CHAPTER

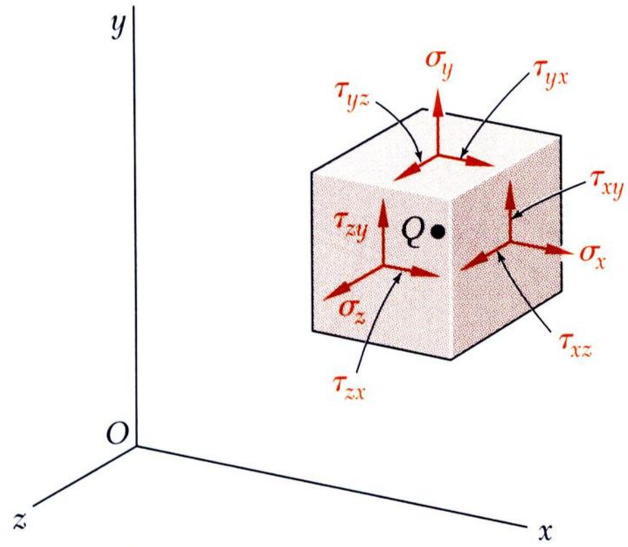
7

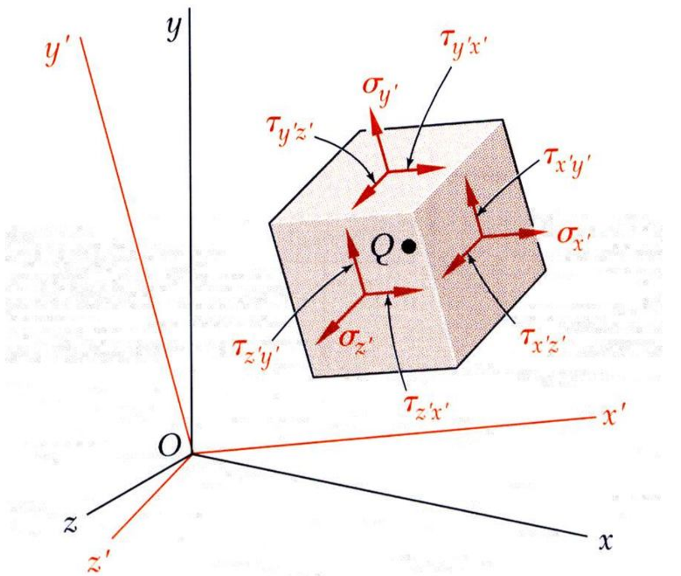
MECHANICS OF MATERIALS

Transformations of Stress and Strain

Transformations of Stress and Strain

Introduction





- General state of stress at a point represented by 6 components,

  $\sigma_x, \sigma_y, \sigma_z$ normal stresses

  $\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

  (Note: $\tau_{xy} = \tau_{yx},\ \tau_{yz} = \tau_{zy},\ \tau_{zx} = \tau_{xz}$)

- If axis are rotated, the same state of stress is represented by a different set of components, i.e., the stress components get transformed.

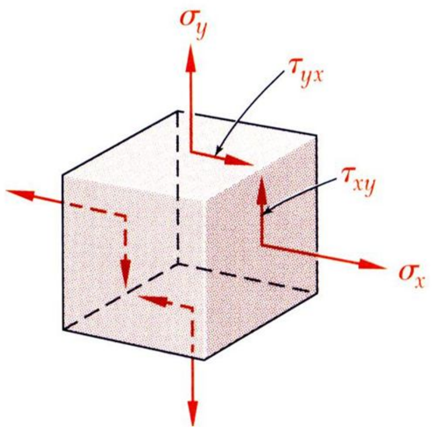
  $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}$ normal stresses

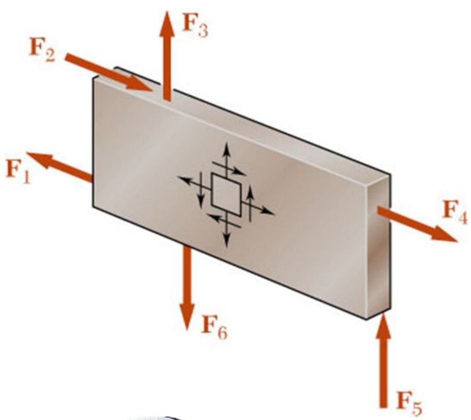
  $\tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ shearing stresses

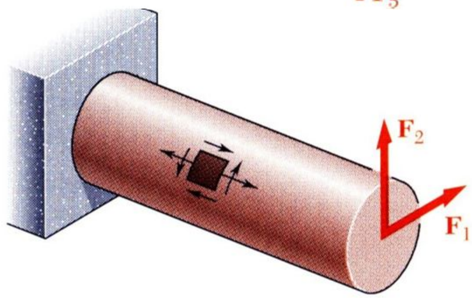
  (Note: $\tau_{x'y'} = \tau_{y'x'},\ \tau_{y'z'} = \tau_{z'y'},\ \tau_{z'x'} = \tau_{x'z'}$)

- First we consider transformation of stress components, due to rotation of coordinate axes. Then we consider a similar transformation of strain components.

Plane Stress







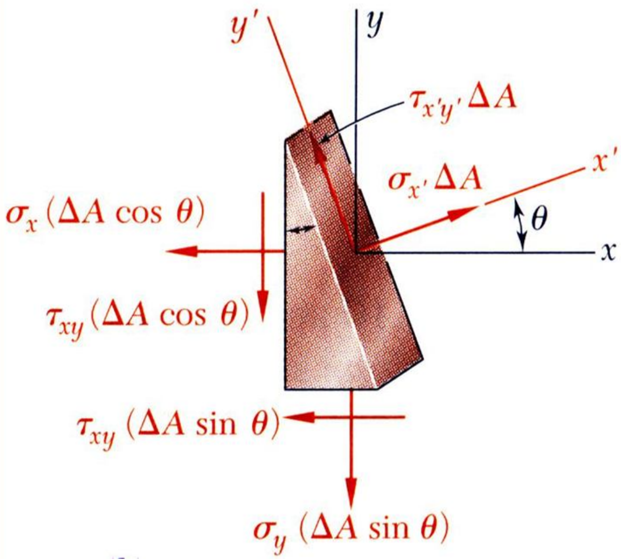
- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. Example,

$$\sigma_x,\ \sigma_y,\ \tau_{xy} \quad \text{are nonzero,} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

- For example, state of plane stress occurs in thin plate subjected to forces acting in midplane of plate.

- Another example of plane stress is on free surface, i.e., unloaded point on surface.

Transformation of Plane Stress



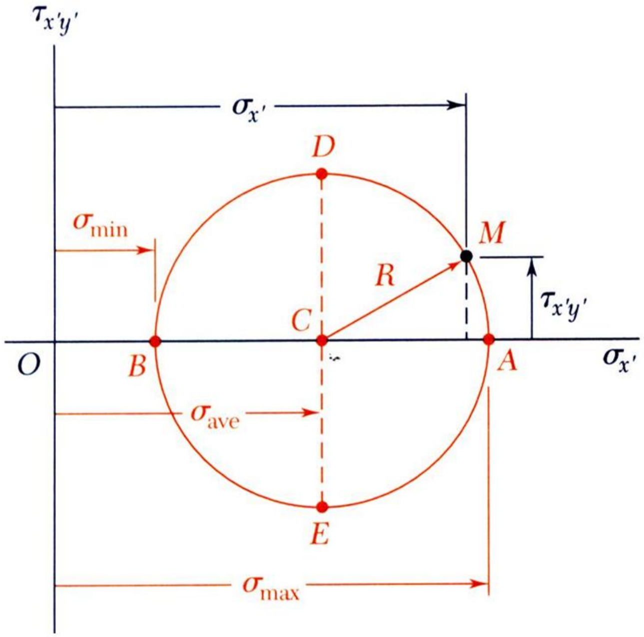
- Consider equilibrium of prismatic element with faces perpendicular to *x*, *y*, and *x'* axes.

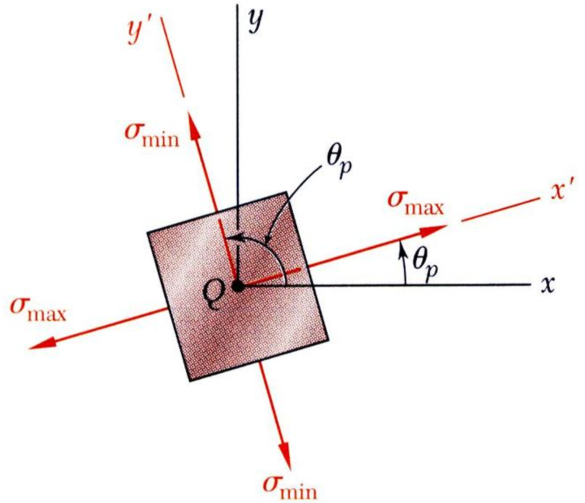
$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta$$
$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta$$

- Solving for transformed stress components,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Principal Stresses





- Eliminating q , this yields equation of a circle,

$$(\sigma_{x'} - \sigma_{ave})^2 + \tau_{x'y'}^2 = R^2$$

where

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

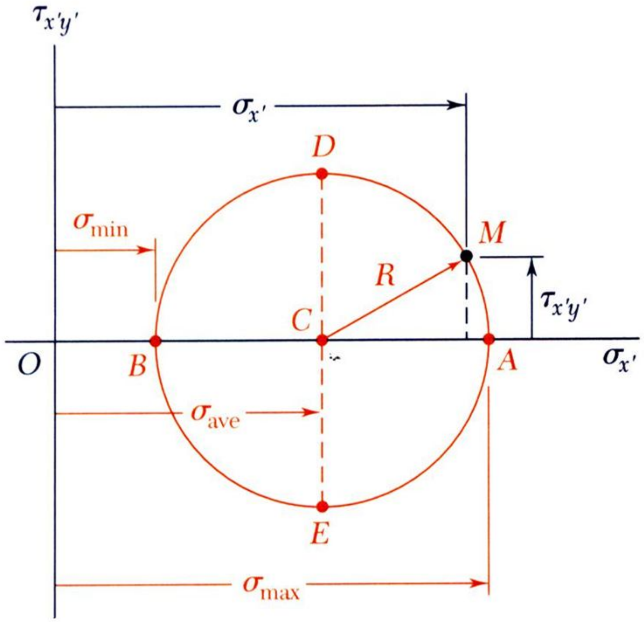
- *Principal stresses* occur on *principal planes* on which there exist *zero shearing stresses*.

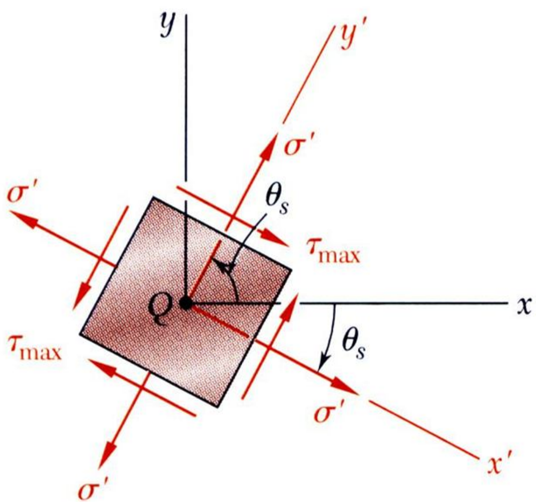
$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \ \text{from } \tau_{x'y'} = 0$$

Note: defines two angles separated by 90°

Maximum Shearing Stress





Maximum shearing stress occurs for $\sigma_{x'} = \sigma_{ave}$

$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \left|\frac{\sigma_{\max} - \sigma_{\min}}{2}\right|$$

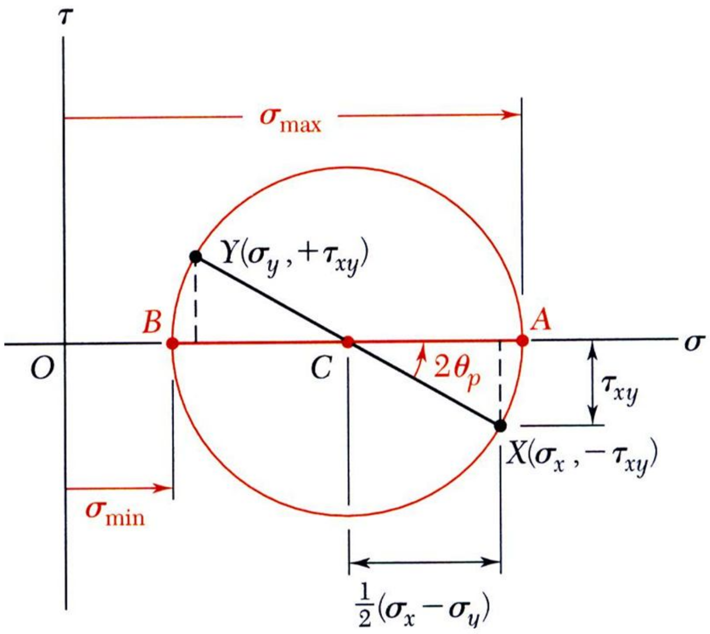
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}, \text{ from } \sigma_{x'} = \sigma_{\text{ave}}$$

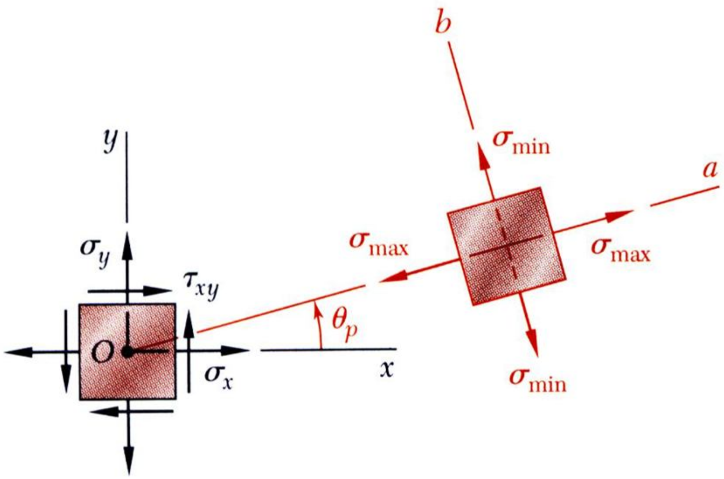
Note : defines two angles separated by 90° and offset from θ_p by 45°

Corresponding normal stresses are :

$$\sigma_{x'} = \sigma_{y'} = \sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

Mohr's Circle for Plane Stress





- Used to graphically find principal stresses and planes and maximum shear stresses and planes

- For known $\sigma_x, \sigma_y, \tau_{xy}$ plot points *X* and *Y* and construct circle centered at *C*.

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

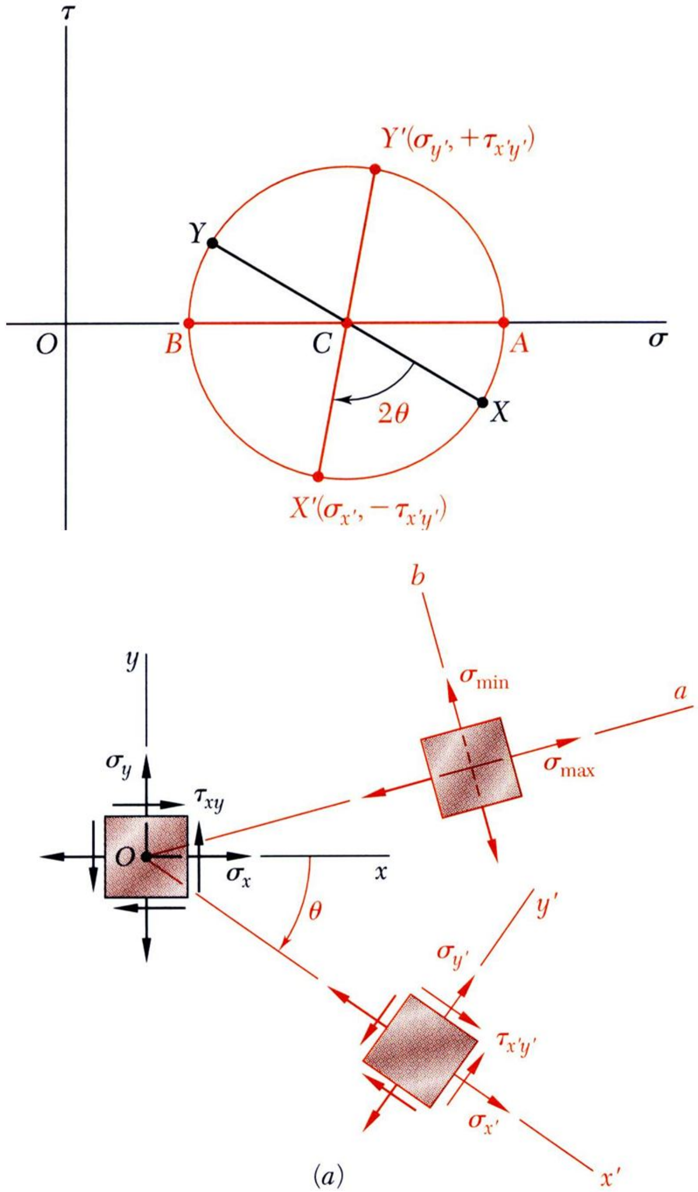
- Principal stresses obtained at *A* and *B*.

$$\sigma_{\max,\min} = \sigma_{ave} \pm R$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Direction of rotation of *Ox* to *Oa* (ie., in physical plane) is same as *CX* to *CA* (ie., in Mohr plane)

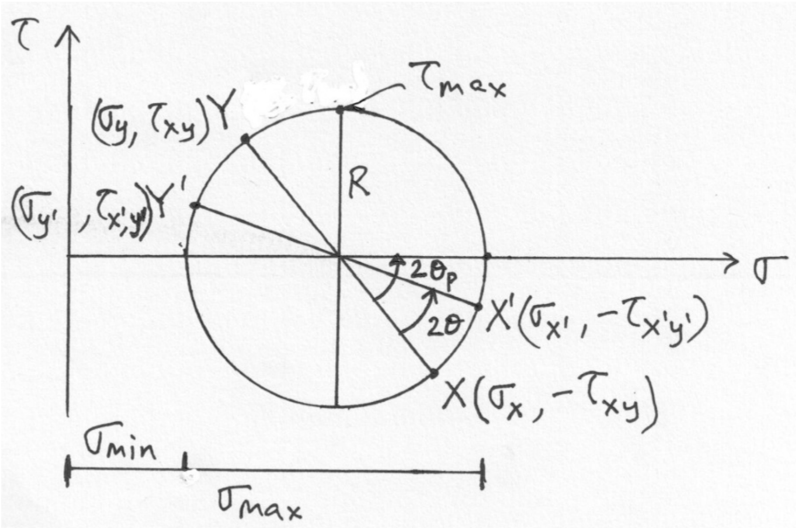
Mohr's Circle for Plane Stress

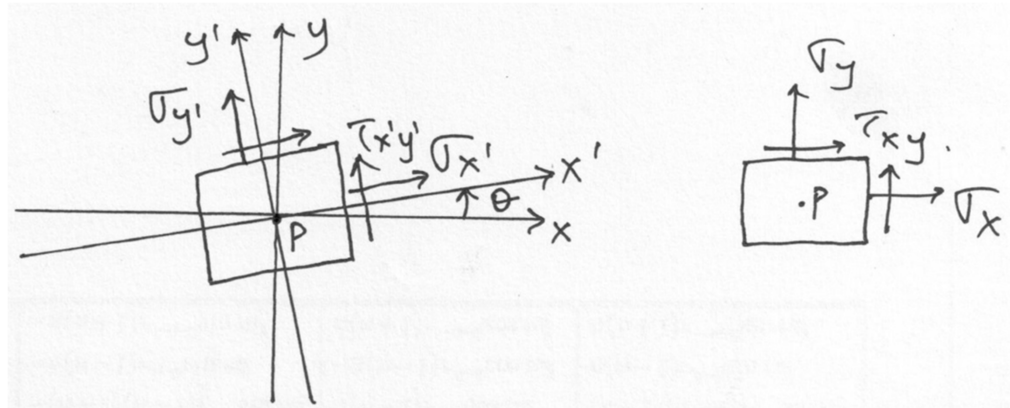


(a)

- From Mohr's circle we can find state of stress at other axes orientations.
- For state of stress at angle θ with respect to the *xy* axes, construct a new diametral line *X'Y'* at angle 2θ with respect to *XY*.
- Coordinates of *X'*, *Y'* are the transformed normal and shear stresses.

Proof of Mohr's Circle construction





$$\sigma_{x'} = \sigma_{ave} + R\cos(2\theta_p - 2\theta) = \sigma_{ave} + R(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)$$

$$= \sigma_{ave} + R\left(\frac{\sigma_x - \sigma_y}{2R}\cos 2\theta + \frac{\tau_{xy}}{R}\sin 2\theta\right) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \sigma_{ave} - R\cos(2\theta_p - 2\theta) = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

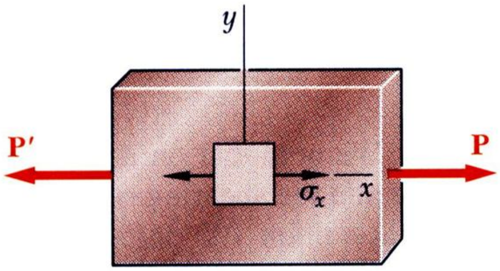
$$\tau_{x'y'} = R\sin(2\theta_p - 2\theta) = R(\sin 2\theta_p \cos 2\theta - \cos 2\theta_p \sin 2\theta)$$

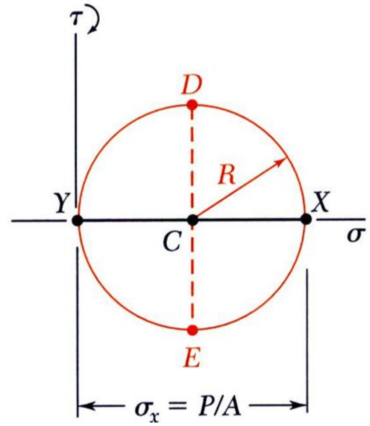
$$= R\left(\frac{\tau_{xy}}{R}\cos 2\theta - \frac{\sigma_x - \sigma_y}{2R}\sin 2\theta\right) = \tau_{xy}\cos 2\theta - \frac{\sigma_x - \sigma_y}{R}\sin 2\theta$$

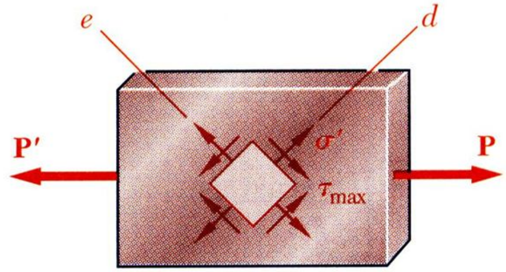
So we get same formulae as before

Mohr's Circle for Plane Stress

- Mohr's circle for centric axial loading:





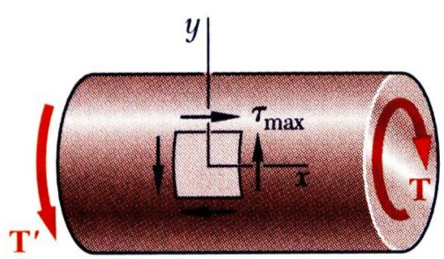


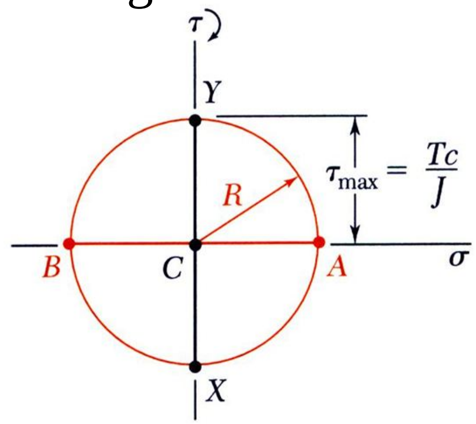
$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \tau_{xy} = 0$$

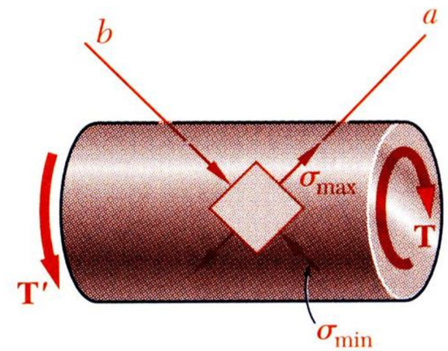
$$\sigma_x = \sigma_y = \tau_{xy} = \frac{P}{2A} = \tau_{max}$$

on 45^0 element (plane)

- Mohr's circle for torsional loading:







$$\sigma_x = \sigma_y = 0 \quad \tau_{xy} = \frac{Tc}{J}$$

$$\sigma_x = \sigma_y = \frac{Tc}{J} = \sigma_{max} \;; \quad \tau_{xy} = 0$$

on 45^0 element (plane)

Points to note when drawing stress block.

Formulae for principal stresses yield their magnitude and sense/sign (+ve or – ve), and the principal planes on which they act ($\sigma_{\max}$, $\sigma_{\min}$, θ_p). However, they do not identify which principal stress acts on which plane.

So once you find principal stresses $\sigma_{\max}$, $\sigma_{\min}$ and associated angles θ_{p} , put the angles in the transformation relations to identify which principal stress acts on which plane.

Points to note when drawing stress block.

Formulae for maximum shear stress yield only magnitude of τ_{max} and planes θ_s on which they act. However, they do not yield the sense/sign (+ve or - ve) of τ_{max} .

So once you find θ_s , put it in the transformation relations to find correct sense/sign (+ve or - ve) of τ_{max} .

So you can just find θ_s and θ_p and use them in transformation relations to find associated σ_{min} , σ_{max} , τ_{max} with correct sense/sign

Points to note when drawing stress block.

Alternately, you can use Mohr circle which gives correct magnitudes and sense of $\sigma_{\max}$, $\sigma_{\min}$, $\tau_{\max}$, and the planes θ_p , θ_s on which they act.

Example1

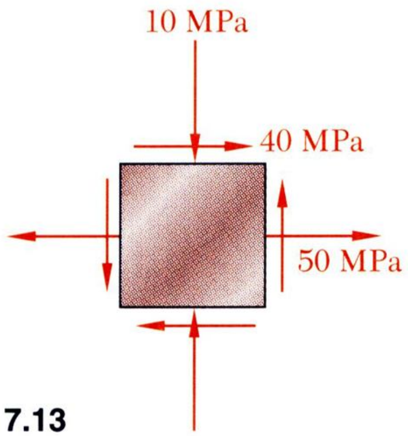


Fig. 7.13

For state of plane stress shown, find (a) principal planes, (b) principal stresses, (c) maximum shearing stress and corresponding normal stress.

Example 1

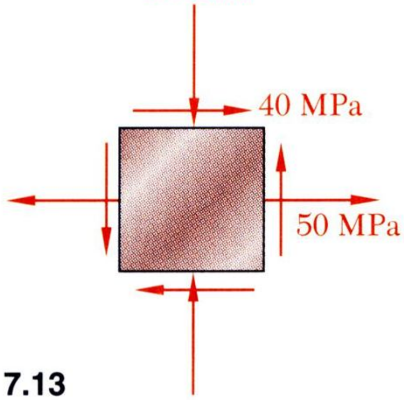


Fig. 7.13

$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

$$\sigma_y = -10\,\text{MPa}$$

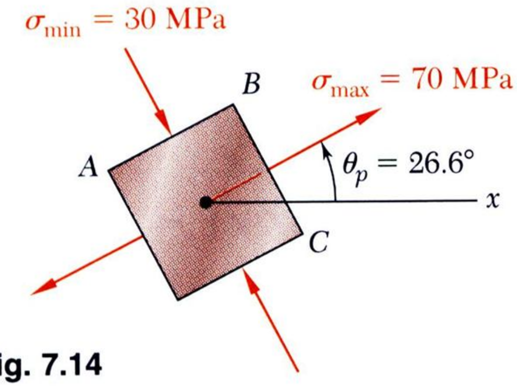


Fig. 7.14

- Find element orientation for principal stresses:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50-(-10)} = 1.333$$

$$2\theta_p = 53.1°,\ 233.1°$$

$$\theta_p = 26.6°,\ 116.6°$$

- Find principal stresses:

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\sigma_{\max} = 70\,\text{MPa}$$
$$\sigma_{\min} = -30\,\text{MPa}$$

Example 1

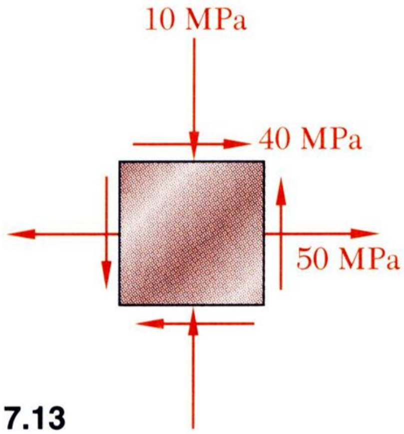


Fig. 7.13

$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

$$\sigma_y = -10\,\text{MPa}$$

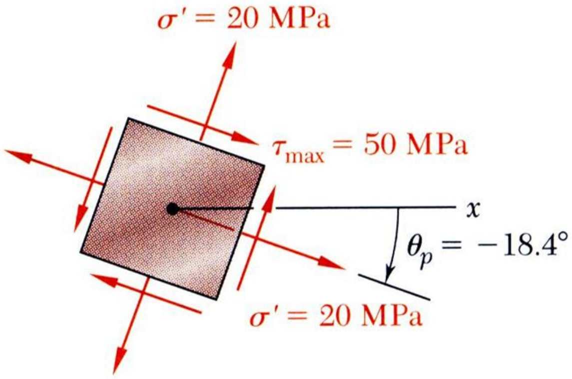


Fig. 7.16

- Find maximum shearing stress:

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\tau_{\max} = 50\,\text{MPa}}$$

$$\theta_s = \theta_p - 45$$

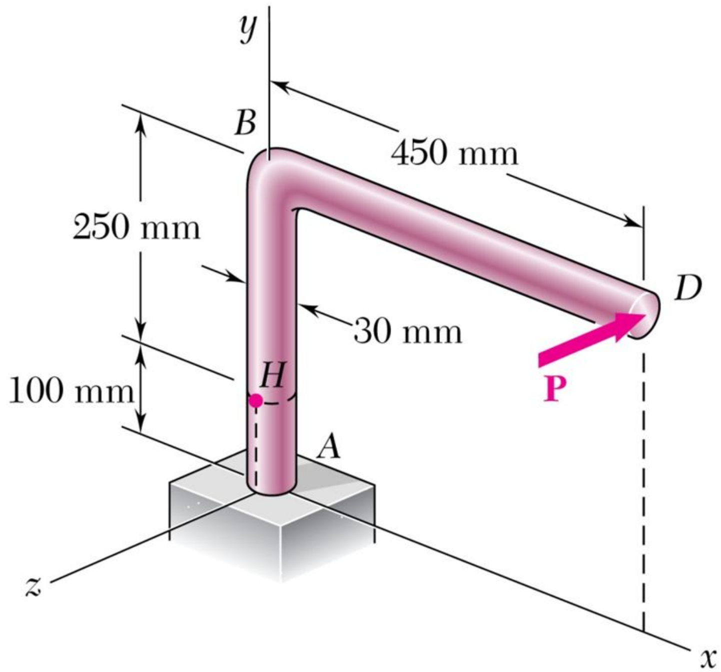
$$\boxed{\theta_s = -18.4^\circ,\ 71.6^\circ}$$

- Find corresponding normal stress:

$$\sigma_{x'} = \sigma_{y'} = \sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50-10}{2}$$

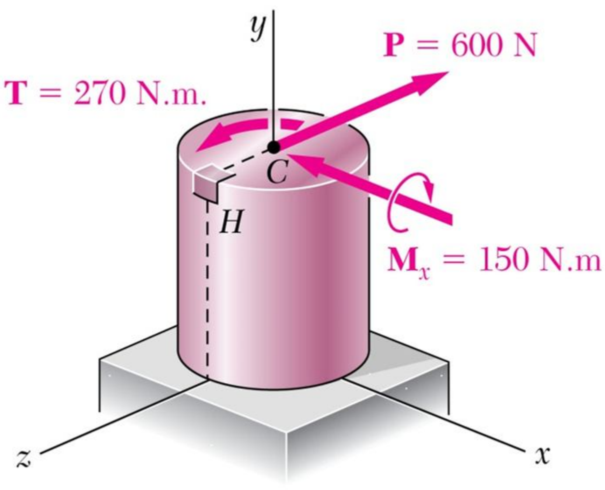
$$\boxed{\sigma' = 20\,\text{MPa}}$$

Example 2



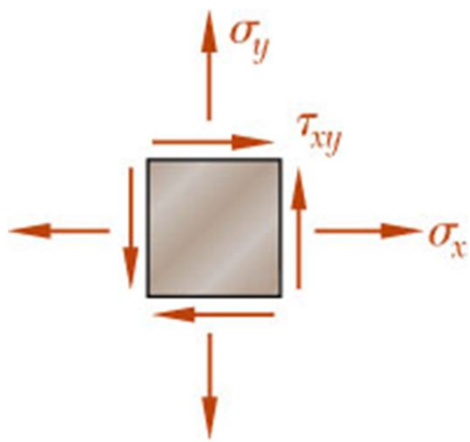
Horizontal force *P* = 600 N magnitude applied to end *D* of lever *ABD*. Find (a) normal and shearing stresses on element at *H* having sides parallel to *x* and *y* axes, (b) principal planes and principal stresses at *H*.

Example 2



Find equivalent force-couple system at center of transverse section passing through *H*.

$$P = 600\ \text{N}$$
$$T = (600\ \text{N})(0.45\ \text{m}) = 270\ \text{Nm}$$
$$M_x = (600\ \text{N})(0.25\ \text{m}) = 150\ \text{Nm}$$



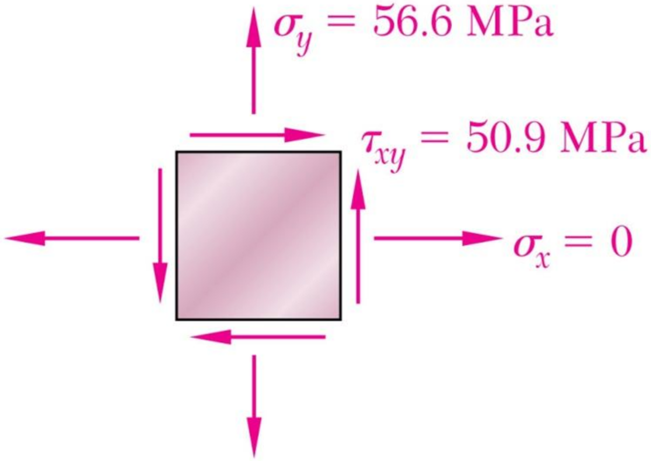
Find normal and shearing stresses at *H*.

$$\sigma_y = +\frac{Mc}{I} = +\frac{(150\ \text{Nm})(0.015\ \text{m})}{\frac{1}{4}\pi(0.015\ \text{m})^4}$$
$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(270\ \text{Nm})(0.015\ \text{m})}{\frac{1}{2}\pi(0.015\ \text{m})^4}$$

$$\sigma_x = 0 \quad \sigma_y = +56.6\ \text{MPa} \quad \tau_y = +50.9\ \text{MPa}$$

Note: t_{yz} due to bending is zero at *H*

Example 2

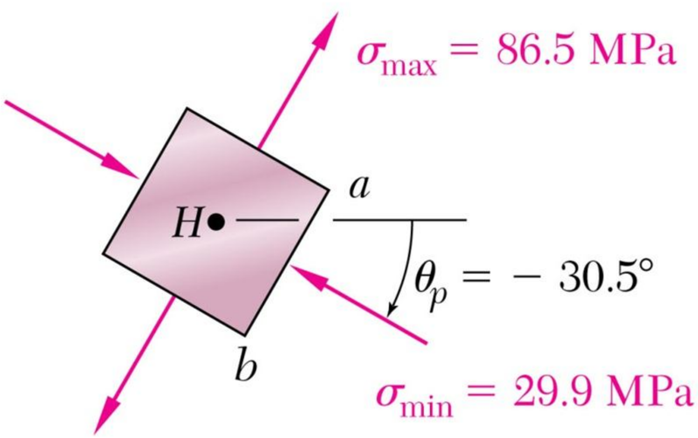


Find principal stresses and planes.

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{0 + 56.6}{2} \pm \sqrt{\left(\frac{0 - 56.6}{2}\right)^2 + (50.9)^2}$$

$$\sigma_{\max} = +86.5\,\text{MPa}$$
$$\sigma_{\min} = -29.9\,\text{MPa}$$



$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(50.9)}{0 - 56.6} = -1.8$$

$$2\theta_p = -61.0°, 119°$$

$$\theta_p = -30.5°,\ 59.5°$$

Example 3

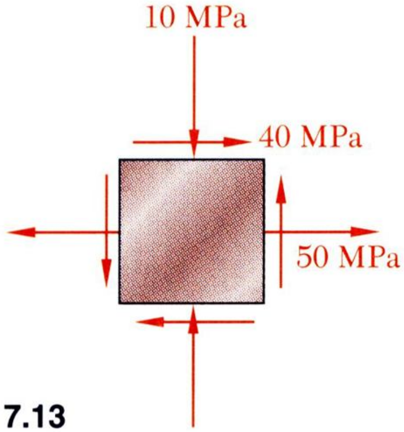
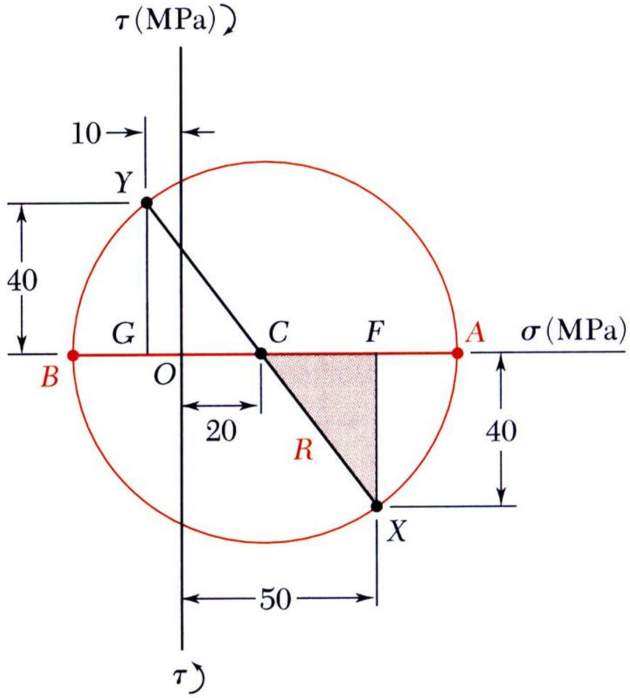


Fig. 7.13



For state of plane stress shown, (a) construct Mohr's circle, find (b) principal planes, (c) principal stresses, (d) maximum shearing stress and corresponding normal stress.

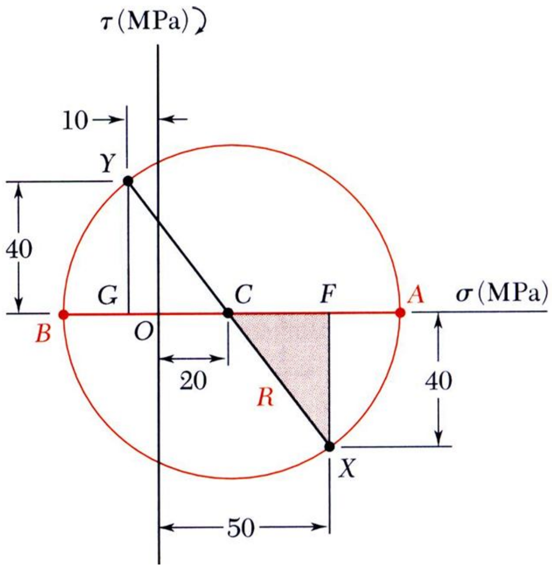
- Construct Mohr's circle

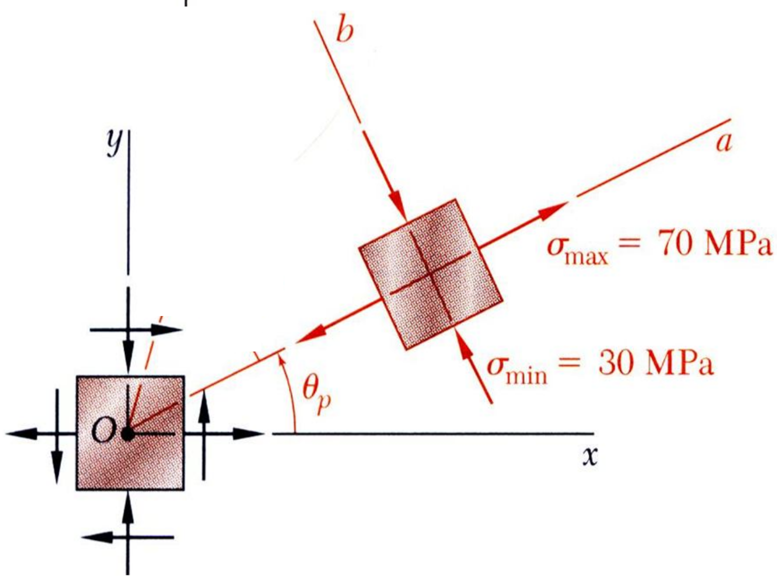
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50) + (-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

Example 3





- Principal planes and stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50$$

$$\sigma_{\max} = 70\,\mathrm{MPa}$$

$$\sigma_{\max} = OB = OC - BC = 20 - 50$$

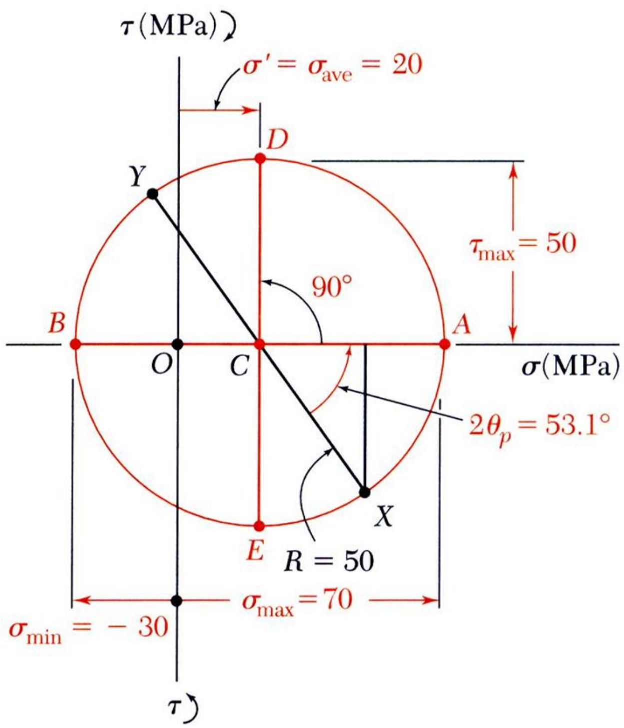
$$\sigma_{\max} = -30\,\mathrm{MPa}$$

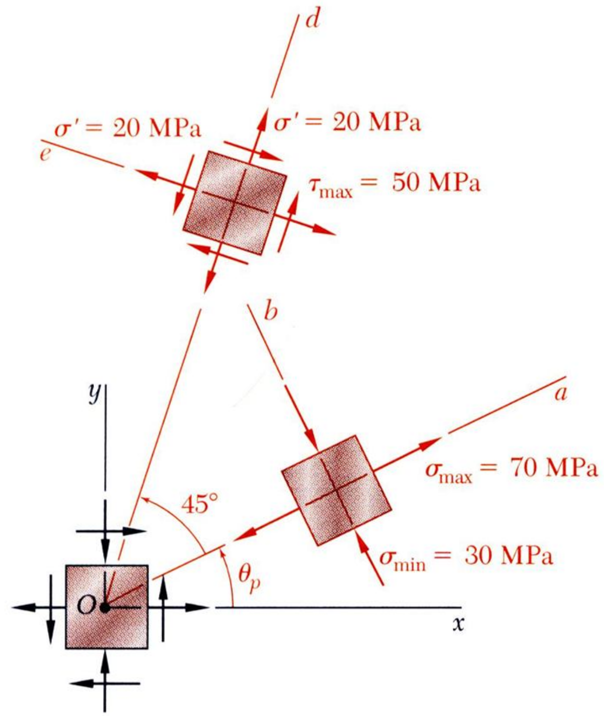
$$\tan 2\theta_p = \frac{FX}{CF} = \frac{40}{30}$$

$$2\theta_p = 53.1°$$

$$\theta_p = 26.6°$$

Example 3



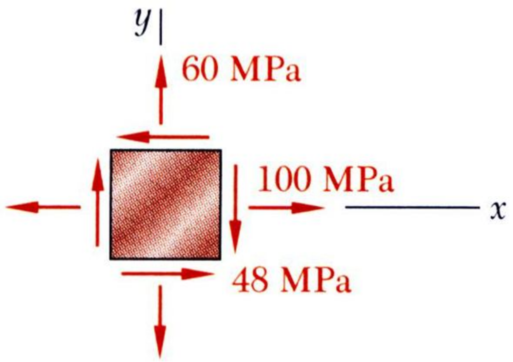


- Maximum shear stress

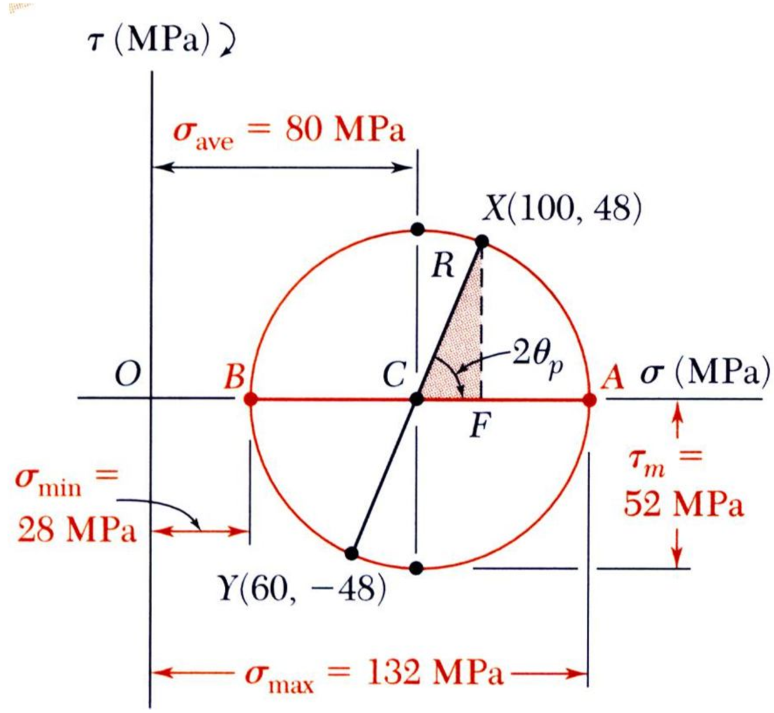
$$\theta_s = \theta_p + 45^\circ \qquad \tau_{max} = R \qquad \sigma' = \sigma_{ave}$$

$$\boxed{\theta_s = 71.6^\circ} \qquad \boxed{\tau_{max} = 50\ \text{MPa}} \qquad \boxed{\sigma' = 20\ \text{MPa}}$$

Example 4



For state of plane stress shown, find (a) principal planes and the principal stresses, (b) stress components on element obtained by rotating given element counterclockwise through 30 degrees.

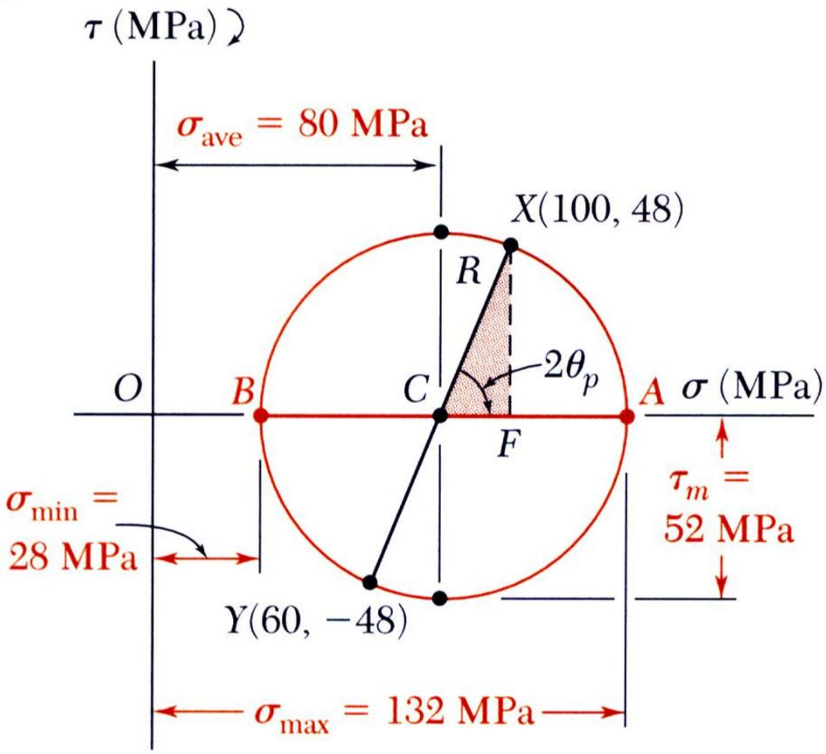


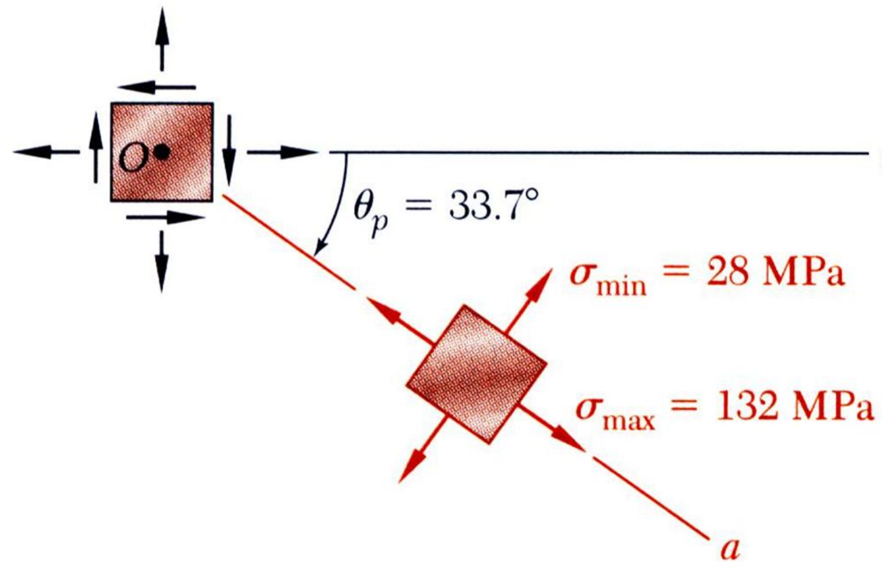
- Construct Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{100+60}{2} = 80\,\text{MPa}$$

$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52\,\text{MPa}$$

Example 4





- Principal planes and stresses

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4$$

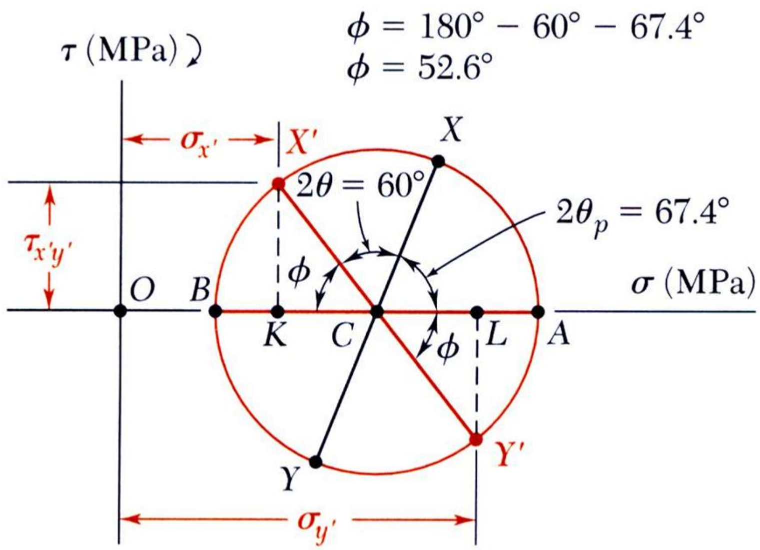
$$2\theta_p = 67.4°$$

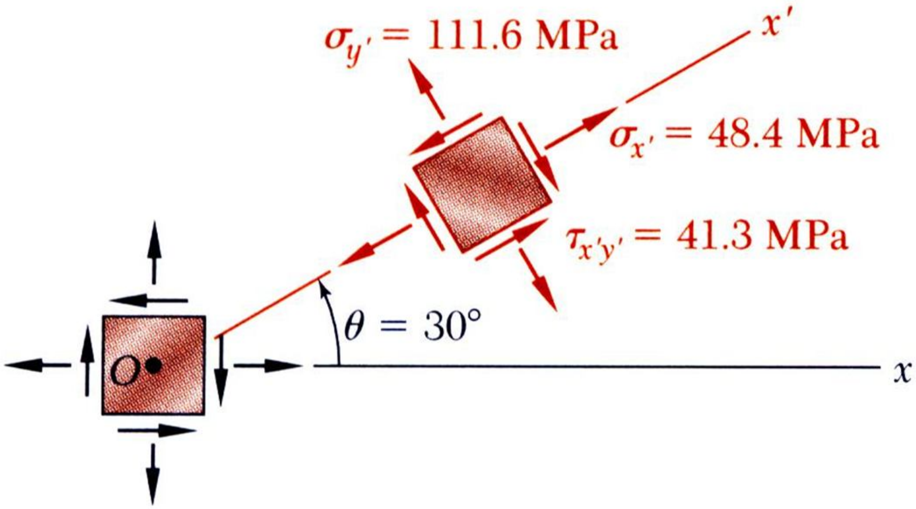
$$\theta_p = 33.7° \text{ clockwise}$$

$$\sigma_{max} = OA = OC + CA$$
$$= 80 + 52$$
$$\sigma_{max} = +132\,\text{MPa}$$

$$\sigma_{max} = OA = OC - BC$$
$$= 80 - 52$$
$$\sigma_{min} = +28\,\text{MPa}$$

Example 4





- Stress components after counterclockwise rotation by 30°

  Points X' and Y' on Mohr's circle, that correspond to stress components on rotated element, are obtained by rotating XY ccw through $2\theta = 60°$

$$\phi = 180° - 60° - 67.4° = 52.6°$$

$$\sigma_{x'} = OK = OC - KC = 80 - 52\cos 52.6°$$

$$\sigma_{y'} = OL = OC + CL = 80 + 52\cos 52.6°$$

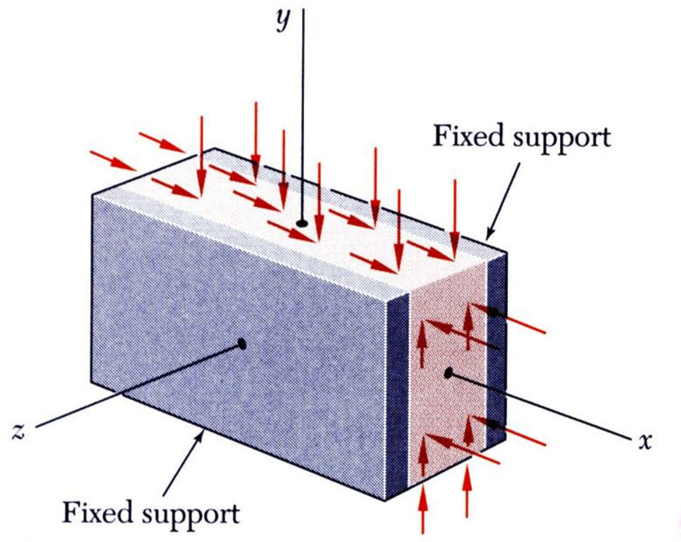
$$\tau_{x'y'} = KX' = 52\sin 52.6°$$

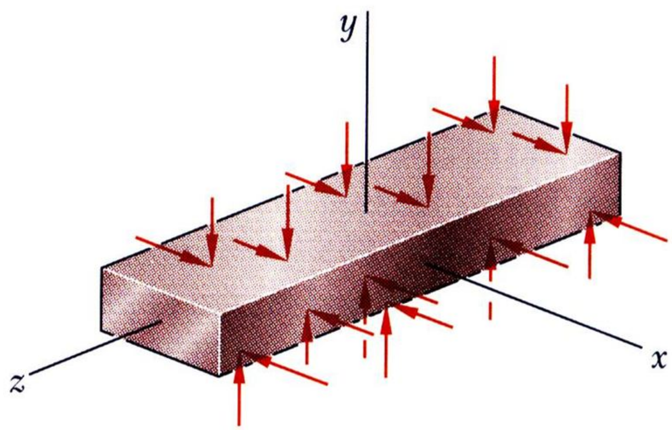
$$\sigma_{x'} = +48.4\,\text{MPa}$$

$$\sigma_{y'} = +111.6\,\text{MPa}$$

$$\tau_{x'y'} = 41.3\,\text{MPa}$$

Transformation of Plane Strain





- *Plane strain* - deformations of the material take place in parallel planes and are the same in each of those planes.

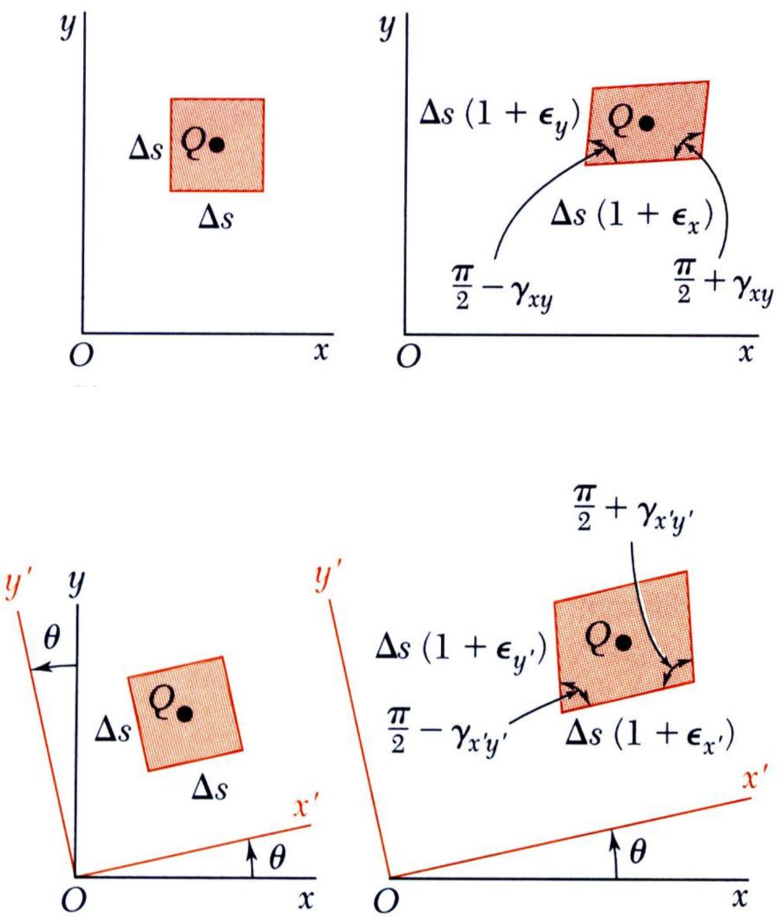
- Plane strain occurs in a plate subjected along its edges to a uniformly distributed (in *z*-direction) load and restrained from expanding or contracting laterally by smooth, rigid and fixed supports

components of strain :

$$\varepsilon_x \; \varepsilon_y \; \gamma_{xy} \quad \left(\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0\right)$$

- Example: Long bar subjected to uniformly distributed transverse loads (ie., normal to *z*-axis). State of plane strain exists in any transverse section not located too close to the ends of the bar.

Transformation of Plane Strain



- State of strain at point *Q* results in different strain components with respect to the *xy* and *x'y'* coordinate systems.
- We get strain transformation relations similar to those for stress transformation (see details in next two slides)

Transformation of Plane Strain

Use cosine rule, neglecting quadratics in strains,

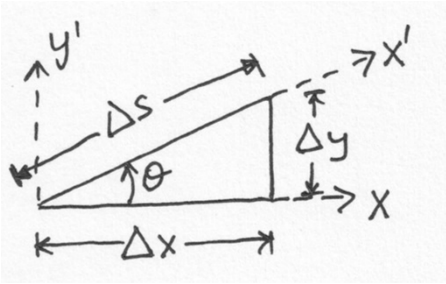
$$(\Delta s)^2(1+\varepsilon_{x'})^2 = (\Delta x)^2(1+\varepsilon_x)^2 + (\Delta y)^2(1+\varepsilon_y)^2$$
$$-2\Delta x(1+\varepsilon_x)\Delta y(1+\varepsilon_y)\cos(\pi/2+\gamma_{xy})$$

$$(\Delta s)^2(1+2\varepsilon_{x'}) = (\Delta x)^2(1+2\varepsilon_x) + (\Delta y)^2(1+2\varepsilon_y) - 2\Delta x\Delta y(-\gamma_{xy})$$

Use $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2,\ \Delta x/\Delta s = \cos\theta,\ \Delta y/\Delta s = \sin\theta,$

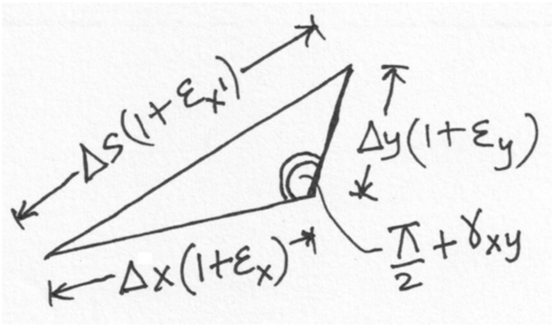
$$\varepsilon_{x'} = \varepsilon(\theta) = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$



put $\theta \to \theta + \pi/2$ in $\varepsilon_{x'}$,

$$\varepsilon_{y'} = \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$



Transformation of Plane Strain

For $\theta = \pi/4$,

$$\varepsilon_{OB} = \varepsilon_{x'}(\pi/4) = \tfrac{1}{2}\left(\varepsilon_x + \varepsilon_y + \gamma_{xy}\right)$$

$$\gamma_{xy} = 2\varepsilon_{OB} - \left(\varepsilon_x + \varepsilon_y\right)$$

where OB is bisector of Ox and Oy

Thus, in $x'y'$ system,

$$\gamma_{x'y'} = 2\varepsilon_{OB'} - \left(\varepsilon_{x'} + \varepsilon_{y'}\right) = 2\varepsilon_{OB'} - \left(\varepsilon_x + \varepsilon_y\right)$$

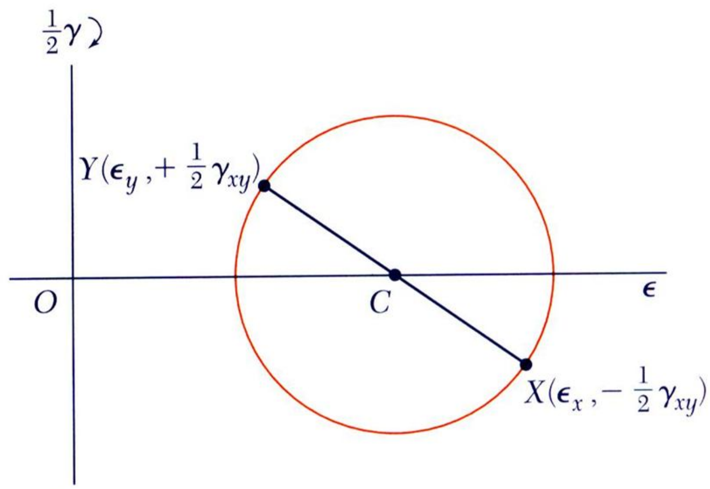
where OB' is bisector of Ox' and Oy'. Thus,

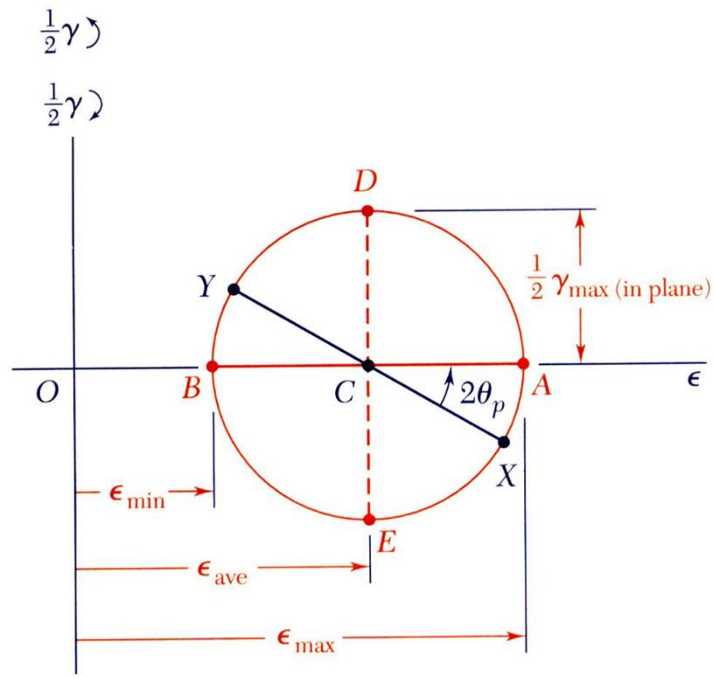
$$\varepsilon_{OB'} = \varepsilon_{x'}(\theta + \pi/4) = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Note: $\varepsilon_{x'} \equiv \sigma_{x'}$; $\varepsilon_{y'} \equiv \sigma_{y'}$; but $\dfrac{\gamma_{x'y'}}{2} \equiv \tau_{x'y'}$

Mohr's Circle for Plane Strain





- Since strain transformation relations are of same form as stress transformation, for plane problems, *Mohr's circle techniques apply*.

- Abscissa for center *C*, and radius *R* , are

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

- Principal axes of strain and principal strains,

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

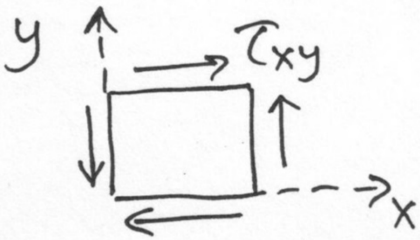
$$\varepsilon_{\max} = \varepsilon_{ave} + R \qquad \varepsilon_{\min} = \varepsilon_{ave} - R$$

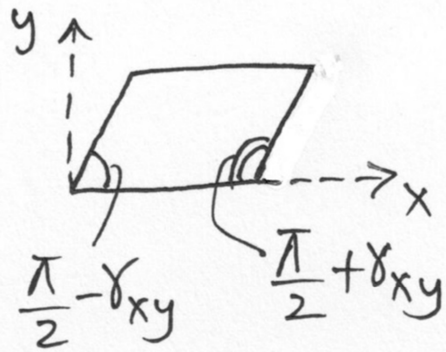
- Maximum in-plane shearing strain,

$$\gamma_{\max} = 2R = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

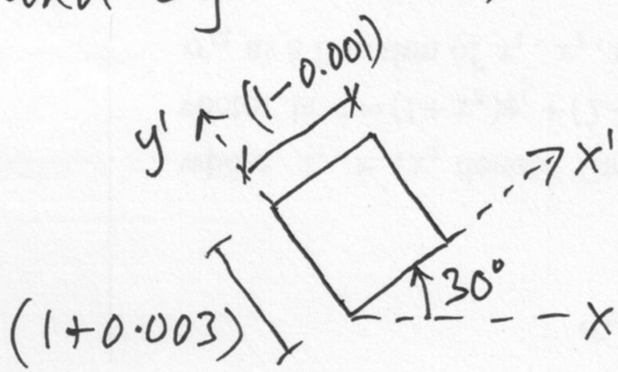
Drawing strain block.

Positive convention for τ_{xy}, γ_{xy}, is,



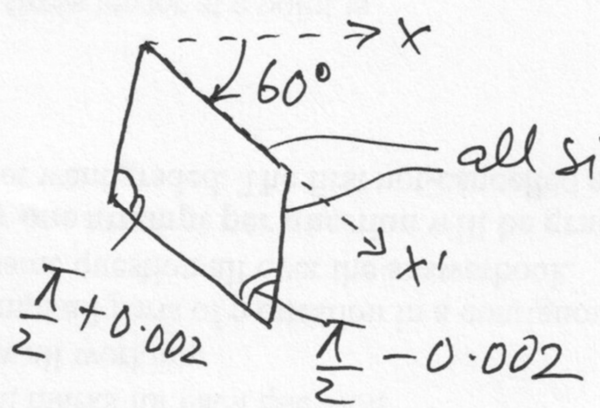


So if, we get solution $\theta_p = 30^\circ$, $\varepsilon_{max} = 0.003$, $\varepsilon_{min} = -0.001$, (or from Mohr circle) and using $\theta_p = 30^\circ$ in transformation relations, we get $\varepsilon_{x'} = -0.001$ and $\varepsilon_{y'} = 0.003$, then strain block is



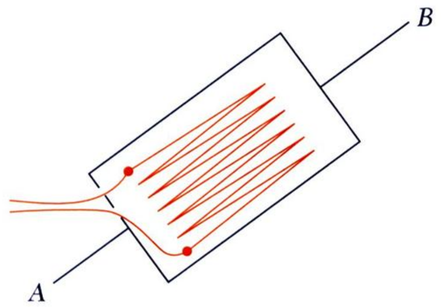
Drawing strain block.

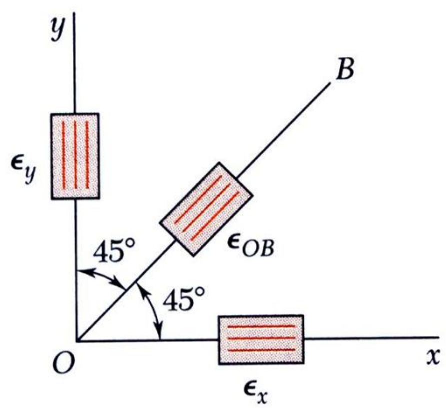
If we get solution $\theta_s = -60^\circ$ and $|\gamma_{max}| = 0.002$, and using $\theta_s = -60^\circ$ in transformation relations, or from Mohr circle, we get $\gamma_{x'y'} = -0.002$ and $\varepsilon_{x'} = \varepsilon_{y'} = 0.0025$, then strain block is

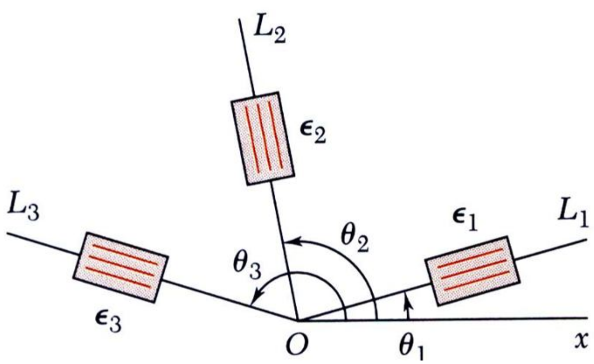


(ie. ∵ $\gamma_{x'y'} < 0$, it shears toward left – see positive γ_{xy} convention above).

Measurements of Strain: Strain Rosette







- Strain gages indicate normal strain through changes in resistance.

- With a 45° rosette, e_x and e_y are measured directly. g_{xy} is obtained indirectly with,

$$\gamma_{xy} = 2\varepsilon_{OB} - \left(\varepsilon_x + \varepsilon_y\right)$$

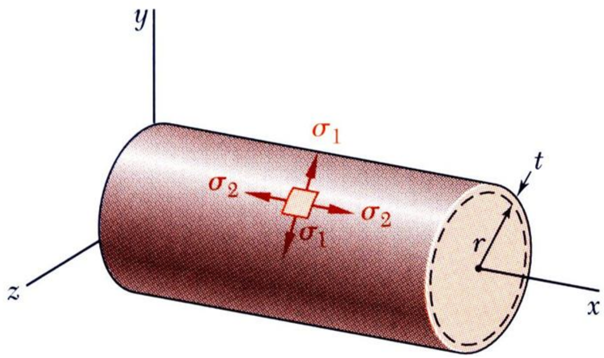
- Normal and shearing strains may be obtained from normal strains in any three directions,

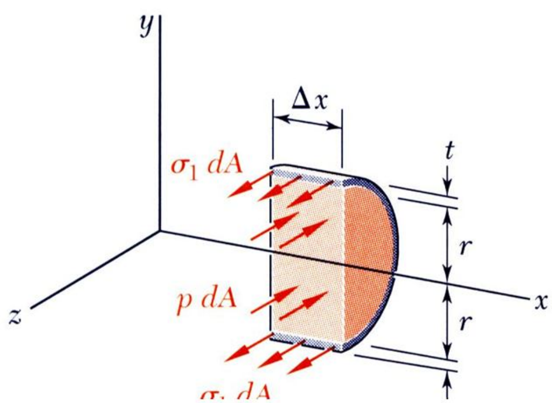
$$\varepsilon_1 = \varepsilon_x \cos^2\theta_1 + \varepsilon_y \sin^2\theta_1 + \gamma_{xy}\sin\theta_1\cos\theta_1$$

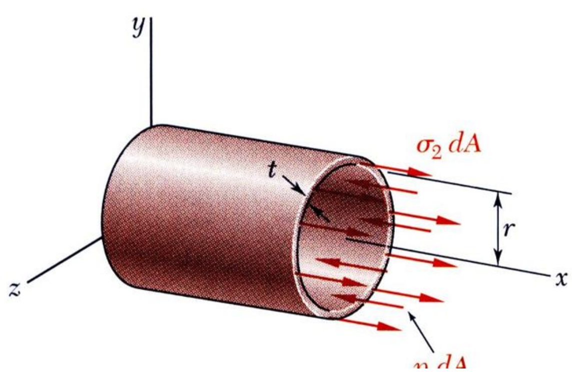
$$\varepsilon_2 = \varepsilon_x \cos^2\theta_2 + \varepsilon_y \sin^2\theta_2 + \gamma_{xy}\sin\theta_2\cos\theta_2$$

$$\varepsilon_3 = \varepsilon_x \cos^2\theta_3 + \varepsilon_y \sin^2\theta_3 + \gamma_{xy}\sin\theta_3\cos\theta_3$$

Stresses in Thin-Walled Cylindrical Pressure Vessels







- Cylindrical vessel with principal stresses

  $s_q = s_1$ = hoop stress

  $s_x = s_2$ = longitudinal stress

- Hoop stress:

$$\sum F_z = 0 = \sigma_1(2t\,\Delta x) - p(2r\,\Delta x)$$

$$\sigma_\theta = \sigma_1 = \frac{pr}{t}$$

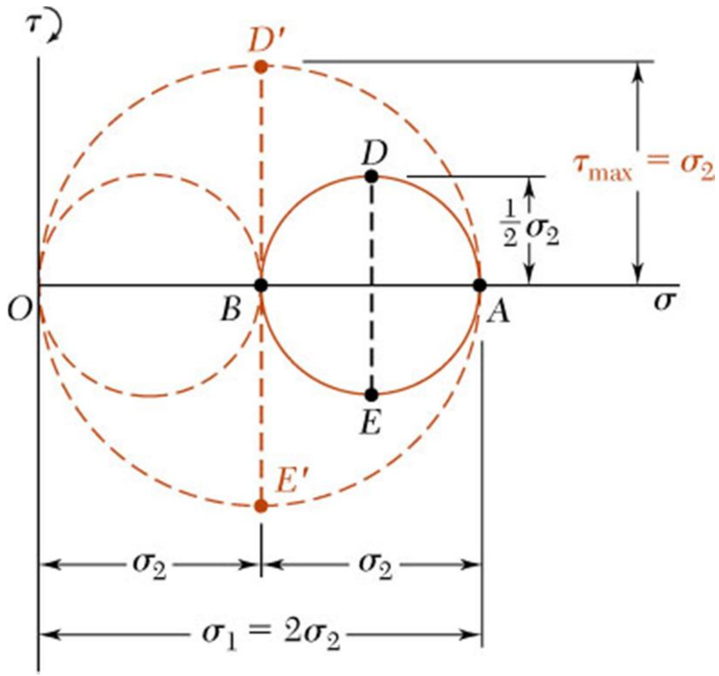
- Longitudinal stress:

$$\sum F_x = 0 = \sigma_2(2\pi r t) - p(\pi r^2)$$

$$\sigma_x = \sigma_2 = \frac{pr}{2t}$$

$$\sigma_1 = 2\sigma_2$$

Stresses in Thin-Walled Cylindrical Pressure Vessels



Points *A* and *B* correspond to hoop stress, s_1, and longitudinal stress, s_2

Maximum in-plane shearing stress:

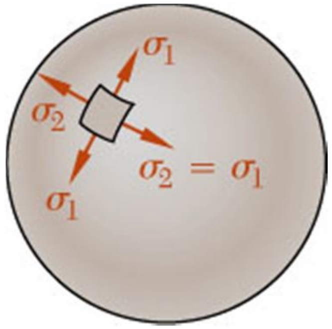
$$\tau_{\max(\text{in-plane})} = \frac{1}{2}\sigma_2 = \frac{pr}{4t}$$

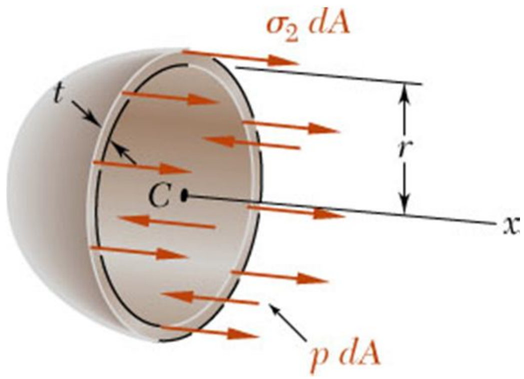
Maximum out-of-plane shearing stress corresponds to a 45° rotation of the plane stress element around a longitudinal axis

$$\tau_{\max} = \sigma_2 = \frac{pr}{2t}$$

Note: Plane stress Mohr's circle is only *ADBE*. The remaining is due to 3-D state of stress with third principal stress being zero. 3-D state of stress is also used in finding maximum out-of-plane shearing stress by using third principal stress as zero. This is not covered in this course.
Only maximum in-plane shearing stress is covered in this course.

Stresses in Thin-Walled Spherical Pressure Vessels





Spherical pressure vessel:

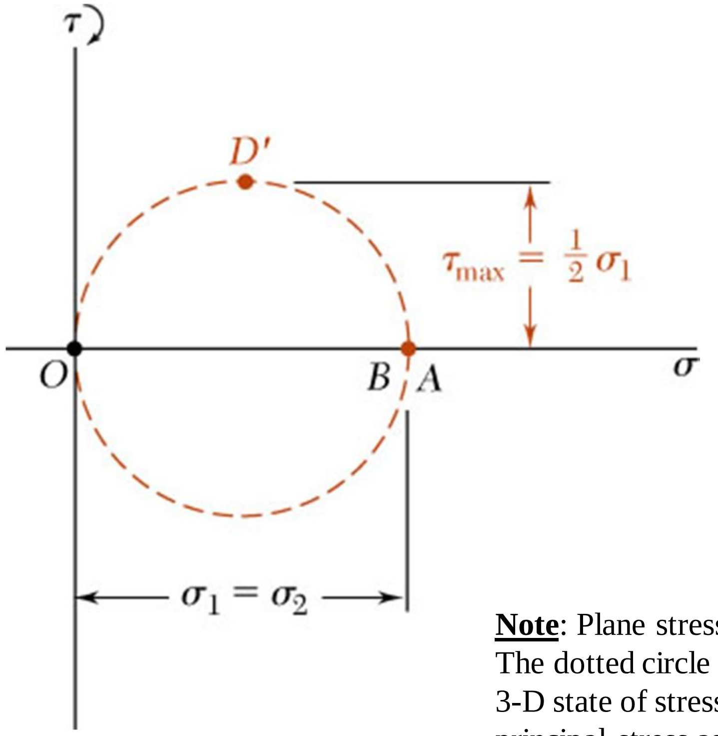
$$\sigma_2 (2\pi r t) = p\pi r^2$$

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Mohr's circle for in-plane transformations reduces to a point

$$\sigma = \sigma_1 = \sigma_2 = \text{constant}$$

$$\tau_{\max(\text{in-plane})} = 0$$

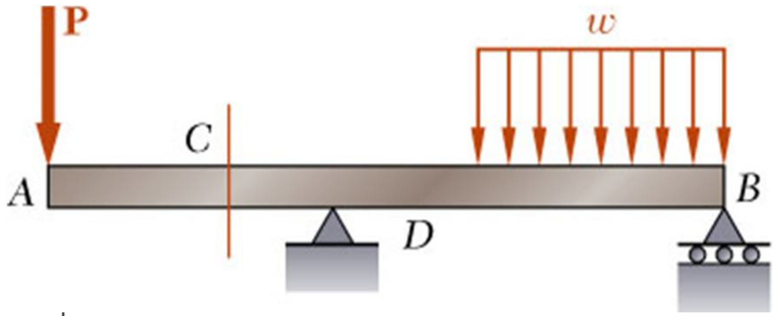


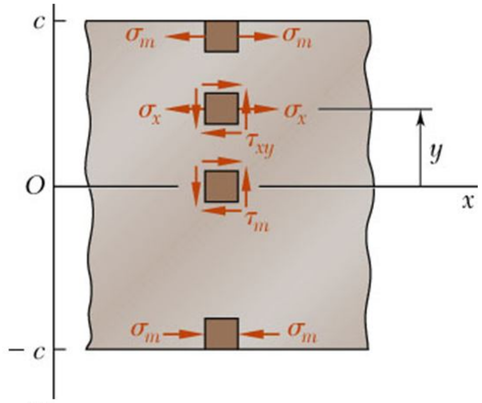
Maximum out-of-plane shearing stress

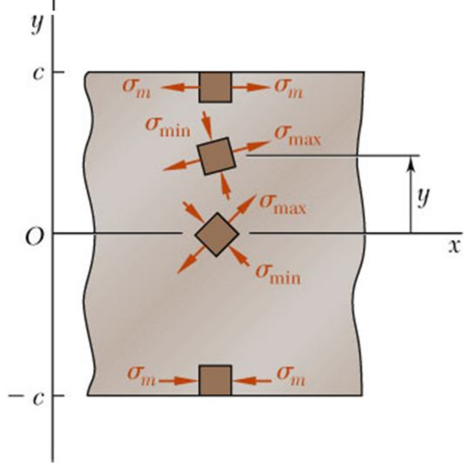
$$\tau_{\max} = \frac{1}{2}\sigma_1 = \frac{pr}{4t}$$

Note: Plane stress Mohr's circle is only *AB*, i.e., circle with zero radius, i.e., a point.
The dotted circle is due to 3-D state of stress with third principal stress being zero.
3-D state of stress is also used in finding maximum out-of-plane shearing stress, by using third principal stress as zero. This is not covered in this course.
Only maximum in-plane shearing stress is covered in this course.

Principle Stresses in a Beam







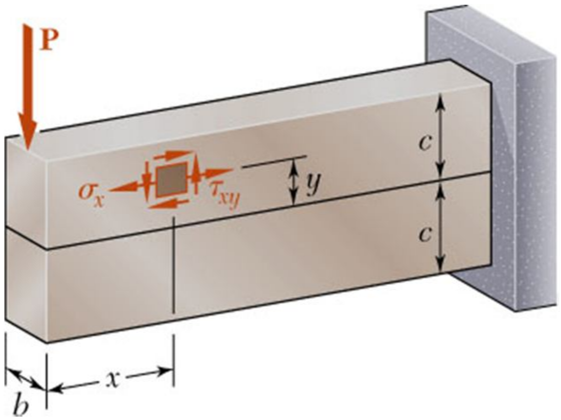
For beam subjected to transverse loading

$$\sigma_x = -\frac{My}{I} \qquad \sigma_m = \frac{Mc}{I}$$
$$\tau_{xy} = -\frac{VQ}{It} \qquad \tau_m = \frac{VQ}{It}$$

Can the maximum normal stress within the cross-section be larger than

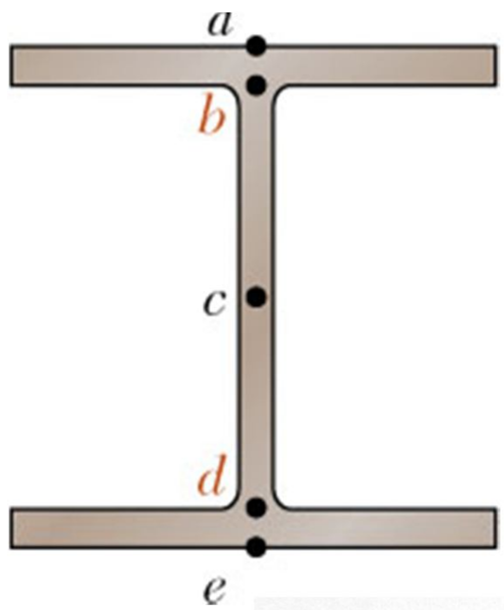
$$\sigma_m = \frac{Mc}{I}$$

Principle Stresses in a Rectangular section Beam



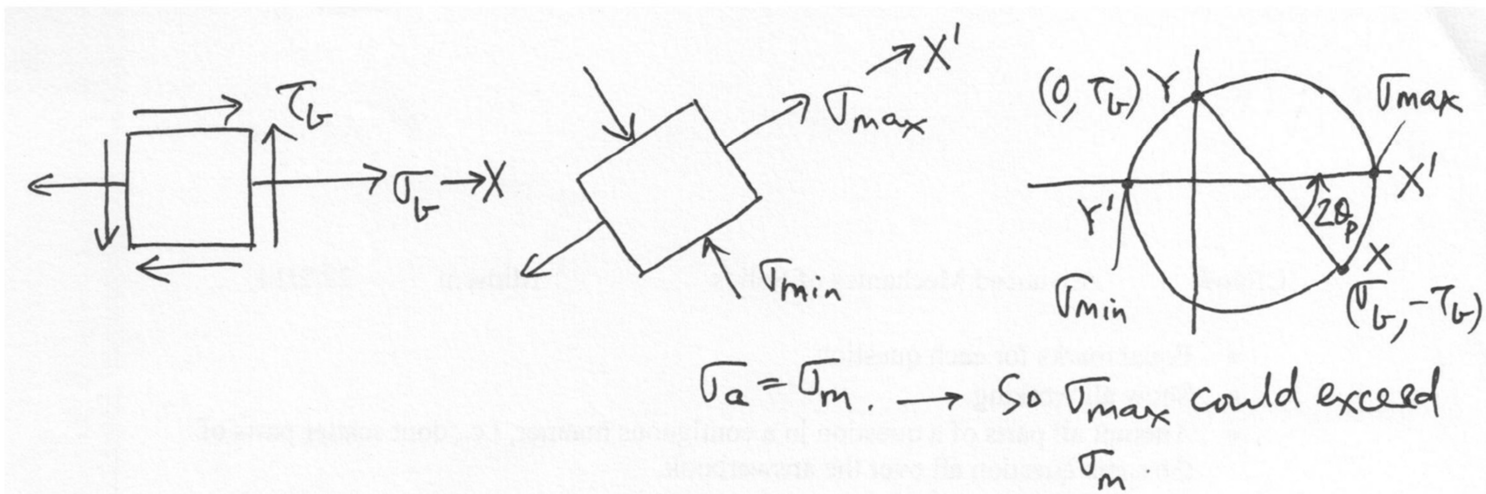
	$x = 2c$		$x = 8c$	
y/c	σ_{min}/σ_m	σ_{max}/σ_m	σ_{min}/σ_m	σ_{max}/σ_m
1.0	0	1.000	0	1.000
0.8	−0.010	0.810	−0.001	0.801
0.6	−0.040	0.640	−0.003	0.603
0.4	−0.090	0.490	−0.007	0.407
0.2	−0.160	0.360	−0.017	0.217
0	−0.250	0.250	−0.063	0.063
− 0.2	−0.360	0.160	−0.217	0.017
− 0.4	−0.490	0.090	−0.407	0.007
− 0.6	−0.640	0.040	−0.603	0.003
− 0.8	−0.810	0.010	−0.801	0.001
− 1.0	−1.000	0	−1.000	0

Principle Stresses in a Beam

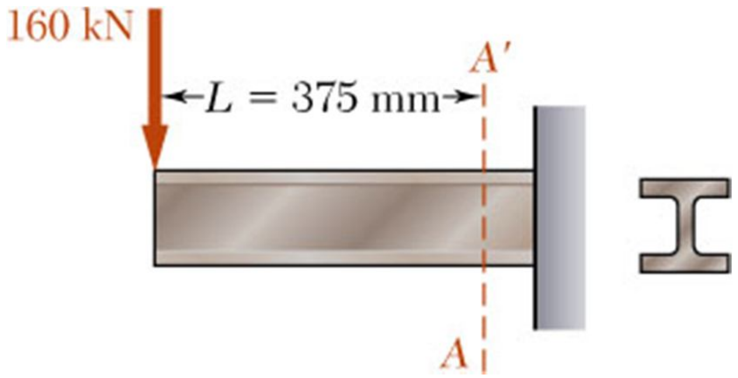


Cross-section shape results in large values of τ_{xy} near the surface where σ_x is also large.

σ_{max} may be greater than σ_m (since τ_b is large)



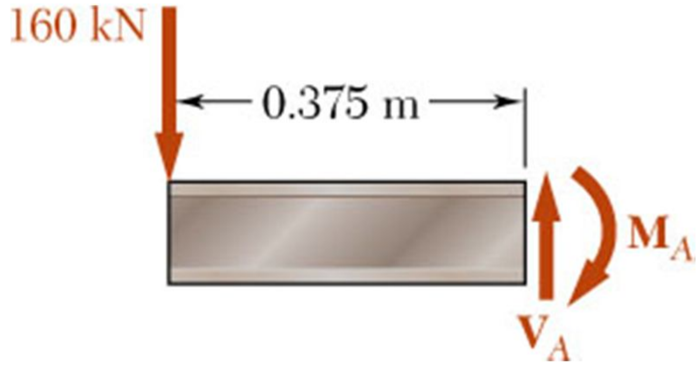
Example 5



160-kN force applied at tip of W200x52 rolled-steel beam.

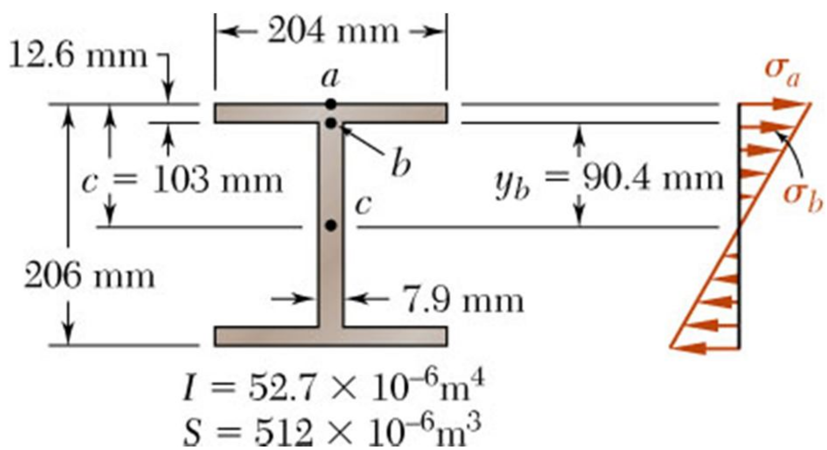
Neglect effects stress concentrations at fillets, determine whether normal stresses at section *A-A’* satisfy σ_{all} =150 MPa

Example 5



Determine shear and bending moment in Section *A-A'*

$$M_A = (160\,\text{kN})(0.375\,\text{m}) = 60\,\text{kN - m}$$
$$V_A = 160\,\text{kN}$$

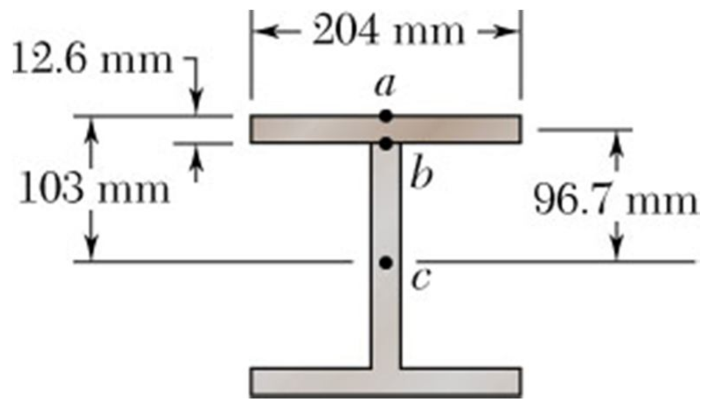


Calculate normal stress at top surface and at flange-web junction.

$$\sigma_a = \frac{M_A}{S} = \frac{60\,\text{kN}\cdot\text{m}}{512\times10^{-6}\,\text{m}^3}$$
$$= 117.2\,\text{MPa}$$

$$\sigma_b = \sigma_a \frac{y_b}{c} = (117.2\,\text{MPa})\frac{90.4\,\text{mm}}{103\,\text{mm}}$$
$$= 102.9\,\text{MPa}$$

Example 5



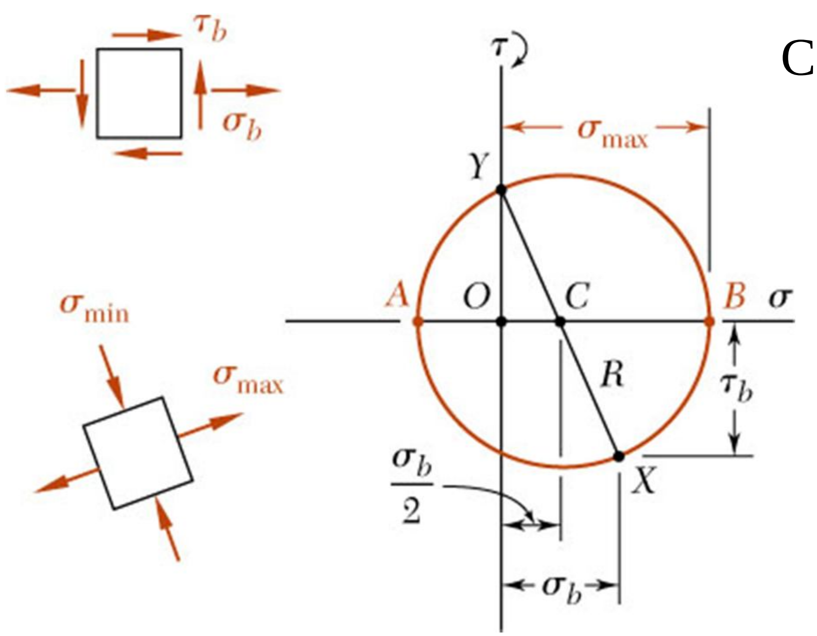
Calculate shear stress at flange-web junction.

$$Q = (204 \times 12.6)96.7 = 248.6 \times 10^3 \mathrm{mm}^3$$

$$= 248.6 \times 10^{-6} \mathrm{m}^3$$

$$\tau_b = \frac{V_A Q}{It} = \frac{(160\,\mathrm{kN})(248.6 \times 10^{-6}\,\mathrm{m}^3)}{(52.7 \times 10^{-6}\,\mathrm{m}^4)(0.0079\,\mathrm{m})}$$

$$= 95.5\,\mathrm{MPa}$$



Calculate principal stresses at flange-web junction

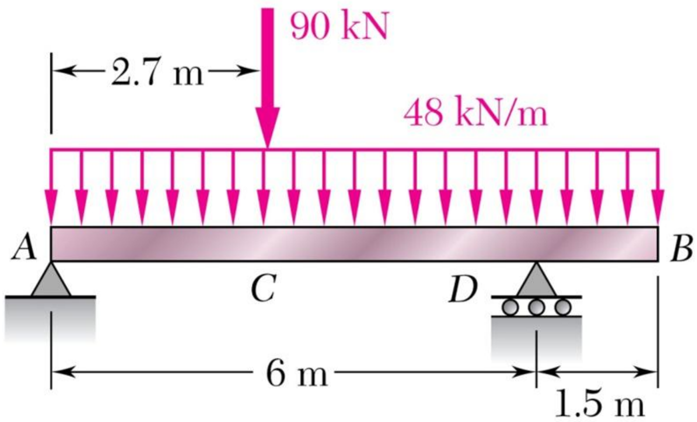
$$\sigma_{max} = \tfrac{1}{2}\sigma_b + \sqrt{\left(\tfrac{1}{2}\sigma_b\right)^2 + \tau_b^2}$$

$$= \frac{102.9}{2} + \sqrt{\left(\frac{102.9}{2}\right)^2 + (95.5)^2}$$

$$= 159.9\,\mathrm{MPa} \qquad (> 150\,\mathrm{MPa})$$

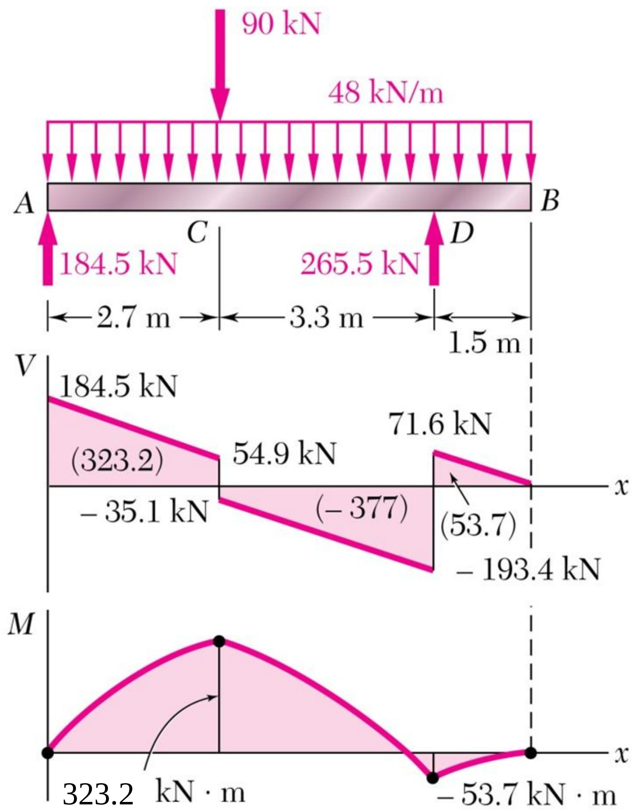
Design specification not satisfied.

Example 6



s_{all} = 165 Mpa, t_{all} = 100 Mpa.
Select wide-flange beam to be used.

Example 6



Shape	S (mm^3)
W610 × 101	2530
W530 × 92	2070
W460 × 113	2400
W410 × 114	2200
W360 × 122	2010
W310 × 143	2150

Reactions at *A* and *D*.

$$\sum M_A = 0 \quad \Rightarrow \quad R_D = 265.5\,\text{kN}$$

$$\sum M_D = 0 \quad \Rightarrow \quad R_A = 184.5\,\text{kN}$$

Maximum shear and bending moment from SFD, BMD.

$$|M|_{\max} = 323.2\,\text{kNm} \quad \text{with} \quad V = 54.9\,\text{kN}$$

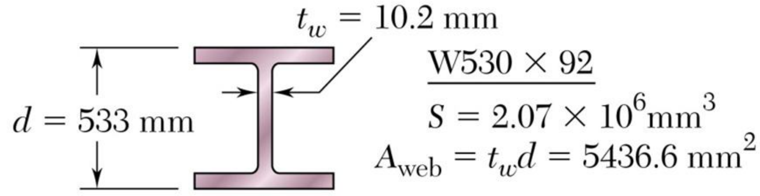
$$|V|_{\max} = 193.4\,\text{kN}$$

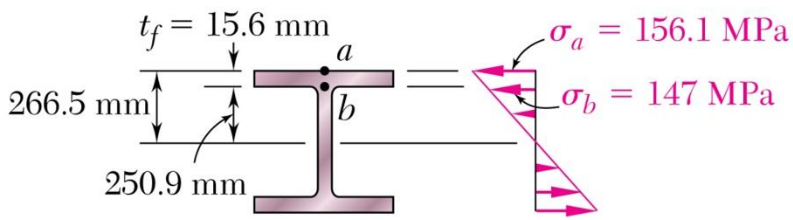
Calculate required section modulus, select appropriate beam section.

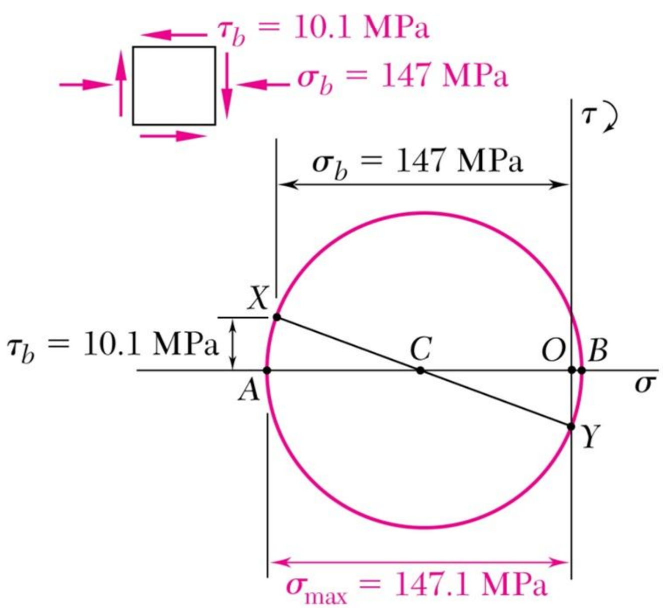
$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}} = \frac{323\times10^3\ \text{Nm}}{165\times10^6\ \text{Pa}} = 1959\,\text{mm}^3$$

select W530 × 92 beam section

Example 6







Find maximum shearing stress.

Assume uniform shearing stress in web (conservative, see shallow parabolic variation, slide 10, shear stresses chapter)

$$\tau_{\max} = \frac{V_{\max}}{A_{web}} = \frac{193.4\times10^3\ \text{N}}{5436.6\times10^{-6}\ \text{m}^2} = 35.6\ \text{MPa} < 100\ \text{MPa}$$

Find maximum normal stress.

$$\sigma_a = \frac{M_{\max}}{S} = \frac{323200\ \text{Nm}}{2.07\times10^{-6}\ \text{m}^3} = 156.1\text{MPa}$$

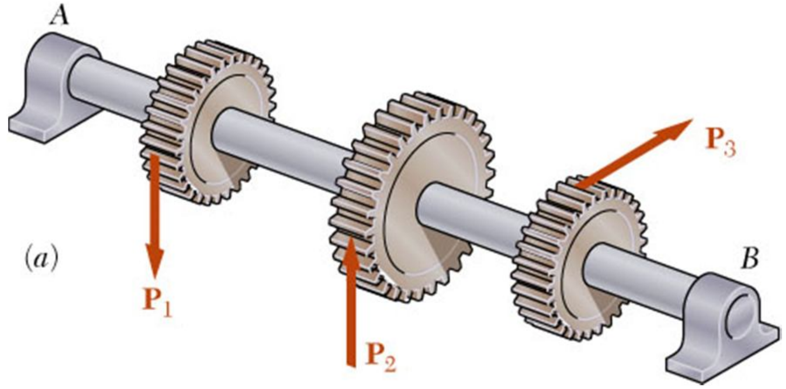
$$\sigma_b = \sigma_a \frac{y_b}{c} = (156.1\text{MPa})\frac{250.9\ \text{mm}}{266.5\ \text{mm}} = 147\ \text{MPa}$$

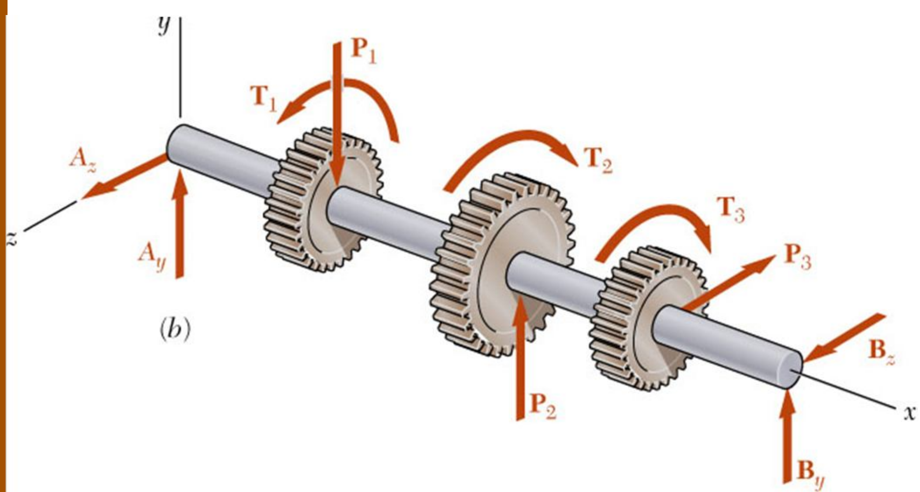
$$\tau_b = \frac{V}{A_{web}} = \frac{54900\ \text{N}}{5436.6\times10^{-6}\ \text{m}^2} = 10.1\text{MPa}$$

$$\sigma_{\max} = \frac{147\ \text{MPa}}{2} + \sqrt{\left(\frac{147\ \text{MPa}}{2}\right)^2 + (10.1\text{MPa})^2}$$

$$= 147.1\text{MPa} < 165\ \text{MPa}$$

Design of Transmission Shaft



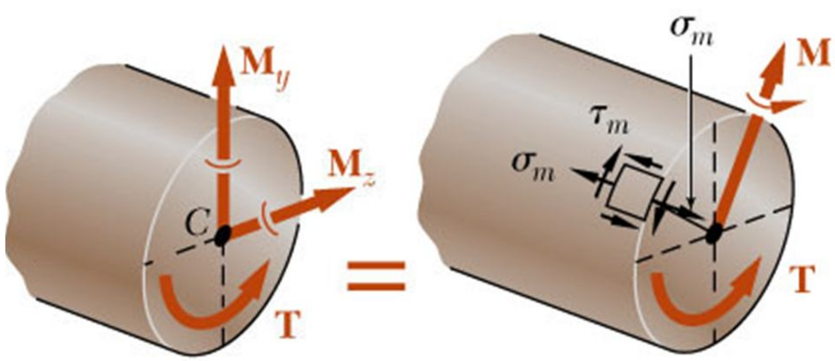


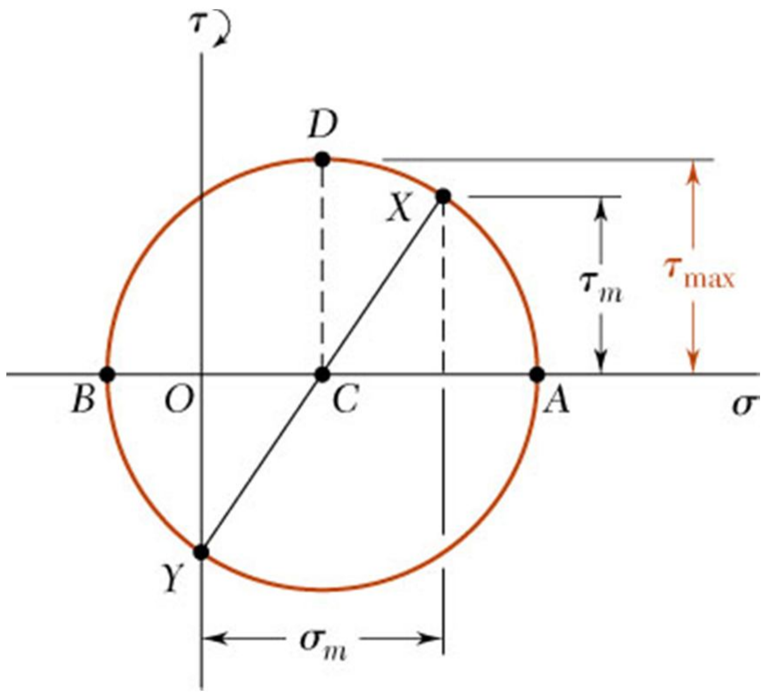
- If power is transferred to and from shaft by gears or sprocket wheels, the shaft is subjected to transverse loading (due to forces in mating gears) as well as shear loading (due to torque from these forces).

- Normal stresses due to transverse loads may be large, should be included in determination of maximum shearing stress.

- Shearing stresses due to transverse loads are usually small and their contribution to maximum shear stress may be neglected.

Design of a Transmission Shaft





At any section,

$$\sigma_m = \frac{Mc}{I} \quad \text{where} \quad M^2 = M_y^2 + M_z^2$$

$$\tau_m = \frac{Tc}{J}$$

Maximum shearing stress,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_m}{2}\right)^2 + (\tau_m)^2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{J}\right)^2}$$

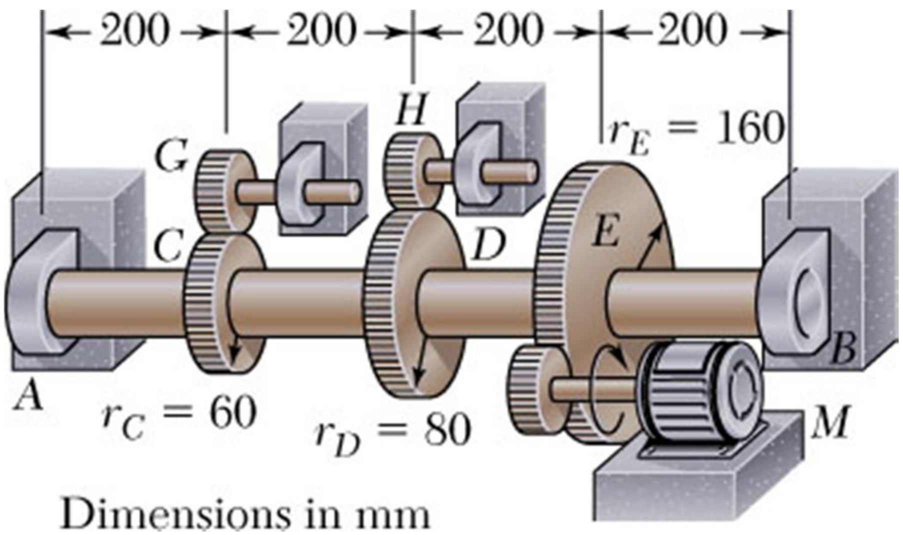
for a circular or annular cross-section, $2I = J$

$$\tau_{\max} = \frac{c}{J}\sqrt{M^2 + T^2}$$

Shaft section requirement,

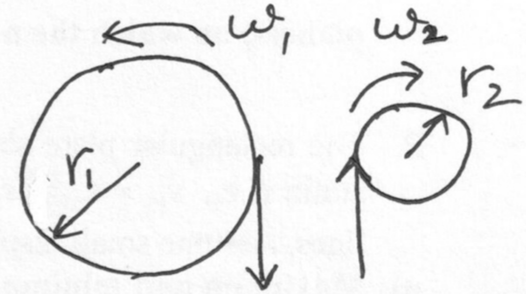
$$\left(\frac{J}{c}\right)_{\min} = \frac{\left(\sqrt{M^2 + T^2}\right)_{\max}}{\tau_{all}}$$

Example 7



Shaft rotates at 480 rpm, transmits 30 kW from motor to gears *G* and *H*; 20 kW is taken off at gear *G* and 10 kW at gear *H*. s_{all} = 50 Mpa. Find smallest permissible diameter for shaft.

Mating gears.



Kinematics $\Rightarrow r_1\omega_1 = r_2\omega_2$

Action-reaction $\Rightarrow F_1 = F_2$

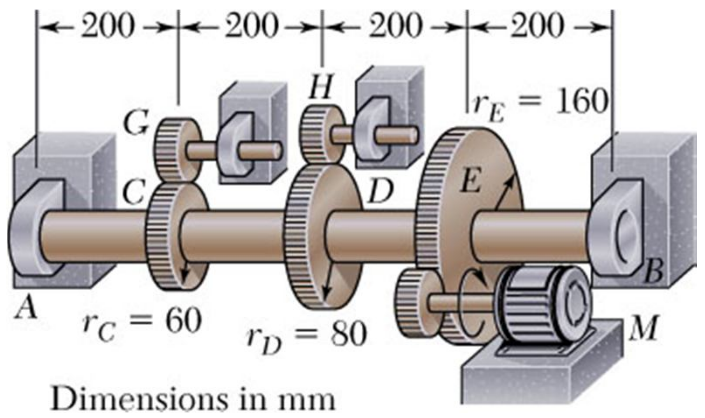
$\Rightarrow F_1 r_1 \omega_1 = F_2 r_2 \omega_2$

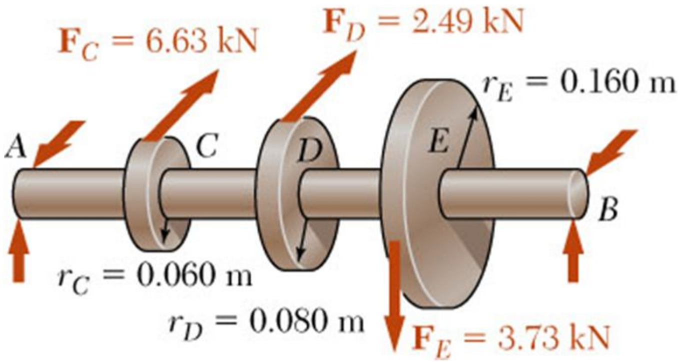
$T_1 \omega_1 = T_2 \omega_2$

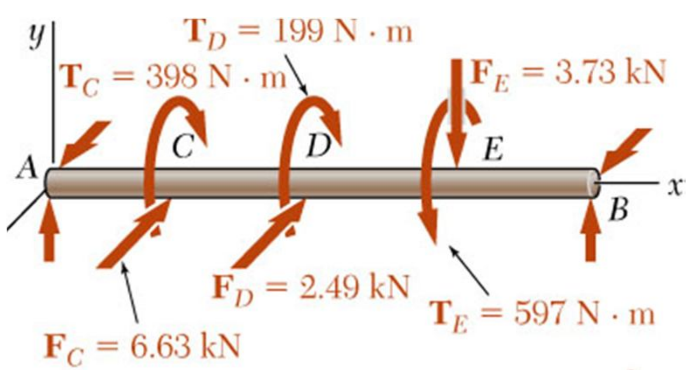
$P_1 = P_2$ (power).

ie power removed from gear 1 is power gained by gear 2.

Example 7







Find gear torques and corresponding tangential forces. Use,

$$T_E\omega_E = P_E = P_M = T_M\omega_M$$
$$T_C\omega_C = P_C = P_G = T_G\omega_G$$
$$T_D\omega_D = P_D = P_H = T_H\omega_H$$

$$T_E = \frac{P}{2\pi f} = \frac{30\,\text{kW}}{2\pi(8\,\text{Hz})} = 597\,\text{N}\cdot\text{m}$$

$$F_E = \frac{T_E}{r_E} = \frac{597\,\text{N}\cdot\text{m}}{0.16\,\text{m}} = 3.73\,\text{kN}$$

$$T_C = \frac{20\,\text{kW}}{2\pi(8\,\text{Hz})} = 398\,\text{N}\cdot\text{m} \qquad F_C = 6.63\,\text{kN}$$

$$T_D = \frac{10\,\text{kW}}{2\pi(8\,\text{Hz})} = 199\,\text{N}\cdot\text{m} \qquad F_D = 2.49\,\text{kN}$$

Find reactions at *A* and *B*.

$$A_y = 0.932\,\text{kN} \qquad A_z = 6.22\,\text{kN}$$
$$B_y = 2.80\,\text{kN} \qquad B_z = 2.90\,\text{kN}$$

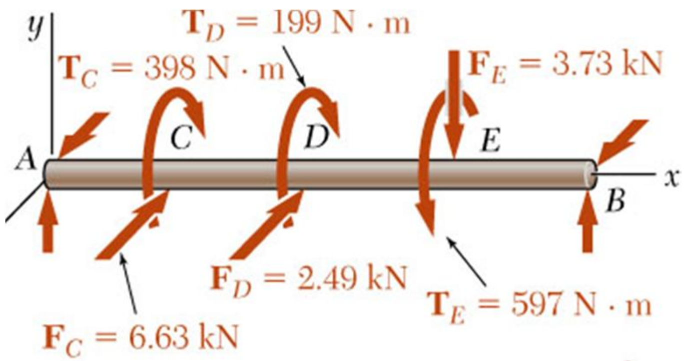
Example 7

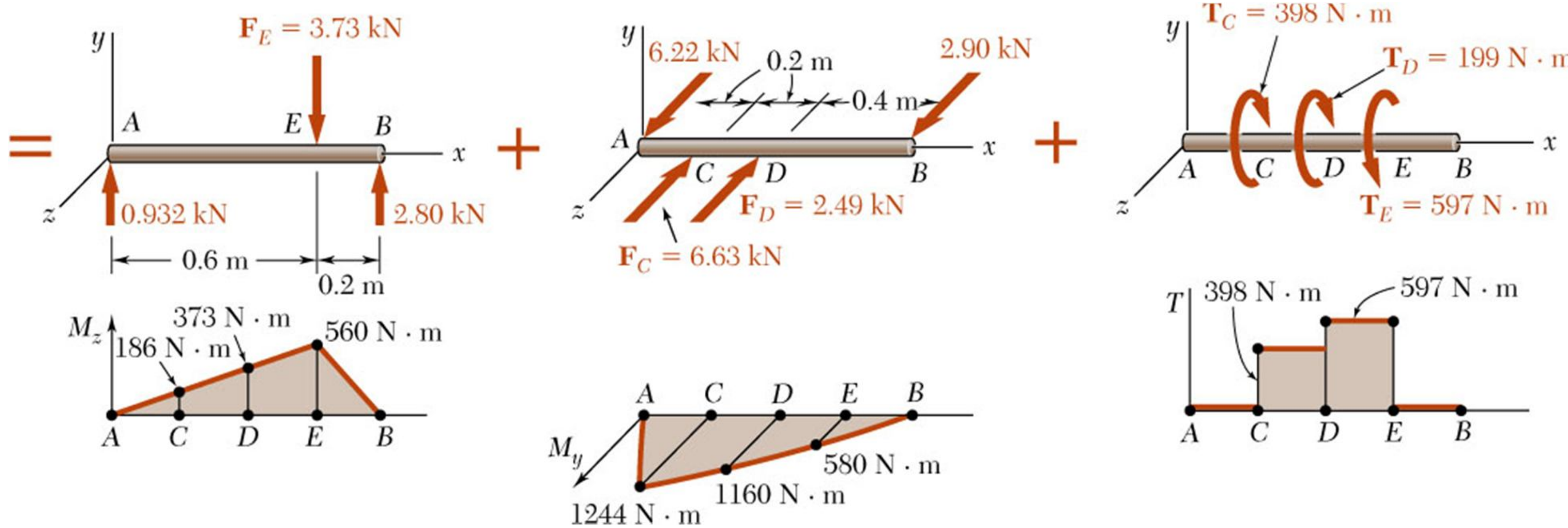
Identify critical shaft section from torque and bending moment diagrams. *D* comes out as critical section.

$$\left(\sqrt{M_y^2 + M_z^2 + T^2}\right)_{max} = \sqrt{1160^2 + 373^2 + 597^2}$$

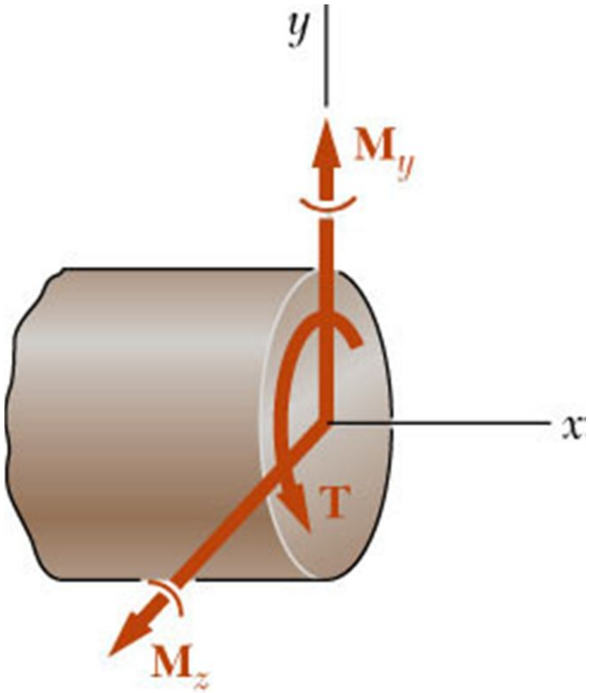
$$= 1357\,\mathrm{N \cdot m}$$

At *C* it is 1319 N.m. At *E* it is 1003 N.m





Example 7



Find minimum allowable shaft diameter.

$$\frac{J}{c} = \frac{\sqrt{M_y^2 + M_z^2 + T^2}}{\tau_{all}}$$

$$= \frac{1357\ \mathrm{N \cdot m}}{50\,\mathrm{MPa}} = 27.14 \times 10^{-6}\,\mathrm{m}^3$$

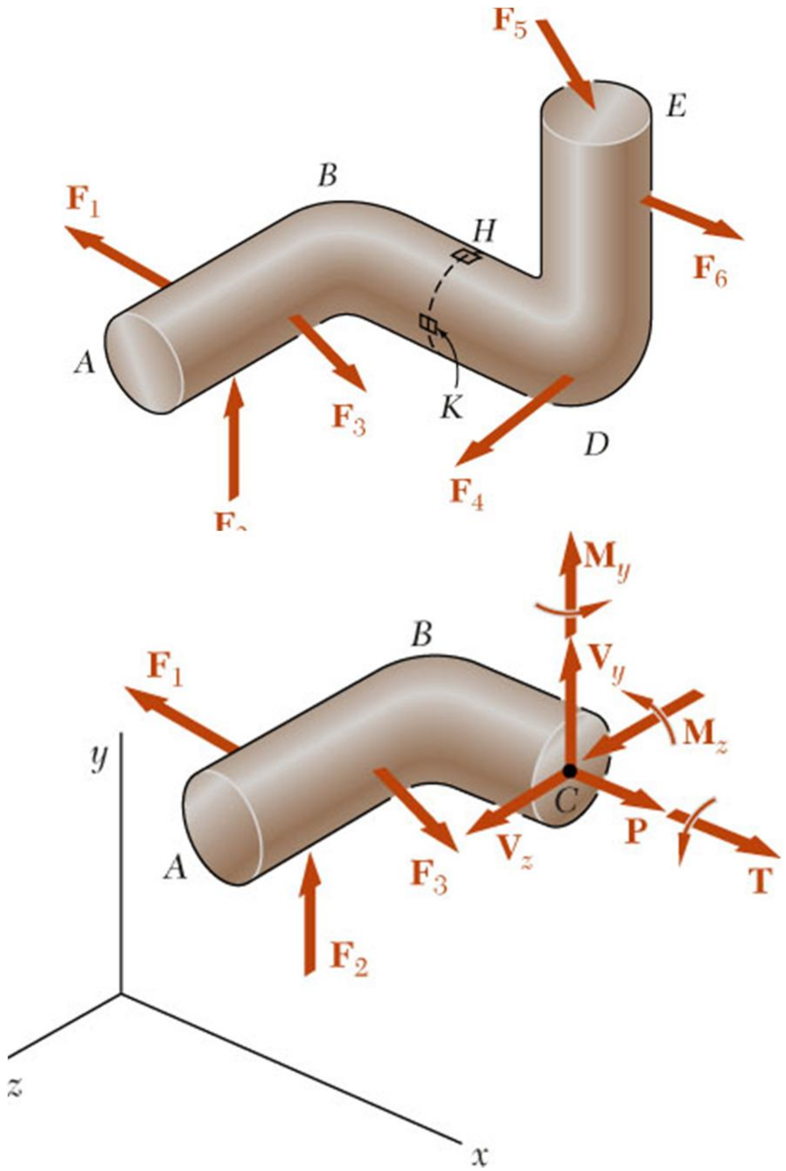
For solid circular shaft,

$$\frac{J}{c} = \frac{\pi}{2} c^3 = 27.14 \times 10^{-6}\,\mathrm{m}^3$$

$$c = 0.02585\mathrm{m} = 25.85\mathrm{m}$$

$$\boxed{d = 2c = 51.7\ \mathrm{mm}}$$

Stresses Under Combined Loadings



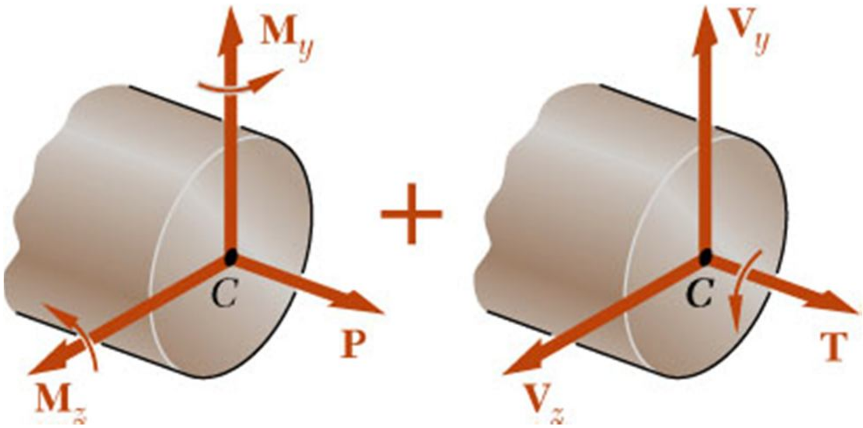
Wish to find stresses in slender structural members subjected to arbitrary loadings.

Pass section through points of interest. Determine force-couple system at centroid of section required to maintain equilibrium.

System of internal forces consist of three force components and three couple vectors.

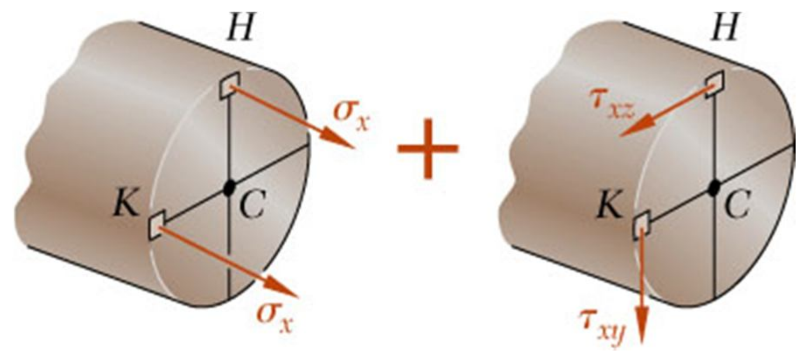
Determine stress distribution by applying the superposition principle.

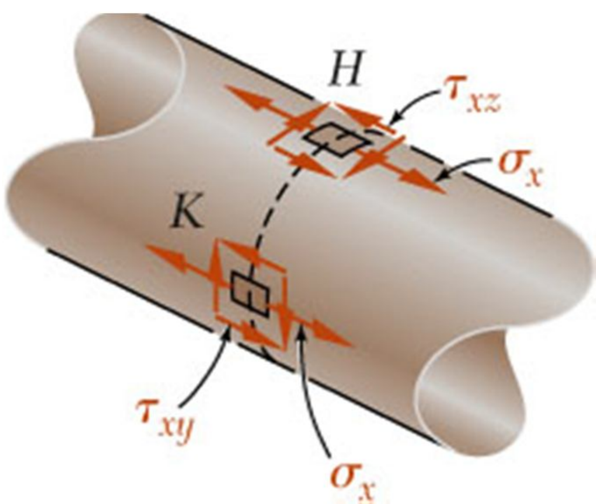
Stresses Under Combined Loadings

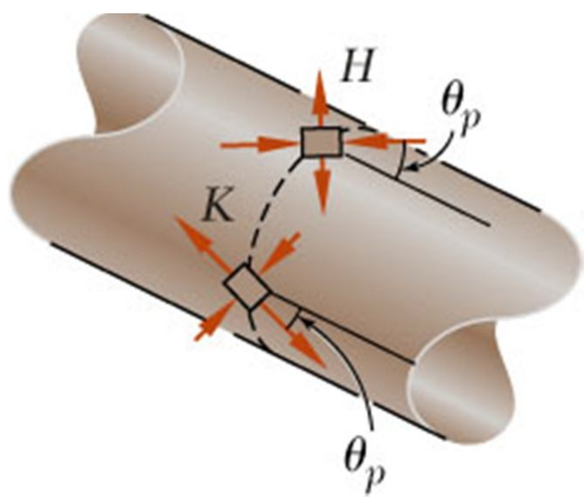


Axial force and bending moments yield normal stresses.

Shear forces and twisting couple yield shearing stresses.

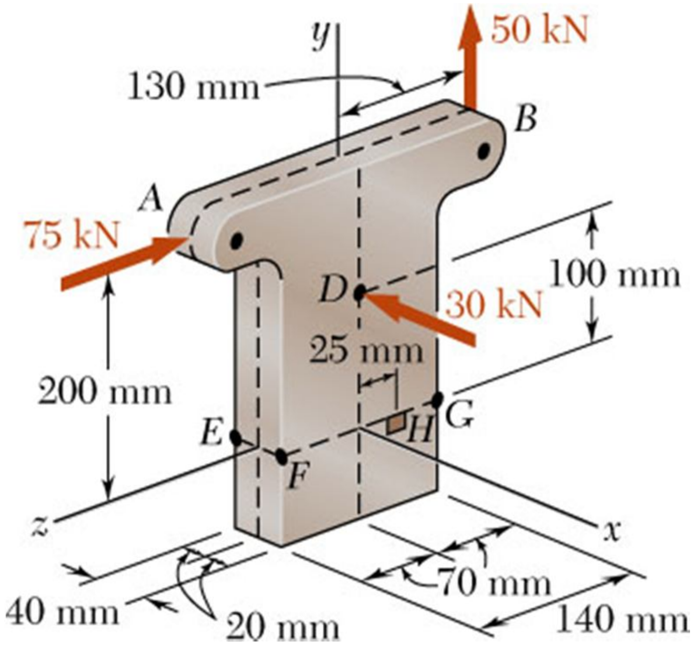






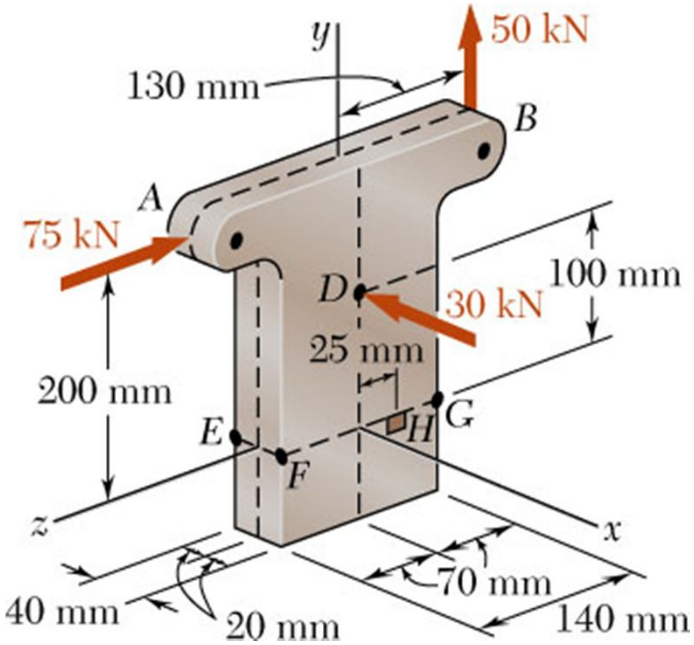
Find principal stresses, maximum shearing stress.

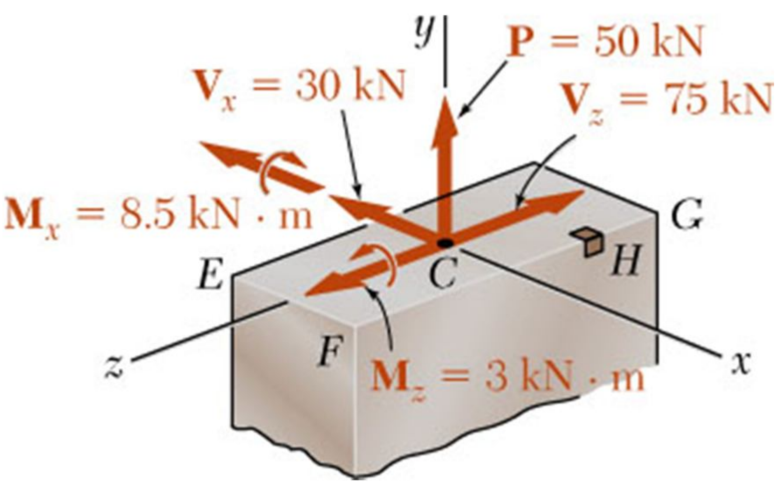
Example 8



Find principle stresses, principal planes, maximum shearing stress, at *H*.

Example 8





Internal forces in Section *EFG*.

$$V_x = -30\,\text{kN} \quad P = 50\,\text{kN} \quad V_z = -75\,\text{kN}$$

$$M_x = (50\,\text{kN})(0.130\,\text{m}) - (75\,\text{kN})(0.200\,\text{m})$$

$$= -8.5\,\text{kN}\cdot\text{m}$$

$$M_y = 0 \quad M_z = (30\,\text{kN})(0.100\,\text{m}) = 3\,\text{kN}\cdot\text{m}$$

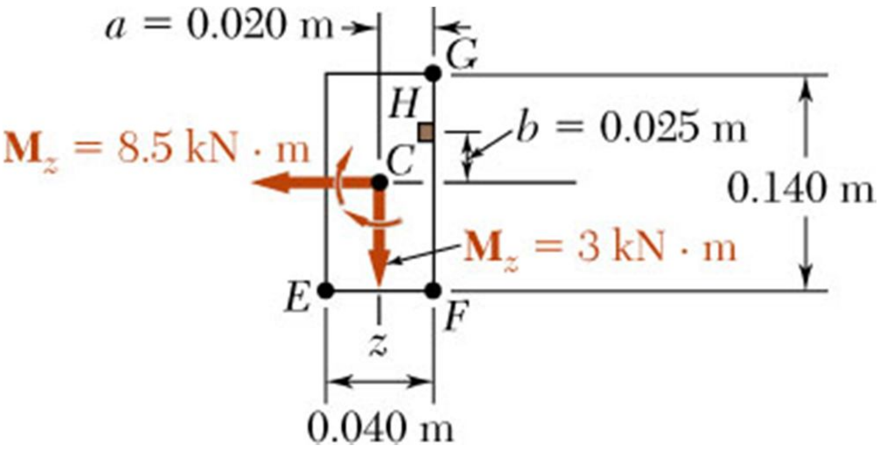
Section properties,

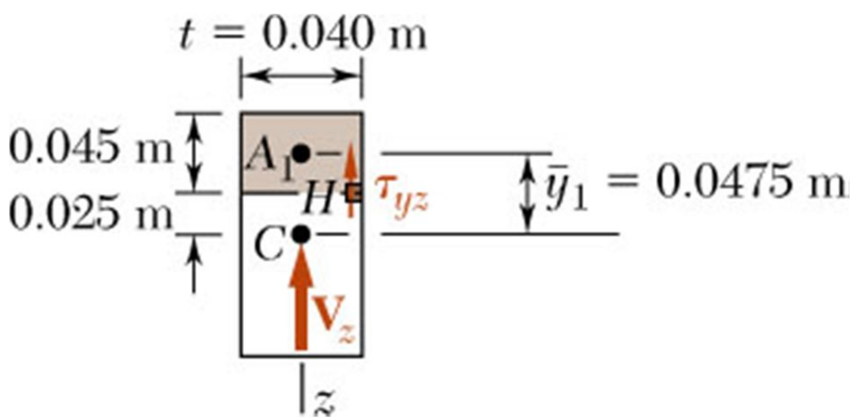
$$A = (0.040\,\text{m})(0.140\,\text{m}) = 5.6\times10^{-3}\,\text{m}^2$$

$$I_x = \tfrac{1}{12}(0.040\,\text{m})(0.140\,\text{m})^3 = 9.15\times10^{-6}\,\text{m}^4$$

$$I_z = \tfrac{1}{12}(0.140\,\text{m})(0.040\,\text{m})^3 = 0.747\times10^{-6}\,\text{m}^4$$

Example 8





Note: 2-D (i.e., plane) state of stress only at H. In interior it is 3-D state of stress

Normal stress at *H*.

$$\sigma_y = +\frac{P}{A} + \frac{|M_z|a}{I_z} - \frac{|M_x|b}{I_x}$$

$$= \frac{50\,\text{kN}}{5.6\times10^{-3}\,\text{m}^2} + \frac{(3\,\text{kN}\cdot\text{m})(0.020\,\text{m})}{0.747\times10^{-6}\,\text{m}^4} - \frac{(8.5\,\text{kN}\cdot\text{m})(0.025\,\text{m})}{9.15\times10^{-6}\,\text{m}^4}$$

$$= (8.93 + 80.3 - 23.2)\,\text{MPa} = 66.0\,\text{MPa}$$

Shearing stresses at *H*.

$$Q = A_1\bar{y}_1 = [(0.040\,\text{m})(0.045\,\text{m})](0.0475\,\text{m})$$

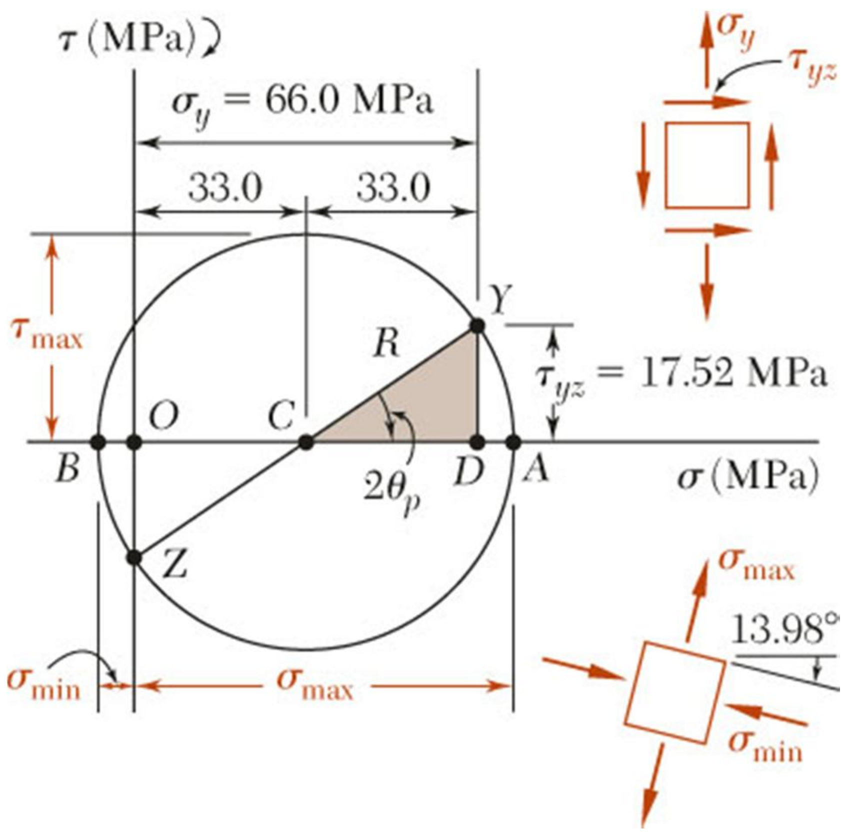
$$= 85.5\times10^{-6}\,\text{m}^3$$

$$\tau_{yz} = \frac{V_z Q}{I_x t} = \frac{(75\,\text{kN})(85.5\times10^{-6}\,\text{m}^3)}{(9.15\times10^{-6}\,\text{m}^4)(0.040\,\text{m})}$$

$$= 17.52\,\text{MPa}$$

$$\tau_{yx} = 0$$

Example 8



Principal stresses and maximum, shearing stress, principal planes.

$$\tau_{\max} = R = \sqrt{33.0^2 + 17.52^2} = 37.4\,\text{MPa}$$

$$\sigma_{\max} = OC + R = 33.0 + 37.4 = 70.4\,\text{MPa}$$

$$\sigma_{\min} = OC - R = 33.0 - 37.4 = -7.4\,\text{MPa}$$

$$\tan 2\theta_{\text{p}} = \frac{CY}{CD} = \frac{17.52}{33.0} \qquad 2\theta_{\text{p}} = 27.96^\circ$$

$$\theta_p = 13.98^\circ$$

$$\tau_{\max} = 37.4\,\text{MPa}$$

$$\sigma_{\max} = 70.4\,\text{MPa}$$

$$\sigma_{\min} = -7.4\,\text{MPa}$$

$$\theta_p = 13.98^\circ$$