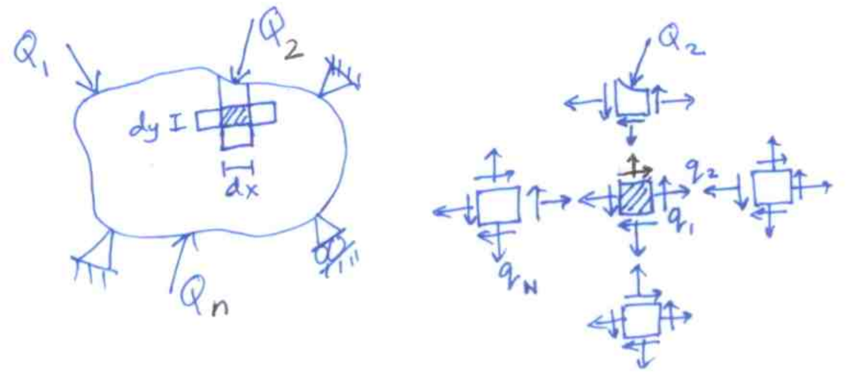
TOPIC - 4

DEFLECTIONS.

(I) DEFLECTIONS by METHOD OF VIRTUAL WORK. (VIRTUAL FORCE METHOD)

(1) PRINCIPLE OF WORK. — DEFORMABLE BODY.



$Q_i \rightarrow$ external applied loads (forces, couples). Can be real or virtual

$q_i \rightarrow$ internally generated stresses due to external loads. So they are, correspondingly, real or virtual.

Internal q's are caused by external Q's such that (Q, q) system is in equilibrium.

Now apply external displacements D's, and corresponding internal displacements d's. Hence d's can be obtained uniquely from D's, i.e., (D, d) system satisfies compatibility. Note that (D, d) can be real or virtual and are, in general, unrelated to (Q, q).

Assumption:

(Q, q) remain constant (in magnitude & direction) w.r.t. undeformed body/structure while (D, d) applied. Hence (Q, q) continue to satisfy equilibrium throughout the application of (D, d).

Note that this in effect implies that (D, d) are small, otherwise (Q, q) will not remain constant (in direction) and wont continue to satisfy equilibrium.

Let work done by force/stress system (Q, q) on a element or particle be denoted dW. This work arises from (Q, q) undergoing displacement (D, d). Now (D, d) comprises a rigid body displacement component and a deformation component.

Thus,

$$dW = dW_R + dW_D$$

dW_R = work done by (Q, q) due to rigid body component of (D, d)

dW_D = work done by (Q, q) due to deformation component of (D, d).

Now, Recall Principle of Virtual Work for a Rigid Body or a Particle (page 5, Topic 3). It states that the VW is zero if particle or rigid body in equilibrium.

$$\Rightarrow dW_R = 0$$

$$dW = dW_D .$$

i.e., total work done by forces/stresses (Q, q) on an element/particle, which arises due to displacement (D, d), is equal to the work done only due to deformation.

Now summing over elements/particles,

$$\sum_{elements} dW = \sum_{elements} dW_d$$ ($\sum_{elements}$ indicates sum over elements)

$$\Rightarrow \quad W = W_d$$

Now W represents sum of work (taken over all elements/particles) of work done by (Q, q) due to total (D, d) (i.e rigid body & deformation). Since adjacent particles/elements have equal & opposite stresses that are acting on planes that undergo identical displacements so as to maintain compatibility (i.e no holes or voids formed inside), we can conclude that the contribution of these equal & opposite stresses in W get cancelled out, i.e., only external Q's contribute to W.

Hence, $W = W_e$

and $\boxed{W_e = W_d}$

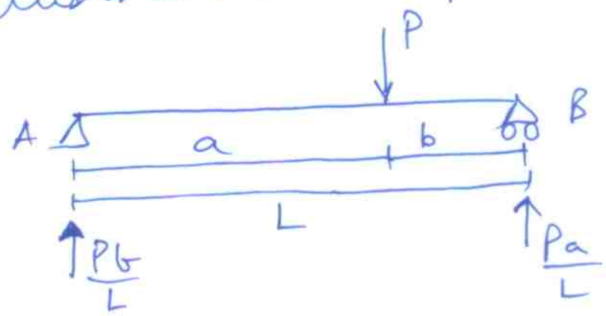
where W_e = work done by external applied loads Q's undergoing displacement D

W_d = work done by internal stresses q's undergoing displacement d

Summarizing the assumptions/limitations & generality of the principle of work,

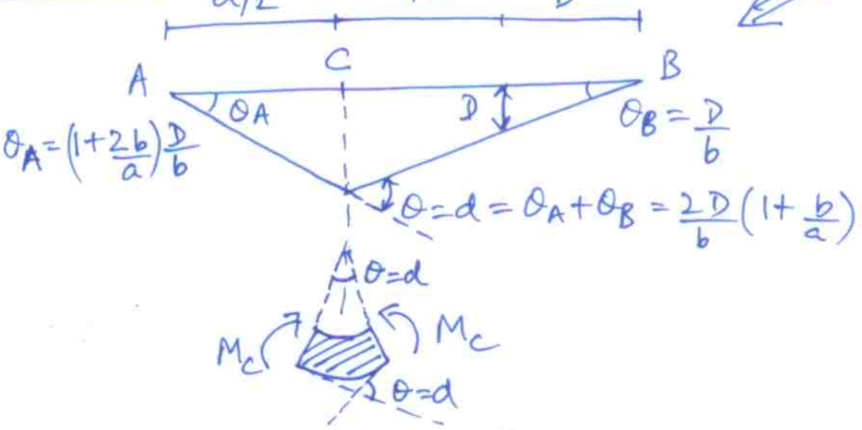
1) (Q, q) must remain constant & in equilibrium throughout application of (D, d). This implies that (D, d) are small.
2) (D, d) must be compatible (no holes/voids formed and support conditions not violated) during application of (D, d)
3) (Q, q) and (D, d) can be real or virtual. They are unrelated (arbitrary), in general. If (D, d) real, it can be due to loads, temperature, misfit in members/components of structural system, and don't need to follow Hooke's law, i.e. nonlinear elastic cases can also be treated by this principle.

Ex Illustrative example for independence of (Q, q) & (D, d)



We will apply two arbitrary but compatible displacements and verify the principle of work.

Case I (No Shearing)



Applied (D, d): D at b from end B, and hinge at $\frac{a}{2}$ from A. Compatibility yields d as shown here.

Deformed Element at C shown. Other elements dont deform.

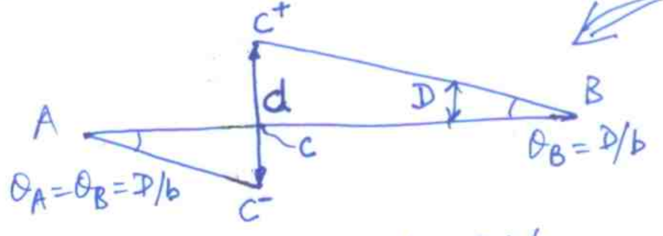
Thus (D, d) applied as above yield work terms,

$$W_e = PD.$$

$$W_d = M_C d = \left(\frac{Pb}{L}\frac{a}{2}\right)\left(\frac{2D}{b}\left[1+\frac{b}{a}\right]\right) = PD$$

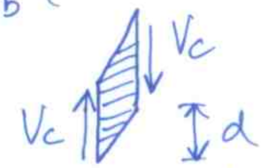
$\Rightarrow W_e = W_d$ verified

Case II (No bending)



Applied D, d: D at b from end B, and pure shear deformation at a/2 from A, ie, AC and C^+B are parallel so AB undergoes no bending.

$$d = \frac{D}{b}(a/2 + a/2 + b) = DL/b.$$



Deformed element at C shown. Other elements dont deform.

Thus (D, d) yield,

$$W_e = -PD$$

$$W_d = -V_C d = -R_A \frac{DL}{b} = \frac{-Pb}{L}\frac{DL}{b} = -PD$$

$\Rightarrow W_e = W_d$ verified.

So (D, d) unrelated to (Q, q). (D, d) must satisfy compatibility,
(Q, q) " " equilibrium.

On the other hand, you can easily recognize that this example is similar to the case when M_c, V_c are unknown and we apply the respective indicated virtual displacements (D, d) and use $W_e = W_d$ to find M_c, V_c. This would be the case of applying the principle of Virtual Work to find real forces (M_c, V_c) by applying Virtual displacements.

Note that when,

1) (Q, q) real, (D, d) virtual we have the Principle of Virtual Work with virtual displacements applied to obtain real forces.
2) (Q, q) virtual, (D, d) real, we have the principle of Virtual Work with virtual forces applied to obtain real displacements.

In this section <u>we are interested</u> in the 2nd application <u>of the principle of Virtual Work</u>.

<u>Need for Principle of Virtual Work to find displacements instead of Principle of Conservation of Energy</u>:

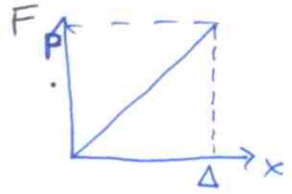
For a conservative system (i.e, not strained beyond the elastic limit), let <u>loads</u> and <u>stresses</u> <u>be increased gradually</u> (<u>quasi-statically</u>) such that they remain in equilibrium throut the increase. Let real displacements (D, d) be caused by these real loads (Q, q). The principle of work applies for every incremental amount of work done during incremental change in load. Then adding incremental work we get $\int dW_e = \int dW_d \Rightarrow W_e = W_d$
where integral is over the loading path (not over elements of the structure). Denoting W_d as U_i (internal energy or strain energy for elastic system), we have

$\boxed{W_e = U_i}$ $\longrightarrow$ Conservation of Energy

W_e (Work of external force): F gradually increased to P.

$$W_e = \int_0^{\Delta} F\,dx = \int_0^{\Delta} kx\,dx = \frac{1}{2}P\Delta \text{, for force applied}$$

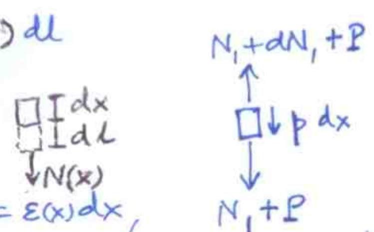
for linear elasticity



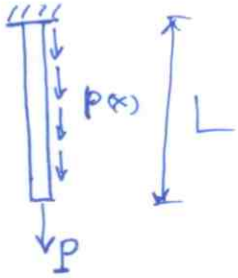
or $W_e = \frac{1}{2}M\theta$ for moment applied.

U_i (Strain energy):

Axial deformation: $dU_i = \frac{1}{2}N(x)\,dl$



$dl = \varepsilon(x)dx$



$$U_i = \int_0^L \frac{1}{2}N(x)\,dl$$

$$= \int \frac{1}{2}(N_1(x)+P)\,\varepsilon\,dx$$

for linear elastic

$$= \int \frac{1}{2}(N_1+P) * \frac{\sigma(x)}{E}\,dx$$

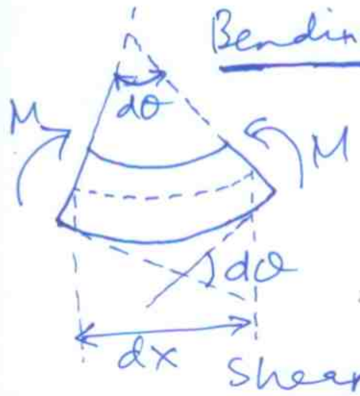
$$= \int_0^L \frac{(N_1+P)^2}{2}\frac{1}{A}\,dx$$

$dN_1 = p(x)\,dx$

$N_1(x) = \int_0^x p\,dx$

$N(x) = N_1(x) + P$.

For $p(x) = 0$, $N_1(x) = 0$, $\Rightarrow$ $\boxed{U_i = \dfrac{P^2L}{2AE}}$

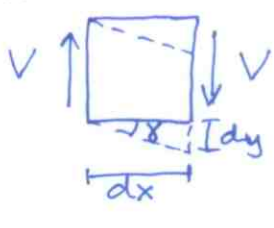


Bending deformation: $dU_i = \frac{1}{2}M(x)\,d\theta$,

$M = EIw'' = EI\theta' \Rightarrow \frac{M}{EI}dx = d\theta$

$$\Rightarrow \boxed{U_i = \int_0^L dU_i = \int_0^L \frac{1}{2}\frac{M^2}{EI}\,dx}$$, L = length of beam.

Shear deformation: This is a secondary deformation for thin (ie, slender) beams or frames made of thin members, bending being the primary deformation in thin members.



$dU_i = \frac{1}{2}V\,dy = \frac{1}{2}V\gamma\,dx$

Linear elastic $\Rightarrow \gamma = \frac{\tau}{G} = \frac{V}{AG}$

$dU_i = \frac{1}{2}\frac{V^2}{AG}\,dx$

Since actually shear stress is not constant over section, i.e $\tau \neq \frac{V}{A}$ we use a shear correction factor, i.e.,

$\tau = K\frac{V}{A}$.

$$\Rightarrow dU_i = \frac{1}{2} K \frac{V^2}{AG} dx$$

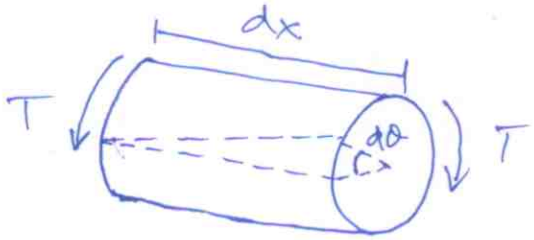
$$\boxed{U_i = \int_0^L \frac{1}{2} K \frac{V^2}{AG} dx}$$

V = shear force, A = crosssection area, G = shear modulus

L = length of beam, K = shear correction factor

K = 1.2 for rectangular section
  = 10/9 " circular "
  = 1 for wide flange and I beams where A = area of web.

Torsional deformation: (Circular sections).



$$\frac{d\theta}{dx} = \frac{T}{GJ}$$

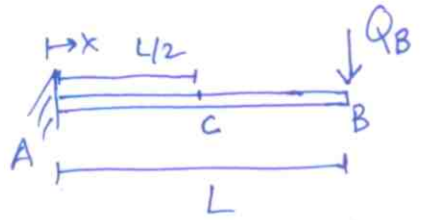
$$dU_i = \frac{1}{2} T d\theta$$

$$\boxed{U_i = \int_0^L \frac{1}{2} \frac{T^2}{GJ} dx}$$

NOTE: The factor of $\frac{1}{2}$ will always appear in We and U_i when loads and ensuing stresses are increased gradually (quasi-statically) to their full value.

Now lets see the need for Virtual Work Method thru an example.

Ex



Find (real) displacement D_B due to Q_B.

$M(x) = Q_B(x-L)$

Consv of energy, $W_e = Q_i \Rightarrow \frac{1}{2} Q_B D_B = \frac{1}{2EI}\int_0^L M^2(x)\,dx$

$$\therefore\ D_B = \frac{1}{Q_B EI} Q_B^2\left(\frac{L^3}{3} + L^3 - 2\frac{L^3}{2}\right) = \frac{Q_B L^3}{3EI}.$$

This wont work if we want displ at C, ie D_C, due to (real) load Q_B. So this method only gives the displacement under the real load applied & in the direction of that load. Further, it works only if one load is applied, since if multiple loads applied then the W_e term contains multiple unknown displacements under each of those loads.

Thus the need for using Principle of Virtual Work (2) on p.

(2) Principle of Virtual Work using Virtual Loads to find Real Displacements.

Step I : Apply virtual Q in direction of desired unknown real displacement D. Find equilibrating virtual q's (internal stresses). For convenience, and without loss of generality, take $Q = 1$.
At this stage the virtual displ due to virtual loads is of no concern, ie, it is as good as treating structure as rigid.

Step II : Apply real loads which cause real displacements d's and D. Relate real loads to real displacements.

Step III : Let virtual Q, q's in Step I do work due to real D, d's in Step II, and apply Principle of (Virtual) Work $W_e = W_i$

$$\boxed{1 \cdot D = \sum q_i d_i}$$

Real displacements

Virtual loads

Compatibility condition.

where,

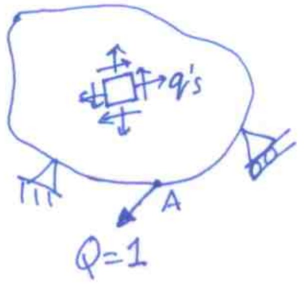
1 → external virtual load in direction of real displ sought

q_i → internal virtual load (stresses).

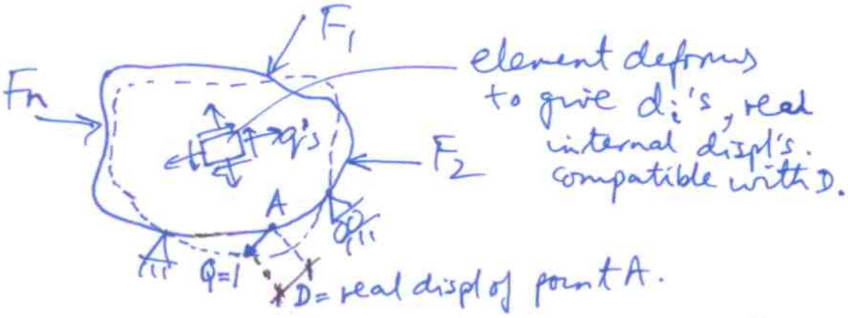
D → external real displ sought due to real loads.

d_i → internal real displ due to real loads.

Note that factor of $\frac{1}{2}$ does not appear in W_e or W_i since virtual loads existed at their full value at the start of the application of real displacements, and these virtual loads were held constant throughout application of the real displ's, i.e, the virtual loads just "ride along" thru the real displ's.



STEP-I Virtual loads applied; equilibrium of rigid body.



STEP-II Real loads applied. Body deforms to give real displ's. Virtual loads do virtual work.

In this form, the VW method represents a Compatibility requirement.

Alternately, using Real loads & Virtual Displ's, as we did in Muller Breslau principle, the VW method represents and equilibrium requirement.

Application to Trusses.

$$\text{VW} \rightarrow \quad \boxed{1 . D = \sum_{i=1}^{b} p_i \frac{P_i L_i}{A_i E_i}} \rightarrow (1) \qquad d_i = P_i L_i / A_i E_i \rightarrow \text{For linear elastic case.}$$

D = real displ of desired joint due to real loads

P_i = member forces (internal forces) due to real loads

p_i = member forces due to unit virtual load externally applied at joint in the direction of desired displ at that joint

1 = external virtual load applied

L_i, E_i, A_i = length, Youngs modulus, Area, of i^{th} member, respectively.

b = nos of members/bars.

<u>Temperature effect</u>: The real displacements could be due to temp change instead of mechanical loading. Hence,

$$\boxed{1 . D = \sum_{i=1}^{b} p_i \, \alpha_i \, \Delta T_i \, L_i} \rightarrow \text{1a}$$

$\alpha_i, \Delta T_i$ → coeff of thermal expansion, temp change (increase, positive), respectively, of i^{th} member.

<u>Fabrication errors or camber.</u>: Real displacements could be due to fabrication errors in member lengths, or deliberate error in member lengths–known as camber. Camber introduced e.g., in bridge deck (ie top or bottom chord of truss, as the case maybe) so that it bows upward convex when unloaded and becomes when loaded, ie deck will be flat in service condition.

$$\boxed{1 . D = \sum_{i=1}^{b} p_i \, \Delta L_i} \rightarrow \text{1b}$$

ΔL_i → error in length of i^{th} member (positive if longer than correct intended length).

<u>Application to Beams & Frames.</u>

Due to real loads, $d_i = d\theta = \left(\frac{M}{EI}\right) dx$ for bending strains, linear elastic case.

The $\sum_{\text{elements}}$ becomes $\int_0^L$

$$\boxed{1 . D = \int_0^L m \frac{M}{EI} \, dx} \rightarrow \text{2}$$

1 = ext virtual load in direction of real displ sought at point of application of ext. virtual load. If real displ sought is linear displ then virtual load is force, else if real

displ sought is angular then virtual load is a couple applied at that point.

$m(x)$ = internal moment due to virtual load 1.

$M(x)$ = internal moment due to real applied loads.

EI = bending rigidity.

Note that if concentrated loads (forces, couples) or discontinuous distributed loads act, integrals would be over several intervals for which $m(x)$, $M(x)$ should be consistent within each interval. Here, we can use Tables shown to compute the integrals.

Here primary straining is due to bending action, secondary straining due to axial, shear, torsion deformations may also occur. For these we have the additional virtual internal work (ie, strain energy) terms as follows, to be added on the RHS of the VW equation (2) pg. 10.

Axial deformation $\longrightarrow$ $\boxed{U_p = p \cdot \frac{PL}{AE}}$

p = internal axial force caused by unit applied virtual load.

P = internal axial force due to real applied external loads.

Shear deformation $\longrightarrow$ $U_s = \int_0^L K\left(v \frac{V}{GA}\right) dx$

v = internal shear due to unit external virtual load

V = internal shear due to real applied external loads

G, A = shear modulus, area of section

K = shear correction factor.

Here $\frac{V}{GA} dx = \gamma dx = dy$, the real displacement thru which the internal virtual shear displaces and does virtual work.

Torsional deformation $\longrightarrow$ $\boxed{U_t = t \frac{TL}{GJ}}$

$t =$ internal torsional moment due to applied ext. virtual unit load

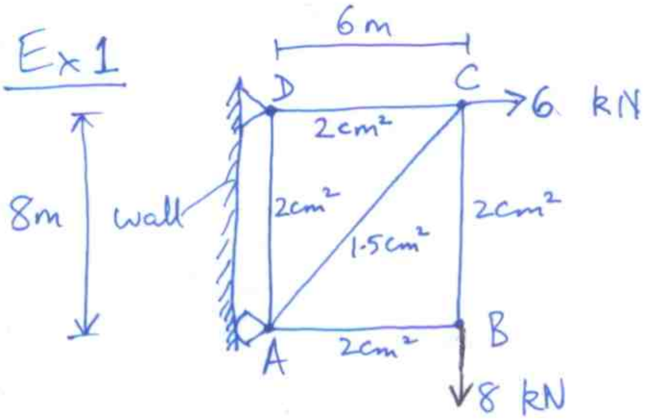
$T =$ internal torsional moment due to applied ext real loads.

Note that the factor $\frac{1}{2}$ is missing from U_p, U_s, U_t as compared to real strain energies on pages 6, 7. Also if applied virtual unit load and applied real loads lead to distributed internal forces/moments, then

$$U_p = \int_0^L p \frac{P}{AE} dx, \quad U_t = \int_0^L t \frac{T}{GJ} dx.$$

Such cases will usually not occur.

Ex 1



Find vertical displ of joint C due to applied loads as shown and temp change

$\Delta T = 120°F$ in member AD (increase)

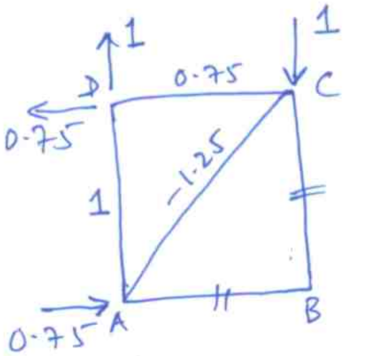
$\Delta T = -60°F$ " " AB (drop)

and length defects

AC short by 5mm

CB long by 5mm.

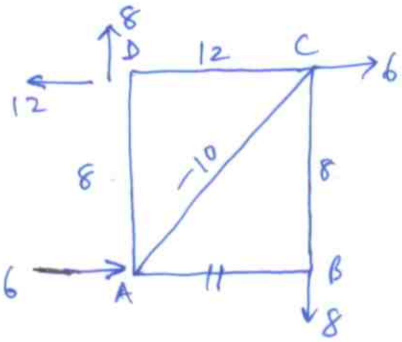
Take $E = 200$ GPa.



$AC = -0.75\left(\frac{1.0}{0.6}\right) = -1.25$

$CB = -1 - (-1.25)(0.8) = 0$ as expected. (zero force member).

$AB = -0.75 - (-1.25)(0.6) = 0$ as expected (zero force mem)



$AC = (-12 + 6)\left(\frac{1}{0.6}\right) = -10$

Member	p	P	L/A	pPL/A	ΔTL	pΔTL	pΔL
AB	0	0	$3*10^4$	0	−360	0	0
BC	0	$8*10^3$	$4*10^4$	0	0	0	0
CD	0.75	$12*10^3$	$3*10^4$	27E7	0	0	0
DA	1	$8*10^3$	$4*10^4$	32E7	960	960	0
AC	−1.25	$-10*10^3$	$\frac{20}{3}*10^4$	83.33E7	0	0	6.25E−3
				142.33E7		960	6.25E−3

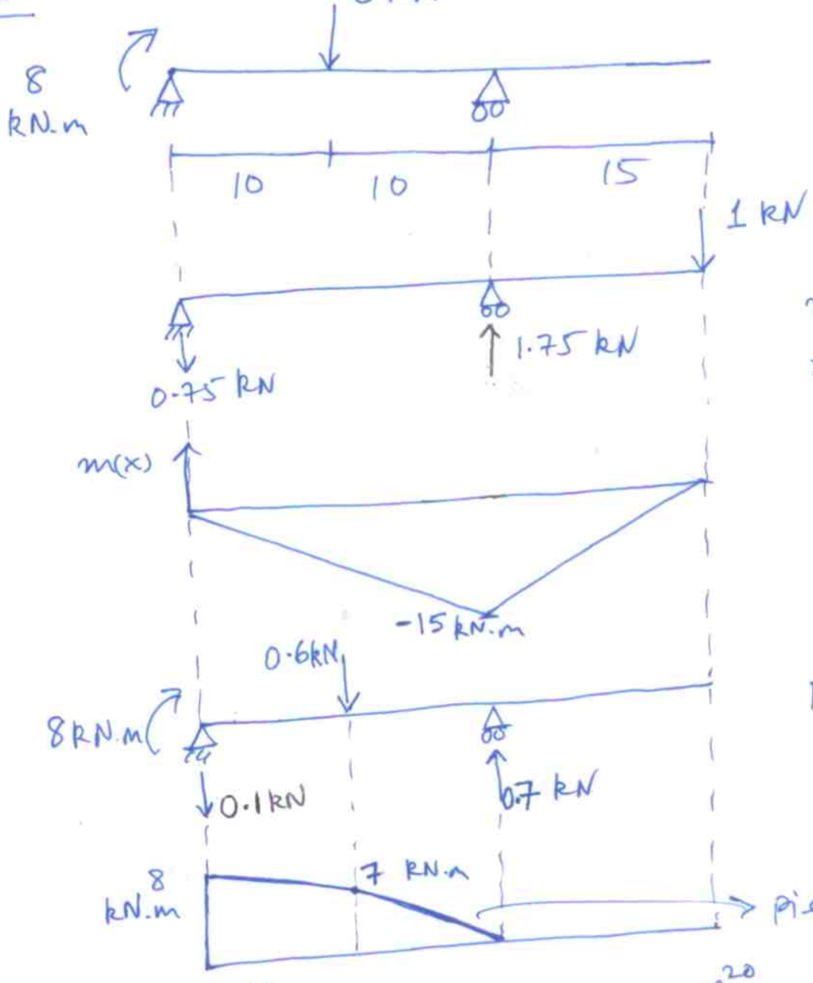
$$1.\ D_c = \frac{142.33*10^7}{200*10^9} + 960*0.6*10^{-5} + 6.25*10^{-3}$$

$$D_c = 7.1165*10^{-3} + 5.76*10^{-3} + 6.25*10^{-3}$$

$$= 19.1265*10^{-3}\,m = 19.1265\ mm.$$

Ex2

Find displacement at D.
$E = 200\ GPa$, $I = 60*10^6\ mm^4$



$m(x) = -0.75x\ ,\quad 0 \le x \le 20$

$m(x) = -0.75x + 1.75(x-20),\quad 20 \le x \le 35$

$M(x) = 8 - 0.1(x)\ ,\quad 0 \le x \le 10$

$= 8 - \frac{x}{10} - \frac{6}{10}(x-10),\quad 10 \le x \le 20$

$= 0\ ,\quad x \ge 20$

→ piecewise linear.

$$1.D = \frac{1}{EI}\left[\int_0^{10} 0.75x\left(\frac{x-80}{10}\right)dx + \int_{10}^{20} -0.75x\left(\frac{140-7x}{10}\right)dx\right]$$

$$= \frac{1}{EI}\left[\frac{0.75}{10}*\frac{10^3}{3} - \frac{60}{10}*\frac{10^2}{2} - \frac{105}{10}\frac{(20^2-10^2)}{2} + \frac{0.75*7*(20^3-10^3)}{10 \cdot 3}\right] = \frac{-625}{EI}$$

$$1\,D\ \text{kN.m} = -\frac{625\ (\text{kN.m})^2}{EI} \Rightarrow D = \frac{-625 * 10^3}{200*10^9 * 60*10^6 * 10^{-12}}$$

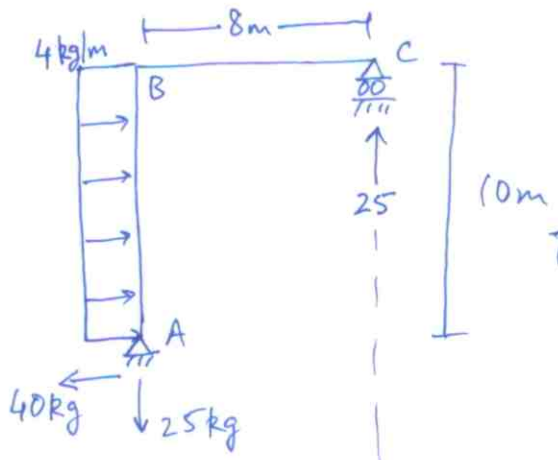
$$= -0.0521\,\text{m} = -52.1\,\text{mm}$$

Alternatively using tables,

$$\int mM\,dx = -\frac{1}{6}(7.5)(8+2*7)(10) - \frac{1}{6}(7)(15+2*7.5)(10)$$

$$= -625 \rightarrow \text{same as by integration.}$$

Ex 3.

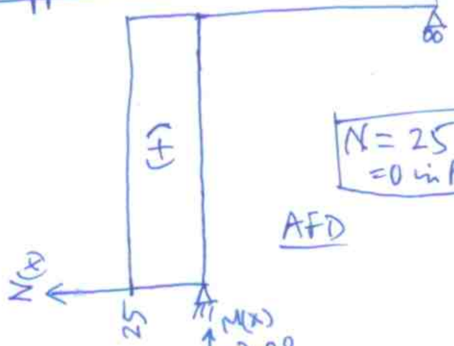


Find horizontal displacement of C. Include virtual strain energy due to axial & shear deformations.

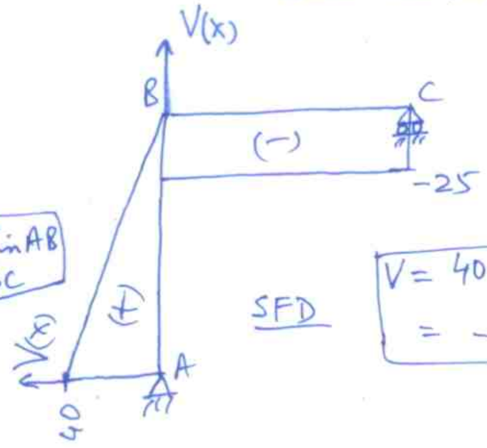
Take: $E = 200\,GPa$, $I = 60*10^6\,mm^4$, $A = 20*10^3\,mm^2$, $G = 160\,GPa$

Consider effects of shear deformation and axial deformation.

Due to applied load.

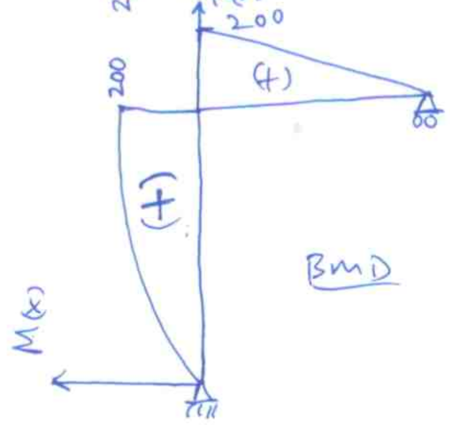


$N = 25$ in AB
$= 0$ in BC



SFD

$V = 40 - 4x$, in AB
$= -25$ in BC

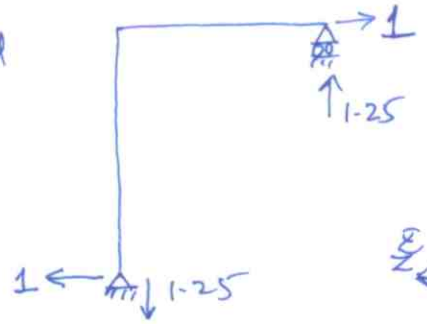


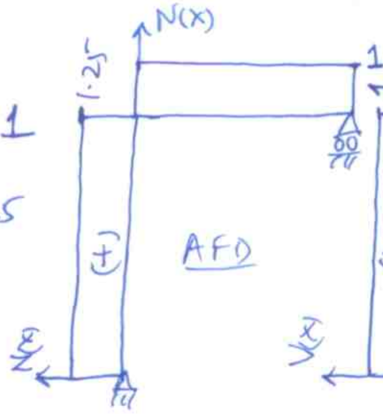
$M(x) = 40x - 2x^2$ in AB
$= 200 - 25x$ in BC.

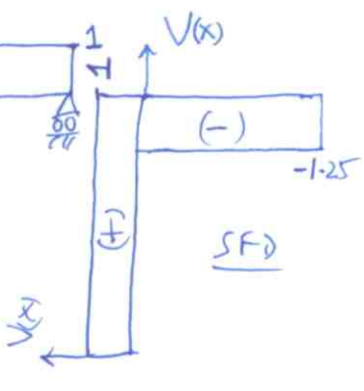
Due to unit virtual load

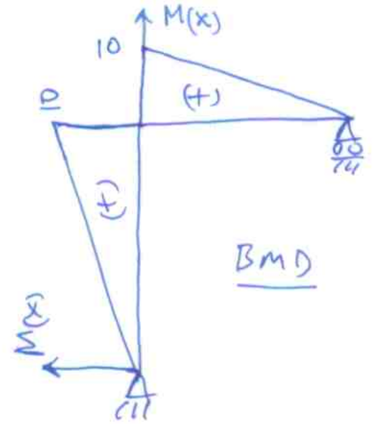
$N(x) = 1.25$ in AB
$= 1$ in BC

$V(x) = 1$ in AB
$= -1.25$ in BC









$$\boxed{\begin{aligned} M(x) &= x \text{ in } AB \\ &= 10 - 1.25x \text{ in } BC \end{aligned}}$$

$$\int m \frac{M}{EI}\,dx = \frac{1}{EI}\left[\int_0^{10} x\,(40x - 2x^2)\,dx + \int_0^{8} (10 - 1.25x)(200 - 25x)\,dx\right]$$

$$= \frac{1}{EI}\left[40\left(\frac{10^3}{3}\right) - 2\left(\frac{10^4}{4}\right) + 2000(8) + 31.25\left(\frac{8^3}{3}\right) - 500\left(\frac{8^2}{2}\right)\right]$$

$$= \frac{41000}{3EI} \quad \frac{(kN.m)(kg.m)}{Nm^2}$$

$$\int K v \frac{V}{GA}\,dx = \frac{K}{GA}\left[\int_0^{10} 1(40 - 4x)\,dx + \int_0^{8} (-1.25)(-25)\,dx\right] = \frac{1}{GA}\left[40(10) - \frac{4}{2}(10)^2 + 31.25(8)\right]$$

$$= \frac{450 * 1.2}{GA} = \frac{540}{GA} \quad \frac{kN.kg}{N}$$

$$\sum_{i=1}^{2} p_i \frac{P_i L_i}{AE} = \frac{(25)(1.25)(10)}{AE} = \frac{312.5}{AE} \quad \frac{kN.kg.m}{N}$$

to convert kg to N.

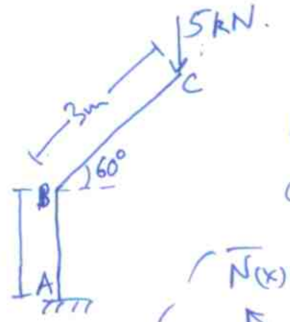
$$1.D\ kN.m = \frac{41000 * 10}{3(200E9)(60E-6)} + \frac{540 * 10}{(160E9)(20E-3)} + \frac{312.5 * 10}{(20E-3)(200E9)}$$

$$= 0.01139 + \underbrace{1.6875 * 10^{-6} + 7.8125 * 10^{-7}}$$

$$= 11.392 \text{ mm}$$

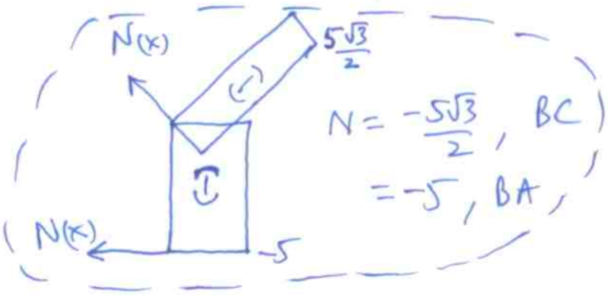
So shear & Axial deformation have negligible contribution in this case. Usually bending is predominant in frames.

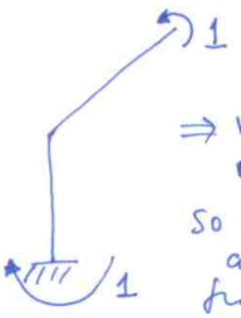
Ex4



Find rotation at C. Take E = 200 GPa, I = 15E6 mm^4

Consider effects of shear & axial deformation.





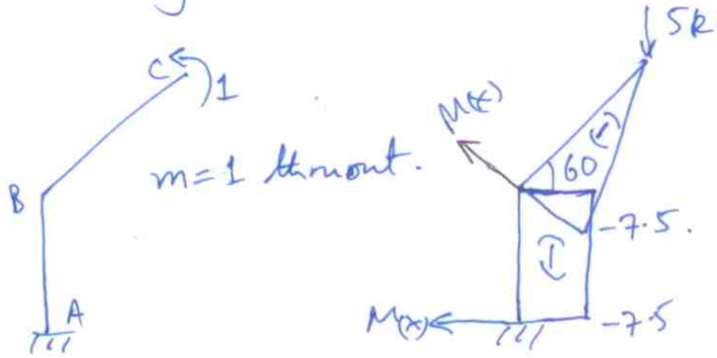
$\Rightarrow n = 0$, $v = 0$ } H moment.

So shear & axial deformation are zero when finding rotation at C.

$\because$ $n=0$, $v=0$, throughout we conclude that shear and axial deformations don't contribute to rotation at C for this problem. This is not true in general, e.g.,

$\Rightarrow$ application of unit couple gives non-zero shear, v, although zero axial force n. So shear def. contributes to rot. at C, although contribution may be small.

Returning to the original problem,



$$M(x) = -2.5x \quad , \text{ in } BC$$
$$= -7.5 \quad , \text{ in } AB$$

$$\int \frac{mM}{EI}\,dx = \frac{1}{EI}\left[\int_0^3 (1)(-2.5x)\,dx + \int_0^2 (1)(-7.5)\,dx\right] = -\frac{\left[(2.5)\left(\frac{9}{2}\right)+(7.5)(2)\right]}{EI}$$

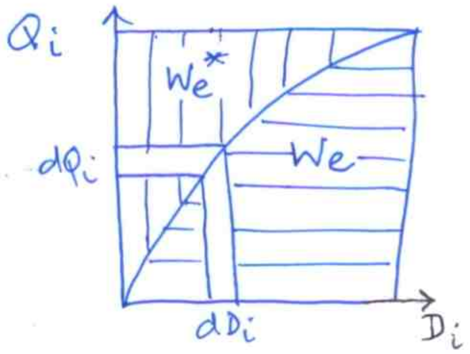
$$= -\frac{26.25}{EI}\;\frac{(kN.kN.m^2)}{N.m^2}$$

$$\underbrace{1.}_{kN.m}D = -\frac{26.25}{EI} * 10^3 \quad \Rightarrow \quad D = \theta_c = -\frac{26.25E3}{(200E9)(15E-6)} = -8.75*10^{-3} \text{ rad CW}$$

Ex 5

CASTIGLIANO's THEOREM. (Second Theorem).

Consider an elastic system. First we define complimentary work (W_e^*) done by applied external loads during real deformation and complimentary internal (or strain) energy (U_i^*) that develops during this process.



$$W_e = \int_0^{D_i} Q_i \, dD_i = \text{work done by real } Q_i \text{ undergoing real } D_i$$

$$W_e^* = \int_0^{Q_i} D_i \, dQ_i = \text{complimentary work done.}$$

Thus, $W_e = W_e[D_1, D_2, \cdots D_n] \rightarrow D_1, \cdots D_n$ are their corresponding displacements.

$W_e^* = W_e^*[Q_1, Q_2, \cdots Q_n] \rightarrow Q_1, \cdots Q_n$ are ext appl. loads.

From Conservation of Energy, we had (ref. p.6)

$$W_e = U_i = U_i[D_1, D_2, \cdots D_n] = \text{internal or strain energy.}$$

Analogously we define complimentary strain energy as,

$$U_i^* = W_e^* = U_i^*[Q_1, Q_2, \cdots Q_n].$$

However, for the present derivation we wont pursue with U_i^*.

Now restrict your attention to linearly elastic material, ie Q_i v/s D_i is straight line, hence

$$W_e = W_e^* = W_e[D_1, D_2 \cdots D_n] = W_e[Q_1, Q_2 \cdots Q_n]$$

and hence, from $W_e = U_i$,

$$U_i = U_i[D_1, D_2 \cdots D_n] = U_i[Q_1, Q_2, \cdots Q_n].$$

Now, first apply loads $Q_1, Q_2, \ldots Q_m \ldots Q_n$. Corresponding external work done, hence internal strain energy, is U_i. Then, apply additional load dQ. Hence, total ext work, hence strain energy, is

$$\underbrace{U_i}_{1^{st}\text{ step}} + \underbrace{\frac{\partial U_i}{\partial Q_m} dQ_m}_{2^{nd}\text{ step}} . \longrightarrow (I)$$

Now, reverse order of load application, ie first apply dQ_m and then apply $Q_1, \ldots Q_m, \ldots Q_n$. Hence, total ext work, hence strain energy, is,

$$\underbrace{\cancel{\tfrac{1}{2} dQ_m \, dD_m}}_{1^{st}\text{ step}}{}^{\nearrow \text{H.O.T (neglect)}} + \underbrace{U_i + dQ_m D_m}_{2^{nd}\text{ step}} \longrightarrow (II) \rightarrow \text{Here \& in (I), } U_i = U_i[Q_1, \cdots Q_n]$$

where dD_m is displacement of point of application of dQ_m (in its direction) in 1^{st} step, and U_i is work (hence strain energy) due to $Q_1, --, Q_m, --Q_n$ in 2^{nd} step and $dQ_m D_m$ is additional work done by dQ_m in second step.

Now for elastic body (linear or nonlinear) sequence of load application is inconsequential since work done is path independent (ie, sequence-of-load-appl independent) for a conservative system. Thus work done by ext loads, hence int energy stored, in (I) and (II) are equal. Thus, equating (I), (II),

$$\text{Compatibility statement} \longleftarrow \boxed{D_m = \frac{\partial U_i}{\partial Q_m}} \longrightarrow \text{CASTIGLIANO'S } 2^{nd} \text{ THEOREM to find DISPLACEMENT.}$$

So, to find D_m, apply force Q_m in its direction, along with all other applied loads. Then find strain energy due to Q_m plus all other applied loads, and carry out the above partial derivative. Then, Substitute actual value of Q_m (could be zero), to get D_m. When finding U_i you must keep Q_m as variable while other applied loads can be substituted as their actual (numerical) values.

Extra: Not for this course.

Now, conservation of energy for general nonlinear elastic case gives $W_e = U_i$, ie $dW_e = dU_i$ during a differential displacement $dD_1, \ldots, dD_m, \ldots dD_n$. Thus,

$$dW_e = dU_i$$

$$\sum_{k=1}^{n} Q_k \, dD_k = \sum_{k=1}^{n} \frac{\partial U_i}{\partial D_k} dD_k \quad , \text{ where } U_i = U_i[D_1, \ldots D_m, \ldots D_n]$$

Let only $dD_m \neq 0$, $dD_k = 0$, $k \neq m$, $k = 1, \ldots n$. Thus,

$$Q_m \cancel{dD_m} = \frac{\partial U_i}{\partial D_m} \cancel{dD_m}$$

Equilibrium statement ← $$\boxed{Q_m = \frac{\partial U_i}{\partial D_m}}$$ → CASTIGLIANOS FIRST THEOREM to find unknown reactions/forces.

So while the 2nd theorem is valid for linearly elastic bodies only, the first theorem is for nonlinearly elastic bodies as well (ie, only conservation of energy required). However 1st theorem has lesser applicability.

Applications of 2nd Theorem

Trusses.

Recall (p.6), $U_i = \sum_{k=1}^{b} \frac{P_k^2 L_k}{2A_k E_k}$, b = nos of bars (members), P_k = bar force in k^{th} bar.

$$\Rightarrow \quad \boxed{D_m = \sum_{k=1}^{b} P_k \left(\frac{\partial P_k}{\partial Q_m}\right) \frac{L_k}{A_k E_k}}$$

Q_m = force applied at joint in direction of D_m sought.

D_m = joint displacement sought

Q_m = corresponding force applied in direction of D_m at joint.

P_k = member force due to all applied loads and also Q_m.

Since there is only one displacement that we determine at a time, you can drop subscript 'm' from D_m, Q_m, ie D, Q, only.

Obtain $P_k = P_k(Q)$ and proceed.

Comparing with principle of virtual work, you see that p_k (page 9) is "analogous" to $\frac{\partial P_k}{\partial Q}$. Since the latter is change in member force P_k per unit load Q, these two terms are in fact "same".

Beams, Frames

Recall (p.6), $U_i = \int_0^L \frac{M^2}{2EI}\,dx$

$$\Rightarrow \boxed{D = \int_0^L M \frac{\partial M}{\partial Q} \frac{1}{EI}\,dx}$$

D = displacement sought. (linear or angular)

Q = Load (force or moment) applied in direction of D.

M = BM due to all applied loads and also Q.

Obtain $M = M(Q)$ and proceed.

The comparison with VW principle holds here also, ie m (pg. 10) is same as $\frac{\partial M}{\partial Q}$ here.

If you include Axial force effect, you must add

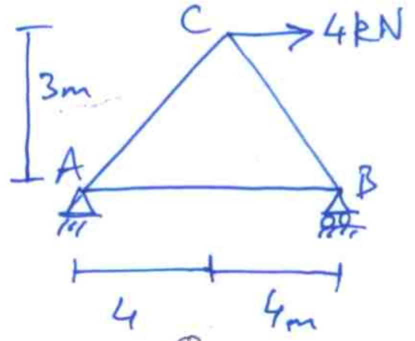
$\boxed{P \frac{\partial P}{\partial Q} \cdot \frac{L}{AE}}$ to D above.

If you include shear force effect you must add

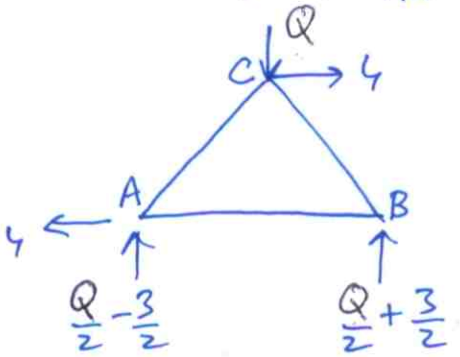
$$\boxed{K\int_0^L V \frac{\partial V}{\partial Q}\frac{1}{GA}\,dx}$$ to D above.

Here $P(Q)$ & $V(Q)$ are axial & shear forces due to all applied loads & Q.

Ex 1



Find vertical displacement of C.
$E = 200\text{ GPa},\ A = 400\text{ mm}^2$.



$$BC = -\left(\frac{Q+3}{2}\right)\left(\frac{5}{3}\right) \quad ; \quad AB = \left(\frac{Q+3}{2}\right)\left(\frac{5}{3}\right)\left(\frac{4}{5}\right)$$

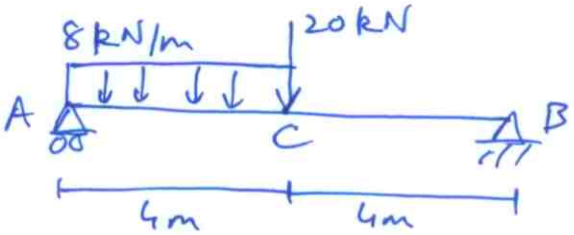
$$AC = \left(\frac{Q+3}{2} - Q\right)\left(\frac{5}{3}\right) = \left(\frac{3-Q}{2}\right)\left(\frac{5}{3}\right)$$

Member	P_k	$\partial P_k/\partial Q$	L_k	$P_k \frac{\partial P_k}{\partial Q} L_k \Big\vert_{Q=0}$
AC	$\frac{5}{6}(3-Q)$	$-5/6$	5	$\left(\frac{15}{6}\right)\left(\frac{-5}{6}\right)(5)$
BC	$-\frac{5}{6}(Q+3)$	$-5/6$	5	$\left(-\frac{15}{6}\right)\left(\frac{-5}{6}\right)(5)$
AB	$\frac{4}{6}(Q+3)$	$4/6$	8	$\left(\frac{12}{6}\right)\left(\frac{4}{6}\right)(8)$
				10.67

$$D_{CV} = \frac{10.67}{AE}(\downarrow) = \frac{10.67 * 10^3}{(200E9)(400E-6)} = 1.33375\,E{-4}\text{ m.} = 0.133375\text{ mm.}$$

If we wanted to find D_{CH} (ie horizontal displ of C) we would replace 4kN by horizontal Q, proceed as usual, and finally evaluate $P_k \frac{\partial P_k}{\partial Q} L_k \Big\vert_{Q=4}$.

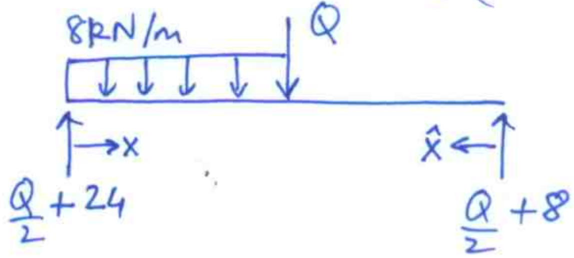
Ex2



Find vertical displacement of C.

$E = 200$ GPa, $I = 150E6$ mm^4

Replace 20 kN with Q (downwards).



$$M = \left(\frac{Q}{2}+24\right)x - 8\frac{x^2}{2}, \quad 0 \le x \le 4$$

$$= \left(\frac{Q}{2}+8\right)\hat{x}, \quad 0 \le \hat{x} \le 4$$

$$\frac{dM}{dQ} = \frac{x}{2}, \quad 0 \le x \le 4.$$

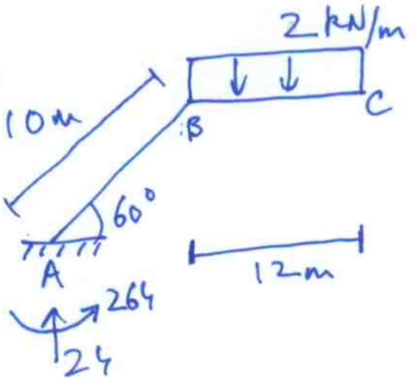
$$= \frac{\hat{x}}{2}, \quad 0 \le \hat{x} \le 4.$$

$$D_{CV} = \frac{1}{EI}\int_0^L M \left.\frac{dM}{dQ}\right|_{Q=20} dx = \frac{1}{EI}\left\{\int_0^4 (34x - 4x^2)\left(\frac{x}{2}\right)dx + \int_0^4 (18\hat{x})\left(\frac{\hat{x}}{2}\right)d\hat{x}\right\}$$

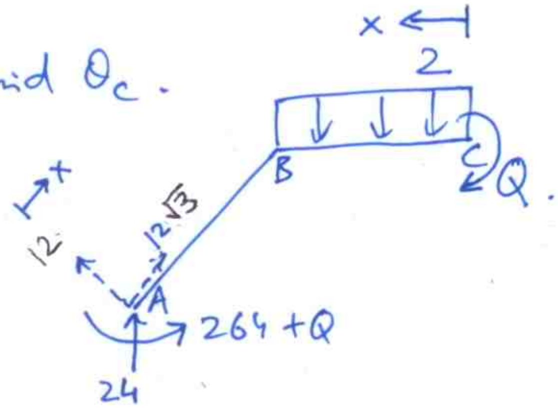
$$= \frac{1}{EI}\int_0^4 (26x^2 - 2x^3)\,dx = \frac{1}{EI}\left[(26)\left(\frac{4^3}{3}\right) - (2)\left(\frac{4^4}{4}\right)\right] = \frac{426.7}{EI}$$

$$= \frac{426.7 * 10^3}{(200E9)(150E6 * 10^{-12})} = 0.0142 \text{ m} = 14.2 \text{ mm}.$$

Ex 3.



Find θ_c.



$$M = -(264+Q) + 12x, \quad \frac{\partial M}{\partial Q} = -1, \text{ in } AB.$$

$$M = -Q - 2\frac{x^2}{2}, \quad \frac{\partial M}{\partial Q} = -1, \text{ in } CB$$

$$D\Big|_{Q=0} = \theta_C = \frac{1}{EI}\left[\int_0^{10}(-264+12x)(-1)\,dx + \int_0^{12}(-x^2)(-1)\,dx\right]$$

$$= \frac{1}{EI}\left[(264)(10) - (12)\left(\frac{10^2}{2}\right) + \left(\frac{12^3}{3}\right)\right] = \frac{2616}{EI}$$

Using EI as in (Ex 2),

$$\theta_C = \frac{2616 * 10^3}{(200E9)(150E-6)} = 0.0872 \text{ rad}$$

CONJUGATE BEAM METHOD

Recall,

$$\frac{dV}{dx} = -w(x) \Rightarrow \frac{d^2M}{dx^2} = -w(x)$$

$$\frac{d\theta}{dx} = \frac{M(x)}{EI} \Rightarrow \frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$

Thus,

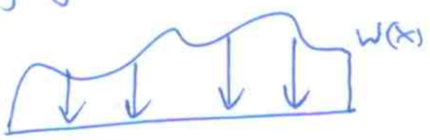
V (SF) analogous to θ (rot)

M (BM) " " v (defl.)

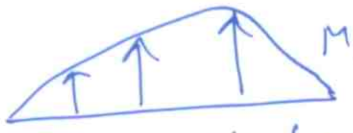
and this 'analogy' can be made an 'equality' by loading the beam with the $\frac{M(x)}{EI}$ diagram obtained for the beam with actual load $w(x)$. Note that if $w(x)$ +ve downward (↓) then the $\frac{M(x)}{EI}$ load is +ve upward(↑).

Real beam = beam loaded with $w(x)$

Conjugate beam = beam loaded with $-\frac{M(x)}{EI}$ of real beam.



Real beam



Conjugate beam.

Then, θ in real beam = SF in conjugate beam

v " " " = BM in conjugate beam

Supports in conjugate beam: We should be careful when applying M/EI of real beam to conjugate beam in the sense that supports should follow the analogy.

Thus,

Real beam | **Conjugate beam**

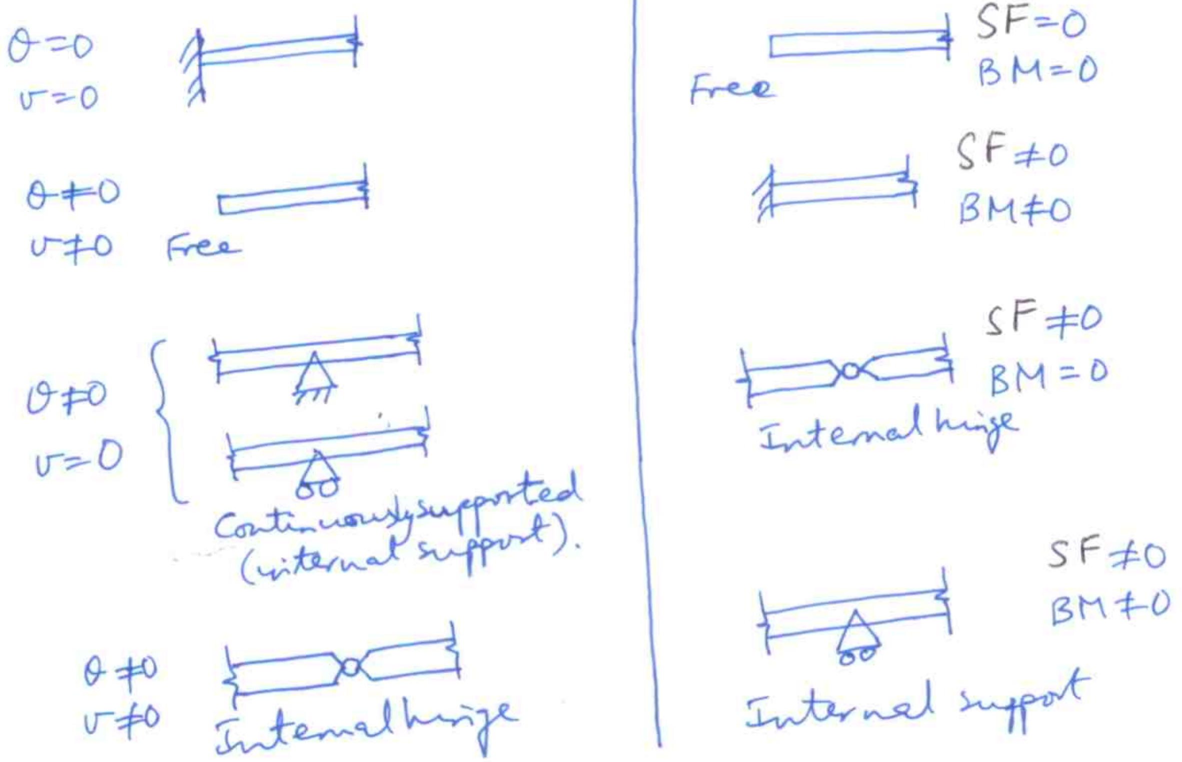
$\theta \neq 0$, $v = 0$

SF $\neq 0$, BM $= 0$

$\theta \neq 0$, $v = 0$

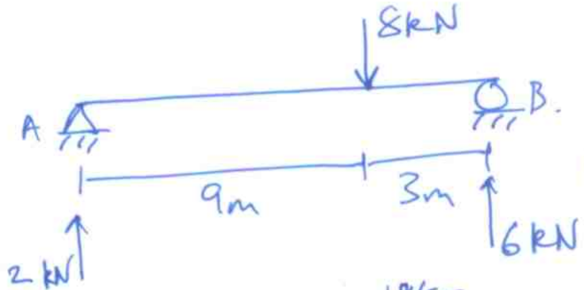
SF $\neq 0$, BM $= 0$

Real beam | **Conjugate beam**

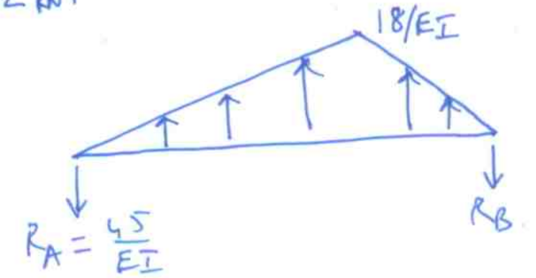


Ex1

Find max defl. $E = 200$ GPa, $I = 60E6\ mm^4$



Real beam with load



M/EI, real beam, applied to conj beam.

Conj. beam with load M/EI

$$R_A = \left(\tfrac{1}{2}\right)\left(\tfrac{18}{EI}\right)\left[(9)(6) + (3)(2)\right]\cdot\left(\tfrac{1}{12}\right) = \frac{45}{EI}$$

Let $\bar{V}$, $\bar{M}$ be SF, BM of conj beam.

Max defl in real beam when $\bar{M}$ max, ie,

$$\frac{d\bar{M}}{dx} = \bar{V} = 0.$$

$$\bar{V} = -\frac{45}{EI} + \left(\tfrac{1}{2}\right)\left(\tfrac{18}{EI}\right)\left(\tfrac{1}{9}\right)(x)(x) = \frac{1}{EI}\left(-45 + x^2\right), \quad 0 \le x \le 9.$$

$\bar{V} = 0$ for $x = 6.708 < 9$

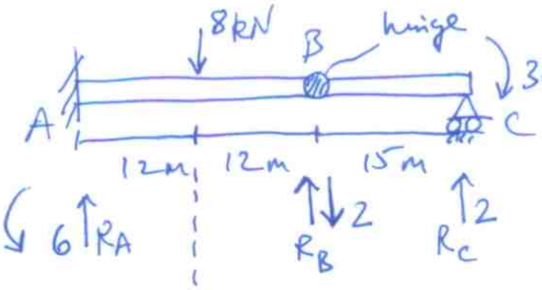
So max defl at $x = 6.708$

$$v_{max} = \bar{M}(6.708) = \frac{1}{EI}\left(-45x + \frac{x^3}{3}\right)_0^{6.708} = -\frac{201.24}{EI}$$

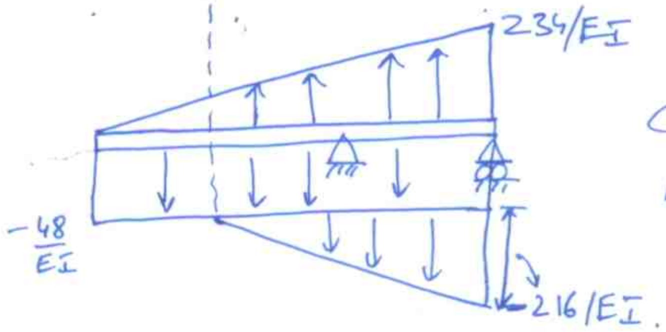
$$v_{max} = \frac{-201.24 * 10^3}{(200\,E9)(60\,E6 * 10^{-12})} = -0.0168\,m.$$

ie 16.8 mm (↓) downward

Ex 2



find v_B and θ_{B+}, θ_{B-}.



Conj beam loaded with M/EI of real beam. Let $\bar{M}$, $\bar{V}$ be BM, SF for conj. beam, and $\bar{R}$ be corresponding reactions.

reaction at B in conj. beam $= \bar{R}_B = \frac{1}{EI}\left\{(48)\frac{(39)^2}{2} + \left(\frac{1}{2}\right)(27)(216)\left(\frac{27}{3}\right) - \left(\frac{1}{2}\right)(39)(234)\left(\frac{39}{3}\right)\right\} * (1/15)$

$$= \frac{228.6}{EI}$$

$$v_B = \bar{M}_B = \frac{1}{EI}\left\{-(48)\frac{(24)^2}{2} - \left(\frac{1}{2}\right)(12)\left(216 * \frac{12}{27}\right)\left(\frac{12}{3}\right) + \left(\frac{1}{2}\right)(24)\left(234 * \frac{24}{39}\right)\left(\frac{24}{3}\right)\right\}$$

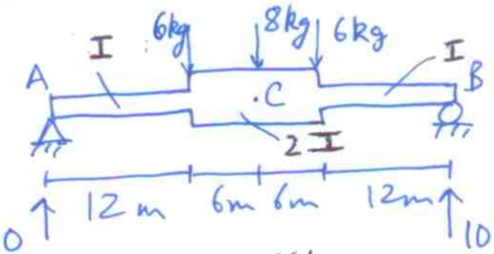
$$= -\frac{2304}{EI} = -\frac{2304 * 10^3}{(200E9)(60E6 * 10^{-12})} = -0.192\,m.$$

$$\theta_{B-} = \bar{V}_{B-} = \frac{1}{EI}\left\{-(48)(24) - \left(\frac{1}{2}\right)(12)\left(216 * \frac{12}{27}\right) - \left(\frac{1}{2}\right)(24)\left(234 * \frac{24}{39}\right)\right\}$$

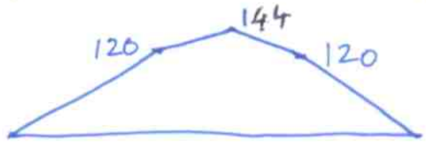
$$= 0$$

$$\theta_{B+} = \bar{V}_{B+} = \bar{V}_B + \bar{R}_B = \frac{228.6}{EI} = \frac{228.6 * 10^3}{(200\,E9)(60E6 * 10^{-12})} = 0.01905 \text{ rad.}$$

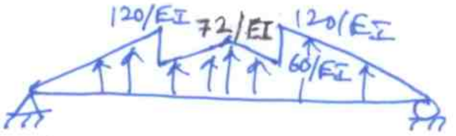
Ex 3.



Stepped girder, I on sides, 2I in middle. Find v_C.
EI = 200 GPa, I = 60 E6 mm^4



BM of real beam.



Conj beam with loading.

$$\bar{R}_A = -\left(\frac{1}{2}\right)\left[(120)(12) + (60+72)(6)\right]\frac{1}{EI} = \frac{-1116}{EI}$$

$$v_c = \bar{M}_c = \frac{1}{EI}\left\{-(1116)(18) + \left(\frac{1}{2}\right)(12)(120)(10) + (60)(6)(3) + \left(\frac{1}{2}\right)(6)(12)(2)\right\}$$

$$= -11736/EI. = \frac{-11736*10}{(200E9)(60E6*10^{-12})} = 0.00978m$$

$$= 9.78mm.$$