DEPARTMENT OF CIVIL ENGINEERING, IIT BOMBAY

CE 201 Solid Mechanics

Tutorial Sheet = 9 Instructor :A. Laskar/ N K Chandiramani

1. For the beams loaded as shown in the Fig. 1 determine the equations of elastic curves using the integration method. Determine the values of maximum deflection, slope and their locations.
2. Determine the deflection of guided roller of the uniform beam shown in the Fig. 2.
3. Determine the deflection of point C of the uniform overhanging beam shown in the Fig. 3.
4. Determine the deflection and slope at the internal hinge of the beam shown in the Fig. 4.
5. Determine the slope and deflection of the point C of the non-uniform beam shown in the Fig. 5.
6. Determine the deflection of the point C of the right angle bent as shown in the Fig. 6.

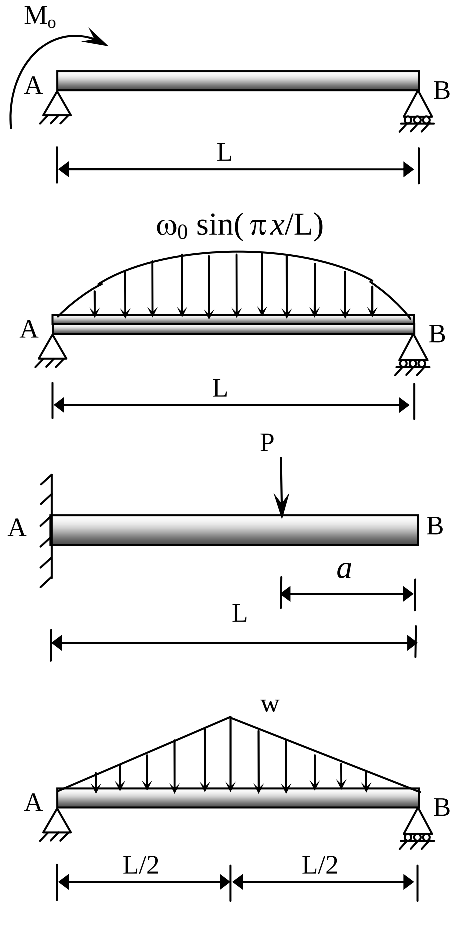


Fig. 1

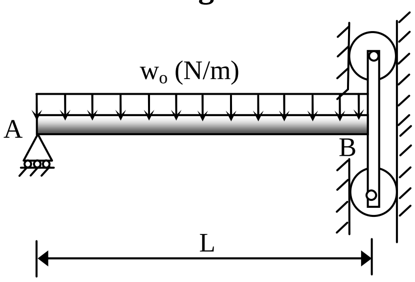


Fig. 2

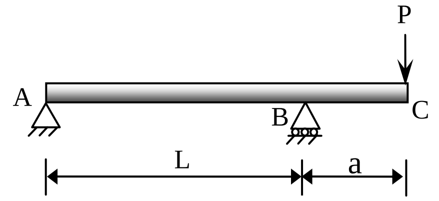


Fig. 3

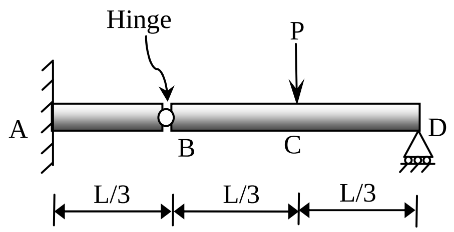


Fig. 4

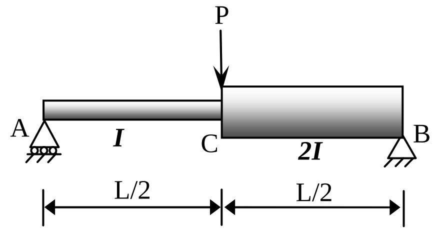


Fig. 5

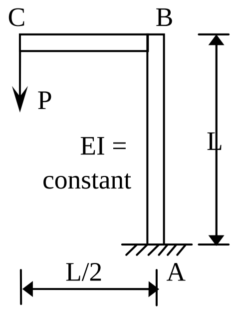
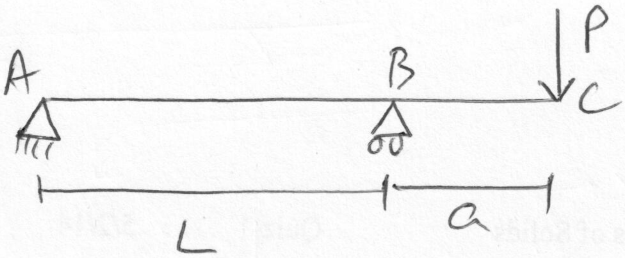
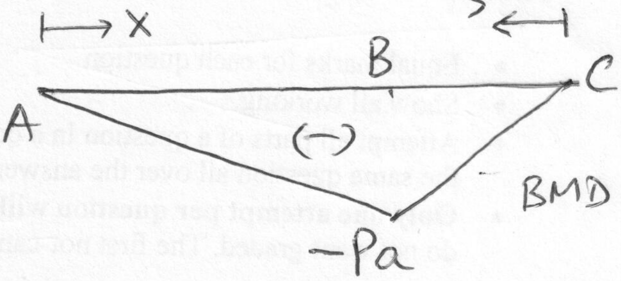


Fig. 6

P3



2nd order method



$$M_{AB} = -\frac{Pa}{L}x$$

$$\Rightarrow EIy_{AB} = -\frac{Pa}{L}\frac{x^3}{6} + C_1 x + C_2$$

$$x=0,\ y_{AB}=0 \Rightarrow C_2 = 0$$

$$x=L,\ y_{AB}=0 \Rightarrow C_1 = \frac{PaL}{6}$$

$$M_{BC} = -P\xi \Rightarrow EIy_{BC} = -\frac{P\xi^3}{6} + C_3\xi + C_4$$

$$\xi=a,\ y_{BC}=0 \Rightarrow C_3 a + C_4 = \frac{Pa^3}{6}$$

$$x=L, \xi=a,\ y'_{AB}\Big|_{x=L} = -y'_{BC}\Big|_{\xi=a} \Rightarrow 0 = -Pa\frac{L}{2} + Pa\frac{L}{6} - \frac{Pa^2}{2} + C_3$$

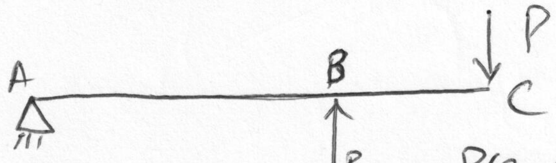
$$\Rightarrow C_3 = Pa\left(\frac{a}{2}+\frac{L}{3}\right)$$

$$C_4 = -Pa^2\left(\frac{a}{3}+\frac{L}{3}\right)$$

$$y_C = y_{BC}\Big|_{\xi=0} = \frac{C_4}{EI} = -\frac{Pa^2}{EI}\left(\frac{a}{3}+\frac{L}{3}\right) = -\frac{Pa^2}{3EI}(a+L)$$

$$\theta_C = y'_{BC}\Big|_{\xi=0} = \frac{C_3}{EI} = Pa\left(\frac{a}{2}+\frac{L}{3}\right)$$

4th order method



$$R_B = P\left(\frac{a}{L}+1\right)$$

$$EIy^{IV} = -W = R_B\langle x-L\rangle^{-1} - P\langle x-(L+a)\rangle^{-1}$$

$$EI\,y^{III} = R_B\langle x-L\rangle^0 - P\langle x-(L+a)\rangle^0 + C_1$$

$$EI\,y^{II} = R_B\langle x-L\rangle^1 - P\langle x-(L+a)\rangle^1 + C_1 x + C_2$$

$$EI\,y^{I} = \frac{R_B}{2}\langle x-L\rangle^2 - \frac{P}{2}\langle x-(L+a)\rangle^2 + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$EI\,y = \frac{R_B}{6}\langle x-L\rangle^3 - \frac{P}{6}\langle x-(L+a)\rangle^3 + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

$$EI\,y\big|_{x=0} = 0 \Rightarrow C_4 = 0$$

$$EI\,y''\big|_{x=0} = 0 \Rightarrow C_2 = 0$$

$$EI\,y\big|_{x=L} = 0 \Rightarrow C_1\frac{L^3}{6} + C_3 L = 0.$$

$$EI\,y''\big|_{x=L+a} = 0 \Rightarrow R_B a + C_1(L+a)$$

$$\Rightarrow C_1 = -\frac{Pa}{L}, \quad C_3 = \frac{PaL}{6}$$

$$y_c = y\big|_{L+a} = \left(\frac{R_B}{6}a^3 - \frac{Pa}{L}\frac{(L+a)^3}{6} + \frac{PaL}{6}(L+a)\right)\frac{1}{EI}$$

$$= -P\frac{(L+a)}{3}\frac{a^2}{EI} \quad \text{(same as by 2nd order method)}$$

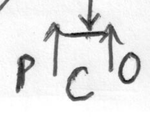
$$\theta_c = y'\big|_{L+a} = \frac{R_B}{2}a^2 - \frac{Pa}{L}\frac{(L+a)^2}{2} + \frac{PaL}{6}$$

$$= Pa\left(-\frac{a}{2} - \frac{L}{3}\right) \quad \text{ie (2)} \rightarrow \text{same as by 2nd order method}$$

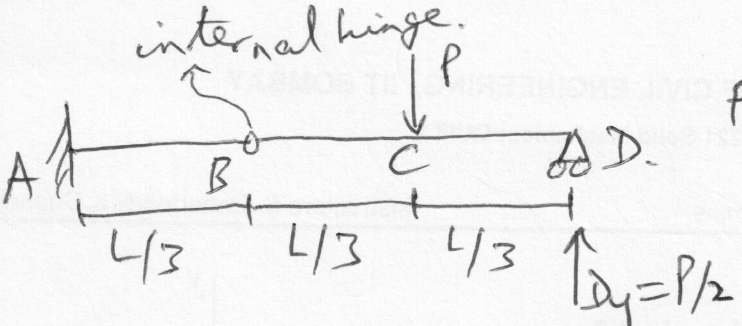
Note that shear force boundary condition is identically satisfied. Check it → (at $x=L+a$)

$$EI\,y^{III}\big|_{x=(L+a)^+} = R_B - P + C_1 = P\left(1 + \frac{a}{L} - 1 - \frac{a}{L}\right) = 0 \checkmark$$

or $$EI\,y^{III}\big|_{x=(L+a)^-} = R_B + C_1 = P\left(1 + \frac{a}{L} - \frac{a}{L}\right) = P \checkmark$$

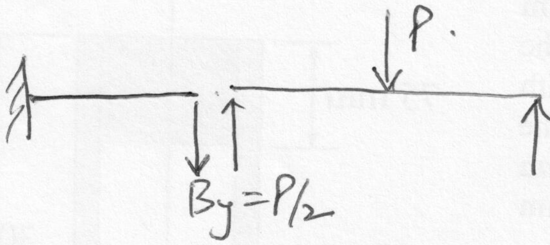


P4



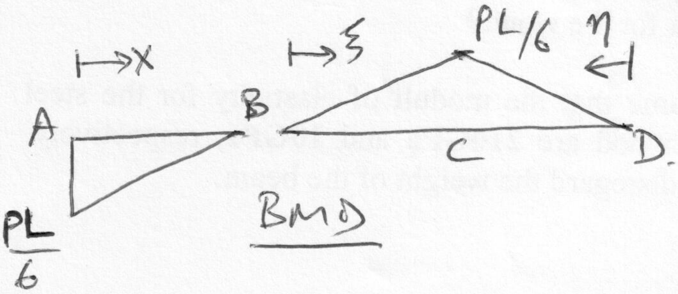
For BCD, $\sum M_B = 0 \Rightarrow D_y = \frac{P}{2}$

$\Rightarrow B_y = \frac{P}{2}$



So AB behaves like cantilever with tip load $B_y = \frac{P}{2}(\downarrow)$

2nd order method



AB: $\left(0 \le x \le \frac{L}{3}\right)$

$$M_{AB} = -P\left(\frac{L}{6} - \frac{x}{2}\right) \Rightarrow P\left(\frac{x^3}{12} - \frac{Lx^2}{12}\right) = EIy_{AB} + C_1x + C_2$$

$$\left.\begin{array}{l} x=0,\ y_{AB}=0 \Rightarrow C_2 = 0 \\ x=0,\ y'_{AB}=0 \Rightarrow C_1=0 \end{array}\right\} \Rightarrow y_{AB}\left(\frac{L}{3}\right) = \boxed{y_B = \frac{PL^3}{EI}\left(-\frac{1}{162}\right)(\downarrow)}$$

$$y'_{AB}\left(\frac{L}{3}\right) = \boxed{\theta_B = \frac{PL^2}{EI}\left(-\frac{1}{36}\right)(\circlearrowright)}$$

BC: $\left(0 \le \xi \le \frac{L}{3}\right)$ $\quad M_{BC} = \frac{P}{2}\xi \Rightarrow \frac{P}{12}\xi^3 = EIy_{BC} + C_3\xi + C_4$

$$x = \frac{L}{3},\ \xi = 0,\ \underbrace{y_{AB}\Big|_{x=L/3}}_{=y_B} = y_{BC}\Big|_{\xi=0} \Rightarrow C_4 = \frac{PL^3}{162}$$

CD: $\left(0 \le \eta \le \frac{L}{3}\right)$ $\quad M_{CD} = \frac{P}{2}\eta \Rightarrow \frac{P}{12}\eta^3 = EIy_{CD} + C_5\eta + C_6$

$$\eta = 0,\ y = 0 \Rightarrow C_6 = 0$$

$$\left.\begin{array}{l} \xi = \frac{L}{3} = \eta,\ y_{BC} = y_{CD} \Rightarrow C_5 - C_3 = \frac{PL^2}{54} \\ \xi = \frac{L}{3} = \eta,\ y'_{BC} = -y'_{CD} \Rightarrow C_3 + C_5 = \frac{PL^2}{18} \end{array}\right\} \Rightarrow C_5 = \frac{PL^2}{27},\quad C_3 = \frac{PL^2}{54}$$

$$\boxed{\theta_B^+ = y'_{BC}\Big|_{\xi=0} = -c_3 = -\frac{PL^2}{EI}\left(\frac{1}{54}\right) \quad (2)}$$

Can also treat BC & CD combined using 4th order method, as follows. Integrating we get

(use $w = \langle \xi - \frac{L}{3} \rangle^{-1} P$)

($EI\, y^{IV}_{BD} = -w$).

$$EI\, y'''_{BD} = -P\langle \xi - \tfrac{L}{3} \rangle^0 + a_1$$

$$EI\, y''_{BD} = -P\langle \xi - \tfrac{L}{3} \rangle^1 + a_1 \xi + a_2$$

$$EI\, y'_{BD} = -\frac{P\langle \xi - \frac{L}{3} \rangle^2}{2} + a_1 \frac{\xi^2}{2} + a_2 \xi + a_3$$

$$EI\, y_{BD} = -\frac{P}{6}\langle \xi - \tfrac{L}{3} \rangle^3 + a_1 \frac{\xi^3}{6} + a_2 \frac{\xi^2}{2} + a_3 \xi + a_4$$

$$x = \frac{L}{3},\ \xi = 0, \quad y_{BD}\Big|_{\xi=0} = y_B = y_{AB}\Big|_{x=\frac{L}{3}} = \frac{PL^3}{EI}\left(-\frac{1}{162}\right)$$

$$\Rightarrow a_4 = -\frac{PL^3}{162}$$

$$\xi = 0, \quad EI\, y'''_{BD} = V_B = \frac{P}{2} = a_1$$

$$\xi = 0, \quad EI\, y''_{BD} = M_B = 0 = a_2$$

$$\xi = \frac{2L}{3}, \quad EI\, y_{BD} = 0 = -\frac{PL^3}{6\cdot 27} + \frac{8PL^3}{2\cdot 6\cdot 27} + a_3 \frac{2L}{3} - \frac{PL^3}{162}$$

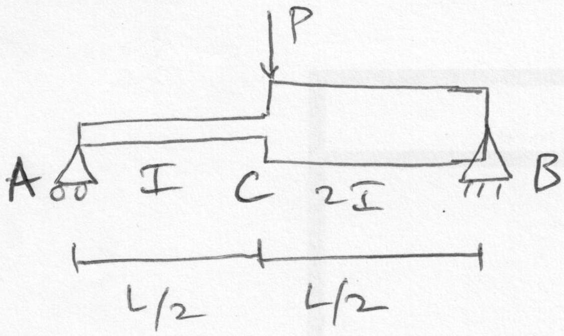
$$\Rightarrow a_3 = PL^2\left(-\frac{1}{54}\right)$$

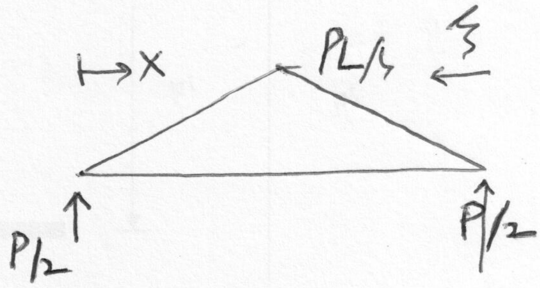
$$\boxed{\theta_B^+ = y'_{BD}\Big|_{\xi=0} = a_3 = -\frac{PL^2}{EI}\left(\frac{1}{54}\right)} \longrightarrow \text{Same as by 2nd order method.}$$

Check:

$$\underbrace{EI\, y''_{BD}\Big|_{\xi=\frac{2L}{3}} = 0, \quad EI\, y'''\Big|_{\xi=\frac{2L}{3}} = -\frac{P}{2}}_{\text{These are identically satisfied.}}$$

P5





2nd order method

$$M_L = \frac{Px}{2} = EIy_L'' \,,\; 0 \le x \le \frac{L}{2} \qquad\Big|\qquad M_R = \frac{P\xi}{2} = 2EIy_R'' \,,\; 0 \le \xi \le \frac{L}{2}$$

$$\frac{Px^3}{12} = EIy_L + C_1 x + C_2 \qquad\Big|\qquad \frac{P\xi^3}{12} = 2EIy_R + C_3\xi + C_4$$

$$x=0,\; y_L=0 \Rightarrow C_2 = 0 \qquad\Big|\qquad \xi=0,\; y_R=0 \Rightarrow C_4=0$$

$$\left.\begin{array}{l} x=\xi=\frac{L}{2},\; y_L = y_R \Rightarrow \frac{PL^3}{96} = C_1 L - C_3\frac{L}{2} \\[2ex] x=\xi=\frac{L}{2},\; y_L' = -y_R' \Rightarrow \frac{PL^2}{8} + \frac{PL^2}{16} = 2C_1 + C_3 \end{array}\right\} \Rightarrow C_1 = \frac{5}{96}PL^2 \quad C_3 = \frac{1}{12}PL^2$$

$$y_C = y_L\left(\frac{L}{2}\right) = y_R\left(\frac{L}{2}\right) = \frac{PL^3}{EI}\left(-\frac{1}{64}\right) \text{ ie } \downarrow$$

$$\theta_C = y_L'\left(\frac{L}{2}\right) = -y_R'\left(\frac{L}{2}\right) = \frac{PL^2}{EI}\left(\frac{1}{96}\right) \curvearrowleft$$

Can also write $M_R = \frac{P}{2}(L-x) \Rightarrow \frac{P}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) = 2EIy_R + C_3'x + C_4'$

$$x=L,\; y_R=0 \Rightarrow \frac{P}{6} = C_3'L + C_4'$$

$$x=\frac{L}{2},\; y_L=y_R \Rightarrow -\frac{1}{32}PL^3 = C_1L - C_3'\frac{L}{2} - C_4'$$

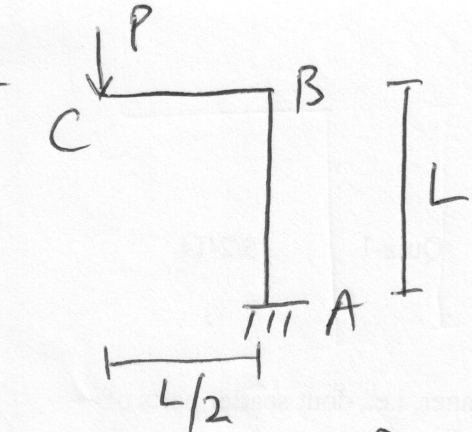
$$x=\frac{L}{2},\; y_L'=y_R' \Rightarrow -\frac{1}{16}PL^2 = 2C_1 - C_3'$$

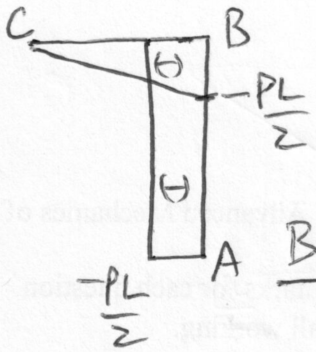
solve for C_1, C_3', C_4'. Then $y_C = y_L\left(\frac{L}{2}\right) = y_R\left(\frac{L}{2}\right)$

$\theta_C = y_L'\left(\frac{L}{2}\right) = y_R'\left(\frac{L}{2}\right)$

→ MORE TEDIOUS!!

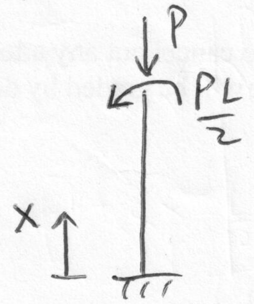
P6





BMD (whole structure)

For AB:



BMD of AB

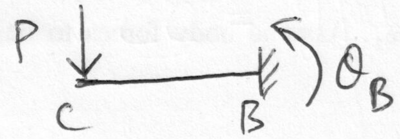
Neglect axial deformation of AB, ie assume axial stiffness $EA \to \infty$

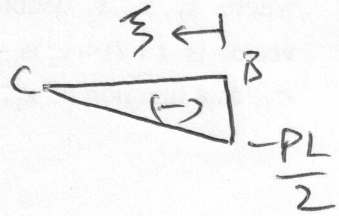
$$M_{AB} = -\frac{PL}{2} \Rightarrow EI\,y_{AB} = -\frac{PL}{2}\frac{x^2}{2} + C_1 x + C_2$$

$$EI\,y_{AB}\Big|_{x=0} = 0 \Rightarrow C_2 = 0 \;;\quad EI\,y'_{AB}\Big|_{x=0} = 0 \Rightarrow C_1 = 0$$

$$y_B = y_{AB}\Big|_{x=L} = -\frac{PL^3}{4EI} \;;\quad y'_{AB}\Big|_{x=L} = \theta_B = -\frac{PL^2}{2EI}$$

For BC:





$$M_{CB} = -P\left(\frac{L}{2} - \xi\right)$$

$$\Rightarrow EI\,y_{CB} = -P\left(\frac{L}{4}\xi^2 - \frac{\xi^3}{6}\right) + C_3\xi + C_4$$

$$EI\,y_{CB}\Big|_{\xi=0} = 0 \Rightarrow C_4 = 0$$

$$EI\,y'_{CB}\Big|_{\xi=0} = EI\,\theta_B \Rightarrow C_3 = -\frac{PL^2}{2}$$

$$y_{CB}\Big|_{\xi=\frac{L}{2}} = \frac{PL^3}{EI}\left(-\frac{7}{24}\right) \;;\quad y'_{CB}\Big|_{\xi=\frac{L}{2}} = \theta_C = \frac{PL^2}{EI}\left(-\frac{5}{8}\right)$$

Deflection of C: $\theta_C = \frac{PL^2}{EI}\left(-\frac{5}{8}\right)(\circlearrowleft)$; $x_C = -\frac{PL^3}{4EI}(\leftarrow)$, $y_C = \frac{PL^3}{EI}\left(-\frac{7}{24}\right)(\downarrow)$