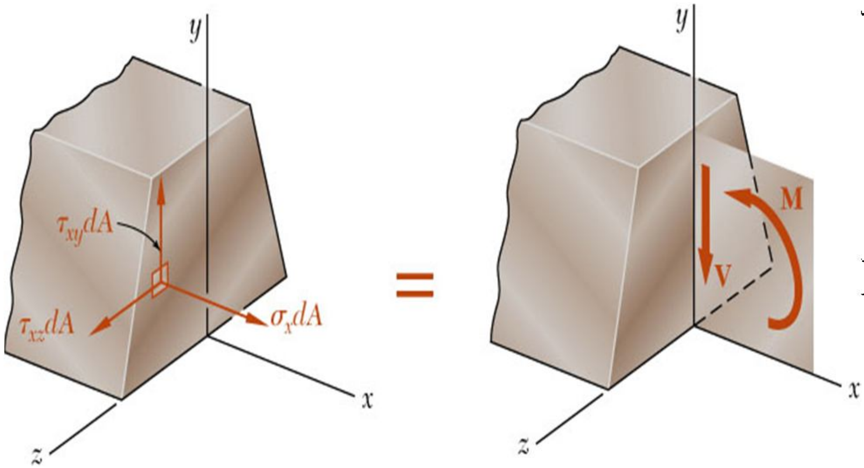
CHAPTER

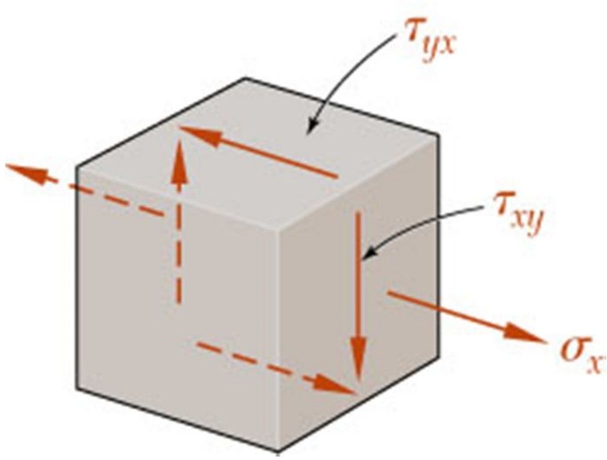
6

MECHANICS OF MATERIALS

Shearing Stresses in Beams and Thin-Walled Members

Introduction





- Transverse loading applied to beam results in normal and shearing stresses in transverse sections.

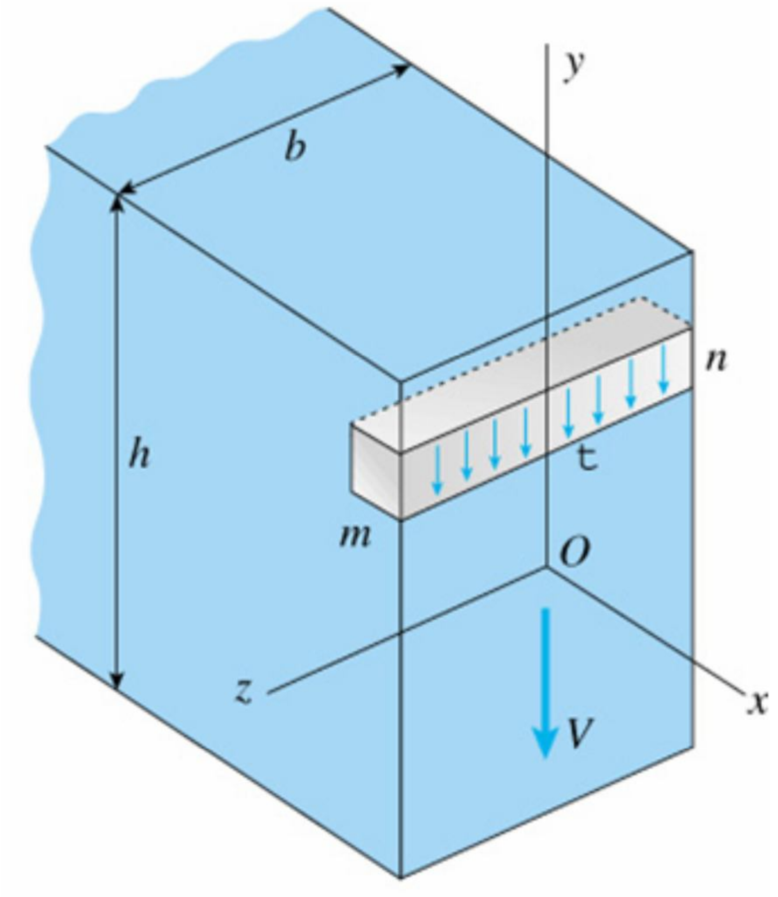
- Distribution of normal and shearing stresses satisfies (from equilibrium)

$$F_x = \int \sigma_x dA = 0 \qquad M_x = \int \left(y\tau_{xz} - z\tau_{xy}\right) dA = 0$$
$$F_y = \int \tau_{xy} dA = -V \qquad M_y = \int z\sigma_x dA = 0$$
$$F_z = \int \tau_{xz} dA = 0 \qquad M_z = \int \left(-y\sigma_x\right) = M$$

- When shearing stresses are exerted on vertical faces of an element, equal stresses exerted on horizontal faces

- Longitudinal shearing stresses must exist in any member subjected to transverse loading.

Vertical and Horizontal Shear Stresses



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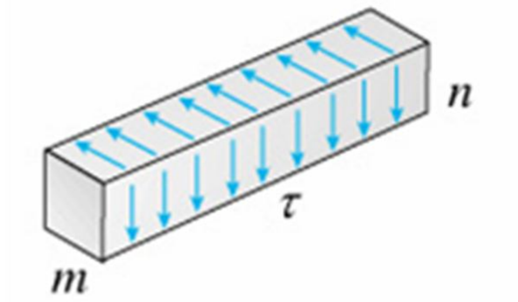

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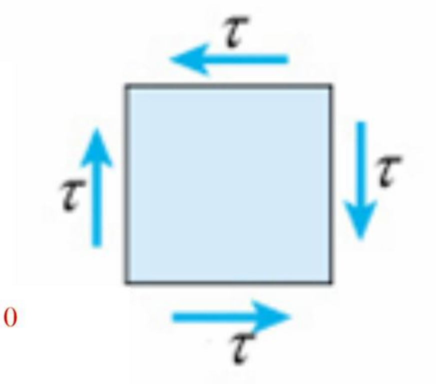
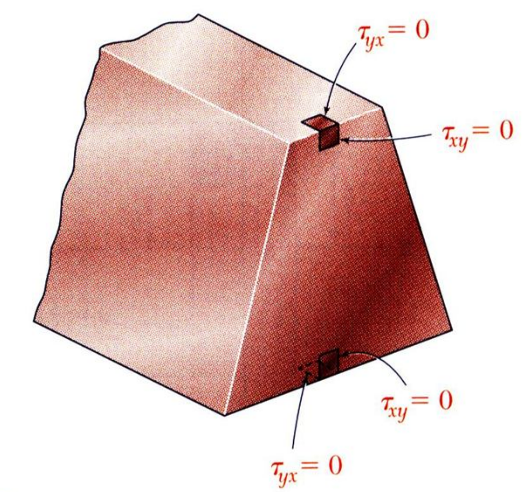
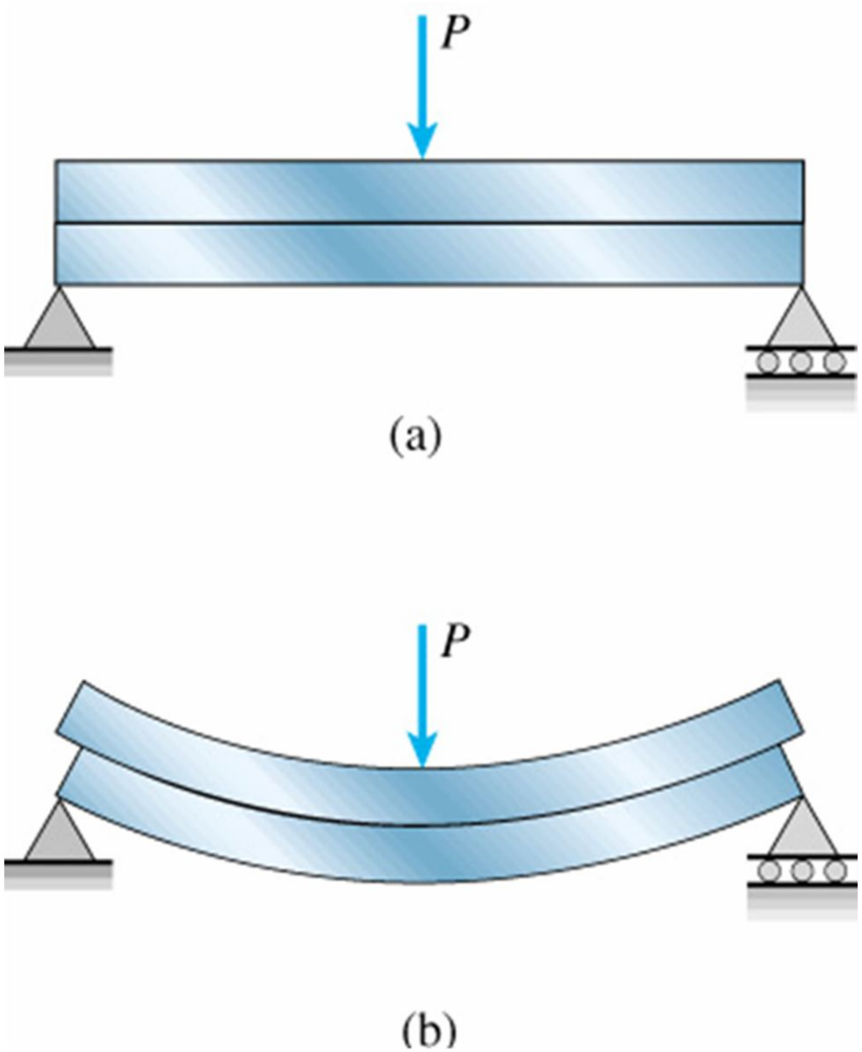




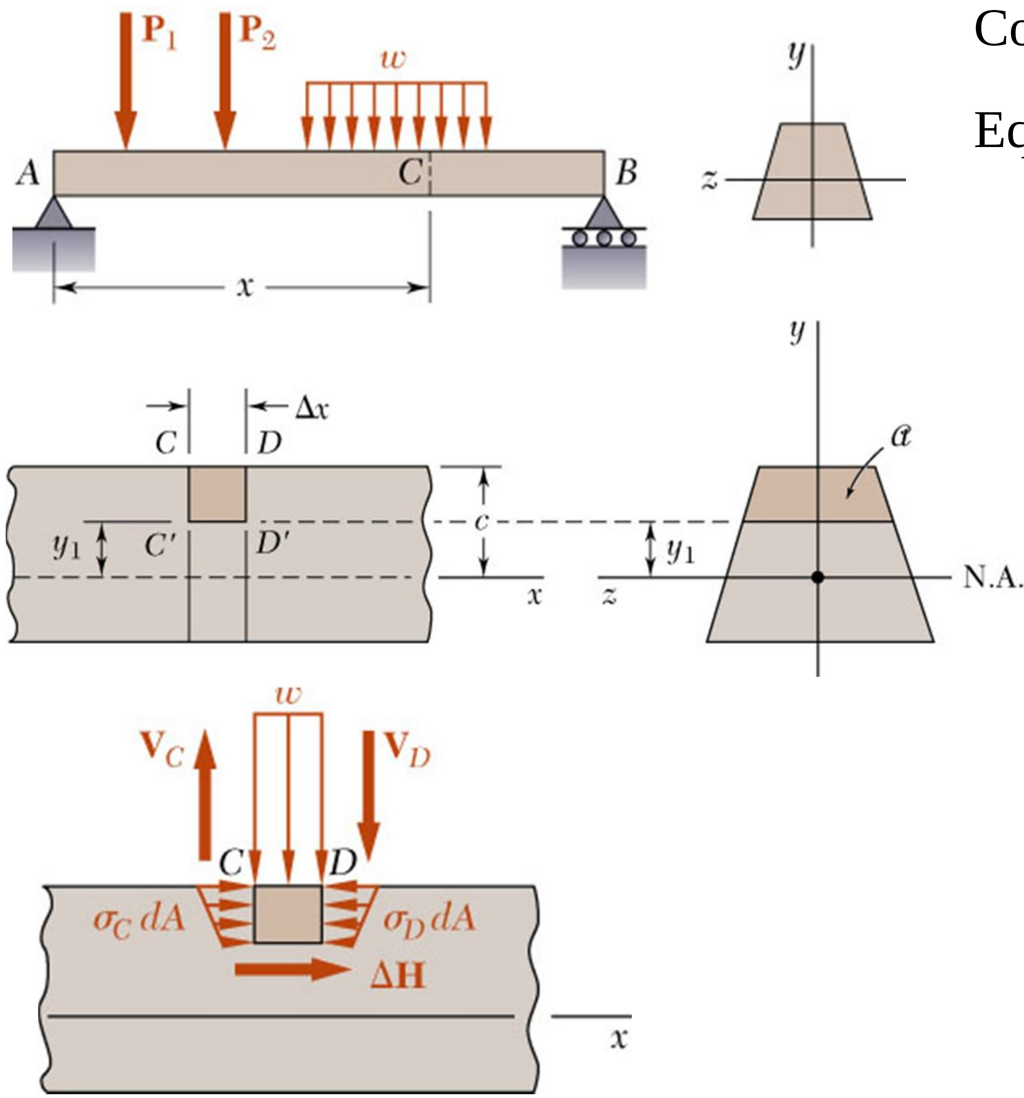
Shear Stress in Beams



Two beams glued together along horizontal surface

When loaded, horizontal shear stress must develop along glued surface **in order to prevent sliding between** the beams.

Shear on Horizontal Face of Beam Element



Consider prismatic beam

Equilibrium of element *CDC'D'*

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_C)\,dA$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y\,dA$$

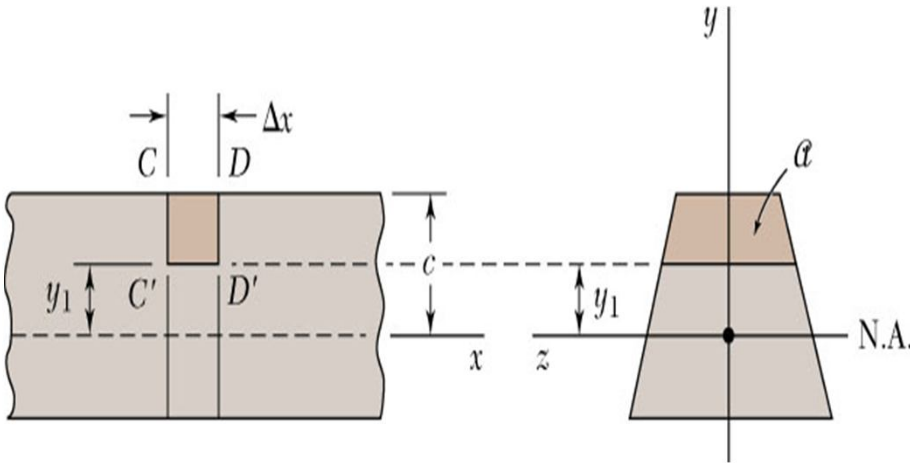
Let, $Q = \int_A y\,dA$

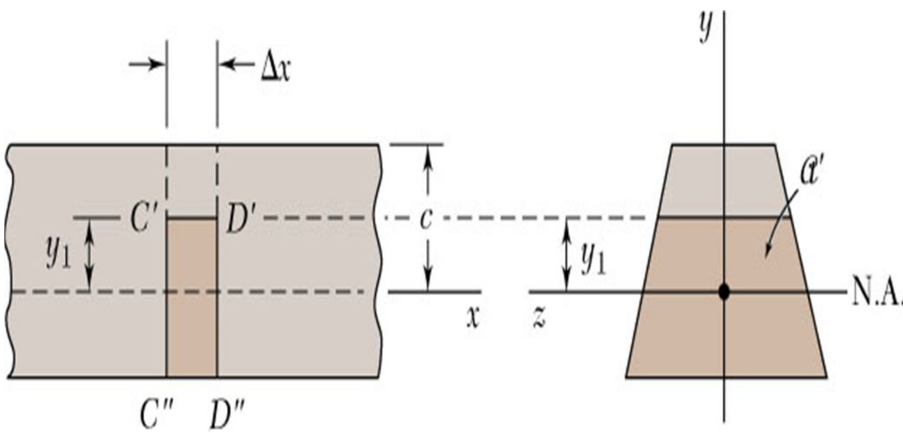
$$M_D - M_C = \frac{dM}{dx}\Delta x = V\,\Delta x$$

$$\Delta H = \frac{VQ}{I}\Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow}$$

Shear on Horizontal Face of Beam Element





Shear flow,

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow}$$

where

$$Q = \int_A y\,dA$$

$$= \text{first moment of area above } y_1$$

$$I = \int_{A+A'} y^2 dA$$

$$= \text{second moment of full cross section}$$

Same result found for lower area

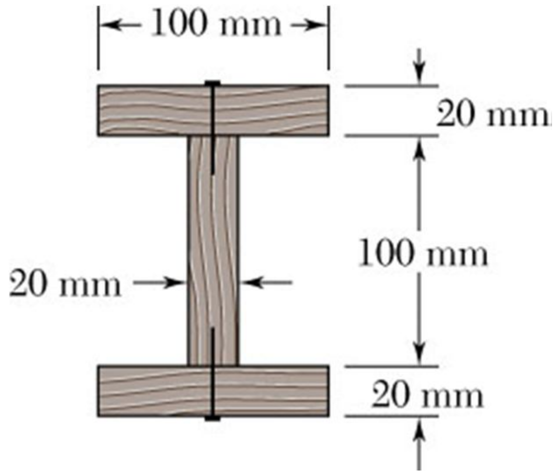
$$Q + Q' = \int_{A_1+A_2} y\,dA = 0$$

$(\because$ first moment of area wrt NA is zero$)$

$$q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = \frac{V(-Q)}{I} = -q$$

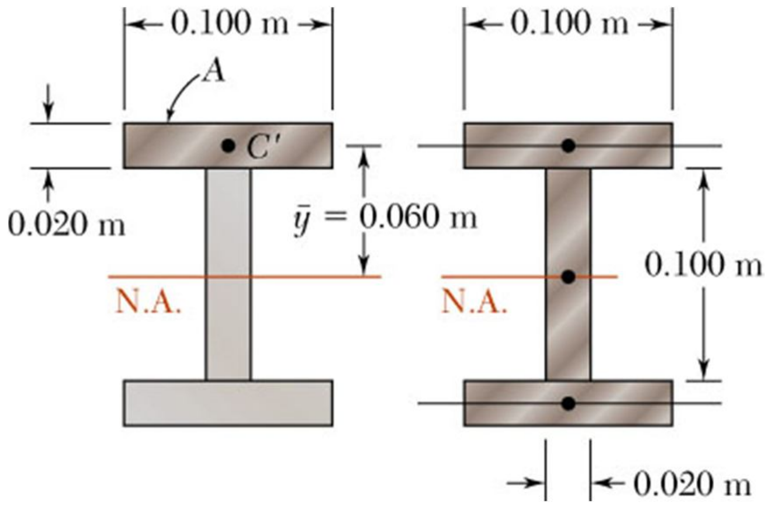
$$\Delta H' = -\Delta H$$

Example 6.1



Beam made of three planks, nailed together. Spacing between nails is 25 mm. Vertical shear in beam is *V* = 500 N. Find shear force in each nail.

Example 6.1



$$Q = A\bar{y}$$
$$= (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})$$
$$= 120 \times 10^{-6}\,\text{m}^3$$

$$I = \tfrac{1}{12}(0.020\,\text{m})(0.100\,\text{m})^3$$
$$+ 2[\tfrac{1}{12}(0.100\,\text{m})(0.020\,\text{m})^3$$
$$+ (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})^2]$$
$$= 16.20 \times 10^{-6}\,\text{m}^4$$

SOLUTION:

Find horizontal force per unit length or shear flow q on lower surface of upper plank.

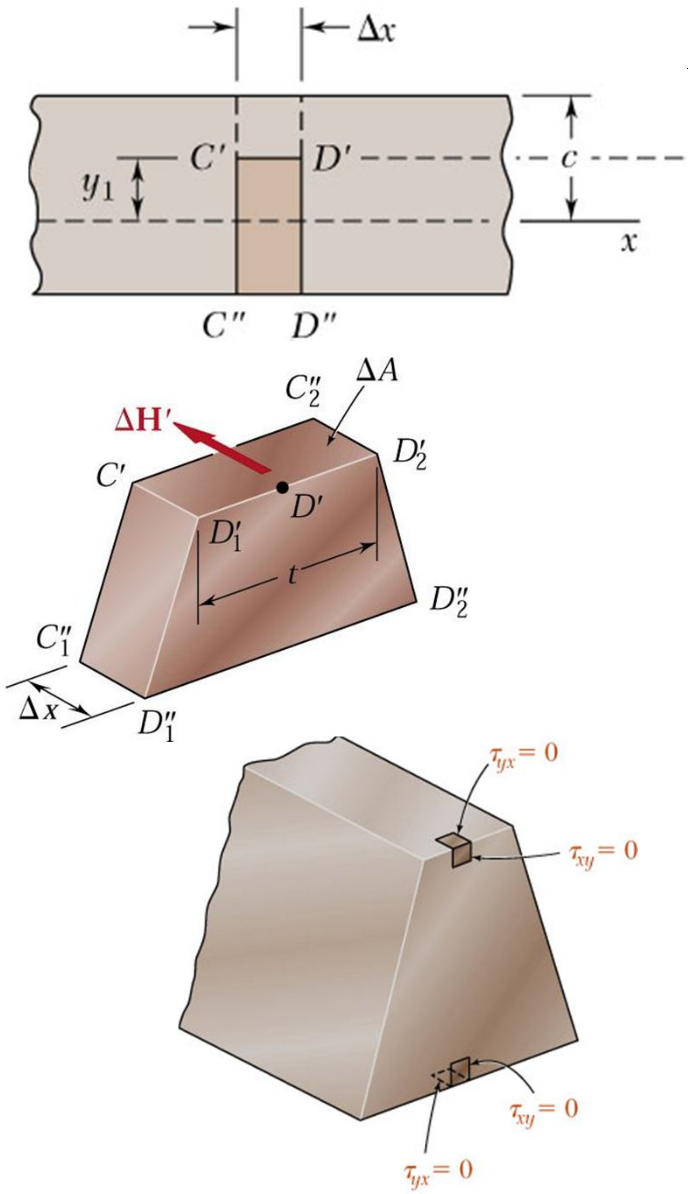
$$q = \frac{VQ}{I} = \frac{(500\text{N})(120 \times 10^{-6}\,\text{m}^3)}{16.20 \times 10^{-6}\,\text{m}^4}$$
$$= 3704\,\text{N}/\text{m}$$

Calculate corresponding shear force in each nail for nail spacing of 25 mm.

$$F = (0.025\text{m})q = (0.025\text{m})(3704\,N/m$$

$$\boxed{F = 92.6\,\text{N}}$$

Determination of Shearing Stress



Average shearing stress on horizontal face of element is shearing force on horizontal face divided by area of horzontal face.

$$\tau_{ave} = \frac{\Delta H}{\Delta A} = \frac{q\,\Delta x}{\Delta A} = \frac{VQ}{I}\,\frac{\Delta x}{t\,\Delta x}$$

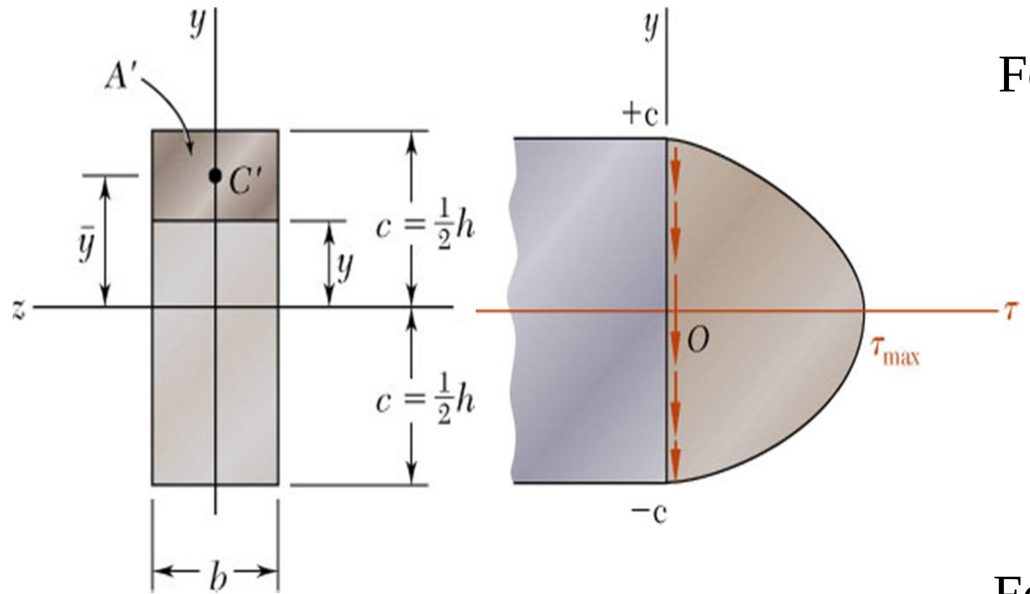
$$\tau = \tau_{ave} = \frac{VQ}{It}; \qquad \tau\, t = q$$

Note averaging is across dimension *t* (width) which is assumed much less than the depth, so this averaging is allowed.

On upper and lower surfaces of beam, τ_{yx}= 0. It follows that τ_{xy}= 0 on upper and lower edges of transverse sections.

If width of beam is comparable or large relative to depth, the shearing stresses at D'_1 and D'_2 are significantly higher than at *D*, i.e., the above averaging is not good.

Shearing Stresses t_{xy} in Common Types of Beams

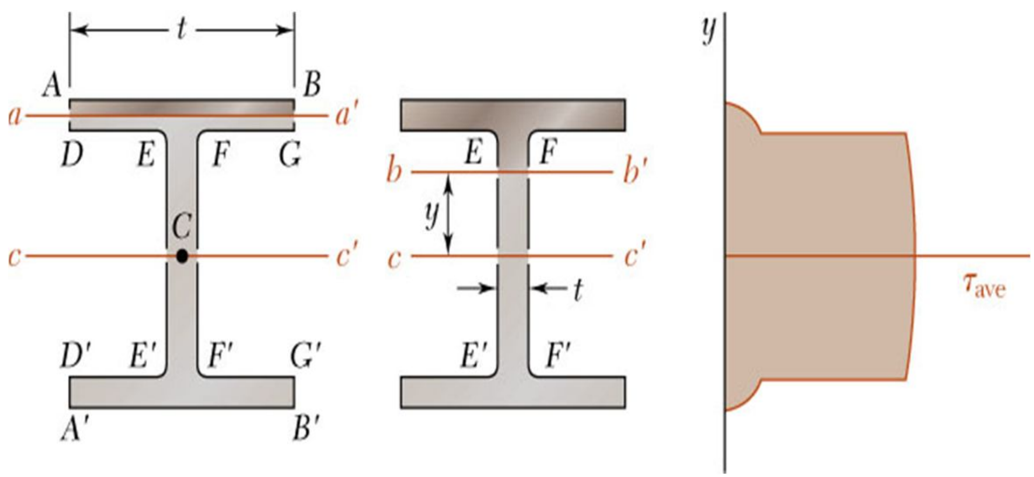


For a narrow rectangular beam,

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right)$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

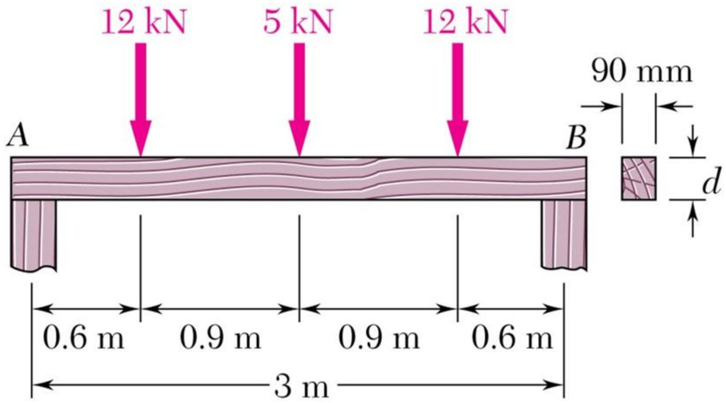
For I beams



$$\tau_{ave} = \frac{VQ}{It}$$

$$\tau_{max} = \frac{V}{A_{web}}$$

Example 6.2

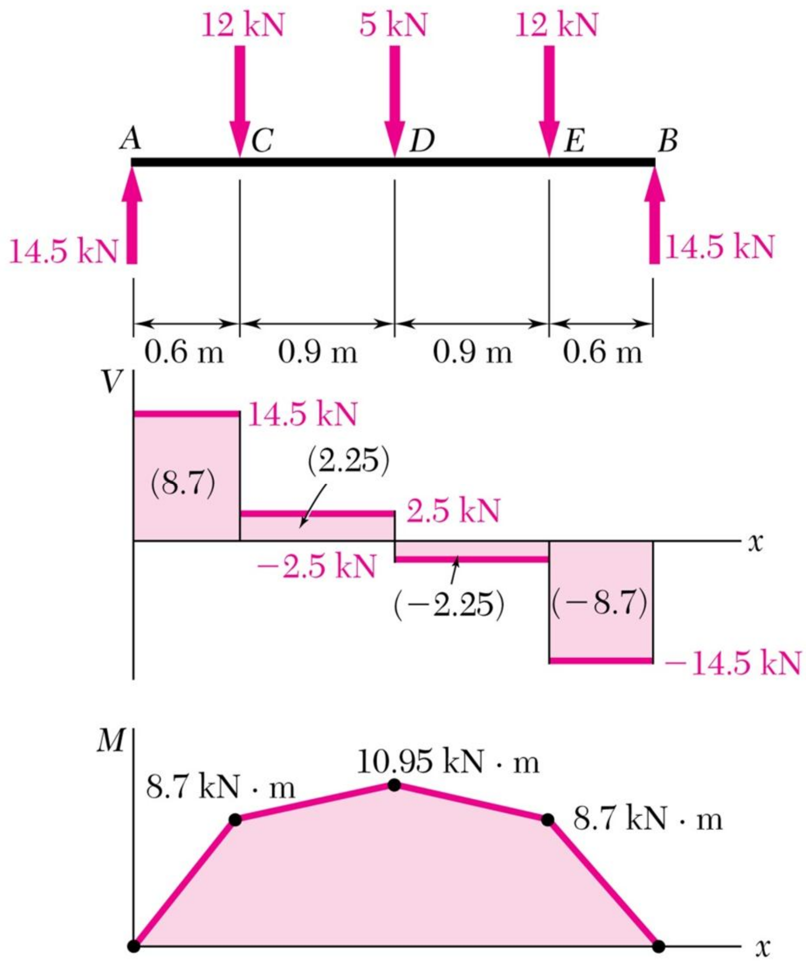


Timber beam supports three concentrated loads.

$$\sigma_{all} = 12\,\text{MPa} \qquad \tau_{all} = 0.8\,\text{MPa}$$

Find minimum required depth *d* of beam.

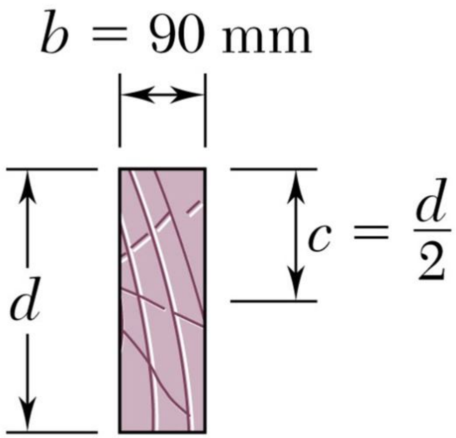
Example 6.2



$$V_{\max} = 14.5\,\text{kN}$$

$$M_{\max} = 10.95\,\text{kNm}$$

Example 6.2



$$I = \tfrac{1}{12} b\, d^3$$

$$S = \frac{I}{c} = \tfrac{1}{6} b\, d^2$$

$$= \tfrac{1}{6}(0.09\,\mathrm{m})d^2$$

$$= (0.015\,\mathrm{m})d^2$$

Determine depth based on allowable normal stress.

$$\sigma_{all} = \frac{M_{\max}}{S}$$

$$12\times10^6\ \mathrm{Pa} = \frac{10.95\times10^3\,\mathrm{Nm}}{(0.015\,\mathrm{m})d^2}$$

$$d = 0.246\,\mathrm{m} = 246\mathrm{mm}$$

Determine depth based on allowable shear stress.

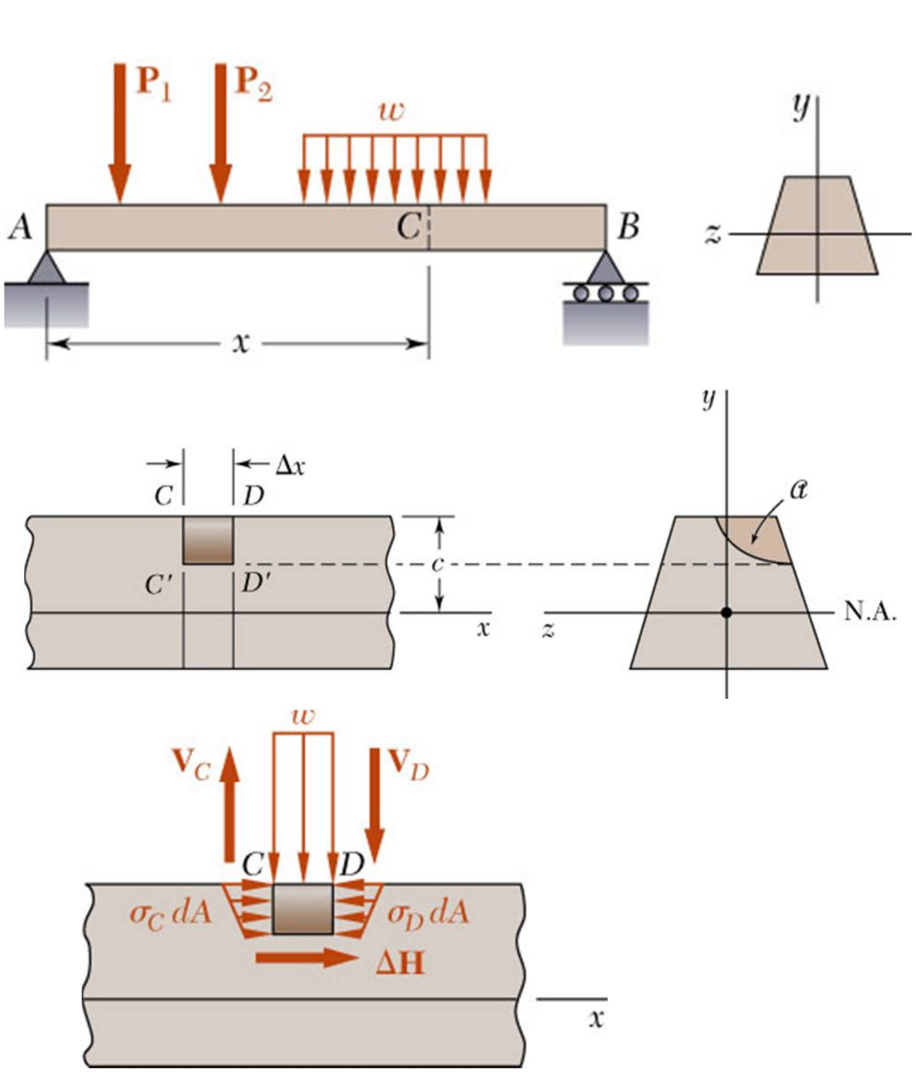
$$\tau_{all} = \frac{3}{2}\frac{V_{\max}}{A}$$

$$0.8\times10^6\ \mathrm{Pa} = \frac{3}{2}\frac{14500}{(0.09\,\mathrm{m})d}$$

$$d = 0.322\,\mathrm{m} = 322\,\mathrm{mm}$$

Required depth $\boxed{d = 322\,\mathrm{mm}}$

Longitudinal Shear Element of Arbitrary Shape



Have examined distribution of vertical components t_{xy} on transverse section. Now consider horizontal components t_{xz} .

Consider element defined by curved surface CDD'C'.

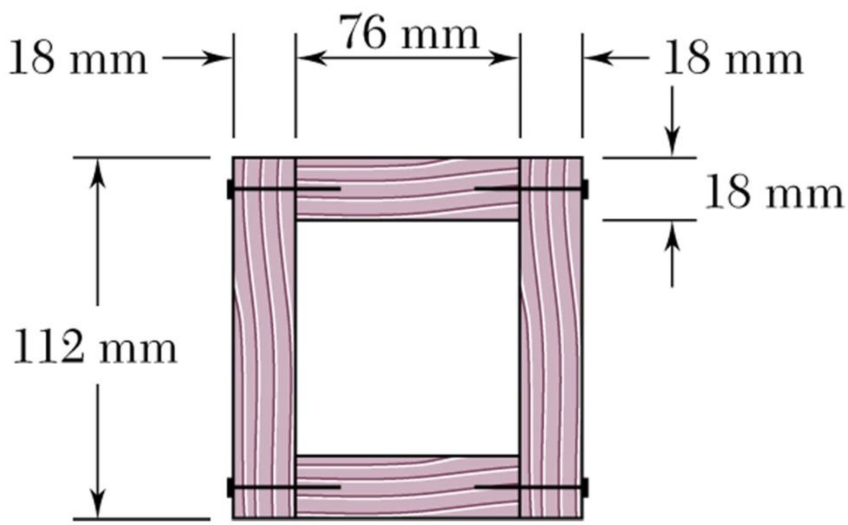
$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_C)\, dA$$

So only the integration area is different, hence result same as before, i.e.,

$$\Delta H = \frac{VQ}{I}\Delta x \qquad q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I}$$

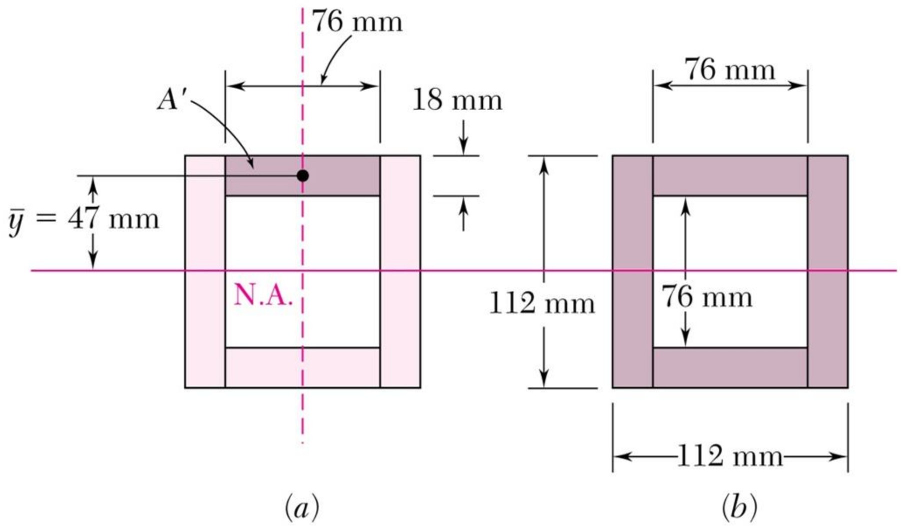
Will use this for thin walled members also

Example 6.3



Square box beam constructed from four planks. Spacing between nails is 44 mm. Vertical shear force $V = 2.5$ kN. Find shearing force in each nail.

Example 6.3



For the upper plank,

$$Q = A'y = (18\,\text{mm})(76\,\text{mm})(47\,\text{mm})$$
$$= 64296\,\text{mm}^3$$

For the overall beam cross-section,

$$I = \tfrac{1}{12}(112\,\text{mm})^4 - \tfrac{1}{12}(76\,\text{mm})^4$$
$$= 10332\,\text{mm}^4$$

SOLUTION:

Determine the shear force per unit length along each edge of the upper plank.

$$q = \frac{VQ}{I} = \frac{(2500\,\text{N})(64296\,\text{mm}^3)}{10332\,\text{mm}^4} = 15.6\,\frac{\text{N}}{\text{mm}}$$

$$f = \frac{q}{2} = 7.8\,\frac{\text{N}}{\text{mm}}$$

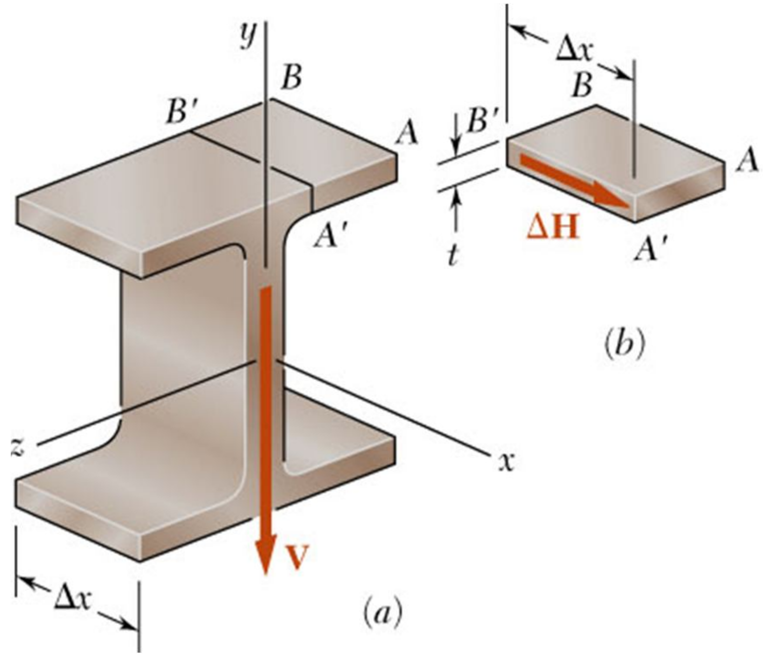
= edge force per unit length

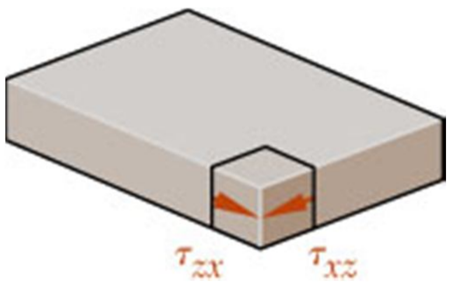
Based on the spacing between nails, determine the shear force in each nail.

$$F = f\,\ell = \left(7.8\,\frac{\text{N}}{\text{mm}}\right)(44\,\text{mm})$$

$$\boxed{F = 343.2\,\text{N}}$$

Shearing Stresses in Thin-Walled Members





Shear stress assumed constant through thickness t, i.e., due to thinnness our averaging is now accurate/exact.

Consider I-beam with vertical shear V.

Longitudinal shear force on element is

$$\Delta H = \frac{VQ}{I}\Delta x$$

Corresponding shear stress is

$$\tau_{zx} = \tau_{xz} \approx \frac{\Delta H}{t\,\Delta x} = \frac{VQ}{It}$$

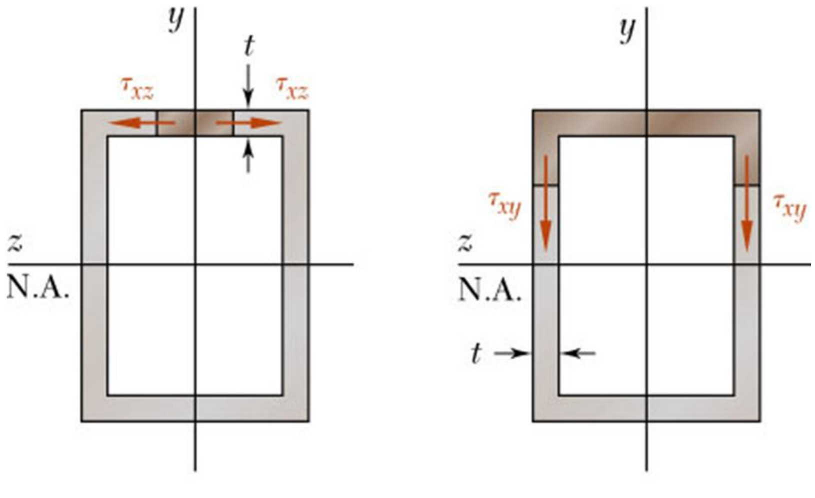
Previously had similar expression for shearing stress web

$$\tau_{xy} = \frac{VQ}{It}$$

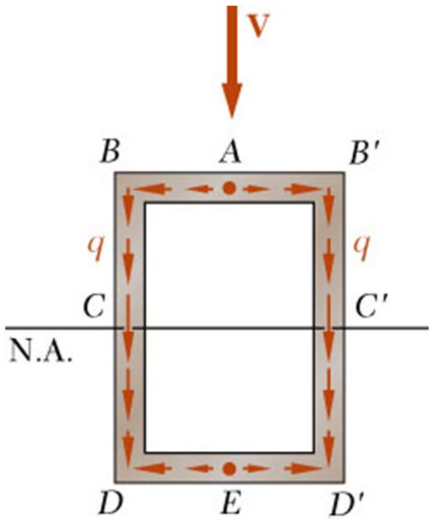
NOTE: $\tau_{xy} \approx 0$ in the flanges

$\tau_{xz} \approx 0$ in the web

Shearing Stresses in Thin-Walled Members



(a) (b)



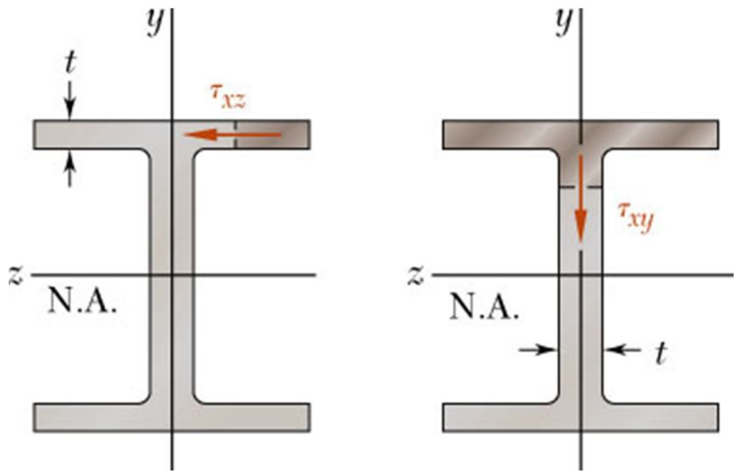
The variation of shear flow across the section depends only on the variation of the first moment.

$$q = \tau t = \frac{VQ}{I}$$

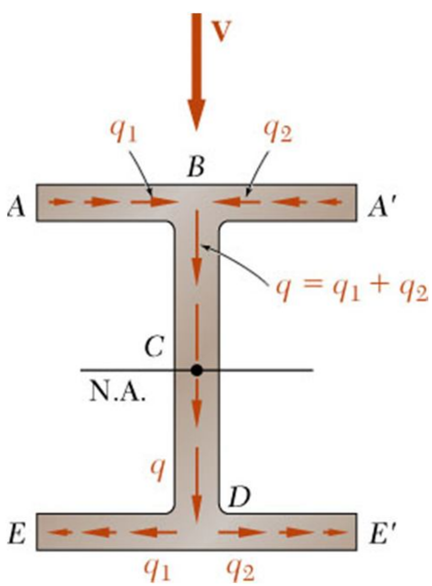
For a box beam, *q* grows smoothly from zero at A to a maximum at *C* and *C'* and then decreases back to zero at *E*.

The sense of *q* in the horizontal portions of the section may be deduced from the sense in the vertical portions or the sense of the shear *V*.

Shearing Stresses in Thin-Walled Members

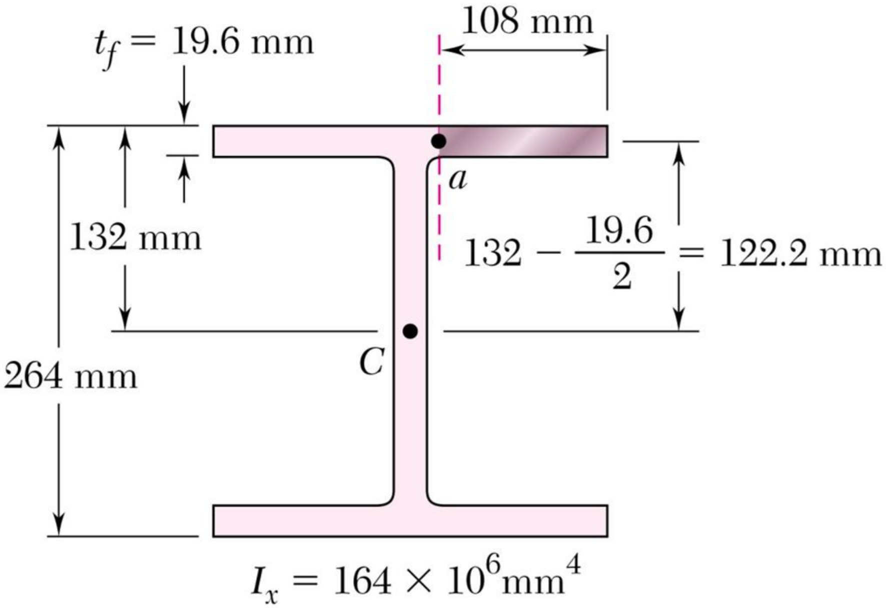


For wide-flange beam, shear flow *q* increases symmetrically from zero at *A* and *A'*, reaches a maximum at *C* and then decreases to zero at *E* and *E'*.



The continuity of the variation in *q* and the merging of *q* from section branches suggests an analogy to fluid flow.

Example 6.4



Vertical shear is 200 kN in a W250x101 rolled-steel beam. Find horizontal shearing stress at *a*.

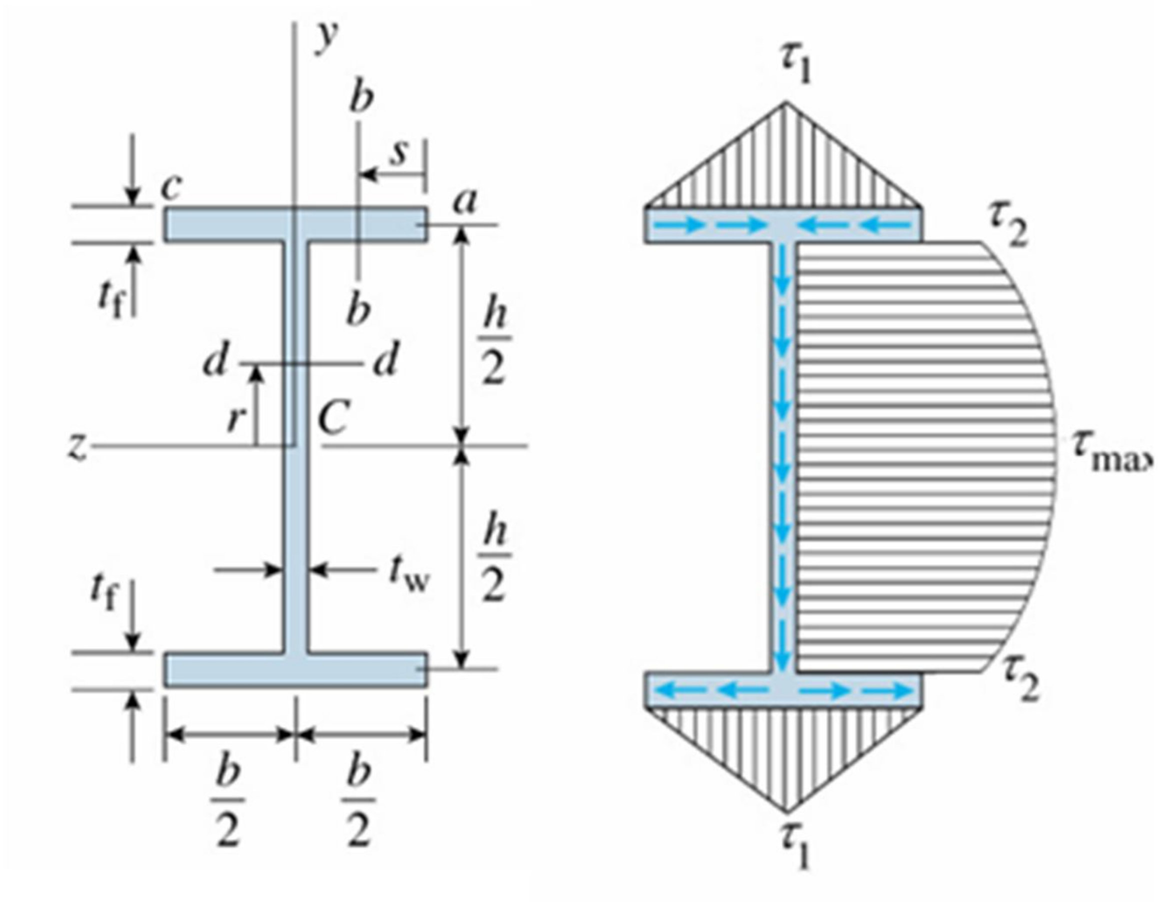
$$Q = (108\,\text{mm})(19.6\,\text{mm})(122.2\,\text{mm})$$
$$= 258700\,\text{mm}^3$$

Shear stress at *a*,

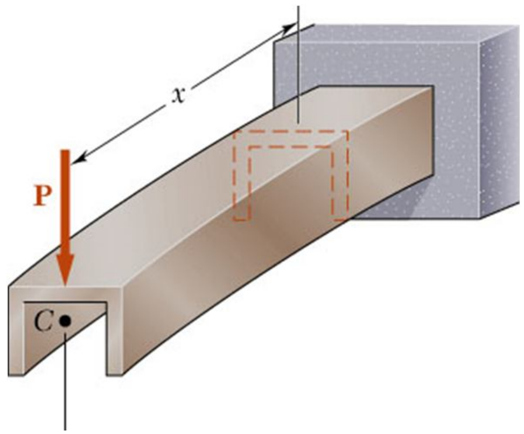
$$\tau = \frac{VQ}{It} = \frac{(200\times10^3\ \text{N})(258.7\times10^{-6}\ \text{m}^3)}{(164\times10^{-6}\ \text{m}^4)(0.0196\,\text{m})}$$

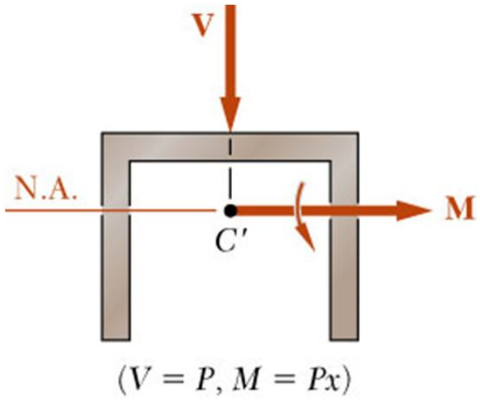
$$\boxed{\tau = 16.1\,\text{MPa}}$$

Work out this example of a wide flange beam (Doubly symmetric)



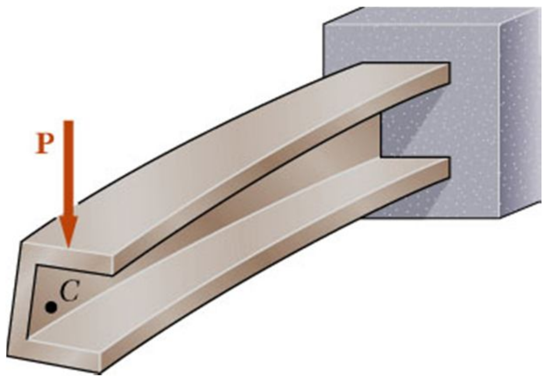
Unsymmetric Loading of Thin-Walled Members

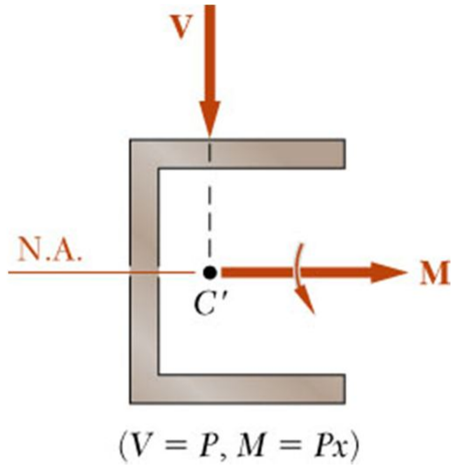




Beam loaded in vertical plane of symmetry, deforms in symmetry plane without twisting.

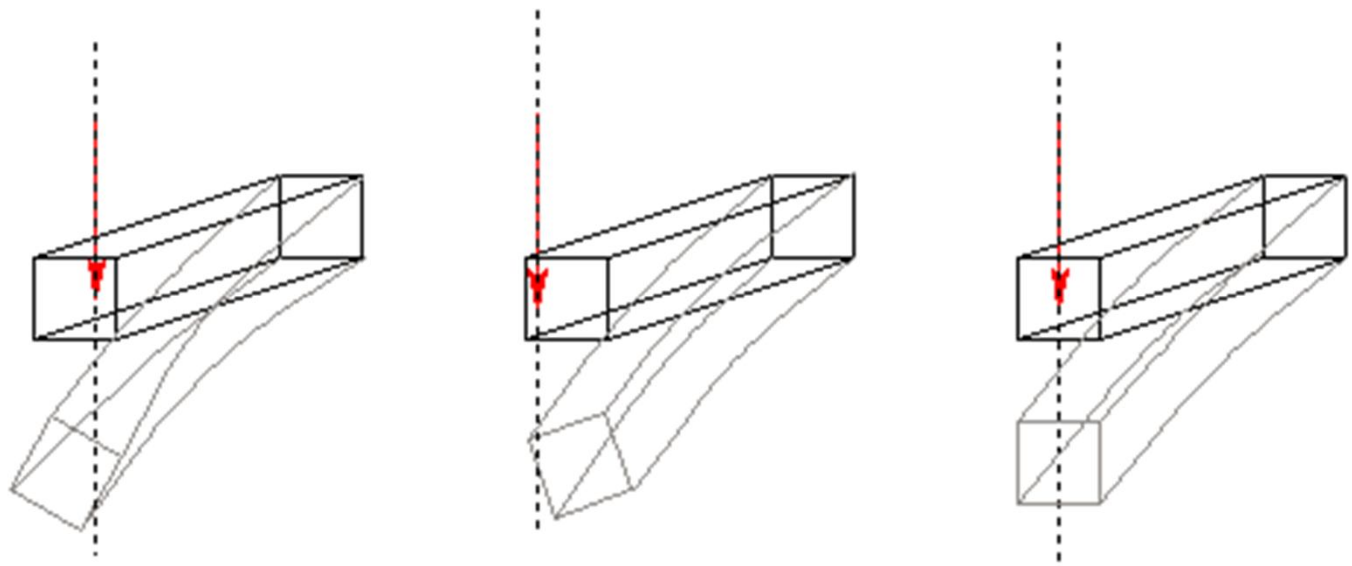
$$\sigma_x = -\frac{My}{I} \qquad \tau_{ave} = \frac{VQ}{It}$$





Beam without vertical plane of symmetry bends and twists under loading.

$$\sigma_x = -\frac{My}{I} \qquad \tau_{ave} \neq \frac{VQ}{It}$$

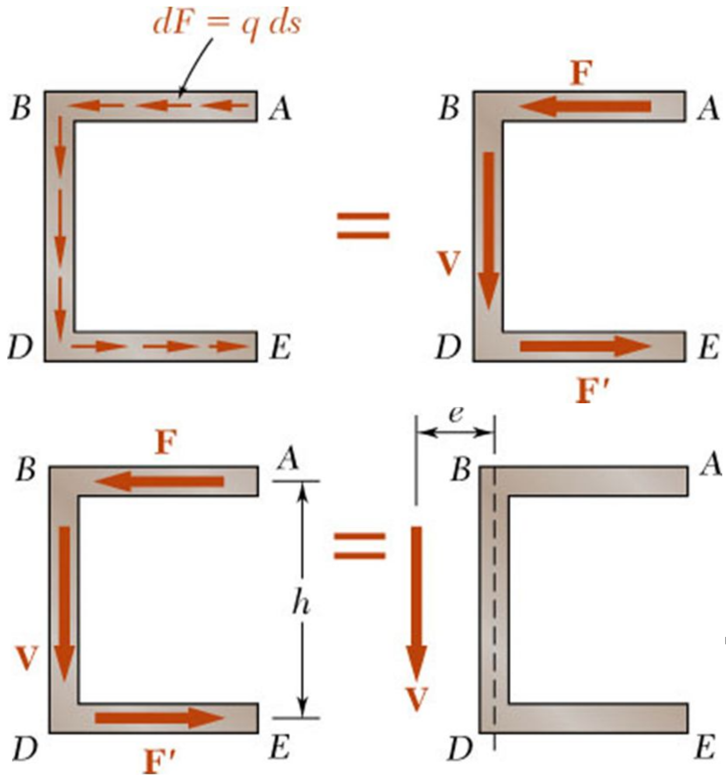


Bending+Torsion effect

Bending+Torsion effect

Pure Bending

Unsymmetric Loading of Thin-Walled Members



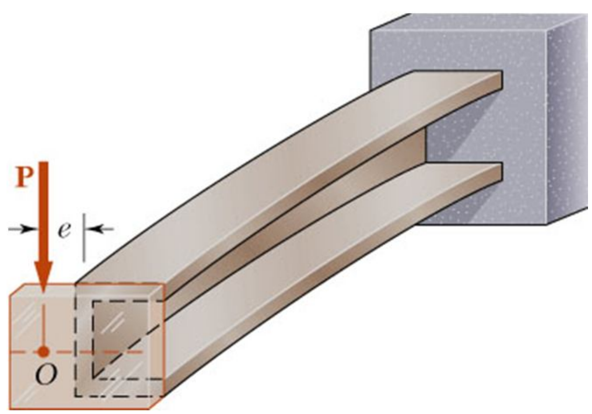
If shear load applied such that beam does not twist, then shear stress distribution satisfies

$$\tau_{ave} = \frac{VQ}{It} \quad V = \int_B^D q\, ds \quad F = \int_A^B q\, ds = -\int_D^E q\, ds = -F'$$

F and *F'* form a couple *Fh*. Thus we have a torque as well as shear load. Static equivalence yields,

$$Fh = Ve$$

Thus if force P applied at distance *e* to left of web centerline, the member bends in vertical plane without twisting. Net torsional moment is *Fh*-*Ve* = 0, so shear stresses due to bending shear only, and not due to torsional shear.



Point *O* is *shear center* of the beam section.

If load not applied thru shear center then net torsional moment exists, so total shear stress due to bending shear & torsional shear (ref. open thin walled torsion)

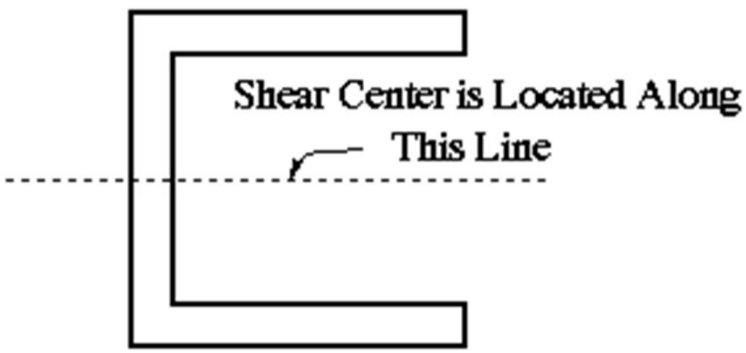
Facts about Shear Center

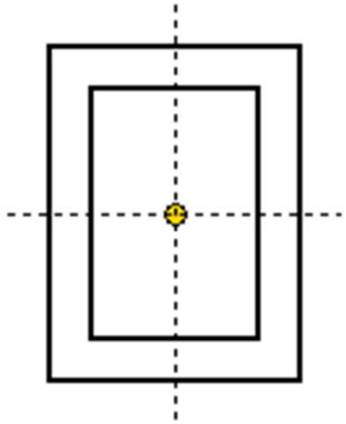
When force applied at shear center, it causes pure bending & no torsion.

Its location depends on cross-sectional geometry only.

If cross-section has axis of symmetry, then shear center lies on the axis of symmetry **(but it may not be at centroid itself).**

If cross section has two axes of symmetry, then shear center is located at their intersection. This is the only case where shear center and centroid coincide.



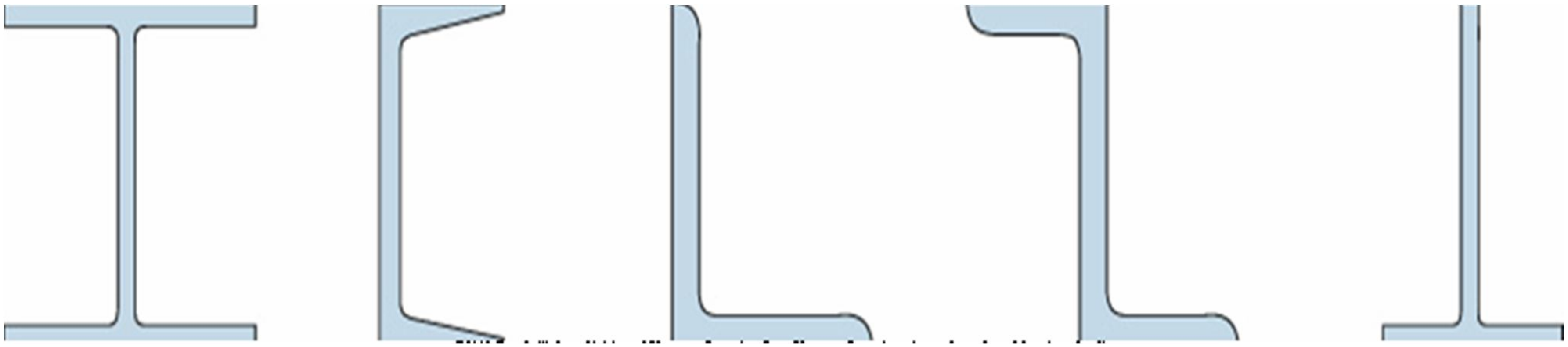


Shear Center is at Centroid

Want to find shear flow and shear center of thin-walled open cross-sections.

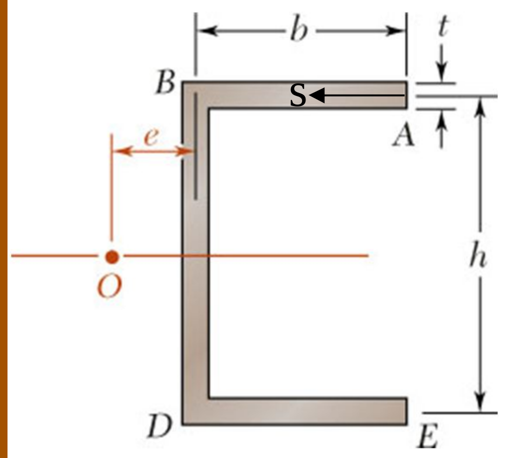
For I and Z -sections s.c. at centroid.

For L and T -sections s.c. at intersection of the two straight limbs, i.e., where bending shear stresses cause zero torsional moment.



Thin-walled cross sections are very weak in torsion, therefore load must be applied through shear center to avoid excessive twisting

Example 6.5

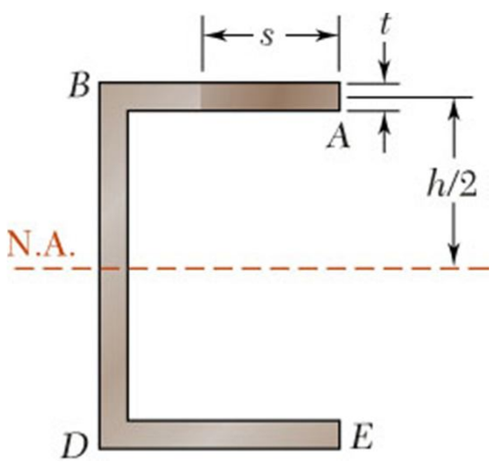


Determine location of shear center of channel section with b = 100 mm, h = 150 mm, and t = 4 mm

$$e = \frac{F\,h}{V}$$

$$F = \int_0^b \tau_{xz} t_f \; ds = \int_0^b q \; ds = \int_0^b \frac{VQ}{I} ds = \frac{V}{I}\int_0^b s t_f \frac{h}{2} ds$$

$$= \frac{V t_f h b^2}{4I}$$



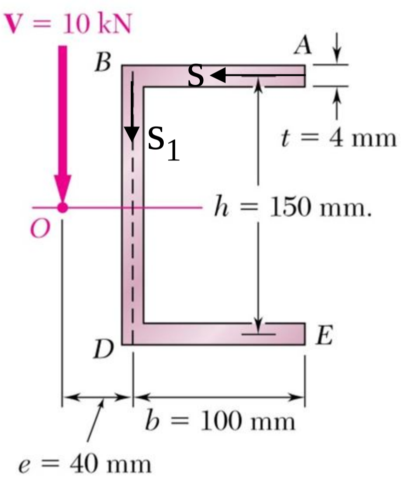
$$I = I_{web} + 2 I_{flange} = \frac{1}{12} t_w h^3 + 2\left[\frac{1}{12} b t_f^{\,3} + b t_f \left(\frac{h}{2}\right)^2\right]$$

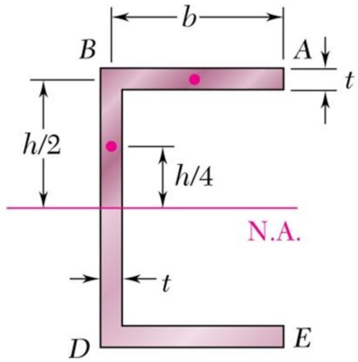
$$\cong \frac{1}{12} t_w h^3 + b t_f \frac{h^2}{2}$$

$$e = \frac{3 t_f b^2}{t_w h + 6 t_f b} = \frac{3 \times 100^{\,2}}{150 + 6 \times 100}$$

$$\boxed{e = 40\,\mathrm{mm}}$$

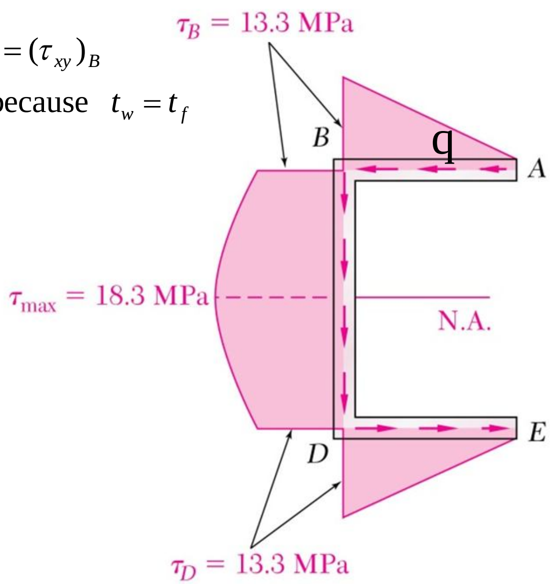
Example 6.6





$(\tau_{xz})_B = (\tau_{xy})_B$

only because $t_w = t_f$



Determine shear stress distribution for

V = 10 kN

$$\tau = \frac{q}{t} = \frac{VQ}{It}$$

Shearing stresses in the flanges,

$$\tau_{xz} = \frac{VQ}{It_f} = \frac{V}{It_f}(st_f)\frac{h}{2} = \frac{Vh}{2I}s$$

$$(\tau_{xz})_B = \frac{Vh}{2I}b = \frac{Vb}{\left(\frac{1}{6}t_w h + bt_f\right)h}$$

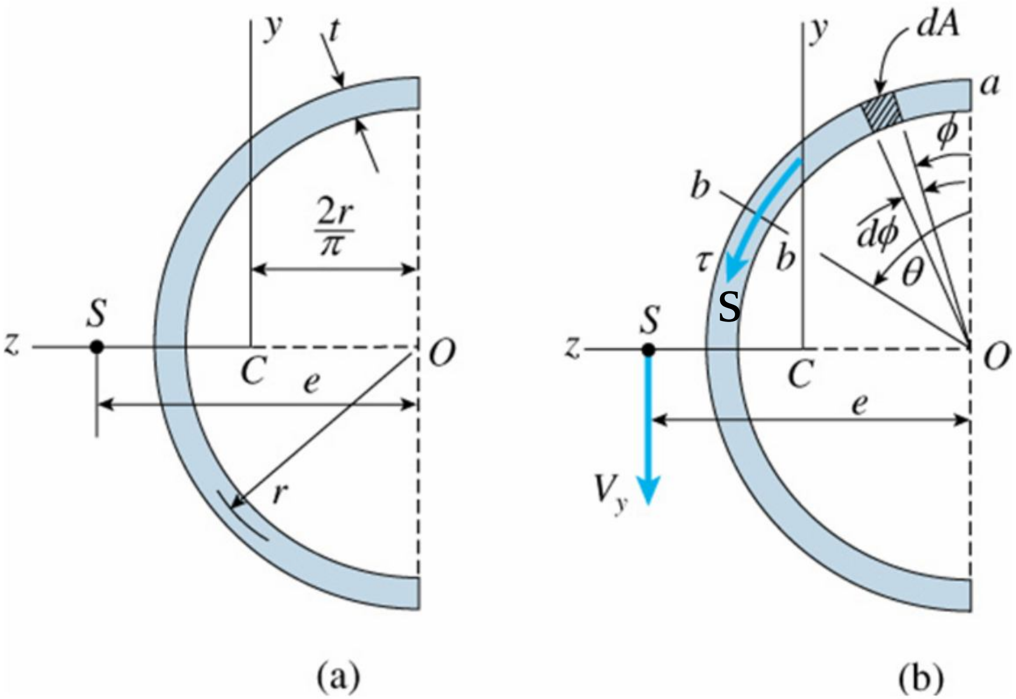
$$= \frac{6(10000\,\text{N})(0.1)}{(0.004)(0.15 + 6\times 0.1)\times 0.15} = 13.3\,\text{MPa}$$

Shearing stress in the web,

$$\tau_{xy} = \frac{VQ}{It_w} = \frac{12V\left(\frac{bt_f h}{2} + s_1 t_w\left(\frac{h}{2} - \frac{s_1}{2}\right)\right)}{(t_w h + 6bt_f)h^2 t_w}$$

$$(\tau_{xy})_{\max} = (\tau_{xy})_{s_1 = h/2} = 18.3\,\text{MPa}$$

Shear center of a thin walled semicircular cross-section



$$V_y = V$$

(a) Find shear stress ($\tau_{x\theta}$) at an angle θ, i.e., at section *bb*

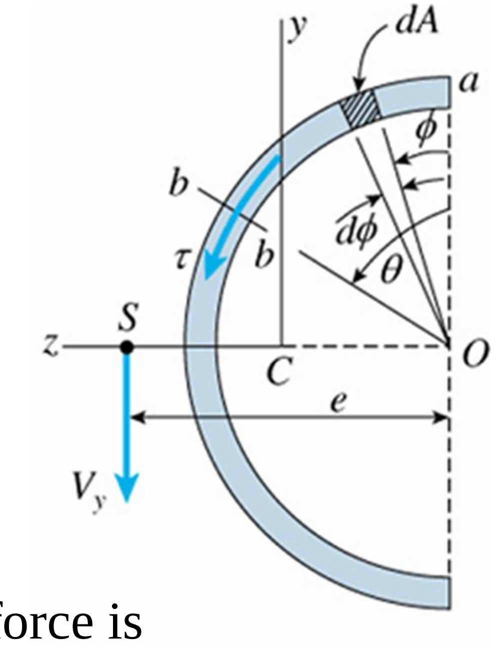
Find the first moment of the cross-sectional area between point *a* and section *bb*

$$Q = \int y\,dA = \int_0^\theta (r\cos\phi)(t\,r\,d\phi) = r^2 t \sin\theta$$

$$\tau_{x\theta} = \frac{VQ}{It} = \frac{V\left(r^2 t\sin\theta\right)}{\left(\pi r^3 t/2\right)t} = \frac{2V\sin\theta}{\pi\, r\, t}$$

(b) Find the shear center (S)

Moment about geometric center of circle *O*, due to the shear force is *Ve*



Shear stress acting on element *dA*

$$\tau_{x\theta} = \frac{2V\sin\phi}{\pi r t}$$

Corresponding force is τ_{xq} *dA* and moment due to this force is

$$dM_o = r(\tau_{x\theta}\, dA) \qquad dA = (rd\phi)t$$

$$M_o = \int dM_o = \int_0^{\pi} r\frac{2V\sin\phi}{\pi r t}(rd\phi)t = \frac{4rV}{\pi}$$

$$M_o = Ve \quad \Rightarrow \quad e = \frac{4r}{\pi}$$