{yz}\frac{\sin 2\theta}{2}$$

For $I_{y'z'} = 0$, i.e., principal axes, $\tan 2\theta = \dfrac{2I_{yz}}{I_{yy} - I_{zz}}$

C=centroid



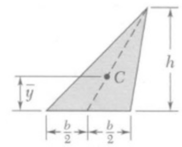
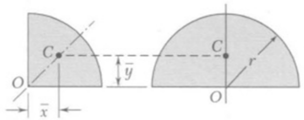
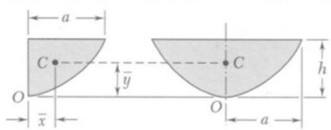
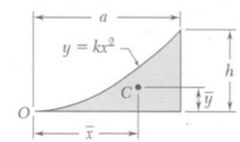
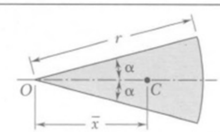
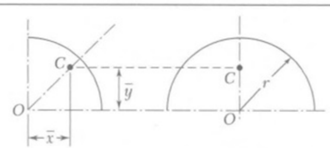
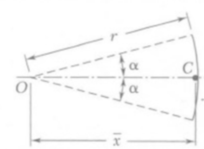
Translation.



Rotation.

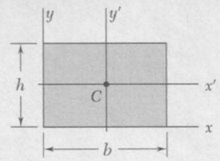
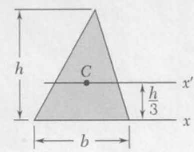
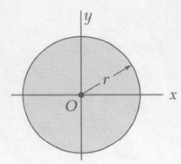
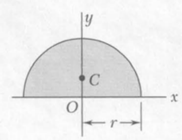
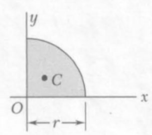
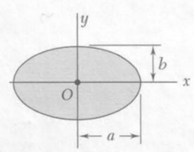
Area centroids and Area Moments of Inertia

Centroids of Common Shapes of Areas and Lines

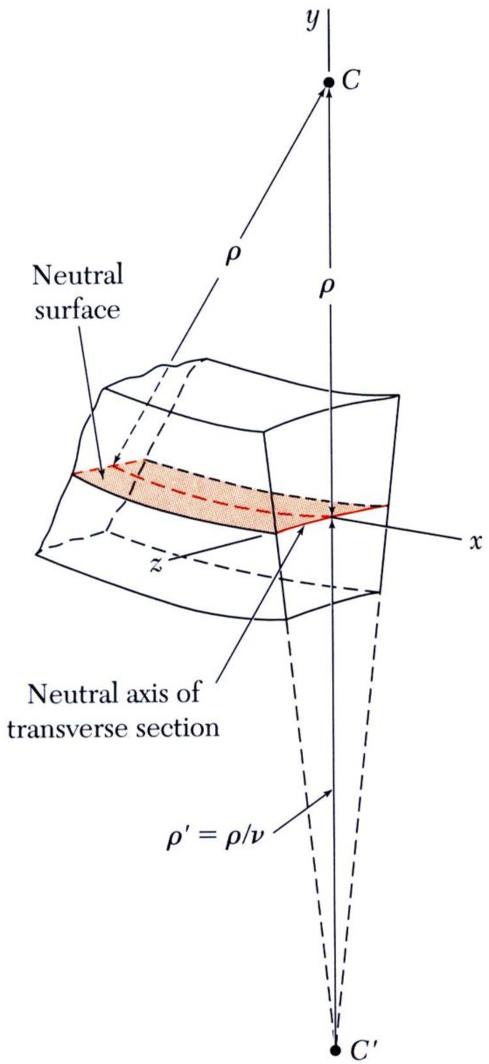
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
Circular sector		$\frac{2r \sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin\alpha}{\alpha}$	0	$2\alpha r$

Area centroids and Area Moments of Inertia

Moments of Inertia of Common Geometric Shapes

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Deformations in a Transverse Cross Section



- Recall: deformation due to bending moment *M* is quantified by curvature of neutral surface

$$\frac{1}{\rho} = -\frac{\varepsilon_x}{y} = -\frac{\sigma_x}{Ey} = \frac{M}{EI}$$

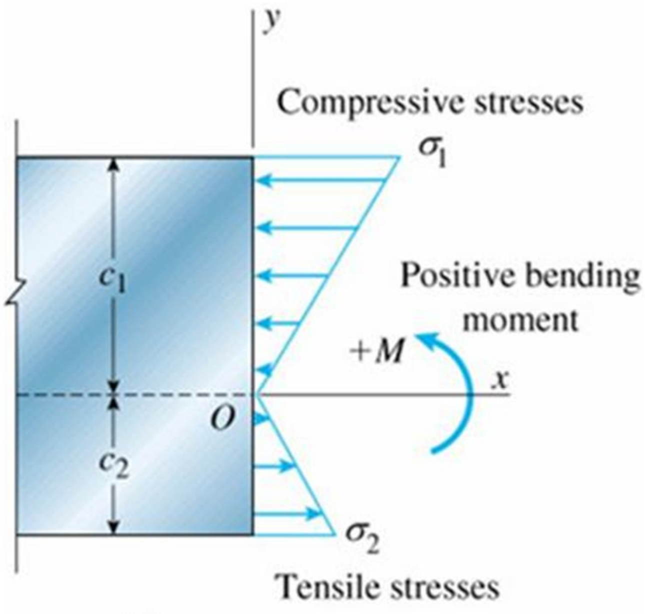
- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

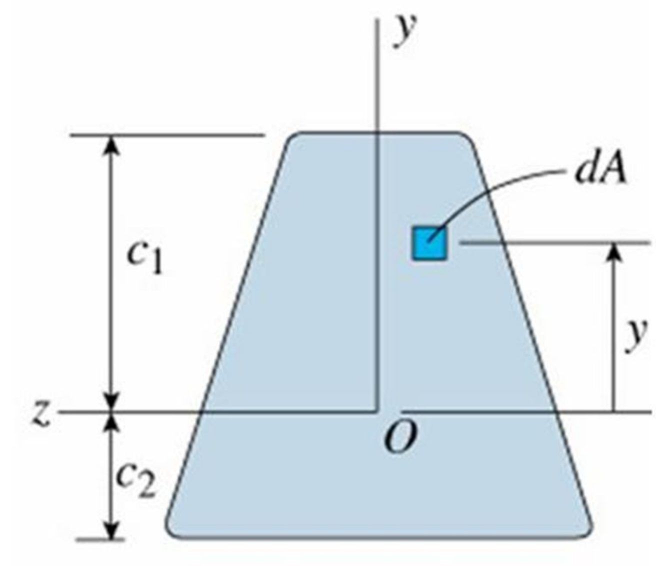
- So expansion occurs above neutral surface and contraction below it, causing in-plane curvature,

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvature}$$

Section Modulii



(a)



$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1}$$

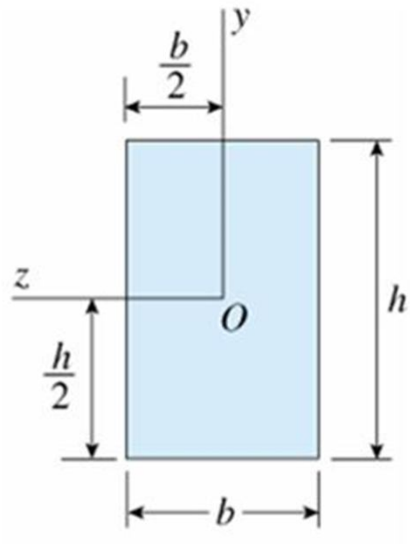
$$\sigma_2 = +\frac{Mc_2}{I} = +\frac{M}{S_2}$$

$$\boxed{S_1 = \frac{I}{c_1}; \quad S_2 = \frac{I}{c_2}}$$

S_1, S_2 section modulii

Doubly Symmetric Shape

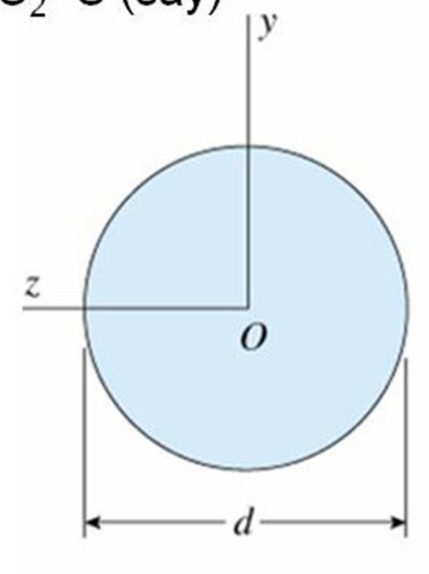
$c_1 = c_2 = c$ (say) therefore $S_1 = S_2 = S$ (say)



$$c = \frac{h}{2}$$

$$I = \frac{bh^3}{12}$$

$$S = \frac{bh^2}{6}$$



(b)

$$c = \frac{d}{2}$$

$$I = \frac{\pi d^4}{64}$$

$$S = \frac{\pi d^3}{32}$$

$$\sigma_1 = \sigma_2 = \sigma = \frac{M}{S}$$

Larger the section modulus lower the max normal stress

Review Calculations of Centroid & I for various cross-sections
Appendix A, Beer and Johnston

For two rectangular beams with same area, the one with larger depth is better

$$S = \frac{bh^2}{6} = \frac{Ah}{6}$$

Square Vs. Circular Cross-Section

Let us assume you have a beam of square cross-section (side h) and a beam of circular cross-section (diameter d) of the same area. Which one is efficient in resisting bending?

$$h^2 = \frac{\pi d^2}{4} \quad \Rightarrow \quad h = \frac{d}{2}\sqrt{\pi}$$

$$S_{square} = \frac{h^3}{6} = \frac{\pi\sqrt{\pi}\,d^3}{48} = 0.116d^3$$

$$S_{circle} = \frac{\pi d^3}{32} = 0.0982d^3$$

Square section is more efficient

Design of Beams for Bending

- The largest normal stress is found at the surface where the maximum bending moment occurs.

$$\sigma_m = \frac{|M|_{\max} c}{I} = \frac{|M|_{\max}}{S}$$

- Safe design requires that maximum normal stress be less than allowable stress for material used. This leads to determination of minimum acceptable section modulus.

$$\sigma_m \leq \sigma_{all}$$

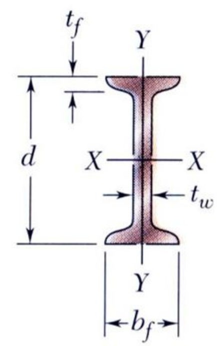
$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}}$$

- Among beam section choices which have an acceptable section modulus, the one with smallest weight per unit length or cross sectional area will be least expensive and hence best choice. See Appendix C of BJ book.

Design of Beams in Bending

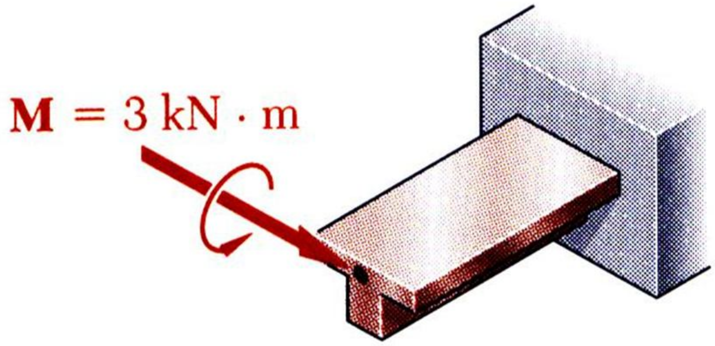
Appendix C. Properties of Rolled-Steel Shapes
(SI Units)

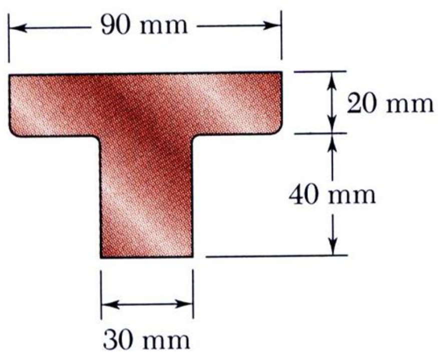
S Shapes
(American Standard Shapes)



			Flange			Axis X-X			Axis Y-Y		
Designation†	Area A, mm^2	Depth d, mm	Width b_f, mm	Thick-ness t_f, mm	Web Thick-ness t_w, mm	I_x 10^6 mm^4	S_x 10^3 mm^3	r_x mm	I_y 10^6 mm^4	S_y 10^3 mm^3	r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

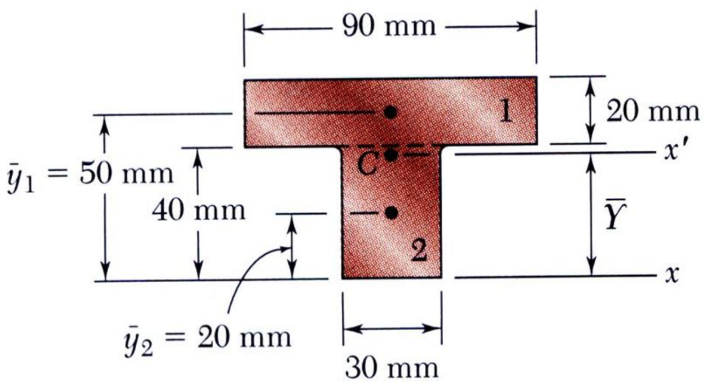
Example 5.1





Cast-iron machine part, acted upon by 3 kN-m couple. *E* = 165 Gpa, neglect effects of fillets, determine (a) max tensile and compressive stresses, (b) radius of curvature.

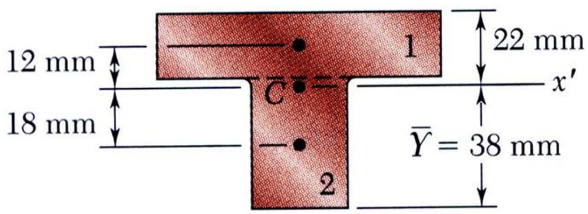
Example 5.1



SOLUTION:

Based on cross section geometry, calculate location of section centroid and moment of inertia.

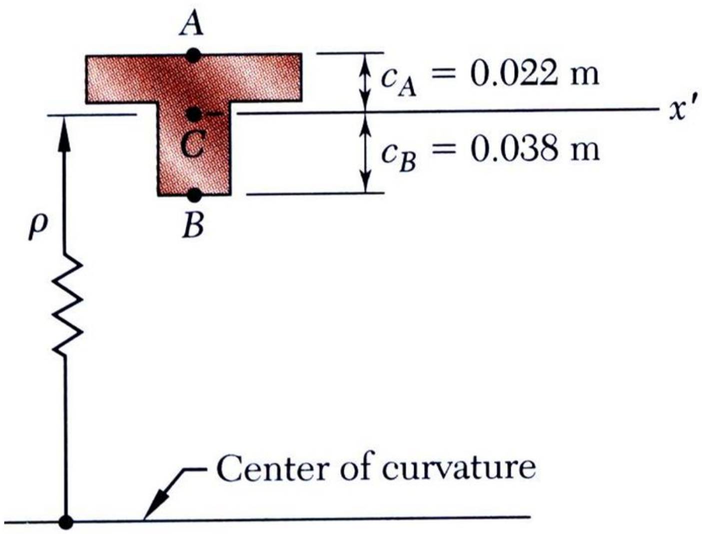
	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20\times 90 = 1800$	50	90×10^3
2	$40\times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114\times 10^3$



$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114\times 10^3}{3000} = 38\,\mathrm{mm}$$

$$I_{x'} = \sum\left(\bar{I} + Ad^2\right) = \sum\left(\tfrac{1}{12}bh^3 + Ad^2\right)$$
$$= \left(\tfrac{1}{12}90\times 20^3 + 1800\times 12^2\right) + \left(\tfrac{1}{12}30\times 40^3 + 1200\times 18^2\right)$$
$$I = 868\times 10^3\,\mathrm{mm} = 868\times 10^{-9}\,\mathrm{m}^4$$

Example 5.1



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{M\,c_A}{I} = \frac{3\,\mathrm{kN\cdot m}\times 0.022\,\mathrm{m}}{868\times 10^{-9}\,\mathrm{mm}^4} \qquad \boxed{\sigma_A = +76.0\,\mathrm{MPa}}$$

$$\sigma_B = -\frac{M\,c_B}{I} = -\frac{3\,\mathrm{kN\cdot m}\times 0.038\,\mathrm{m}}{868\times 10^{-9}\,\mathrm{mm}^4} \qquad \boxed{\sigma_B = -131.3\,\mathrm{MPa}}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

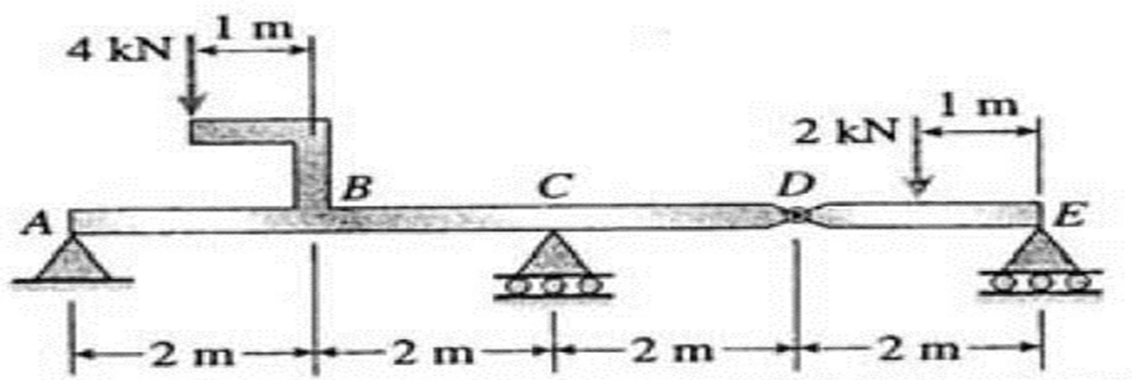
$$= \frac{3\,\mathrm{kN\cdot m}}{(165\,\mathrm{GPa})(868\times 10^{-9}\,\mathrm{m}^4)}$$

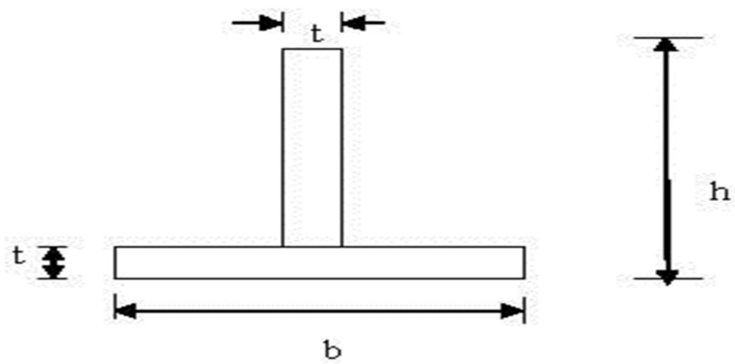
$$\boxed{\begin{array}{l}\dfrac{1}{\rho} = 20.95\times 10^{-3}\,\mathrm{m}^{-1} \\ \rho = 47.7\,\mathrm{m}\end{array}}$$

Example 5.2

The compound beam ABCDE shown in the figure consists of two beams (AD and DE) joined by a hinged connection at D. The hinge can transmit a shear force but not a bending moment. The loads on the beam consist of a 4-kN force at the end of a bracket attached at point B and 2kN force at the midpoint of beam DE.

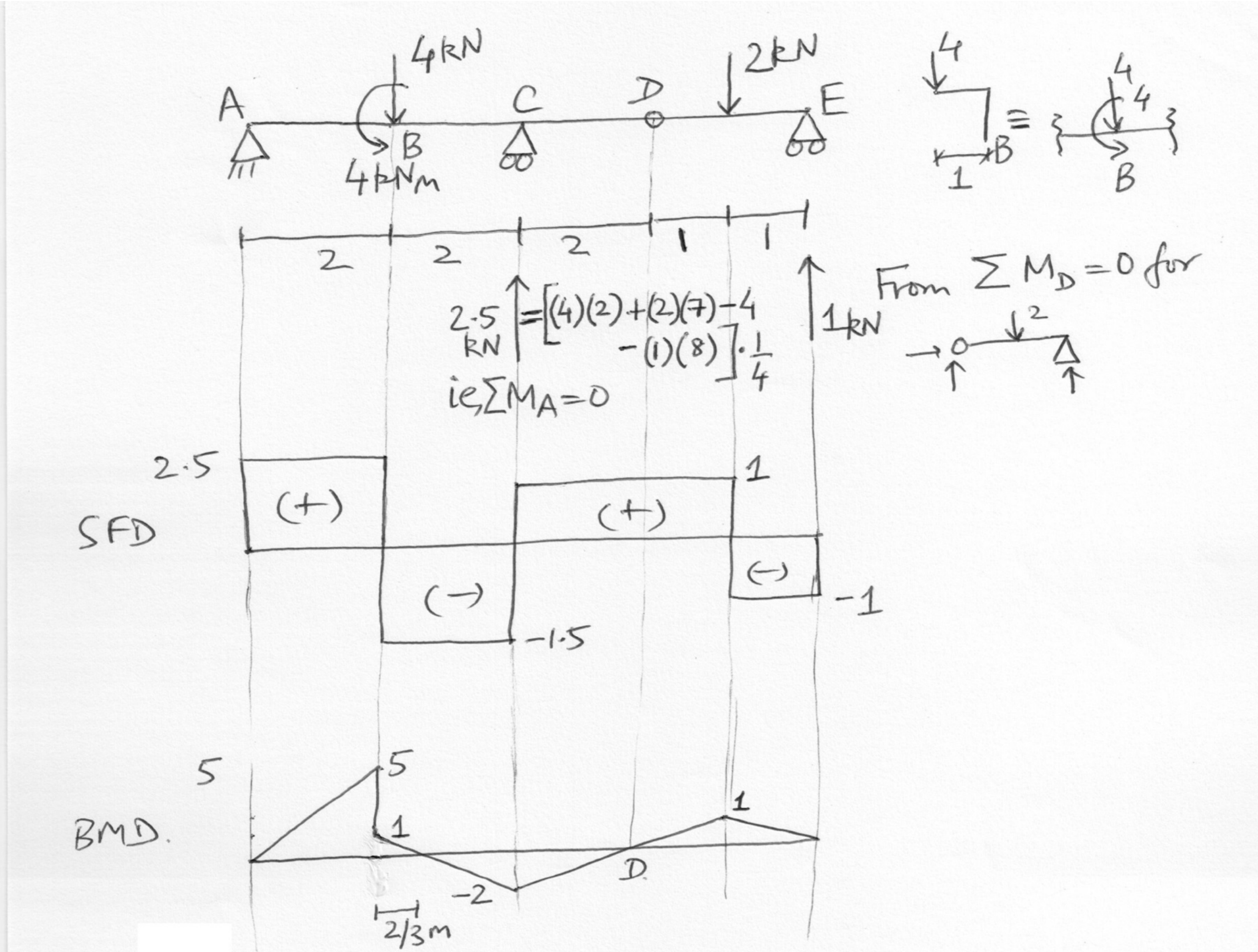
1. Draw the shear-force and bending-moment diagram for the compound beam
2. Calculate the maximum compressive and maximum tensile bending stress in the beam for the cross-section shown.



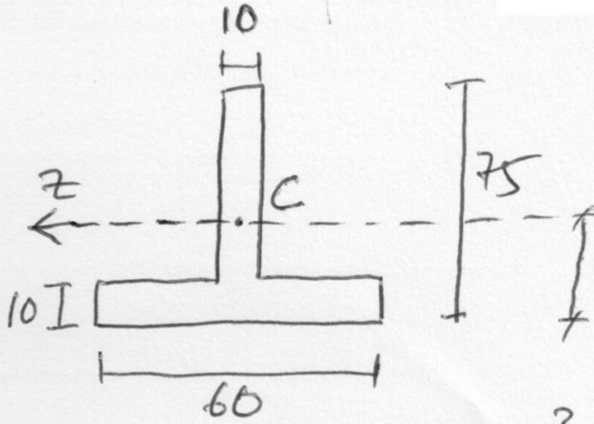


b= 60 mm, h= 75mm, t=10 mm

Example 5.2


4kN
2kN
A
B
C
D
E
4kNm
2
2
2
1
1
2.5 kN $= [(4)(2)+(2)(7)-4 - (1)(8)]\cdot\frac{1}{4}$
ie, $\sum M_A = 0$
1kN
From $\sum M_D = 0$ for
4
1
B
≡
4
4
B
2
2.5
1
SFD
(+)
(+)
(−)
(−)
−1.5
−1
5
5
1
1
BMD.
−2
D
2/3m

Example 5.2



$$\bar{y} = \frac{(10)(60)(5) + (10)(65)(42.5)}{(10)(60) + (10)(65)} = 24.5\,\text{mm}$$

$$I = I_{zz} = \frac{(60)(10)^3}{12} + (60)(10)(24.5-5)^2 + \frac{(10)(65)^3}{12} + (10)(65)\left(\frac{65}{2} + 10 - 24.5\right)^2$$

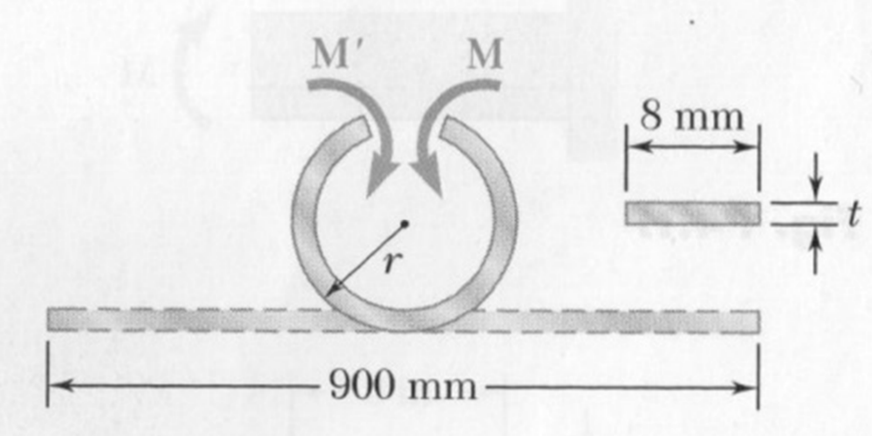
$$= 672\,604\ \text{mm}^4$$

max compr & tensile.

For $M = 5$ kN.m, $\sigma_{top} = -\dfrac{M(75-\bar{y})}{I} = -375.4\ \text{MPa}$, $\sigma_{bot} = -\dfrac{M}{I}(-24.5) = 182.1\ \text{MPa}$

For $M = -2$ kNm, $\sigma_t = 150.2$ MPa, $\sigma_b = -72.84$ MPa.

Example 5.3



A 900-mm strip of steel is bent into a full circle by two couples applied as shown. Determine (*a*) the maximum thickness *t* of the strip if the allowable stress of the steel is 420 MPa, (*b*) the corresponding moment *M* of the couples. Use *E* = 200 GPa.

(a) $\sigma = -\dfrac{Ey}{\rho}$, $\rho = r = \dfrac{900}{2\pi}$, $y = \dfrac{t}{2}$ for max Compressive

$-420 = -\dfrac{200\times10^3(t/2)}{(900/2\pi)} \Rightarrow t = 0.6016\,mm$

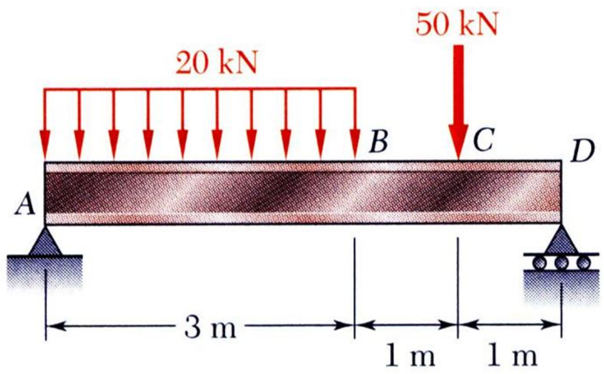
Max compressive

(b) $\sigma = -\dfrac{My}{I}$, $I = \dfrac{8(0.6016)^3}{12}$, $y = \dfrac{0.6016}{2}$,

$\sigma = -420$

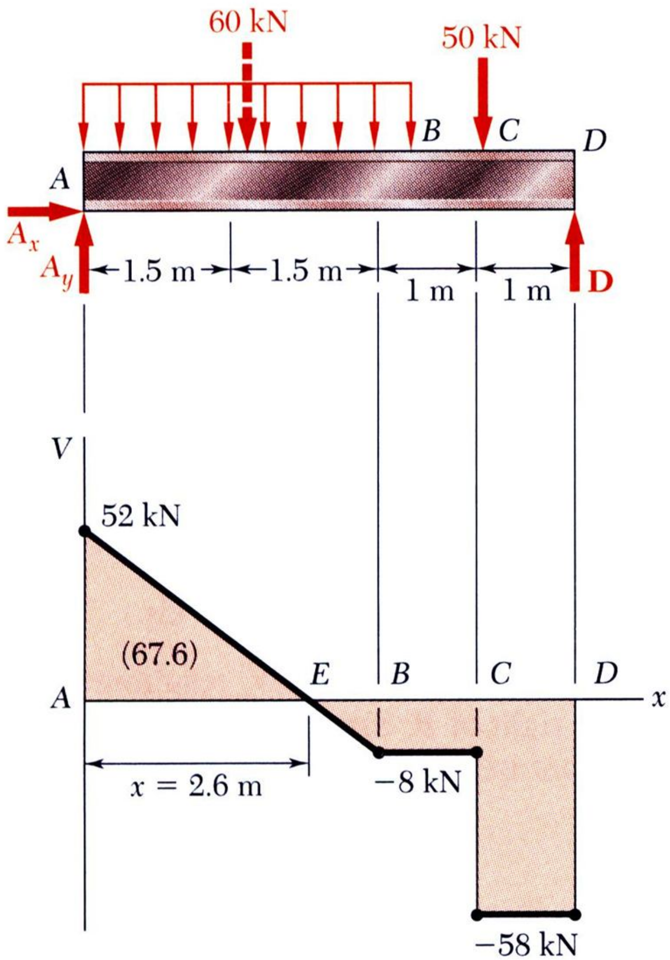
get $M = 202.7$ N.mm

Example 5.4



Simply supported steel beam is to carry loads as shown. Allowable normal stress for steel used is 160 Mpa. Select the wide-flange shape that should be used (section properties of wide flange sections are given).

Example 5.4



- Determine reactions at A and D.

$$\sum M_A = 0 = D(5\,\mathrm{m}) - (60\,\mathrm{kN})(1.5\,\mathrm{m}) - (50\,\mathrm{kN})(4\,\mathrm{m})$$

$$D = 58.0\,\mathrm{kN}$$

$$\sum F_y = 0 = A_y + 58.0\,\mathrm{kN} - 60\,\mathrm{kN} - 50\,\mathrm{kN}$$

$$A_y = 52.0\,\mathrm{kN}$$

- Develop shear force diagram and determine maximum bending moment.

$$V_A = A_y = 52.0\,\mathrm{kN}$$

$$V_B - V_A = -(\mathit{area\ under\ load\ curve}) = -60\,\mathrm{kN}$$

$$V_B = -8\,\mathrm{kN}$$

- Maximum bending moment occurs at $V = 0$ or $x = 2.6$ m.

$$|M|_{\max} = (\mathit{area\ under\ shear\ curve}, A\ to\ E)$$

$$= 67.6\,\mathrm{kN}$$

Example 5.4

- Determine minimum acceptable section modulus.

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}} = \frac{67.6\,\mathrm{kN \cdot m}}{160\,\mathrm{MPa}}$$

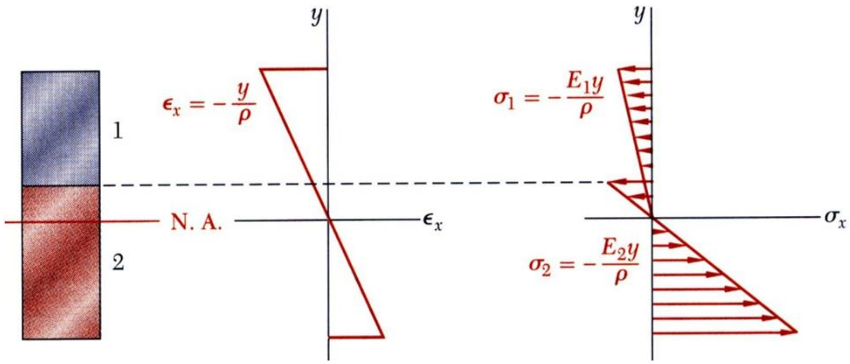
$$= 422.5 \times 10^{-6}\,\mathrm{m}^3 = 422.5 \times 10^3\,\mathrm{mm}^3$$

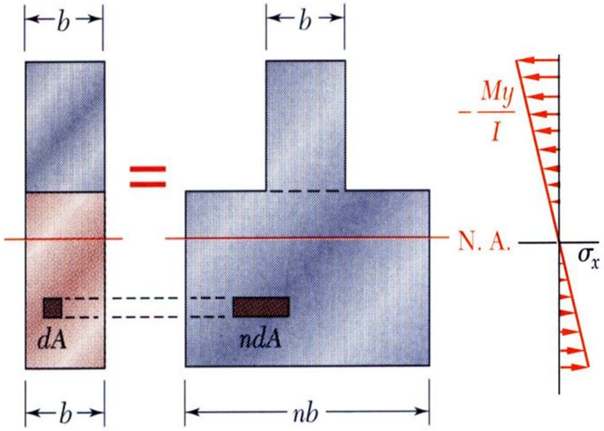
- Choose best standard section which meets this criteria.

Shape	$S \times 10^3$, A
W410×38.8	629, 4950
W360×32.9	475, 4190
W310×38.7	547, 4940
W250×44.8	531, 5700
W200×46.1	451, 5880

$\boxed{W360 \times 32.9}$

Bending of Members Made of Several Materials





$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2.
- Normal strain (still) varies linearly (cannot violate kinematics).

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \qquad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

Alternate derivation - neutral axis for composite beam

$$F = 0 = \int_{A_1} \sigma_{x1} dA + \int_{A_2} \sigma_{x2} dA = \int_{A_1} E_1 \varepsilon_{x1} dA + \int_{A_2} E_2 \varepsilon_{x2} dA$$

$$= -\int_{A_1} E_1 \frac{y}{\rho} dA - \int_{A_2} \frac{E_2}{E_1} E_1 \frac{y}{\rho} dA =$$

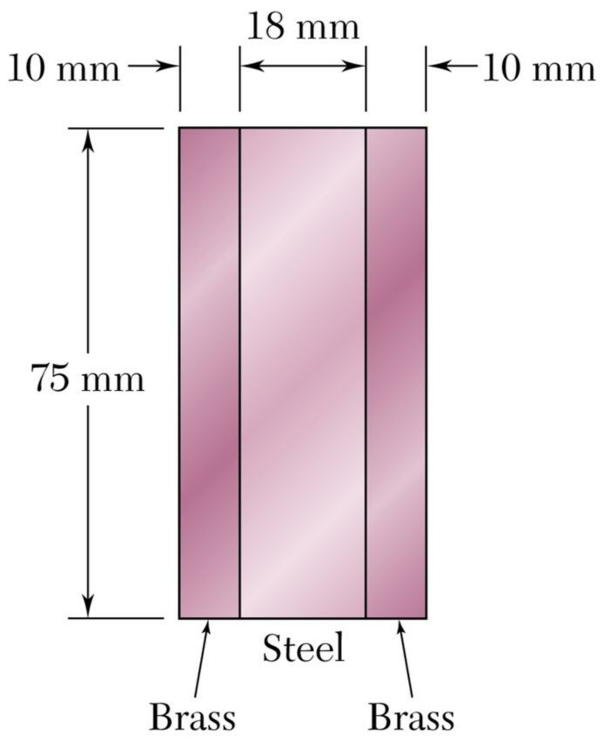
$$= -\int_{A_1 \cup \frac{E_2}{E_1} A_2} E_1 \frac{y}{\rho} dA = -\int_{A_1 \cup nA_2} E_1 \frac{y}{\rho} dA$$

$$\Rightarrow \int_{A_1 \cup nA_2} y \, dA = 0$$

So neutral axis is centroid of $A_1 \cup nA_2$

Note: widening ($n > 1$) or narrowing ($n < 1$) done parallel to neutral axis, so that *y* and hence ε_x remain unaltered.

Example 5.5



Bar made from bonded pieces of steel (E_s = 200 GPa) and brass (E_b = 100 GPa). Determine maximum stress in steel and in brass when a moment of 4.5 KNm is applied.

SOLUTION:

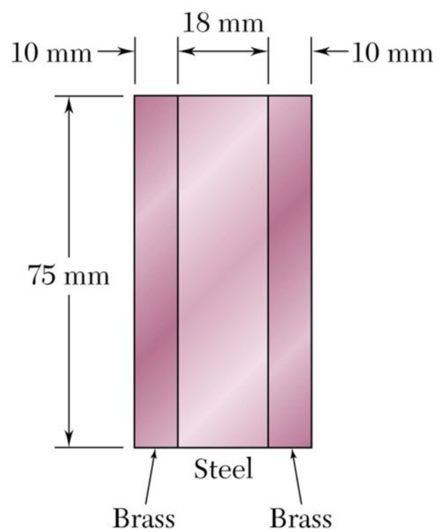
Transform to an equivalent cross section made entirely of (say) brass

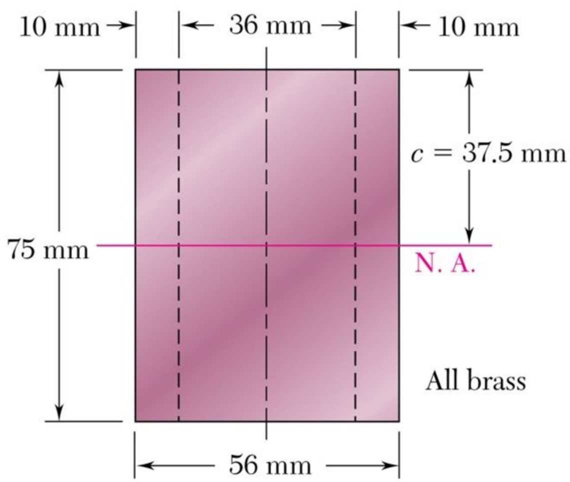
Evaluate cross sectional properties of transformed section

Calculate maximum stress in transformed section. This is the correct maximum stress for brass portion of the bar.

Determine the maximum stress in steel portion of by multiplying maximum stress for transformed section by the modular ratio. Don't forget this important step!!

Example 5.5





SOLUTION:

Transform to equivalent brass section.

$$n = \frac{E_s}{E_b} = \frac{200\text{GPa}}{100\text{GPa}} = 2.0$$

$$b_T = 10\text{ mm} + 2\times 18\text{ mm} + 10\text{ mm} = 56\text{ mm}$$

Evaluate transformed cross sectional properties

$$I = \tfrac{1}{12} b_T h^3 = \tfrac{1}{12}(56\text{ mm})(75\text{ mm})^3$$
$$= 1.96875\times 10^6\text{ mm}^4$$

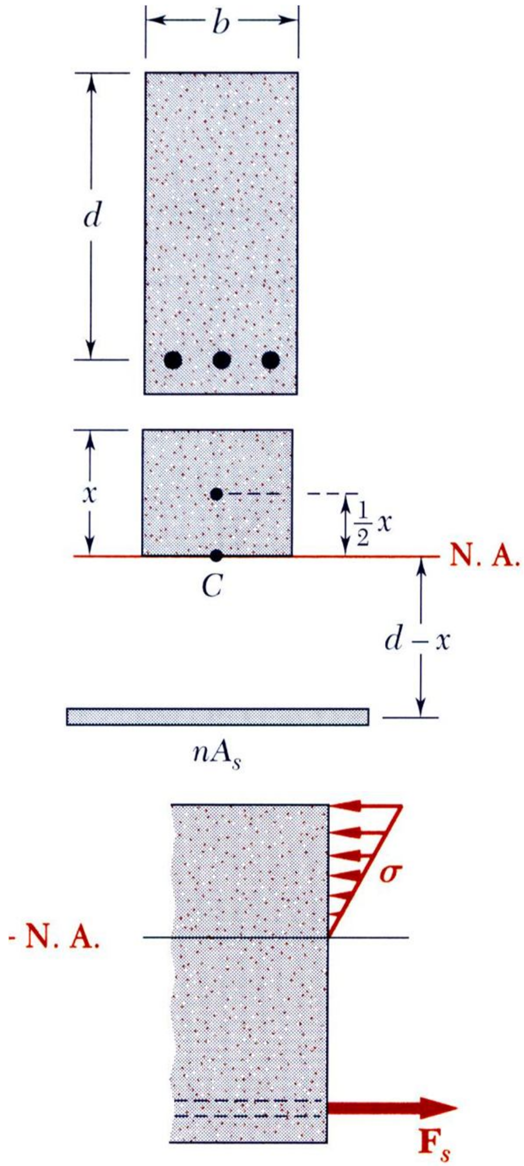
Calculate maximum stresses

$$\sigma_m = \frac{Mc}{I} = \frac{(4500\text{ Nm})(0.0375\text{ m})}{1.96875\times 10^{-6}\text{ m}^4} = 85.7\text{ MPa}$$

$$(\sigma_b)_{\max} = \sigma_m$$
$$(\sigma_s)_{\max} = n\sigma_m = 2\times 85.7\text{ MPa}$$

$$\boxed{\begin{aligned}(\sigma_b)_{\max} &= 85.7\text{ MPa}\\ (\sigma_s)_{\max} &= 171.4\text{ MPa}\end{aligned}}$$

Reinforced Concrete Beams



- Concrete beams subjected to bending moments are reinforced by steel rods.
- The steel rods assumed to carry the entire tensile load below neutral axis (since concrete is weak in tension). Upper part of the concrete beam carries the compressive load.
- In the transformed section, cross sectional area of steel, A_s, replaced by equivalent area nA_s where $n = E_s/E_c$.
- To determine the location of the neutral axis,

$$(bx)\frac{x}{2} - nA_s(d - x) = 0$$

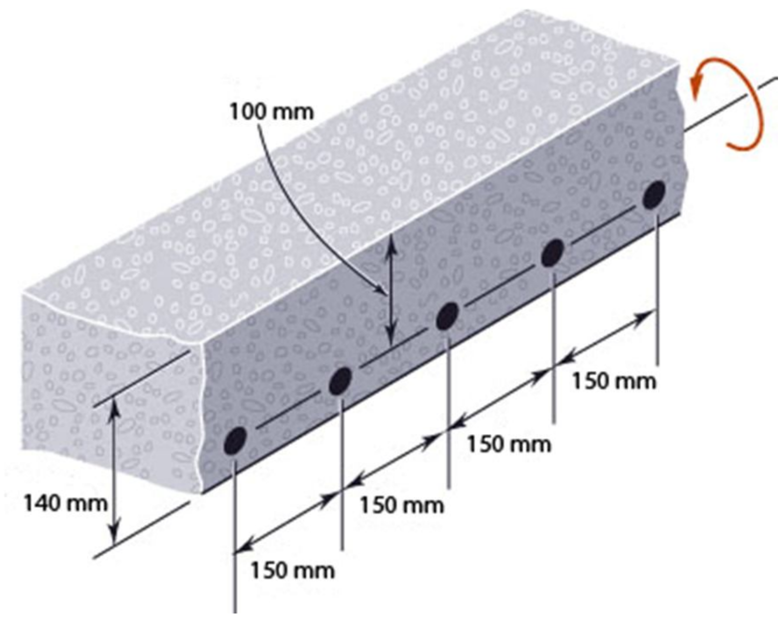
$$\tfrac{1}{2}bx^2 + nA_s x - nA_s d = 0$$

- The normal stress in the concrete and steel

$$\sigma_x = -\frac{My}{I}$$

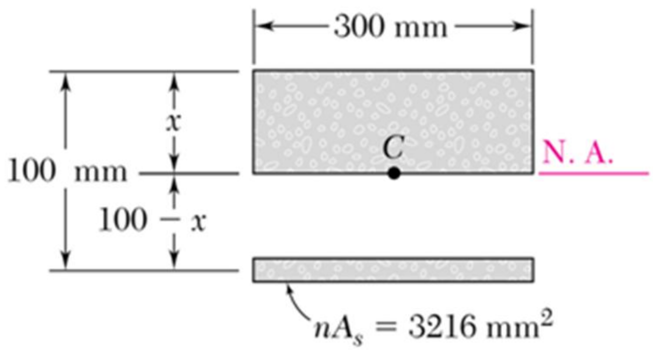
$$\sigma_c = \sigma_x \qquad \sigma_s = n\sigma_x$$

Example 5.6



Concrete floor slab reinforced with 16-mm-diameter steel rods. Modulus of elasticity is 200 GPa for steel and 25 GPa for concrete. Applied bending moment is 4.5 kNm per 0.3 m width of slab. Find maximum stress in concrete and in steel.

Example 5.6



SOLUTION:

Transform to section made entirely of concrete.

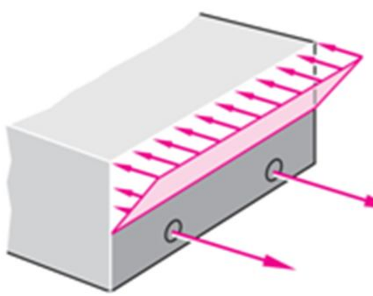
$$n = \frac{E_s}{E_c} = \frac{200\,\text{GPa}}{25\,\text{GPa}} = 8.0$$

$$nA_s = 8.0 \times 2\left[\tfrac{\pi}{4}(16\text{mm})^2\right] = 3216\,\text{mm}^2$$

Evaluate geometric properties of transformed section.

$$300x\left(\frac{x}{2}\right) - 3216(100 - x) = 0 \qquad x = 36.8\,\text{mm}$$

$$I = \tfrac{1}{3}(300\,\text{mm})(36.8\,\text{mm})^3 + (3216\,\text{mm}^2)(63.2\,\text{mm})^2 = 17.83\times10^6\ \text{mm}^4$$



Calculate maximum stresses.

$$\sigma_c = \frac{Mc_1}{I} = \frac{4500\,\text{Nm}\times 0.0368\,\text{m}}{17.83\times10^{-6}\ \text{m}^4} \qquad \boxed{\sigma_c = 9.29\,\text{MPa}}$$

$$\sigma_s = n\frac{Mc_2}{I} = 8.0\frac{4500\,\text{Nm}\times 0.0632\,\text{m}}{17.83\times10^{-6}\ \text{m}^4} \qquad \boxed{\sigma_s = 127.61\,\text{MPa}}$$

Check: $F_s = F_c$

Example 5.6

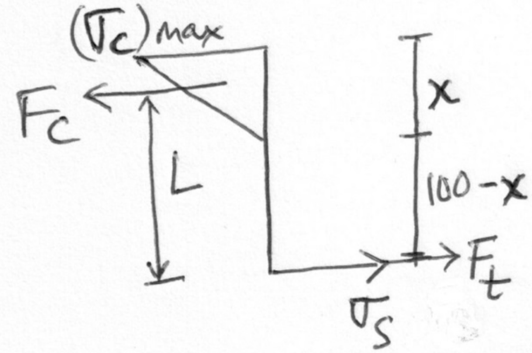
Alternate way to find max stresses without using I.

$F_c = F_t$ (tensile force = compr force).

$M = F_c L = F_t L$ (L = lever arm

$F_c = \frac{1}{2}(\sigma_c)_{max}\, x b, \quad F_t = \sigma_s A_s, \quad L = \left(\frac{2}{3}x + 100 - x\right)$

$$\Rightarrow (\sigma_c)_{max} = \frac{2M}{x\left(100 - \frac{x}{3}\right) b}$$



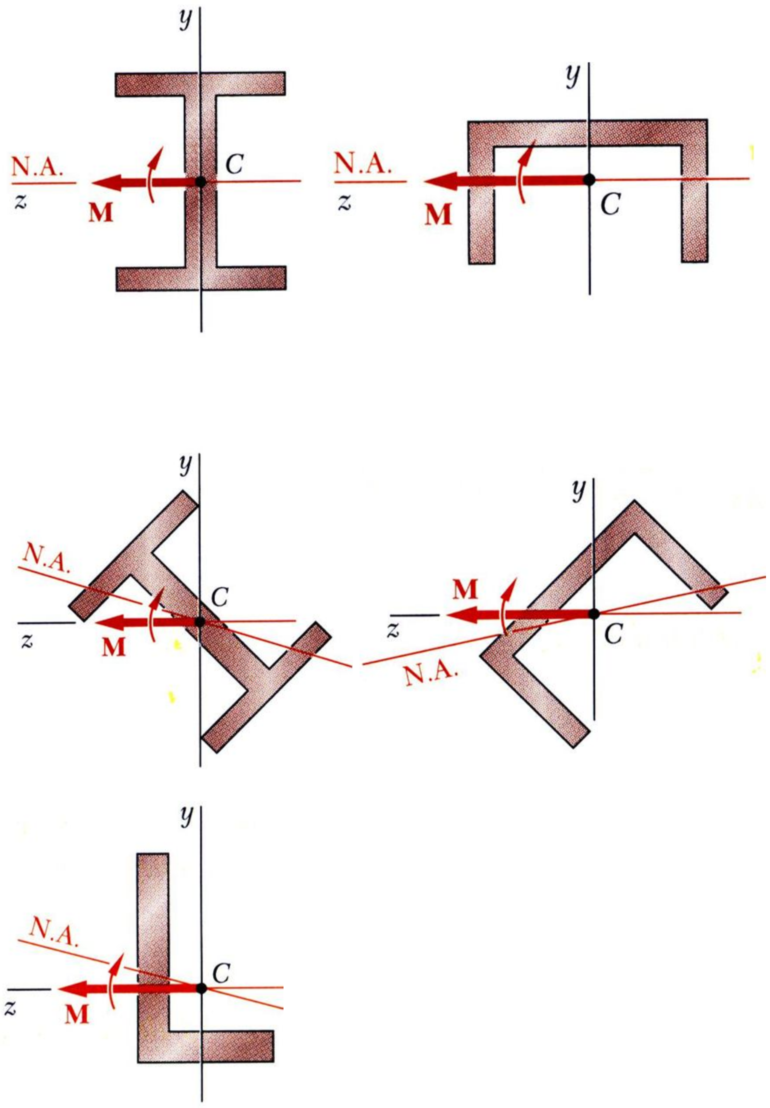
$$\sigma_t = \frac{M}{A_s\left(100 - \frac{x}{3}\right)}$$

Substitute values, get, $(\sigma_c)_{max} = 9.29\,\text{MPa}$

$\left(M = 4500,\ A_s = 2\left(\frac{\pi}{4}(16)^2\right),\ x = 36.8\right)$ $\quad \sigma_t = 127.6\ \text{MPa}$

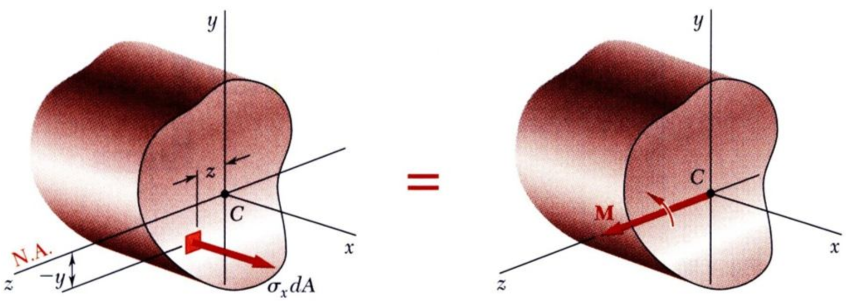
$b = 300$

Unsymmetric Bending



- Thus far analysis of pure bending limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- Neutral axis of cross section coincides/parallel with axis of couple
- Now consider situations in which bending couple does not act in a plane of symmetry.
- Cannot assume that member will bend in the plane of the couples.
- In general, neutral axis of section will not coincide with axis of couple.

Unsymmetric Bending



Wish to determine conditions under which neutral axis of section of arbitrary shape coincides with axis of couple, as shown above.

- Resultant force and moment from the stress distribution in the section must satisfy:

$$F_x = 0 = M_y \quad M_z = M = \text{applied couple}$$

- $$0 = F_x = \int \sigma_x dA = \int \left(-E \frac{y}{\rho} \right) dA$$

  or $0 = \int y \, dA$

  neutral axis passes through centroid

- $$M = M_z = -\int y \left(-E \frac{y}{\rho} \right) dA$$

  or $M = \dfrac{EI}{\rho} = -\dfrac{\sigma_x I}{y}, \quad I = I_z = \text{moment of inertia}$

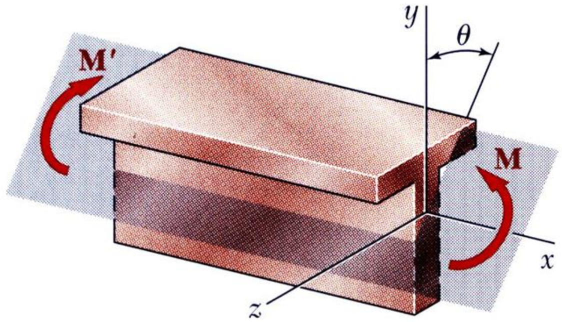
  defines stress distribution

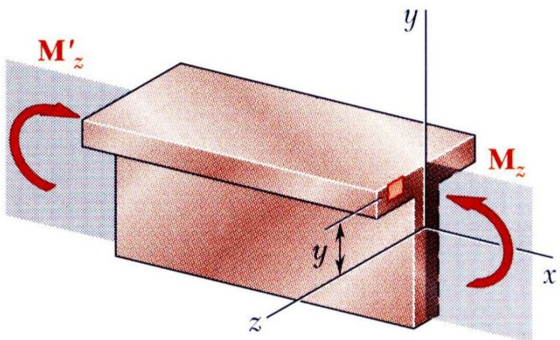
- $$0 = M_y = \int z \sigma_x dA = \int z \left(-\frac{M}{I} y \right) dA$$

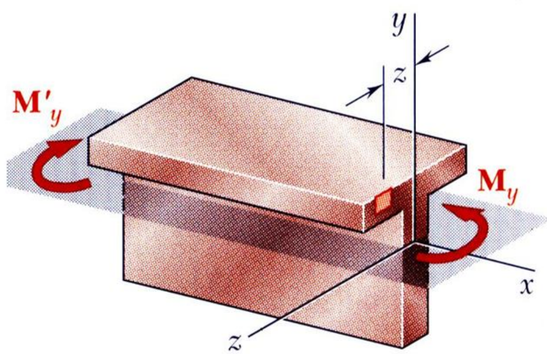
  or $0 = \int yz \, dA = I_{yz} = \text{product of inertia}$

  So couple vector must be directed along a principal centroidal axis.

Unsymmetric Bending







Superposition applied to determine stresses in the most general case of unsymmetric bending.

- Resolve couple vector into components along principle centroidal axes.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

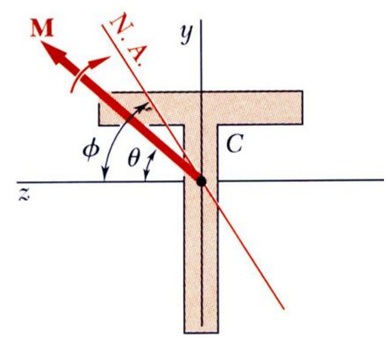
- Superpose stresses due to M_y and M_z

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

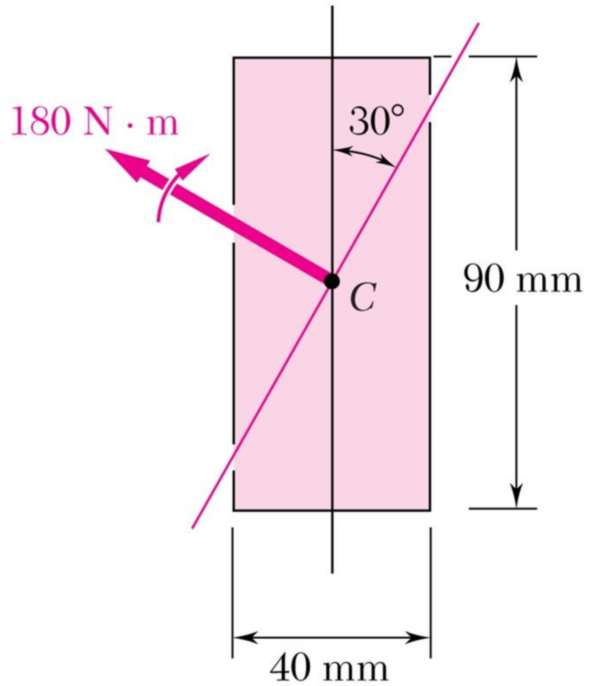
- Along neutral axis we have,

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = -\frac{(M\cos\theta)\,y}{I_z} + \frac{(M\sin\theta)\,z}{I_y}$$

$$\tan\varphi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$

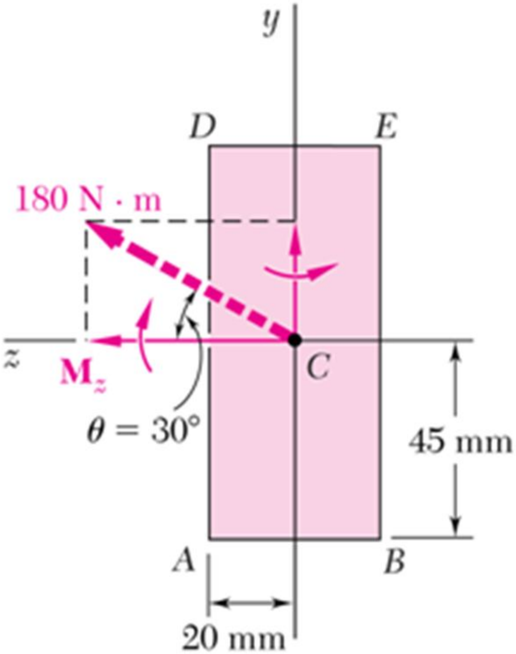


Example 5.7



180 Nm couple applied to rectangular wooden beam in a plane at 30 deg. to vertical. Find (a) maximum stress in beam, (b) angle that neutral axis makes with horizontal plane.

Example 5.7



Resolve couple vector along principal axes and calculate corresponding maximum stresses.

$$M_z = (180\ \text{Nm})\cos 30 = 155.9\ \text{Nm}$$
$$M_y = (180\ \text{Nm})\sin 30 = 90\ \text{Nm}$$

$$I_z = \tfrac{1}{12}(0.04\ \text{m})(0.09\ \text{m})^3 = 2.43\times10^{-6}\ \text{m}^4$$
$$I_y = \tfrac{1}{12}(0.09\ \text{m})(0.04\ \text{m})^3 = 0.48\times10^{-6}\ \text{m}^4$$

The largest tensile stress due to M_z occurs along AB

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(155.9\ \text{Nm})(0.045\ \text{m})}{2.43\times10^{-6}\ \text{m}^4} = 2.89\ \text{MPa}$$

The largest tensile stress due to M_z occurs along AD

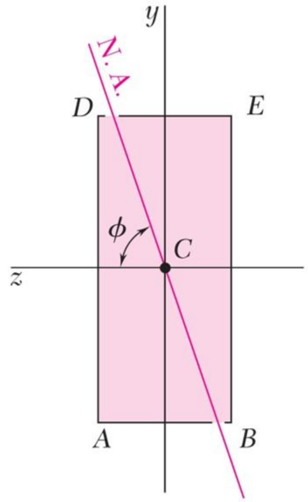
$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(90\ \text{Nm})(0.02\ \text{m})}{0.48\times10^{-6}\ \text{m}^4} = 3.75\ \text{MPa}$$

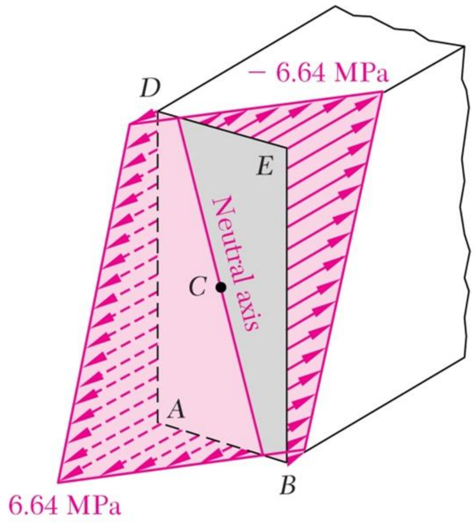
Largest tensile stress due to combined M_y and M_z occurs at A.

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 2.89 + 3.75$$

$$\boxed{\sigma_{\max} = 6.64\ \text{MPa}}$$

Example 5.7





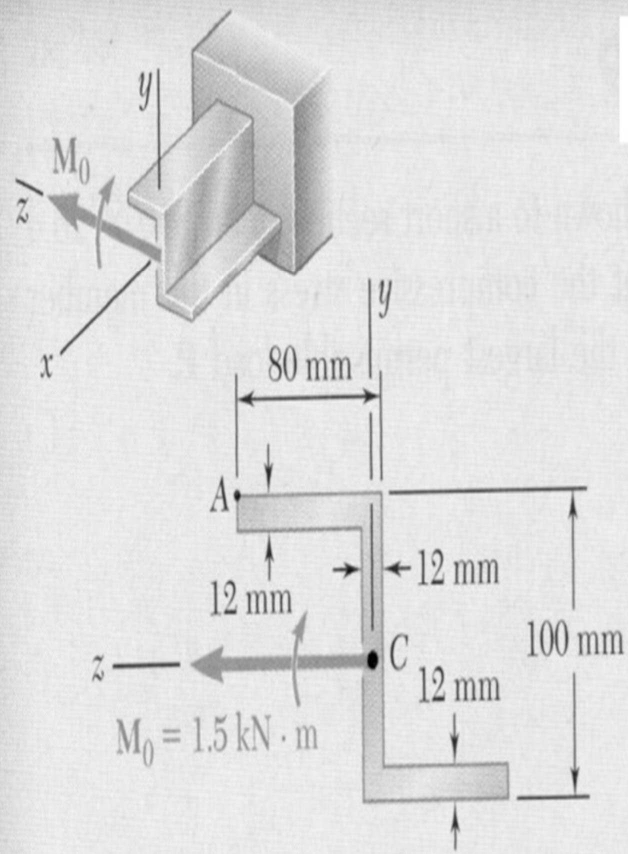
Determine angle of neutral axis.

$$\tan\phi = \frac{I_z}{I_y}\tan\theta = \frac{2.43\times10^{-6}\ \mathrm{m}^4}{0.48\times10^{-6}\ \mathrm{m}^4}\tan 30$$

$$= 2.9$$

$$\boxed{\phi = 71^o}$$

Example 5.8



A couple of magnitude $M_0 = 1.5$ kN · m acting in a vertical plane is applied to a beam having the Z-shaped cross section shown. Determine (*a*) the stress at point A, (*b*) the angle that the neutral axis forms with the horizontal plane. The moments and product of inertia of the section with respect to the *y* and *z* axes have been computed and are as follows:

$$I_y = 3.25 \times 10^{-6}\ \text{m}^4$$
$$I_z = 4.18 \times 10^{-6}\ \text{m}^4$$
$$I_{yz} = 2.87 \times 10^{-6}\ \text{m}^4$$

Example 5.8

Although I_{zy} given, if we wanted to find it,

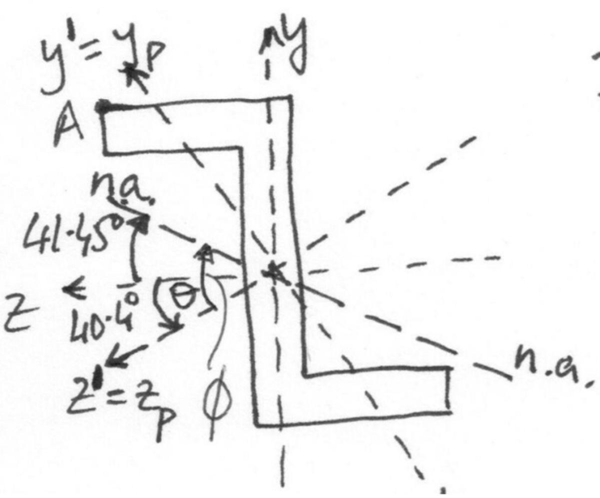
$$I_{zy} = 2\left([80-12][12][-(50-6)]\left[-\left(\frac{80-12}{2}+6\right)\right]\right) = 2Ad_yd_z$$

$$= 2872320 \text{ mm}^4 = 2.87E{-6} \text{ m}^4.$$

Find p.a. (principal axes)

$$\tan 2\theta = \frac{2I_{yz}}{I_y - I_z} = \frac{2(2.87)}{(3.25-4.18)} \Rightarrow 2\theta = -80.8$$

$$\theta = -40.4°.$$



$$I_{y'y'} = I_y\cos^2\theta + I_z\sin^2\theta + I_{zy}\sin 2\theta$$

$$= 0.8075 \text{ m}^4$$

$$I_{z'z'} = I_y\sin^2\theta + I_z\cos^2\theta - I_{zy}\sin 2\theta$$

$$= 6.6224 \text{ m}^4$$

Example 5.8

For pt. A: $y' = -z\sin\theta + y\cos\theta = -(80-6)\sin(-40.4) + 50\cos(-40.4)$

$= 86.04\text{ mm}$

$z' = z\cos\theta + y\sin\theta = 74\cos(-40.4) + 50\sin(-40.4)$

$= 23.95\text{ mm.}$

$M_{z'} = M\cos 40.4 = 1.5\cos 40.4\text{ kN.m}, \quad M_{y'} = 1.5\sin 40.4\text{ kN.m.}$

(a) $(\sigma_x)_A = -\dfrac{M_{z'}}{I_{z'}}(y_A) + \dfrac{M_{y'}}{I_{y'}}(z_A) = 1.5 * 10^6\left(-\dfrac{\cos 40.4 * 86.04}{6.6224\text{E}12} + \dfrac{\sin 40.4 * 23.95}{0.8076\text{E}12}\right)$

y' z'

$= 1.399\text{E}-5\text{ MPa} = 13.99\text{ N/m}^2$

(b) Location of n.a.

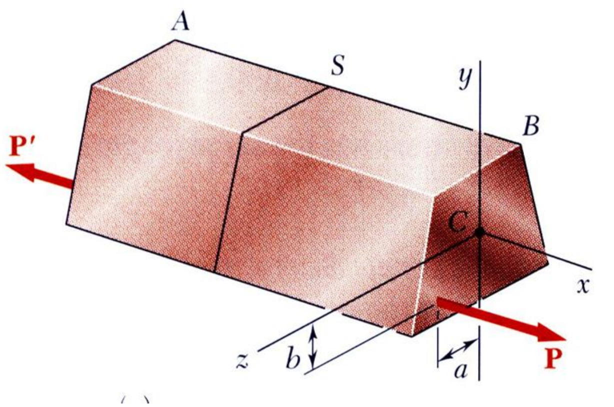
$\sigma_x = 0 \Rightarrow \dfrac{y'}{z'} = \dfrac{M_{y'}}{M_{z'}}\dfrac{I_{z'}}{I_{y'}} = \tan 40.4 * 8.2 = \tan\phi$

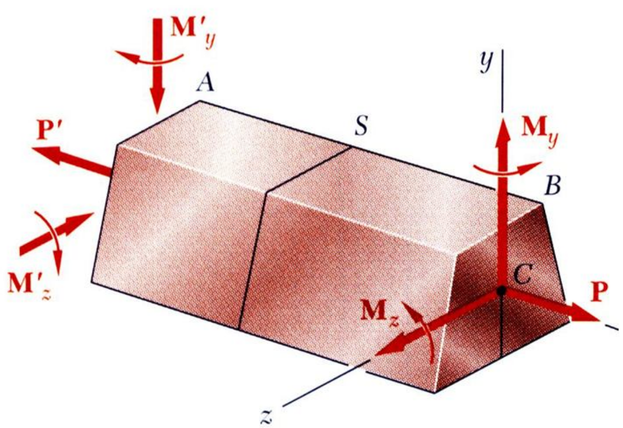
$\Rightarrow \phi = 81.85$ w.r.t z' axis.

$= (81.85 - 40.4)$

$= 41.45^\circ$ wrt z axis

General Case of Eccentric Axial Loading





- Consider straight member subject to equal and opposite eccentric forces.
- Eccentric force equivalent to system comprising centric force and two couples.

$$P = \text{centric force}$$

$$M_y = Pa \qquad M_z = Pb$$

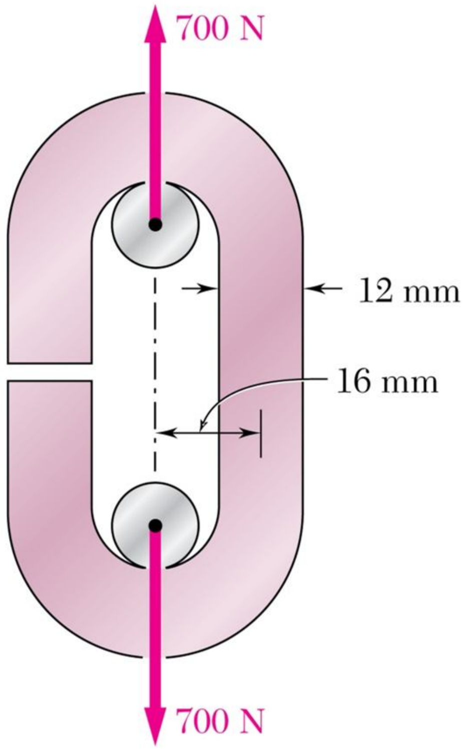
- By principle of superposition, combined stress distribution is

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- If neutral axis lies on section, it is found from

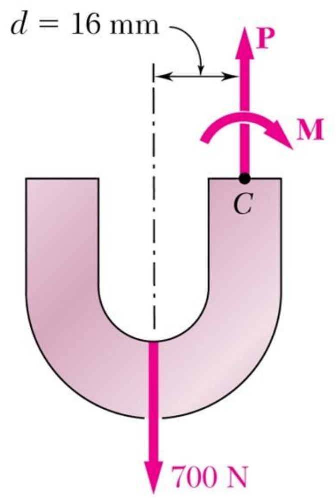
$$0 = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

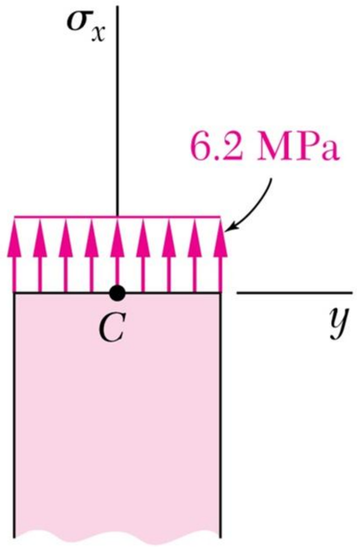
Example 5.9



Open-link chain obtained by bending low-carbon steel rods into shape shown. For 700 N load, find (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

Example 5.9



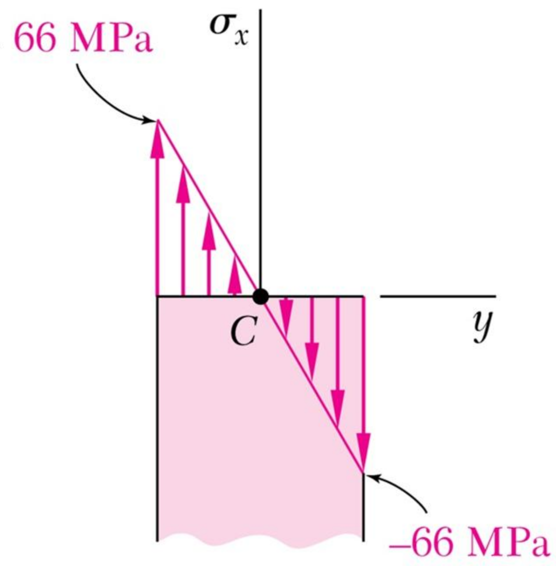


Normal stress due to a centric load

$$A = \pi c^2 = \pi (6\,\mathrm{mm})^2$$
$$= 113.1\,\mathrm{mm}^2$$
$$\sigma_0 = \frac{P}{A} = \frac{700\,\mathrm{N}}{113.1\times10^{-6}\,\mathrm{m}^2}$$
$$= 6.2\,\mathrm{MPa}$$

Equivalent centric load and bending moment

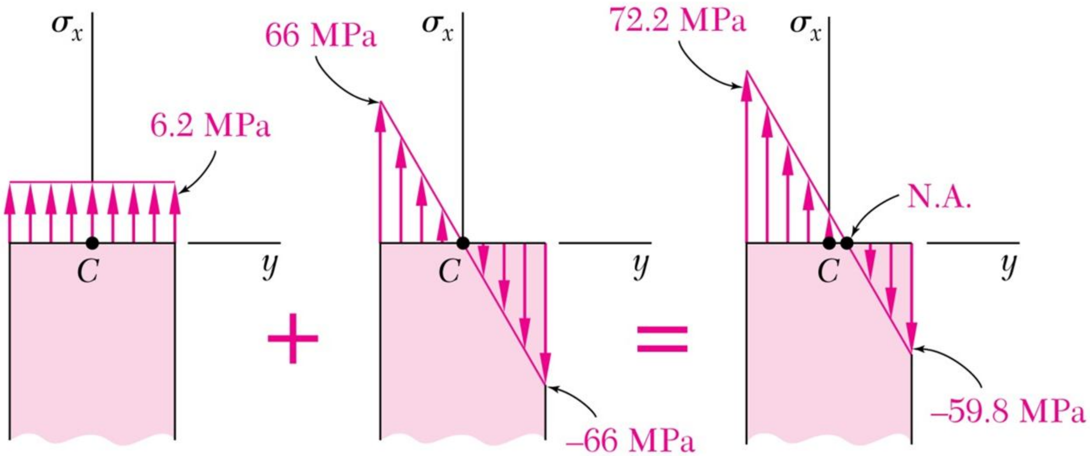
$$P = 700\,\mathrm{N}$$
$$M = Pd = (700\,\mathrm{N})(0.016\,\mathrm{m})$$
$$= 11.2\,\mathrm{Nm}$$



Normal stress due to bending moment

$$I = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi (6\,\mathrm{mm})^4$$
$$= 1017.9\,\mathrm{mm}^4$$
$$\sigma_m = \frac{Mc}{I} = \frac{(11.2\,\mathrm{Nm})(0.006\,\mathrm{m})}{1017.9\times10^{-12}\,\mathrm{m}^4}$$
$$= 66\,\mathrm{MPa}$$

Example 5.9



Maximum tensile and compressive stresses

$$\sigma_t = \sigma_0 + \sigma_m = 6.2 + 66$$

$$\boxed{\sigma_t = 72.2\,\text{MPa}}$$

$$\sigma_c = \sigma_0 - \sigma_m = 6.2 - 66$$

$$\boxed{\sigma_c = -59.8\,\text{MPa}}$$

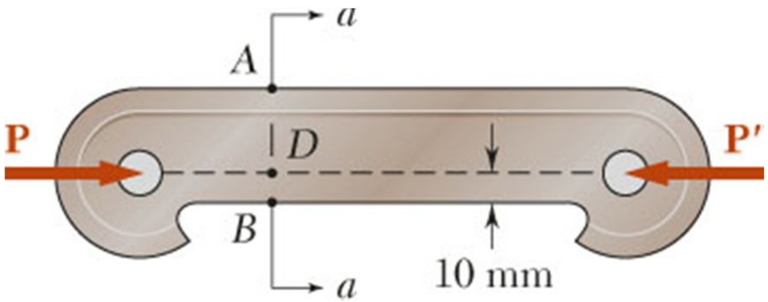
Neutral axis location

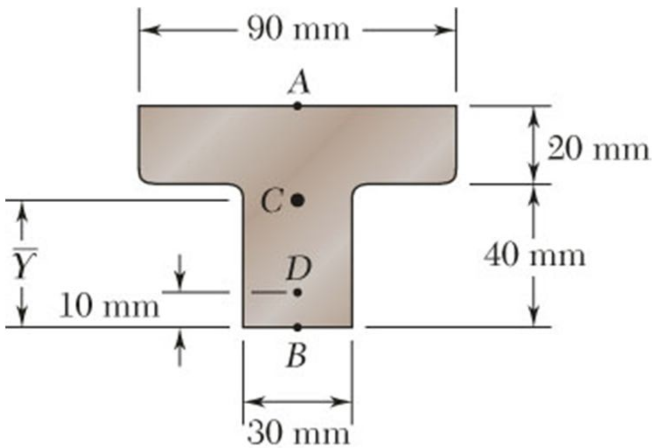
$$0 = \frac{P}{A} - \frac{My_0}{I}$$

$$y_0 = \frac{P}{A}\frac{I}{M} = \left(6.2\times10^6\ \text{Pa}\right)\frac{1017.9\times10^{-12}\,\text{m}^4}{11.2\,\text{Nm}}$$

$$\boxed{y_0 = 0.56\,\text{mm}}$$

Sample Problem 5.10



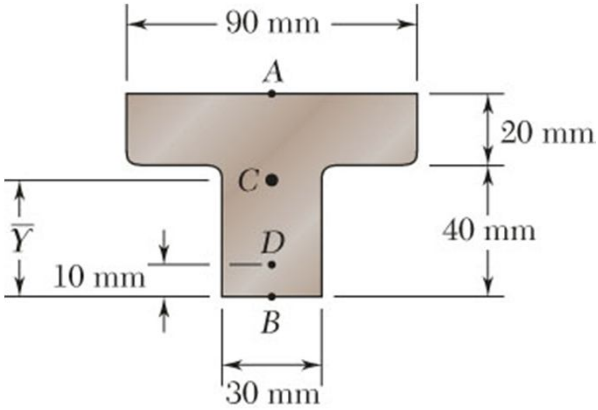


Section *a–a*

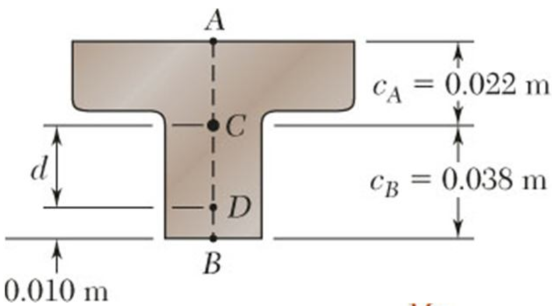
Largest allowable stresses for cast iron link are 30 MPa in tension and 120 MPa in compression. Determine largest force *P* which can be applied.

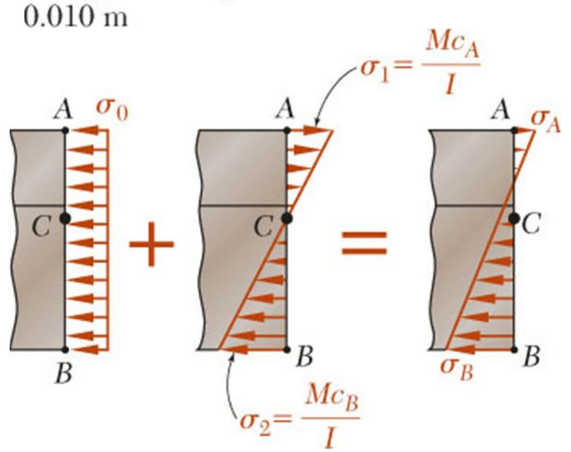
$$A = 3\times 10^{-3}\,\mathrm{m}^2$$
$$\bar{Y} = 0.038\,\mathrm{m}$$
$$I = 868\times 10^{-9}\,\mathrm{m}^4$$

Sample Problem 5.10



Section *a–a*





Determine equivalent centric load & bending moment.

$$d = 0.038 - 0.010 = 0.028\,\text{m}$$

$$P = \text{centric load}$$

$$M = Pd = 0.028P = \text{bending moment}$$

Superpose stresses due to *P* and *M*,

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} + \frac{(0.028P)(0.022)}{868\times10^{-9}} = +377P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} - \frac{(0.028P)(0.022)}{868\times10^{-9}} = -1559P$$

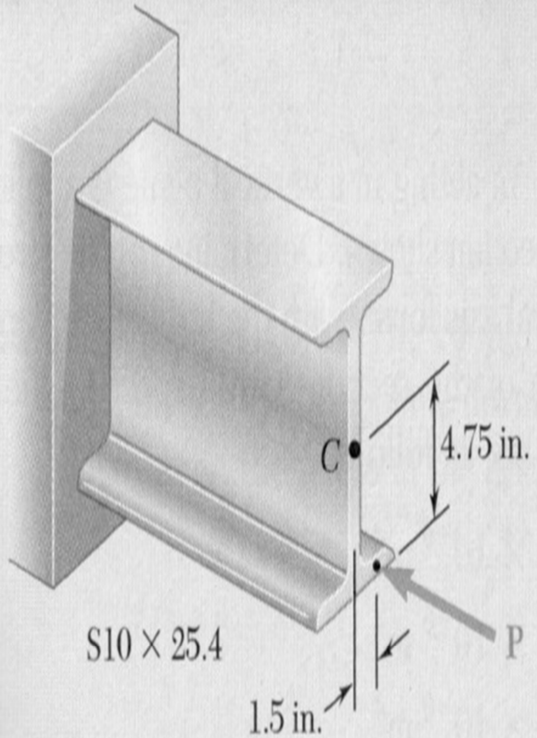
Evaluate critical loads for allowable stresses.

$$\sigma_A = +377P = 30\,\text{MPa} \qquad P = 79.6\,\text{kN}$$

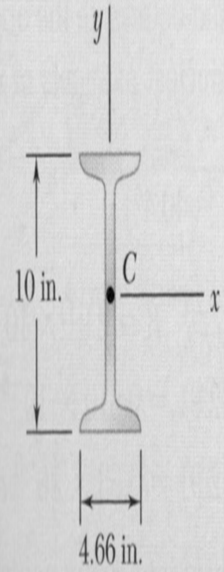
$$\sigma_B = -1559P = -120\,\text{MPa} \qquad P = 77.0\,\text{kN}$$

Largest allowable load is $\boxed{P = 77.0\ \text{kN}}$

Example 5.11



A horizontal load **P** is applied as shown to a short section of an S10 × 25.4 rolled-steel member. Knowing that the compressive stress in the member is not to exceed 12 ksi, determine the largest permissible load **P**.



Also locate N.A.

Properties of Cross Section. The following data are taken from Appendix C.

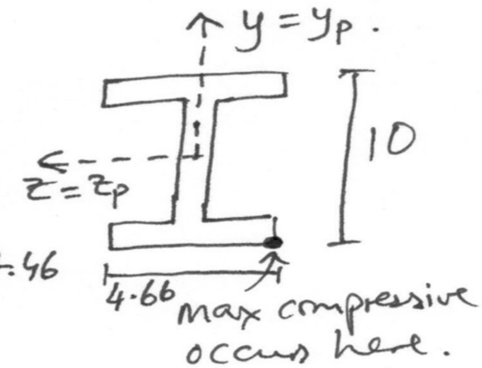
Area: $A = 7.46\ \text{in}^2$

Section moduli: $S_x = 24.7\ \text{in}^3 \quad S_y = 2.91\ \text{in}^3$

Example 5.11

$$M_z = -4.75P \quad , \quad M_y = 1.5P$$

$$\underbrace{(\sigma_c)_{max}}_{\text{compressive}} = 12 = \frac{4.75P}{\underbrace{S_z}_{24.7}} + \frac{1.5P}{\underbrace{S_y}_{2.91}} + \frac{P}{\underbrace{A}_{7.46}}$$



$$P = 14.25 \text{ kips} = P_{max}$$

Location of n.a. $\quad \sigma_x = 0 \Rightarrow \frac{4.75 \cancel{P} y}{I_z} + \frac{1.5 \cancel{P} z}{I_y} - \frac{\cancel{P}}{A} = 0$

$$\left(\frac{4.75}{24.7 * 5}\right) y + \left(\frac{1.5}{2.91 * 2.33}\right) z - \frac{1}{7.46} = 0 .$$

$$y = mz + C$$

$$m = 5.752 \Rightarrow \theta = 80.14^\circ .$$

$$c = 3.485 \text{ in.}$$