TOPIC 5

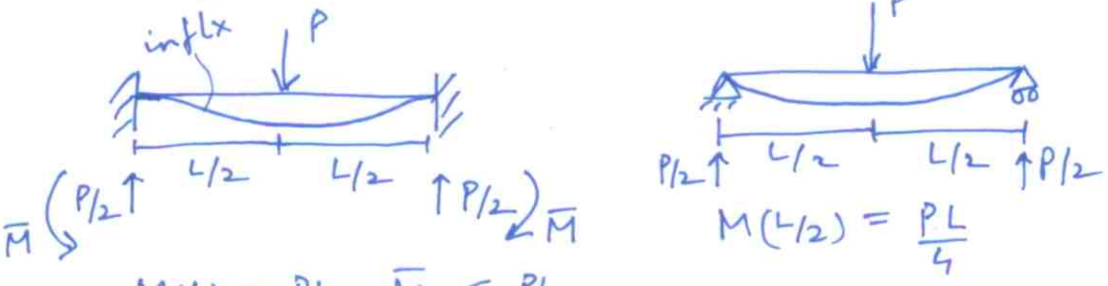
STATICALLY INDETERMINATE STRUCTURES — ANALYSIS BY FORCE METHOD

[a.k.a Flexibility method, Maxwell's method, method of Consistent Displacements, Superposition-equation method].

General.

Advantages of SID structures

(i) For given loading, maximum stresses, displacements usually smaller in SID structures.



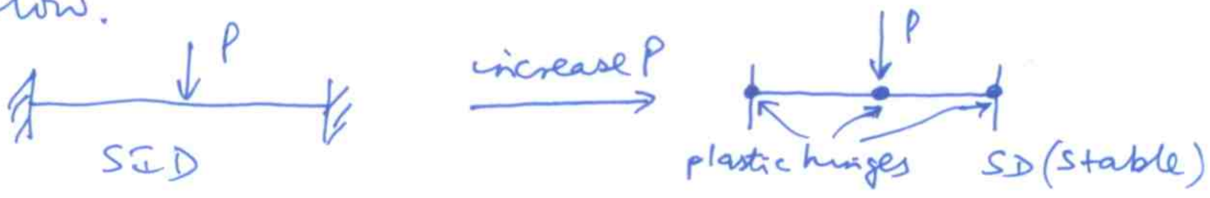
$$M\left(\tfrac{L}{2}\right) = \frac{PL}{4} - \bar{M} < \frac{PL}{4}$$

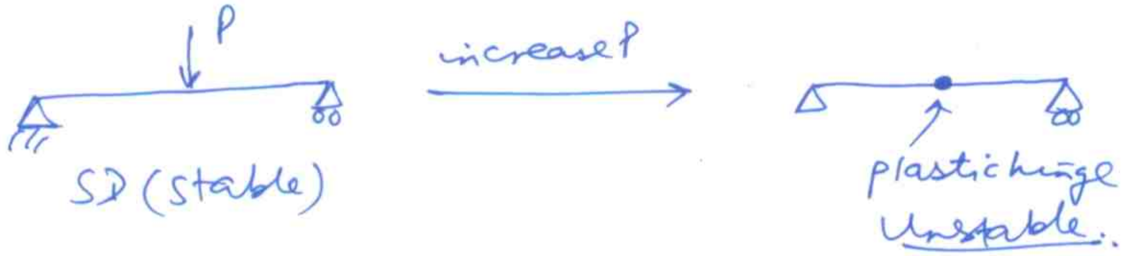
In fact $M\left(\tfrac{L}{2}\right) = \frac{PL}{8}$

as we shall see later.

Thus BM at center in fixed-fixed beam is half that of BM at center in simply-supported beam, ie stresses at $\frac{L}{2}$ are also half in fixed-fixed beam. Also deflections are $1/4^{th}$ that of S.S. beam.

(ii) SID structures have redundancy, ie redundant reactions, that when removed still yield a stable SD. Thus, when overloading or faulty design occurs, since loads are redistributed to redundant supports, at some point plastic hinges will form due to excessive loading. These hinges will form at points of max stress, ie max BM in case of beams shown below.





So you see how the SD structure finally collapses.

Disadvantages of SID structures.

(i) Additional stresses introduced due to settlement of support, temperature changes, fabrication errors. This does not happen in SD structures.

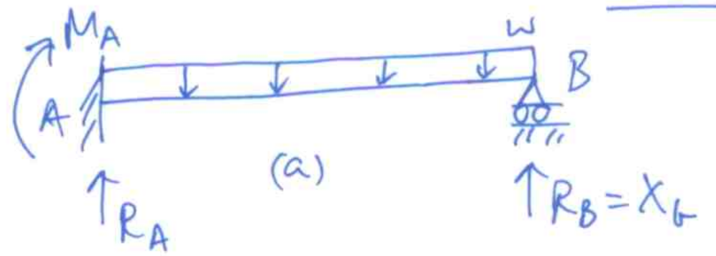
Methods of Analysis.

(i) Force method: First write compatibility equations (ie, consistent displacement equations) to solve for unknown redundant forces/reactions. Then use equilibrium to solve for remaining forces/reactions.

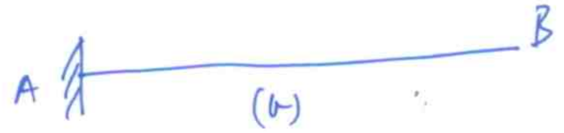
(ii) Displacement method (a.k.a. Stiffness method - next course): First write equilibrium equations in terms of unknown displacements and solve these unknowns. Then use the displacements to obtain forces thru force-displ relations.

FORCE METHOD

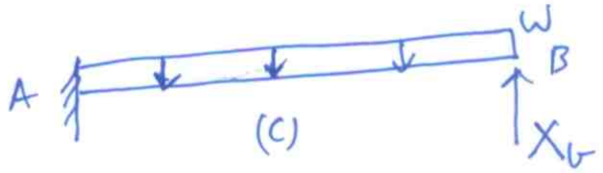
1-DOI Problem.



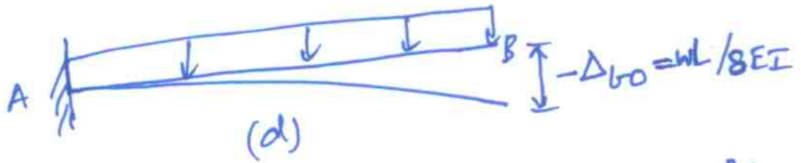
Actual Structure



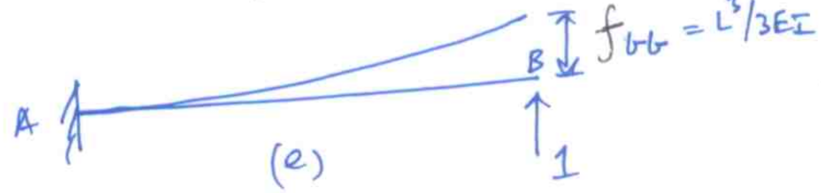
Primary structure
(ie, redundant support removed)



Primary struct with applied load & redundant reaction acting.



Primary struct with only applied load w acting.
(ie $X_b = 0$).



Primary struct with only $X_b = 1$ acting.
(ie $w = 0$).

$\boxed{(a) = (c) = (d) + X_b * (e)}$

Δ_b = upward deflection of point B in primary structure due to all causes (ie applied load w and redundant reaction X_b) (Fig (c)).

Δ_{b0} = upward defl. of point B in primary structure due to applied load w only, ie redundant reaction $X_b = 0$, hence the '0' in the subscript (Fig. (d)).

Δ_{bb} = upward defl of point B in primary structure due to redundant X_b only.

f_{bb} = Δ_{bb} due to $X_b = 1$ = displ at B due to unit load at B = flexibility coefficient.

Compatibility (ie consistent displacements) demands,

$$\Delta_b = \Delta_{b0} + \Delta_{bb}$$

From linearity, ie superposition valid, we have

$\Delta_{bb} = X_b f_{bb}$

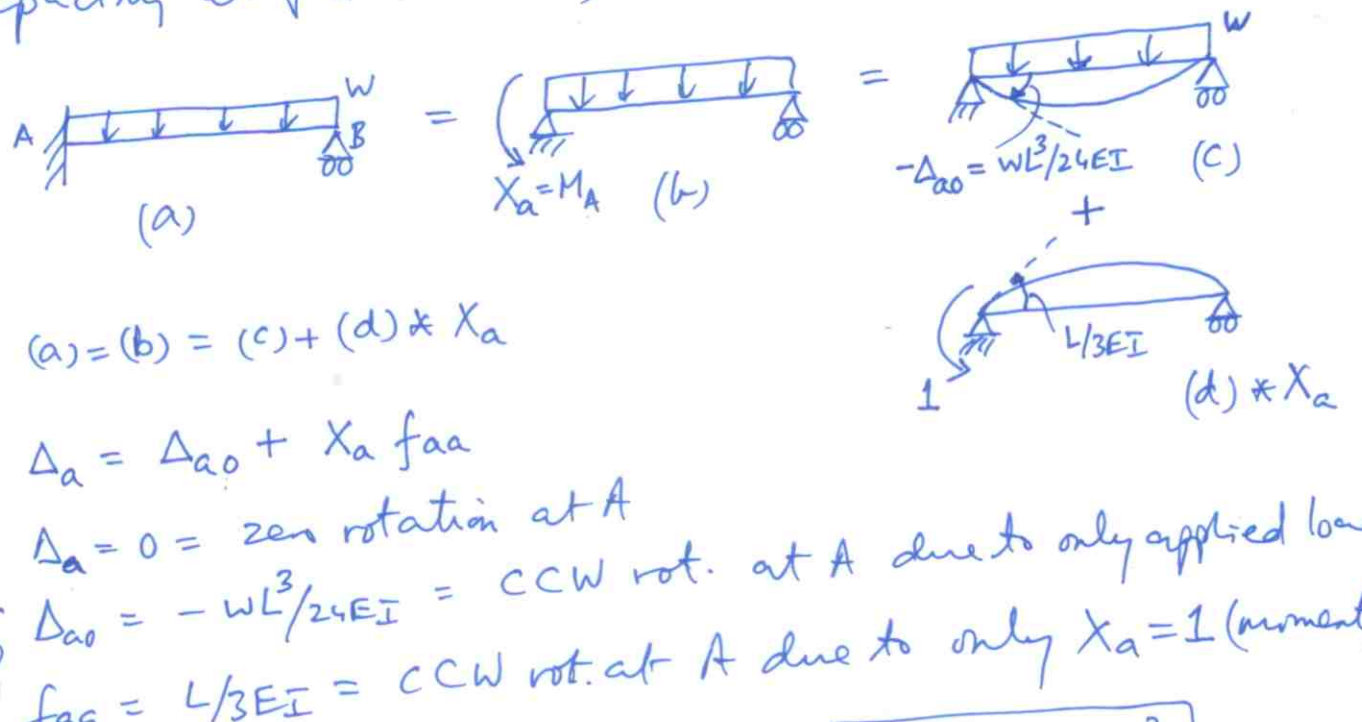
$\Rightarrow \boxed{\Delta_b = \Delta_{b0} + X_b f_{bb}}$

Here, $\Delta_b = 0$, $\underbrace{\Delta_{b0} = -wL^4/8EI, \; f_{bb} = L^3/3EI}_{\text{From tables or any method to find displ. in SD structure.}}$

$\Rightarrow 0 = -\frac{wL^4}{8EI} + X_b \frac{L^3}{3EI} \Rightarrow \boxed{X_b = \frac{3}{8} wL}$

Knowing $X_b (= R_B)$, solve remaining reactions as we do in SD structure. Then solve displacements by your favorite method (ie VW, Castigliano's, Conjugate beam, Direct integration, Moment Area).

Instead we could have removed the redundant BM at left support (ie remove moment bearing capacity at fixed end).



$(a) = (b) = (c) + (d) * X_a$

$\Delta_a = \Delta_{a0} + X_a f_{aa}$

$\Delta_a = 0 =$ zero rotation at A

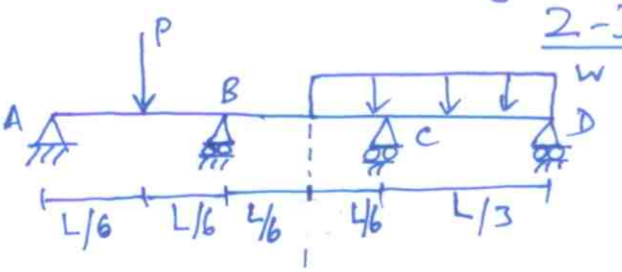
from Tables $\begin{cases} \Delta_{a0} = -wL^3/24EI = \text{CCW rot. at A due to only applied load} \\ f_{aa} = L/3EI = \text{CCW rot. at A due to only } X_a = 1 \text{ (moment).} \end{cases}$

$\Rightarrow 0 = -\frac{wL^3}{24EI} + X_a\left(\frac{L}{3EI}\right) \Rightarrow \boxed{X_a = M_a = \frac{wL^2}{8}}$

In the previous eg. we had D.O.I = 1 (deg. of indet.). Now lets see an eg. with D O I = 2.

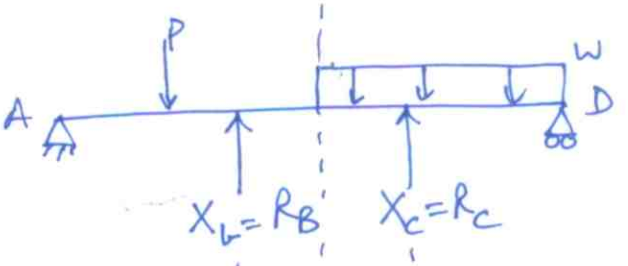
2-DOI Problem.

(a)



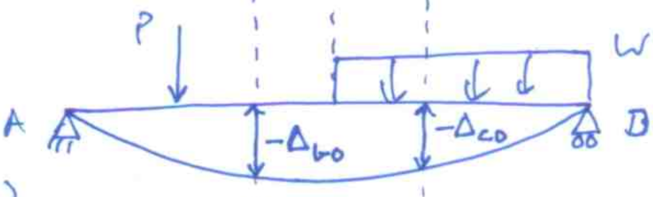
Actual Structure

(b)



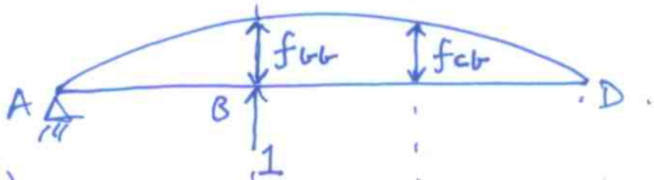
Primary struct with applied loads and redundancies X_b, X_c

(c)



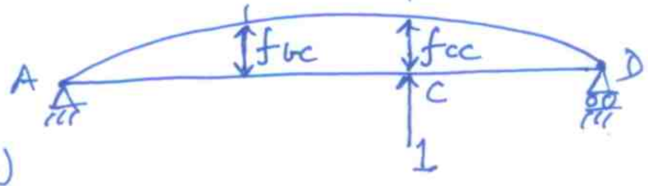
Primary struct with applied loads only $(X_b = X_c = 0)$

(d)



Primary struct with $X_b = 1$ only (applied loads $= 0$, $X_c = 0$)

(e)



Primary struct with $X_c = 1$ only (applied loads $= 0$, $X_b = 0$)

$$(a) = (b) = (c) + X_b * (d) + X_c * (e)$$

Δ_{bo}, Δ_{co} = upward displ of B, C, respectively for applied loads with redundants removed $(X_b = X_c = 0)$

f_{ij} = flexibility coefficient = displacement at point i when unit load applied at point j in same direction as displacement sought at i, in primary structure, ie., $X_j = 1$, no other loads applied.

Δ_b, Δ_c = overall displ (upward) of pts B, C, due to all causes (ie applied loads, redundants X_b, X_c) as in Fig (b) or Fig (a).

Compatibility gives,

$$\Delta_b = \Delta_{b0} + X_b f_{bb} + X_c f_{bc} \longrightarrow ①$$

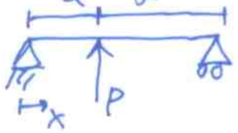
$$\Delta_c = \Delta_{c0} + X_b f_{cb} + X_c f_{cc} \longrightarrow ②$$

Here $\Delta_b = \Delta_c = 0$,

$$f_{bb} = f_{cc} = \frac{\frac{2L}{3}\cdot\frac{L}{3}\left(L^2 - \frac{4}{9}L^2 - \frac{L^2}{9}\right)}{6LEI} = \frac{\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)}{6}\frac{L^3}{EI} = \frac{4}{243}\frac{L^3}{EI}$$

$$f_{cb} = \frac{\frac{L}{3}\cdot\frac{L}{3}\left(L^2 - \frac{L^2}{9} - \frac{L^2}{9}\right)}{6LEI} = \frac{7}{486}\frac{L^3}{EI} = f_{bc}$$

where we used Tables, ie



, $v = \frac{Pbx}{6LEI}\left(L^2 - b^2 - x^2\right)$, $0 \le x \le a$

Δ_{b0}, Δ_{c0} to be found by any of the displacement calculation methods or Tables (use Tables here).

$$\Delta_{b0} = \frac{-P\left(\frac{L}{6}\right)\left(\frac{2L}{3}\right)}{6LEI}\left(L^2 - \frac{L^2}{36} - \frac{4L^2}{9}\right) - \frac{wL}{384EI}\left((8)\left(\frac{2L}{3}\right)^3 - 24L\left(\frac{2L}{3}\right)^2 + 17L^2\left(\frac{2L}{3}\right) - L^3\right)$$

$$= -\frac{19}{1944}\frac{PL^3}{EI} - \frac{55}{10368}\frac{wL^4}{EI}$$

$$\Delta_{c0} = \frac{-P\left(\frac{L}{6}\right)\left(\frac{L}{3}\right)}{6LEI}\left(L^2 - \frac{L^2}{36} - \frac{L^2}{9}\right) - \frac{w\frac{L}{3}}{384EI}\left(9L^3 - 24L\left(\frac{L}{3}\right)^2 + 16\left(\frac{L}{3}\right)^3\right)$$

$$= -\frac{31}{3888}\frac{PL^3}{EI} - \frac{187}{31104}\frac{wL^4}{EI}$$

Put $\Delta_b = \Delta_c = 0$, ①, ② written in matrix form as,

$$\begin{bmatrix} f_{bb} & f_{bc} \\ f_{cb} & f_{cc} \end{bmatrix}\begin{Bmatrix} X_b \\ X_c \end{Bmatrix} = \begin{Bmatrix} \cancel{\Delta_b}^{0} - \Delta_{b0} \\ \cancel{\Delta_c}^{0} - \Delta_{c0} \end{Bmatrix}$$

ie,

$$\frac{L^3}{486EI}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix}\begin{Bmatrix} X_b \\ X_c \end{Bmatrix} = \frac{L^3}{EI}\begin{Bmatrix} \frac{19}{1944}P + \frac{55}{10368}wL \\ \frac{31}{3888}P + \frac{187}{31104}wL \end{Bmatrix}$$

$$\begin{Bmatrix} X_b \\ X_c \end{Bmatrix} = \frac{1}{15}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} * 486 \begin{bmatrix} \frac{19}{1944}P + \frac{55}{10368}wL \\ \frac{31}{3888}P + \frac{187}{31104}wL \end{bmatrix} = \begin{Bmatrix} \frac{87}{120}P + \frac{11}{960}wL \\ \frac{-18}{120}P + \frac{341}{960}wL \end{Bmatrix}$$

Generalizing the Force Method

This method is the most general one available for indet. structures. Other methods, like Displacement Method (a.k.a. stiffness method) may be better for certain applications, but no method matches the generality and flexibility of the Force (a.k.a Superposition) method. It applies to cases of explicit mechanical loads, support settlement, temperature loads, fabrication errors, applied to structure. Generalizing, let us have a n-DOI structure, thus we have n redundants, ie, $X_1, X_2, \ldots X_n$. Choice of redundants is arbitrary, but their removal must yield a SD primary structure (ie make sure that you dont choose redundants that cause the resulting primary structure to be unstable). Later we will discuss "good" choices of redundants based on symmetry considerations. Thus redundants acting along with applied loads on the primary structure is equivalent to original SID structure with applied loads. Let, for a point m,

Δ_m = total deflection due to all causes.

Δ_{m0} = defl due to $X_1 = \ldots = X_n = 0$, ie applied loads only.

Δ_{mT} = defl. due to change in temperature

Δ_{mS} = defl due to settlement of supports of primary structure

Δ_{mE} = defl due to fabrication errors.

$f_{ma}, f_{mb}, \ldots, f_{mm}, \ldots, f_{mn}$ = defl due to $X_a = 1, X_b = 1, \ldots, X_m = 1, \ldots, X_n = 1$, respectively, ie, no applied loads.

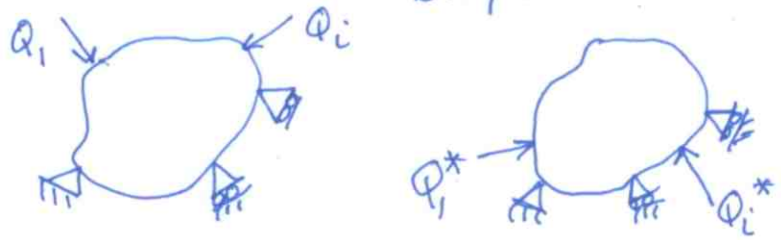
$f_{ma}, f_{mb}, \cdots, f_{mm}, \cdots, f_{mn}$ = defl. due to $X_a = 1$ only, $X_b = 1$ only, … $X_m = 1$ only, … $X_n = 1$ only, respectively, ie no applied loads.

Then the compatibility equations, in matrix form are,

$$\begin{Bmatrix} \Delta_{a0} + \Delta_{aT} + \Delta_{aS} + \Delta_{aE} \\ \Delta_{b0} + \Delta_{bT} + \Delta_{bS} + \Delta_{bE} \\ \vdots \\ \Delta_{m0} + \Delta_{mT} + \Delta_{mS} + \Delta_{mE} \\ \vdots \\ \Delta_{n0} + \Delta_{nT} + \Delta_{nS} + \Delta_{nE} \end{Bmatrix} + \underbrace{\begin{bmatrix} f_{aa} & f_{ab} & \cdots & f_{am} & \cdots & f_{an} \\ f_{ba} & f_{bb} & \cdots & f_{bm} & \cdots & f_{bn} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ f_{ma} & f_{mb} & \cdots & f_{mm} & \cdots & f_{mn} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ f_{na} & f_{nb} & & f_{nm} & & f_{nn} \end{bmatrix}}_{\text{Flexibility matrix.}} \begin{Bmatrix} X_a \\ X_b \\ \vdots \\ X_m \\ \vdots \\ X_n \end{Bmatrix} = \begin{Bmatrix} \Delta_a \\ \Delta_b \\ \vdots \\ \Delta_m \\ \vdots \\ \Delta_n \end{Bmatrix}$$

Reciprocity Laws.

(i) Betti's Law. — Consider linearly elastic structure, no temperature change or support settlement.



Consider two systems of ext. forces Q_i, Q_i^*, with corresponding int. forces q_i, q_i^*, respectively. Let Q_i system undergo displacements caused by q_i^* system. Work principle gives,

$$\sum Q_i D_i^* = \sum q_i d_i^*$$

where D_i^*, d_i^* are external & internal displacements of the Q_i, q_i forces, respectively, caused by the Q_i^*, q_i^* forces, respectively. Thus,

$$\sum Q_i D_i^* = \sum_{i=1}^{b} P_i \cdot \frac{P_i^* L_i}{A_i E_i}$$

for trusses, P_i, P_i^* are member forces due to Q_i, Q_i^*, resp^ly.

$\sum Q_i D_i^* = \int M . \frac{M^*}{EI} dx$, — for Beams, frames M, M* are BM's due to Q_i, Q_i^*, resp'ly.

Now let Q_i^* system undergo displacements caused by Q_i system. Work principle gives,

$$\sum Q_i^* D_i = \sum q_i^* d_i$$

where D_i, d_i, are ext & int displ's of the Q_i^*, q_i^* forces, respectively, caused by Q_i, q_i, forces, resp'ly.

Thus,

$$\sum Q_i^* D_i = \sum P_i^* \frac{P_i L_i}{A_i E_i} \quad \text{for trusses}$$

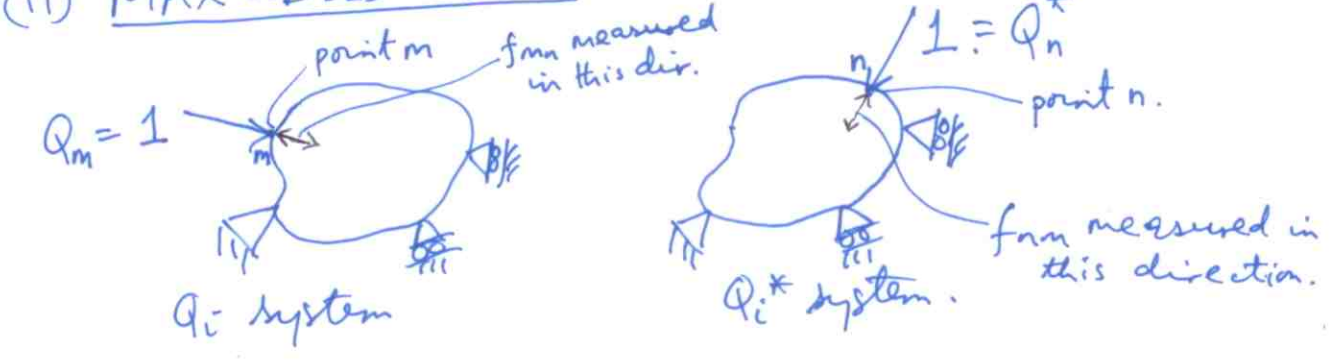
$$= \int M^* \frac{M\,dx}{EI} \quad \text{for beams, frames}$$

Comparing the RHS's,

$$\boxed{\sum Q_i D_i^* = \sum Q_i^* D_i} \rightarrow \text{BETTI'S LAW.}$$

BETTI'S LAW : For linearly elastic structure without temperature change or support settlement, the ext. virtual work done by Q_i system due to displ's caused by Q_i^* system equals ext virtual work done by Q_i^* system due to displ's caused by Q_i system.

(ii) MAXWELLS LAW.



Consider the Q_i system as a single unit load $Q_m=1$ applied at point m, and Q_i^* system as a single unit load $Q_n^*=1$ applied at point n. Let f_{mn} be defl at m in direction of Q_m due to $Q_n^*=1$ and f_{nm} be defl. at n in direction of Q_n^* due to $Q_m=1$.

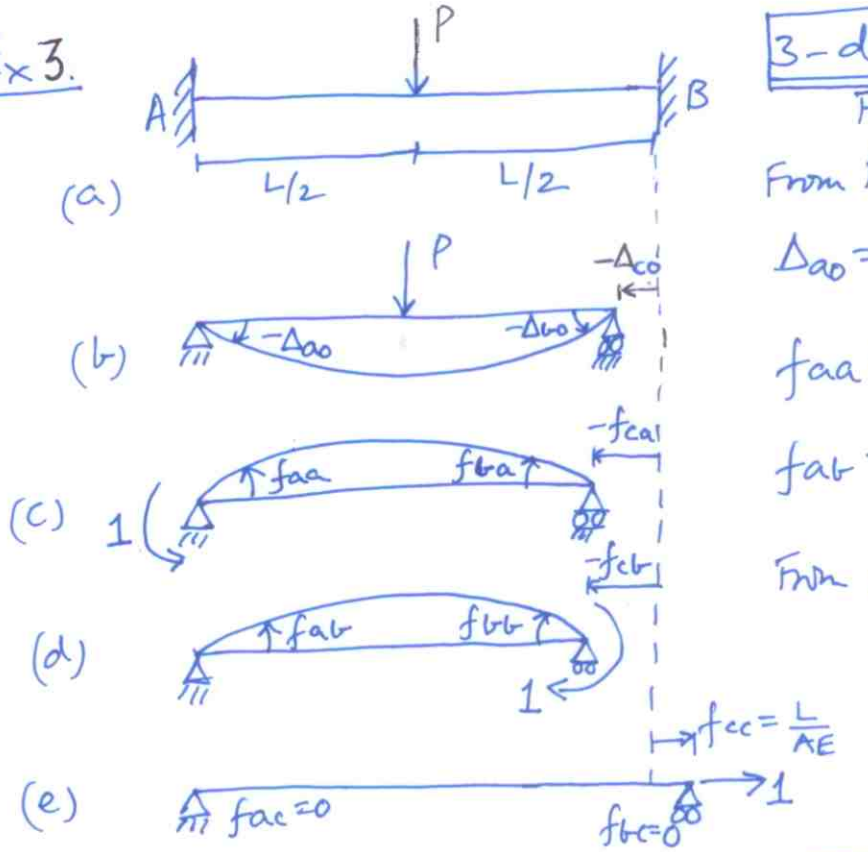
Applying Betti's law,

$$1.f_{mn} = 1.f_{nm} \Rightarrow \boxed{f_{mn} = f_{nm}} \rightarrow \text{Maxwells Law.}$$

MAXWELL'S LAW: For linearly elastic, no temperature or settlement effect, the displacement at point m in the direction of unit load at m but due to unit load at n (ie, f_{mn}) is equal to displ at point n in the direction of unit load at n but due to unit load at m (ie. f_{nm}).

Thus Maxwell's law implies symmetry of the flexibility matrix.

Ex 3.



$\boxed{\text{3-d.o.i. - ext indet.}}$

Find M_A, M_B, B_x.

From tables

$$\Delta_{ao} = \Delta_{bo} = -\frac{PL^2}{16EI}$$

$$f_{aa} = f_{bb} = \frac{L}{3EI}$$

$$f_{ab} = f_{ba} = \frac{L}{6EI}$$

From virtual work (by observation)

$$\Delta_{co} = f_{ca} = f_{cb} = 0$$

Also $f_{ac} = f_{bc} = 0$

$$f_{cc} = \frac{L}{AE}$$

$$\boxed{(a) = (b) + X_a * (c) + X_b * (d) + X_c * (e)}$$

$$\begin{Bmatrix}\Delta_a-\Delta_{a0}\\ \Delta_b-\Delta_{b0}\\ \Delta_c-\Delta_{c0}\end{Bmatrix} = \begin{bmatrix} f_{aa} & f_{ab} & f_{ac}\\ f_{ba} & f_{bb} & f_{bc}\\ f_{ca} & f_{cb} & f_{cc}\end{bmatrix}\begin{Bmatrix}X_a\\ X_b\\ X_c\end{Bmatrix}, \quad \Delta_a=\Delta_b=\Delta_c=0$$

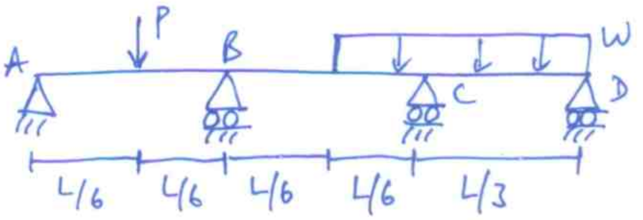
$$\Rightarrow \frac{PL^2}{16EI}\begin{Bmatrix}1\\ 1\\ 0\end{Bmatrix} = \begin{bmatrix} 2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & \frac{L}{AE}\cdot\frac{6EI}{L}\end{bmatrix} * \frac{L}{6EI} * \begin{Bmatrix}X_a\\ X_b\\ X_c\end{Bmatrix}$$

Solution is, $\underset{B_x}{\underbrace{X_c}} = 0$, $\underset{M_A}{\underbrace{X_a}} = \underset{M_B}{\underbrace{X_b}} = \frac{1}{3}\cdot\frac{PL^2}{16EI}\cdot\frac{6EI}{L} = \frac{PL}{8}$

(Ex4) Re-look at (Ex 2)

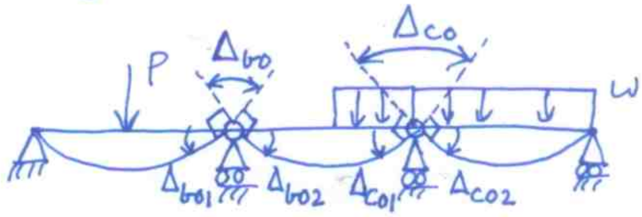
2-DOI (ext) Problem – done as Internal indet

(a)



Suppose we want to find M_B, M_c. An alternative way is to take $M_B = X_B$, $M_c = X_C$ as redundants.

(b)



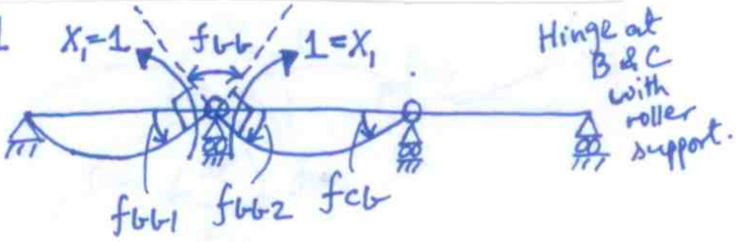
$\Delta_{b0} = \Delta_{b01} + \Delta_{b02}$

$\Delta_{c0} = \Delta_{c01} + \Delta_{c02}$

Only applied loads, $X_1 = X_2 = 0$.

$X_1 = 1$

(c)

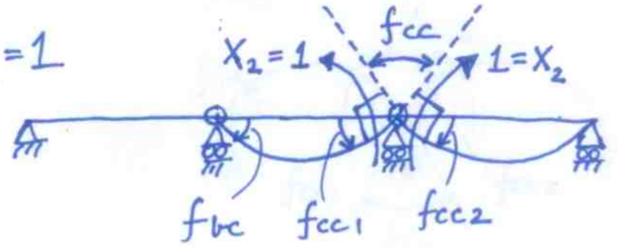


$f_{bb} = f_{bb1} + f_{bb2}$

$X_1 = 1$ applied, $X_2 = 0$, no "applied" loads

$X_2 = 1$

(d)



$f_{cc} = f_{cc1} + f_{cc2}$

$X_2 = 1$ applied, $X_1 = 0$, no "applied" loads.

$$\boxed{(a) = (b) + X * (c) + X_c * (d)}$$

$$\Delta_{b0} = \frac{P\left(\frac{L}{3}\right)^2}{16EI} + \frac{7w(L/3)^3}{384EI} \quad ; \quad \Delta_{c0} = \frac{3w(L/3)^3}{128EI} + \frac{w(L/3)^3}{24EI}$$

$$f_{bb} = \frac{L/3}{3EI} + \frac{L/3}{3EI} = \frac{2}{9}\frac{L}{EI} = f_{cc} \quad ; \quad f_{cb} = f_{bc} = \frac{(L/3)}{6EI}$$

$$\begin{Bmatrix} \Delta_b - \Delta_{b0} \\ \Delta_c - \Delta_{c0} \end{Bmatrix} = \begin{bmatrix} f_{bb} & f_{bc} \\ f_{cb} & f_{cc} \end{bmatrix} \begin{Bmatrix} X_b \\ X_c \end{Bmatrix}, \quad \Delta_b = \Delta_c = 0$$

for continuity in beam at B & C

Compatibility

$$\begin{Bmatrix} X_b \\ X_c \end{Bmatrix} = \frac{18EI}{L} * \frac{1}{15} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -PL^2/144EI - 7wL^3/10368EI \\ -25wL^3/10368EI \end{bmatrix}$$

Check with result of (Ex2)

$$\Rightarrow M_b = -\frac{PL}{30} - \frac{1}{2880}wL^2 ; \quad M_c = \frac{PL}{120} - \frac{31}{2880}wL^2$$

From (Ex2)

$$R_A L = \frac{5}{6}PL + \frac{wL^2}{8} - \frac{87}{120}\cdot\frac{2}{3}PL - \frac{11}{960}\cdot\frac{2}{3}wL^2 - \frac{(-18)}{120}\cdot\frac{1}{3}PL - \frac{341}{960}\cdot\frac{1}{3}wL^2$$

$$\Rightarrow R_A = \frac{2}{5}P - \frac{1}{960}wL$$

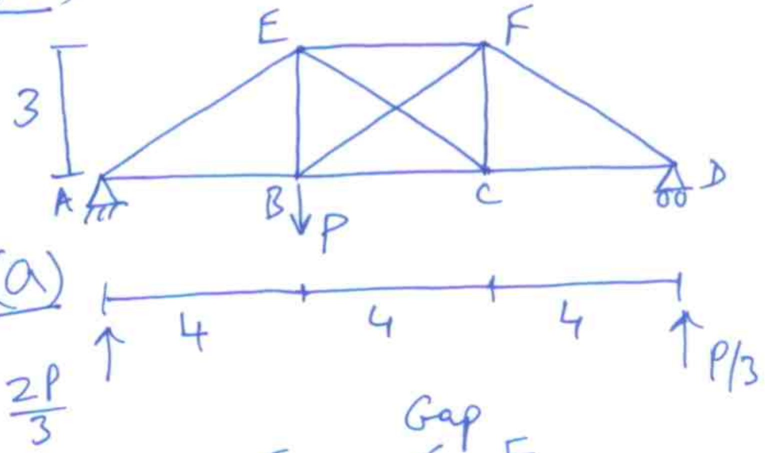
$$\Rightarrow M_B = R_A\frac{L}{3} - P\frac{L}{6} = -\frac{1}{30}PL - \frac{1}{2880}wL \quad \checkmark \text{ checks out}$$

$$R_D L = \frac{1}{6}PL + w\frac{L}{2}\cdot\frac{3}{4}L - \frac{87}{120}\cdot\frac{1}{3}PL - \frac{11}{960}\cdot\frac{1}{3}wL^2 - \left(\frac{-18}{120}\right)\cdot\frac{2}{3}PL - \frac{341}{960}\cdot\frac{2}{3}wL^2$$

$$\Rightarrow R_D = \frac{1}{40}P + \frac{43}{320}wL$$

$$\Rightarrow M_c = R_D\frac{L}{3} - \frac{wL^2}{18} = \frac{1}{120}PL - \frac{31}{2880}wL^2 \quad \checkmark \text{ checks out.}$$

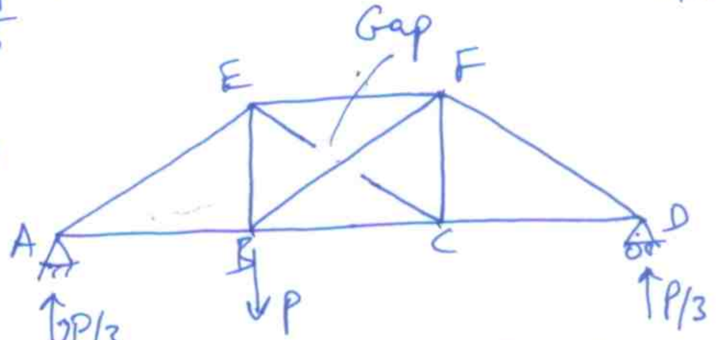
(Ex 5)



$\boxed{\text{1-DOI (Internal)}}$

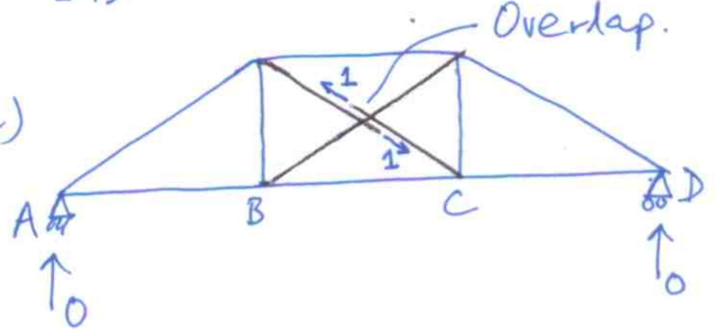
Let X_1 = EC.
i.e, cut EC to form primary structure.

(b)



Primary structure with only applied load P, ie $X_1=0$. This creates a Gap, ie, redundant X_1 assumed tensile positive

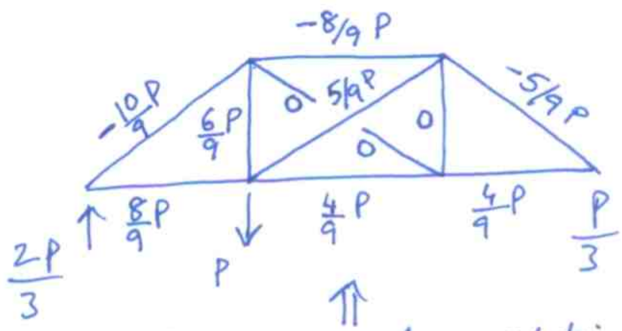
(c)



Primary struct with $X_1=1$, no applied load.
This creates an Overlap, ie redundant X_1 assumed tensile positive

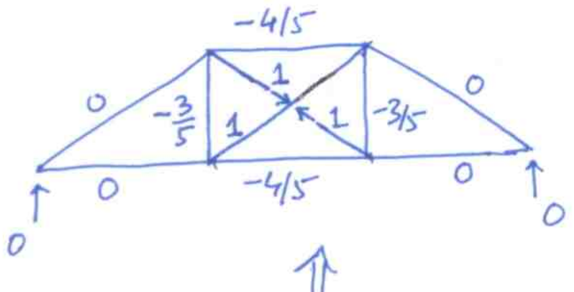
$$\boxed{\begin{array}{l}\text{Compatibility} \Rightarrow \text{Gap} + \text{Overlap} = 0. \\ \text{ie,} \quad (a) = (b) + X_1 * (c)\end{array}}$$

Gap: This is the relative displacement of E w.r.t. C, considered positive when they move closer since we assume tensile positive convention. Only applied loads cause gap.



⇑

Real loads. → solution P_i



⇑

Virtual Load in EC = 1
↳ solution p_i

Note that we applied tensile load of +1 in EC at each end of the cut member. For +1 applied only at end of cut closer to E, in direction E to C, it gives displ. of E in dir. E to C. Similarly for +1 applied at end of cut closer to C, in dir. C to E, it gives displ of C in dir. C to E. The sum of these two gives rel. displ. of E wrt C considered positive when they come closer. Thus, superposition implies that we apply a tensile load at each end of cut.

Overlap: Definition same as gap, ie rel. displ of E w.r.t. C, +ve when they move closer. Here, only Redundant $X_1 = 1$ causes overlap. Thus the same truss solution due to virtual loads in Gap case is applicable here.

Compatibility $\Rightarrow \boxed{\sum_{i=1}^{b} P_i p_i \frac{L_i}{A_i E_i} + X_1 \sum_{i=1}^{b} p_i^2 \frac{L_i}{A_i E_i} = 0 = \Delta_{10} + f_{11} X_1}$

b = total number of members/bars.

Mem	P_i	p_i	L_i	$P_i p_i L_i$	$p_i^2 L_i$
EF	$-8/9\,P$	-0.8	4	$128/45\,P$	2.56
BC	$4/9\,P$	-0.8	4	$-64/45\,P$	2.56
EB	$6/9\,P$	-0.6	3	$-54/45\,P$	1.08
FC	0	-0.6	3	0	1.08
EC	0	1	5	0	5
BF	$5/9\,P$	1	5	$125/45\,P$	5
				$3\,P$	17.28

$\Delta_{10} = \sum_{i=1}^{b} P_i p_i \frac{L_i}{A_i E_i}$ = gap.

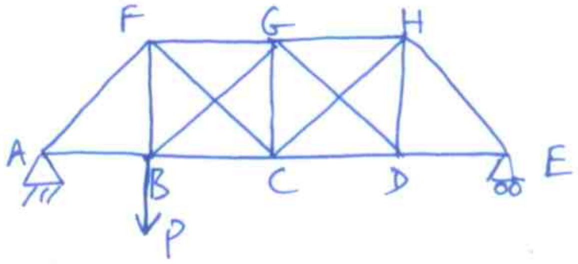
$f_{11} = \sum_{i=1}^{b} p_i^2 \frac{L_i}{A_i E_i}$ = overlap for $X_1 = 1$.

$\boxed{\Delta_{10} + f_{11} X_1 = 0.} \rightarrow$ COMPAT

$\Rightarrow \quad 3P + X_1 (17.28) = 0 \quad \Rightarrow X_1 = \frac{25}{144} P$

Ex 6.

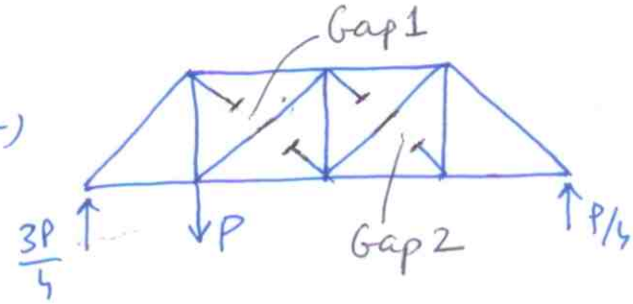
(a)



$\boxed{\text{2-DoI (Internal)}}$

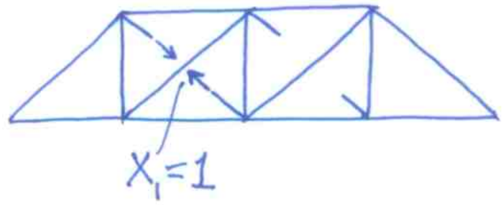
Let $X_1 = FC$, $X_2 = GD$ be redundants.

(b)



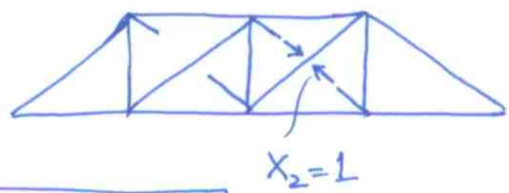
Primary, applied load, $X_1 = X_2 = 0$
$\longrightarrow$ solve member forces P_i, $i = 1, \cdots b$, i.e, $FC = GD = 0$.

(c)



Primary, $X_1 = 1$, $X_2 = 0$, no applied load.
$\longrightarrow$ solve mem. forces p_{i1}, $i = 1, \cdots b$, i.e, $FC = 1$, $GD = 0$.

(d)



Primary, $X_2 = 1$, $X_1 = 0$, no applied load
$\longrightarrow$ solve member forces p_{i2}, $i = 1, \cdots b$, i.e, $FC = 0$, $GD = 1$.

$\boxed{(a) = (b) + X_1 * (c) + X_2 (d)}$

$$\Delta_{10} = \text{Gap 1} = \sum_{i=1}^{b} P_i \, p_{i1} \frac{L_i}{A_i E_i}$$

$$\Delta_{20} = \text{Gap 2} = \sum_{i=1}^{b} P_i \, p_{i2} \frac{L_i}{A_i E_i}$$

$$f_{11} = \sum_{i=1}^{b} p_{i1}^2 \frac{L_i}{A_i E_i} \; ; \; f_{22} = \sum_{i=1}^{b} p_{i2}^2 \frac{L_i}{A_i E_i} \; ; \; f_{12} = f_{21} = \sum_{i=1}^{b} p_{i1} p_{i2} \frac{L_i}{A_i E_i}$$

$$\text{Overlap 1} = f_{11} X_1 + f_{12} \; ; \; \text{Overlap 2} = f_{21} X_1 + f_{22} X_2$$

$$\text{Compatibility} \Rightarrow \begin{Bmatrix} \Delta_{10} \\ \Delta_{20} \end{Bmatrix} + \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0 \longrightarrow \text{2X2 system of linear eqns.}$$

$\boxed{\text{For general K-DoI case}}$,

$$f_{mn} = \sum_{i=1}^{b} p_{im} \, p_{in} \frac{L_i}{A_i E_i}$$

$m, n, = 1, \cdots, K$

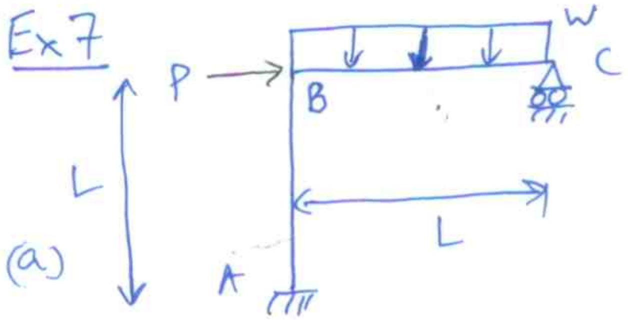
So you get $K \times K$ system of eqns.

$$\Delta_{m0} = \sum_{i=1}^{b} P_i \, p_{im} \frac{L_i}{A_i E_i}$$

$$\begin{Bmatrix} \Delta_{10} \\ \vdots \\ \Delta_{K0} \end{Bmatrix} + \underbrace{\begin{bmatrix} f_{11} & \cdots & f_{1K} \\ \vdots & \ddots & \vdots \\ f_{1K} & \cdots & f_{KK} \end{bmatrix}}_{\text{symmetric } K \times K \text{ matrix}} \begin{Bmatrix} X_1 \\ \vdots \\ X_K \end{Bmatrix} = 0 \rightarrow$$

$K \times K$ system of linear equations (ie, K linear eqns in K unknowns $X_1, \cdots X_K$)

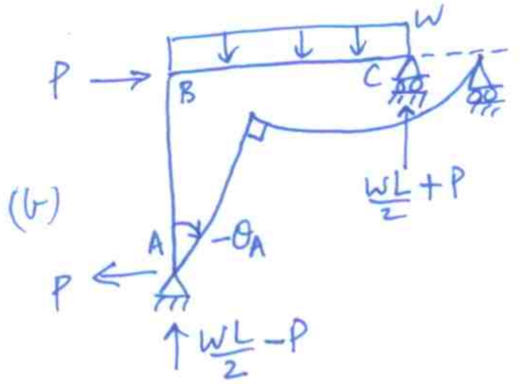
Ex 7



Find moment at fixed support A.

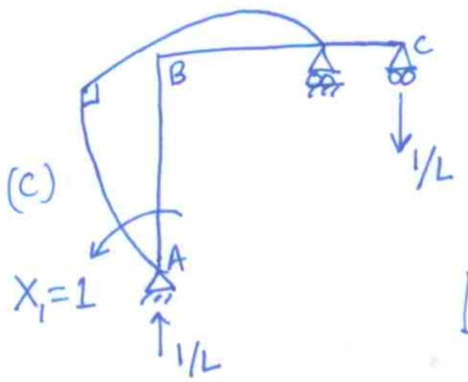
$\boxed{\text{1-DOI (ext)}}$

Redundant is $X_1 = M_A$.



Primary, $X_1 = 0$, applied loads.

$$M = Px, \text{ in } AB$$
$$= \left(\frac{WL}{2}+P\right)x - \frac{Wx^2}{2}, \text{ in } CB.$$



Primary $X_1 = 1$, no applied loads.

$$m = -1, \text{ in } AB$$
$$= -\frac{x}{L}, \text{ in } CB.$$

$\boxed{(a) = (b) + X_1 * (c)}$

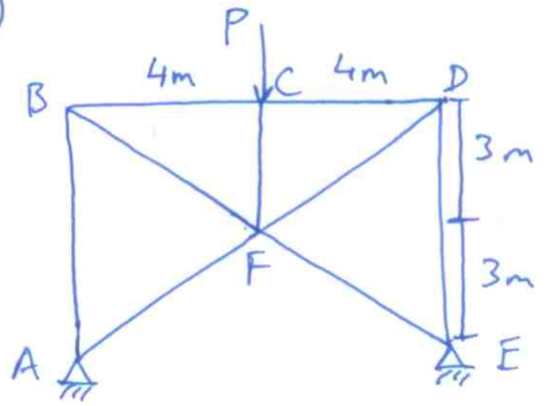
$$\Delta_{10} = \theta_A = \int \frac{Mm}{EI}dx = \frac{1}{EI}\left[\int_0^L Px(-1)\,dx + \int_0^L \left[\left(\frac{WL}{2}+P\right)x - \frac{Wx^2}{2}\right]\left(-\frac{x}{L}\right)dx\right]$$

$$= -\frac{1}{EI}\left[\frac{PL^2}{2} + \left(\frac{W}{2}+\frac{P}{L}\right)\frac{L^3}{3} - \frac{W}{2L}\frac{L^4}{4}\right] = -\frac{1}{EI}\left[\frac{5}{6}PL^2 + \frac{1}{24}WL^3\right]$$

$$f_{11} = \int_0^L \frac{m^2 dx}{EI} = \frac{1}{EI}\left[L + \frac{L^3}{3}\cdot\frac{1}{L^2}\right] = \frac{1}{EI}\frac{4L}{3}$$

$$\Delta_{10} + f_{11}X_1 = 0 \Rightarrow X_1 = M_A = \frac{5}{8}PL + \frac{1}{32}WL^2$$

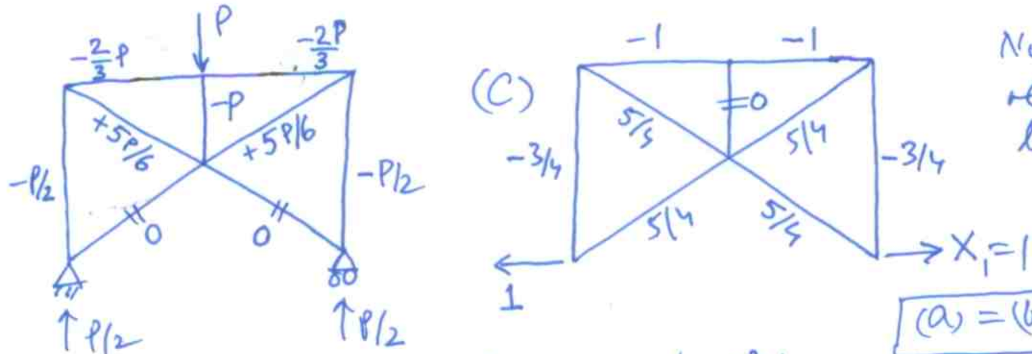
(Ex 8.)



1-DOI (external) - Truss.

Find reactions.

$X_1 = E_x$ = redundant.



Note symmetry in real & virtual load

$\boxed{(a) = (b) + X_1 \times (c)}$

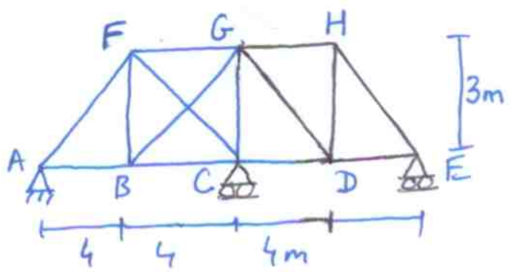
mem	P_i	p_i	L_i	$P_i p_i L_i$	$p_i^2 L_i$
AB	$-P/2$	$-3/4$	6	$2.25P$	3.375
BC	$-2P/3$	-1	4	$8P/3$	4
BF	$5P/6$	$5/4$	5	$125P/24$	7.8125
AF	0	$5/4$	5	0	7.8125
				$10.125P$	23

$\Delta_{10} + f_{11} X_1 = 0$

$\Rightarrow 10.125P + 23X_1 = 0$

$\Rightarrow X_1 = -\frac{81}{184}P = -0.4402P$

(Ex 9.)



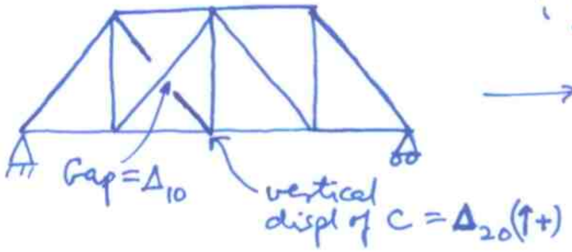
2-DOI (Int + Ext) with temperature and settlement and fabrication error.

Members AF, FG, GH, HE undergo increase in temp of 60°F,
$\alpha = 1/1.5\times10^5\ /^\circ F$.

$X_1 = FC,\ X_2 = C_y$

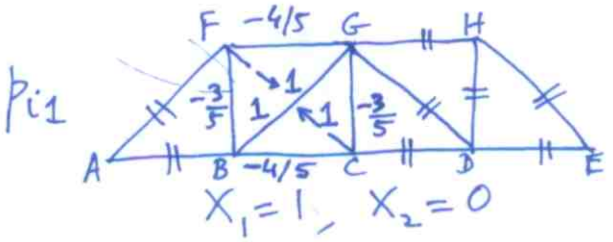
Settlement: A 0.24 cm down
C 0.48 cm down
E 0.36 cm down.

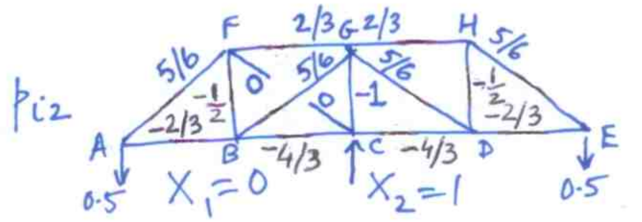
Fabrication error: FB 0.01 m short.



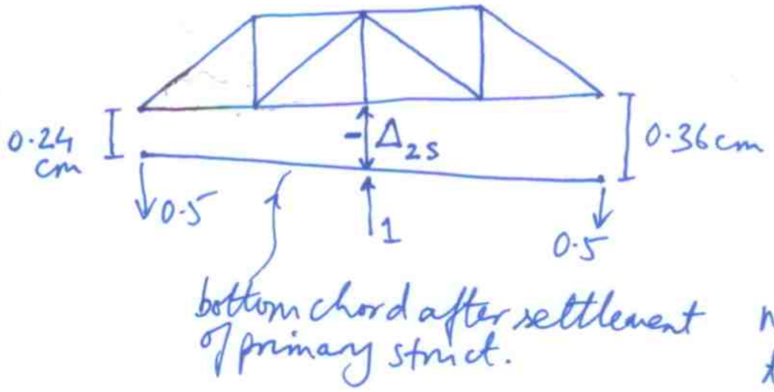
→ ~~Primary~~, temperature & fabrication error loads only (i.e applied loads), $X_1 = X_2 = 0$.

Find $\alpha \Delta T L_i$ and e_i, i.e real displacements (internal) of members.





Primary, no temp or fabr. error loads.



<u>Finding Δ_{2s}</u>, ie, settlement of point C <u>due to settlement of primary structure only</u>. Use virtual work. Apply virtual unit load (↑) at C. Since primary structure has no redundants, no internal forces generated due to settlement. So no internal work done (due to zero int. displs d_i).

$$\Rightarrow 1(\Delta_{2s}) + (0.5)(0.24) + (0.5)(0.36) = \sum p_i \cancel{d_i} = 0.$$

$d_i = \frac{P_i L_i}{A_i E_i}$, $P_i = 0$ due to settlement of Primary Struct.

$$\Rightarrow \boxed{\Delta_{2s} = -0.30\,\text{cm.}}$$

Alternately, ∵ primary structure behaves rigid under settlement (∵ no int forces due to settlement, hence no int. displ's), we can get the same result simply by geometry, ie $\Delta_{2s} = \dfrac{0.36 + 0.24}{2} = 0.30$ cm.

mem	p_{i1}	p_{i2}	L_i	$\alpha \Delta T L_i$	e_i	$p_{i1} * (e_i + \alpha\Delta T L_i)$	$p_{i2} * (e_i + \alpha\Delta T L_i)$	$p_{i1}^2 L_i$	$p_{i2}^2 L_i$	$p_{i1} p_{i2} L_i$
FG	-0.8	$2/3$	4	1.6E-3	0	-1.28E-3	$\frac{16}{15} * 10^{-3}$	2.56	16/9	$-32/15$
BC	-0.8	$-4/3$	4	0	0	0	0	2.56	64/9	64/15
FB	-0.6	-0.5	3	0	-0.01	6E-3	5E-3	1.08	0.75	0.9
GC	-0.6	-1.0	3	0	0	0	0	1.08	3.0	1.8
FC	1	0	5	0	0	0	0	5	0	0
GB	1	5/6	5	0	0	0	0	5	125/36	25/6
GH	0	2/3	4	1.6E-3	0	0	$\frac{16}{15} * 10^{-3}$	0	16/9	0
CD	0	$-4/3$	4	0	0	0	0	0	64/9	0

mem	P_{i1}	P_{i2}	L_i	$\alpha\Delta T L_i$	e_i	$P_{i1}*(e_i+\alpha\Delta T L_i)$	$P_{i2}*(e_i+\alpha\Delta T L_i)$	$P_{i1}^2 L_i$	$P_{i2}^2 L_i$	$P_{i1}P_{i2}L_i$
GD	0	5/6	5	0	0	0	0	0	125/36	0
HD	0	-0.5	3	0	0	0	0	0	0.75	0
HE	0	5/6	5	2E-3	0	0	$\frac{5}{3}*10^{-3}$	0	125/36	0
DE	0	-2/3	4	0	0	0	0	0	16/9	0
AF	0	5/6	5	2E-3	0	0	$\frac{5}{3}*10^{-3}$	0	125/36	0
AB	0	-2/3	4	0	0	0	0	0	16/9	0
						4.72E-3	$\frac{157}{15}*10^{-3}$	17.28	$\frac{715}{18}$	9
						$= \Delta_{1T}+\Delta_{1E}$	$= \Delta_{2T}+\Delta_{2E}$	$= f_{11}$	$= f_{22}$	$= f_{12}=f_{21}$

Also note $\boxed{\Delta_{1S}=0}$, $\boxed{\Delta_1=0,\ \Delta_2=0.0048m}$ $\boxed{\Delta_{10}=\Delta_{20}=0}$

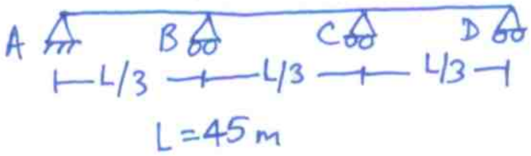
Assembly (Compatibility)

$$\begin{Bmatrix} 0 \\ -0.0048 \end{Bmatrix} = \begin{Bmatrix} 0+4.72*10^{-3}+0 \\ 0+\frac{157}{15}*10^{-3}-0.003 \end{Bmatrix} + \begin{bmatrix} 17.28 & 9 \\ 9 & 715/18 \end{bmatrix} \cdot \frac{1}{AE} \cdot \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = AE \cdot \frac{1}{605.4} \begin{bmatrix} 715/18 & -9 \\ -9 & 17.28 \end{bmatrix} \begin{Bmatrix} -4.72E-3 \\ -23/1875 \end{Bmatrix} = AE \begin{Bmatrix} -1.273*10^{-4} \\ -2.8*10^{-4} \end{Bmatrix}$$

(Ex 10)

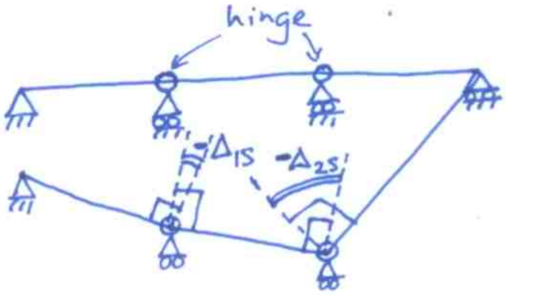
$\boxed{2-DOI}$



L=45m

Support settlement :
A : 0.02m ↓ ; C : 0.05m ↓
B : 0.04m ↓ ; D : 0.0m.

First we do by $X_1=M_b$, $X_2=M_c$ as redundants.



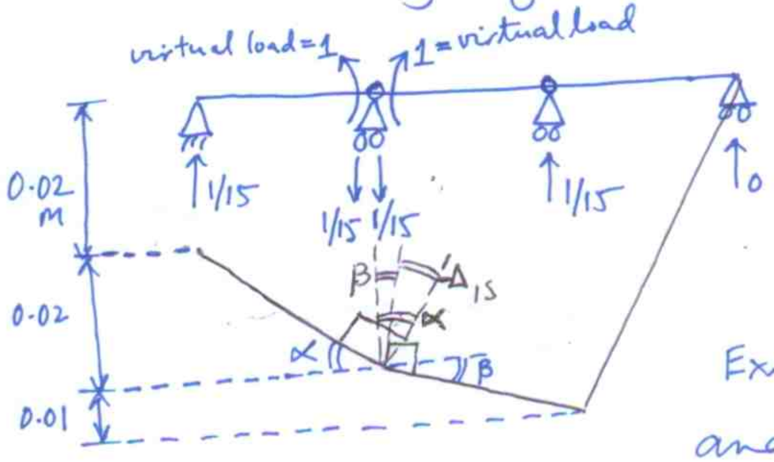
Primary structure with support settlement shown.

From geometry,

$$-\Delta_{1S} = \sin^{-1}\left(\frac{0.04-0.02}{15}\right) - \sin^{-1}\left(\frac{0.05-0.04}{15}\right) \simeq \frac{0.01}{15} = \frac{1}{1500}$$

$$-\Delta_{2S} \cong \frac{0.05}{15} + \frac{0.01}{15} = \frac{1}{250}$$

or alternately by virtual work,



$$\left(-\frac{1}{15}\right)(0.02) + 2\cdot\left(\frac{1}{15}\right)(0.04) - \left(\frac{1}{15}\right)(0.05)$$
$$-(1)(\alpha) + (1)(\beta) = \text{ext V.W}$$

Int VW = 0 ∵ primary struct does not deform under settlement.

Ext VW = Int VW $\Rightarrow \alpha - \beta = -\Delta_{1S} = \frac{1}{1500}$

and do similarly for Δ_{2S}.

From (Ex 4), $f_{11} = f_{22} = \frac{2}{9}\left(\frac{L}{EI}\right)$; $f_{12} = f_{21} = \frac{1}{18}\left(\frac{L}{EI}\right)$

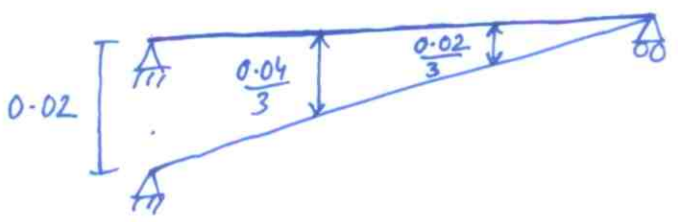
Also $\boxed{\Delta_1 = \Delta_2 = 0}$, ie no slope discontinuity at B & C in real structure. Thus compatibility gives,

$$\begin{Bmatrix} -\frac{1}{1500} \\ -\frac{1}{250} \end{Bmatrix} + \frac{L}{EI}\cdot\frac{1}{18}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \begin{matrix} \Delta_1 \\ \Delta_2 \end{matrix}$$

where X_1, X_2 are according to the "standard" positive convention (see Figs (c), (d), Ex 4).

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{18EI}{L}\cdot\frac{1}{15}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{Bmatrix} 1/1500 \\ 1/250 \end{Bmatrix} = \begin{Bmatrix} -1/625 \\ 23/1250 \end{Bmatrix}\frac{EI}{45}$$

Next we do by $X_1 = R_B$, $X_2 = R_C$, both (↑) positive.



Primary structure with settlement.

$\Delta_{1S} = \frac{-0.04}{3}$, $\Delta_{2S} = \frac{-0.02}{3}$

(get from geometry or V.W)

From (Ex 2), $f_{11} = f_{22} = \frac{L^3}{EI}\left(\frac{4}{243}\right)$; $f_{12} = f_{21} = \frac{7}{486}\frac{L^3}{EI}$

also, $\Delta_1 = -0.04$, $\Delta_2 = -0.05$

Compatibility :

$$\begin{Bmatrix} -0.04 \\ -0.05 \end{Bmatrix} = \begin{Bmatrix} -0.04/3 \\ -0.02/3 \end{Bmatrix} + \frac{L^3}{EI}\cdot\frac{1}{486}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{EI}{L^3}\frac{486}{15}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}\begin{Bmatrix} -2/75 \\ -13/300 \end{Bmatrix} = \begin{Bmatrix} 2.916 \\ -5.184 \end{Bmatrix}\frac{EI}{L^3}$$

check M_B, M_C :

$$R_A \cdot L + \frac{EI}{L^3}\left(2.916 * \frac{2L}{3} - 5.184 * \frac{L}{3}\right) = 0 \Rightarrow R_A = -0.216\frac{EI}{L^3}$$

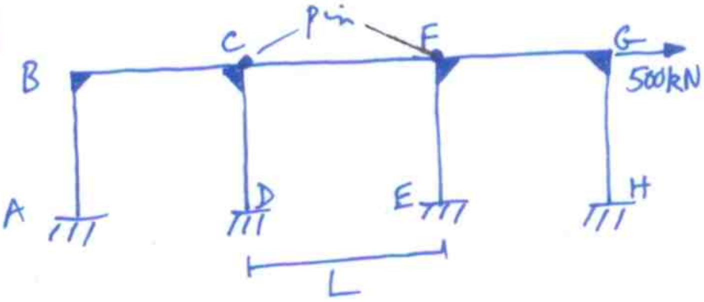
$$\Rightarrow M_B = R_A * \frac{L}{3} = -\frac{0.072}{45^2}EI = \frac{1}{625}\cdot\frac{1}{45}EI \quad \checkmark \text{ checks out.}$$

$$R_D L - \frac{EI}{L^3}\left(5.184 * \frac{2L}{3} - 2.916 * \frac{L}{3}\right) = 0 \Rightarrow R_D = 2.484\frac{EI}{L^3}$$

$$\Rightarrow M_C = R_D\frac{L}{3} = \frac{0.828}{45^2}EI = \frac{23}{1250}\cdot\frac{1}{45}EI \quad \checkmark \text{ checks out.}$$

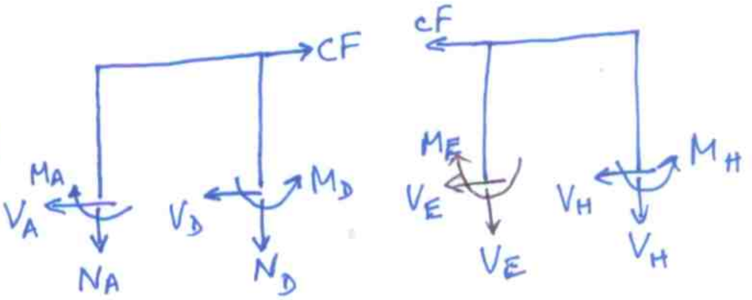
(Ex 11)

(a)



Frames ABCD and EFGH have rigid joints at B, C and F, G, respectively. Rod CF is pin-connected to these frames at its ends C and F.

(b)

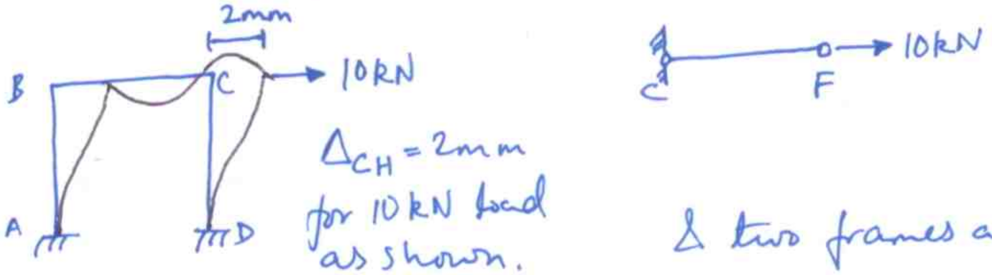


→ 2 FBD's, 13 unknowns,

DOI = 13 − 6 = 7.

However, lets assume we are given the following additional information.

(c)

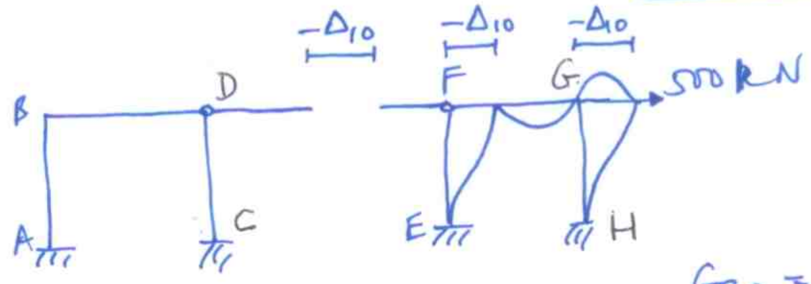


$\Delta_{CH} = 2$mm for 10 kN load as shown.

$\Delta_{FH} = 5$mm for 10kN load as shown.

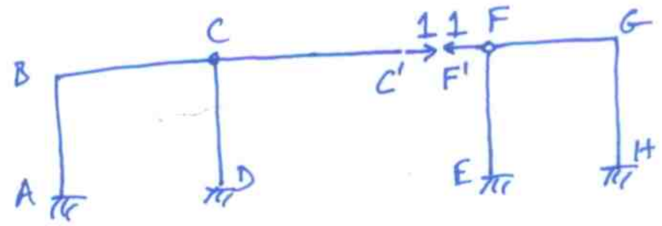
& two frames are identical.

Now we want to find force CF for 500 kN load as shown. So choose $\underline{X_1 = CF}$.



Primary with applied load, $X_1 = 0$.
Only EFGH deforms

Gap $= \Delta_{10} = -500 * \frac{2}{10} = -100$ mm



Primary with $X_1 = 1$ only. Note it does not matter where in CF we make the cut. Let C', F' be the ends of the cut as shown.

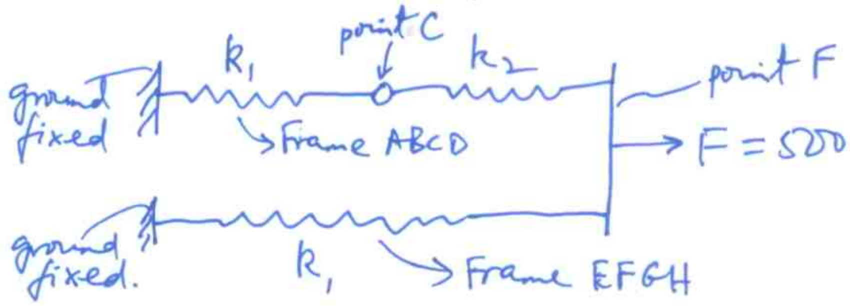
$$\left.\begin{aligned} \Delta_{C'H} &= (1)\left(\frac{2}{10}\right) + 1\left(\frac{5}{10}\right) = 0.7\text{mm} \rightarrow \\ \Delta_{F'H} &= (1)\left(\frac{2}{10}\right) = 0.2\text{mm} \leftarrow . \end{aligned}\right\} \Rightarrow f_{11} = 0.7 + 0.2 = 0.9$$

overlap $= 0.9 * X_1$

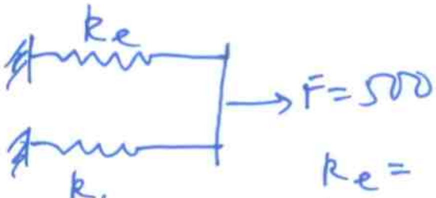
Compatibility $\Rightarrow$ gap+overlap $= 0$ $\Rightarrow f_{11} X_1 + \Delta_{10} = 0$.
$\Rightarrow 0.9 X_1 - 100 = 0$

$$\boxed{X_1 = 100/0.9}$$

Easier way, by stiffness method (Next course), not included in present course, is as follows. The structure with given info can be modelled as springs.

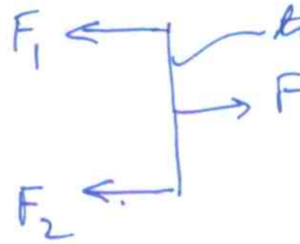


$k_1 = \frac{2}{10} = 0.2$ mm/kN
= stiffness of frames for horizontal caused by horz load as shown in Fig (c).



$k_e = k_1 k_2 / (k_1 + k_2)$
$\hookrightarrow$ springs in series.

$k_2 = \frac{5}{10} = 0.5$ mm/kN
= stiffness of rod CF due to axial load as shown in Fig (c)



treat it as point, ie., F

<u>Equilibrium</u> $\Rightarrow$ $\boxed{F_1 + F_2 = F = 500.}$

<u>Compatibility</u> $\Rightarrow$ point F displaces same amount whether obtained from upper two springs in series or lower spring.

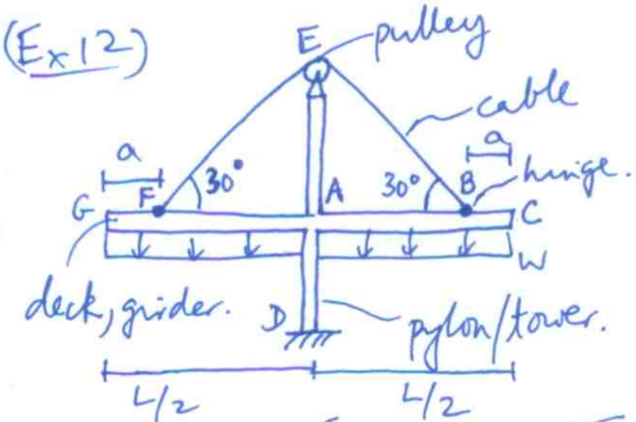
$$\Rightarrow \boxed{\frac{F_1}{k_e} = \frac{F_2}{k_1}} = \Delta_{FH}$$

$$\Rightarrow F_1 + F_1\left(\frac{k_1}{k_e}\right) = 500 \Rightarrow F_1 = \frac{500}{1+\left(\frac{k_1+k_2}{k_1}\right)} = \frac{500}{1+\frac{0.7}{0.2}} = \frac{100}{0.9}$$

<u>Same result.</u>

Now Note that $\boxed{F_1 = X_1}$ and F_2 = total horz reaction from ground on frame EFGH.

(<u>Ex 12</u>)



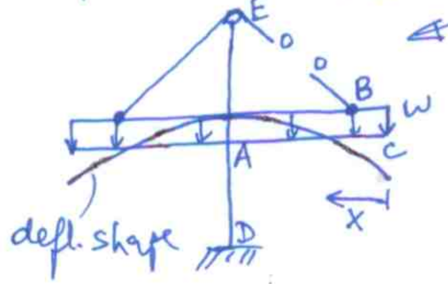
<u>Model of a cable stayed bridge.</u>

<u>1-DOI.</u>

$\boxed{X_1 = \text{cable tension.}}$

<u>Neglect axial, shear def in frame part.</u> for simplicity only.

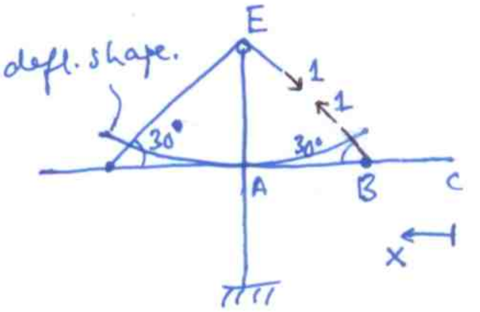
Primary with applied load, $X_1 = 0$.



$M = -w\frac{x^2}{2}$ in CA. ; $P = 0$ in cable BEF.

No BM in pylon/tower DE due to symmetry of applied load (otherwise we would just need to find it the usual way)

Primary with $X_1 = 1$, no applied load.



$m = (1 \sin 30)(x-a) = 0.5(x-a)$ in BA, x measured from C

$p = 1$ in cable BEF.

$m = 0$ in CB.

$$\Delta_{10} = \text{gap} = 2 * \left[\int_0^{L/2} m \frac{M}{EI} dx + p \frac{\cancel{P}^{\,0}}{AE} \right]$$

$$= \frac{2}{EI}\left[-\int_a^{L/2} w \frac{x^2}{2} \frac{(x-a)}{2} dx \right] = -\frac{1}{EI}\frac{w}{2}\left[\frac{L^4}{64} - \frac{a^4}{4} - \frac{aL^3}{24} + \frac{a^4}{3} \right]$$

$$f_{11} = \frac{\text{overlap}}{X_1} = 2\left[\int_0^{L/2} \frac{m^2}{EI} dx + \frac{p^2}{AE} \right] = 2\left[\frac{1}{EI} \int_a \frac{(x-a)^2}{4} dx + \frac{1}{AE} \right]$$

$$= 2\left[\frac{1}{4EI} \frac{(L/2-a)^3}{3} + \frac{1}{AE} \right]$$

Compatibility $\Rightarrow$ $\Delta_{10} + f_{11} X_1 = 0$

$$\Rightarrow \boxed{X_1 = \frac{\frac{w}{2EI}\left[\frac{L^4}{64} - \frac{aL^3}{24} + \frac{a^4}{12} \right]}{2\left[\frac{(L/2-a)^3}{12EI} + \frac{1}{AE} \right]}}$$

Denominator in $X_1 > 0$ for $\frac{L}{2} > a$

Numerator in $X_1 \rightarrow$ examine when it is zero. Put $a = cL$

$\Rightarrow \frac{L^4}{64} - \frac{aL^3}{24} + \frac{a^4}{12} = 0 \Rightarrow \frac{1}{64} - \frac{c}{24} + \frac{c^4}{12} = 0 \Rightarrow c = \underbrace{\frac{1}{2}, \frac{1}{2}}, \frac{-1 \pm \sqrt{2}i}{2}$

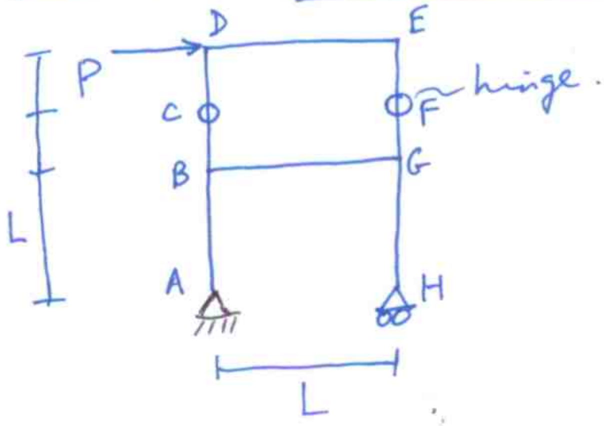
repeated real roots — complex roots. (obtained from MATLAB)

$\Rightarrow$ Numerator > 0 for $a < cL$, ie $a < \frac{L}{2}$ which is always true.

$\Rightarrow X_1 > 0$ always as long as $a < L/2$ (Note: check necessary since cable carries Tension not compression).

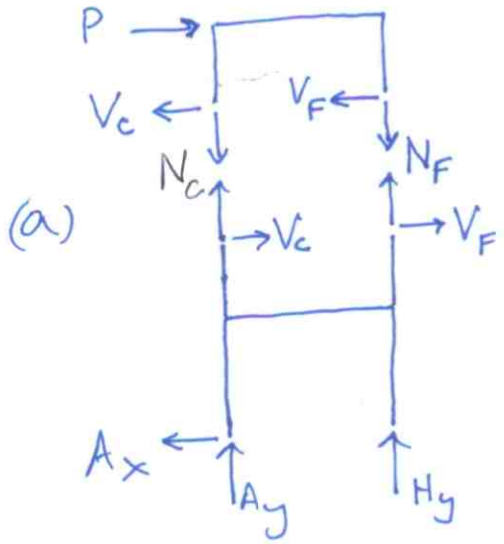
<u>Note</u>: If we had unsymmetric structure, i.e, loading or geometry (like cable angles, lengths of the cantilevers AC, AG, being unequal), the pylon would also carry BM which needs to be included. But, procedure remains same. If we choose to consider deflections due to shear & axial forces in the frame, these can be included in the usual manner by adding terms like $\int^{L/2} \frac{vV}{AG} dx$, $\int_0^{L/2} \frac{nN}{AE} dx$ in Δ_{10} and $\int_0^{L/2} \frac{v^2}{AG} dx$, $\int_0^{L/2} \frac{n^2}{AE} dx$ in f_{11}. But, procedure remains same.

(Ex 13) Frame with 1-DOI (Internal)



Externally determinate.

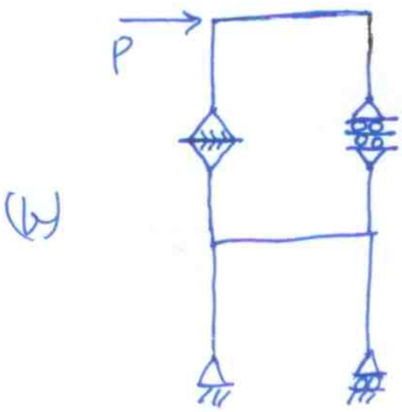
Neglect deflection due to shear & axial force, for convenience (can always include them in the usual manner, procedure remains same).



7 unknowns. 2 FBD's. Once these unknowns are found, BM, SF, AF throughout (including leg BG) is known thru statics.

$\Rightarrow$ DOI $= 7 - 2*3 = \underline{1 \text{ (Internal)}}$

Take $X_1 = V_F$ as redundant.

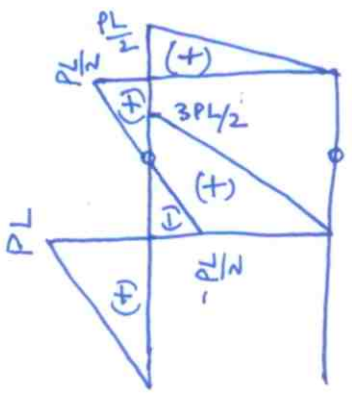


Primary structure, applied load,

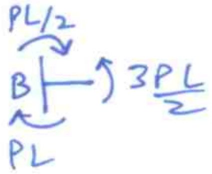
$X_1 = 0. = V_F$

$P\left(\frac{L}{2}\right) + N_F(L) = 0 \Rightarrow N_F = -\frac{P}{2} = -N_C$

$V_C = P = A_x \quad ; \quad H_y = 2P = -A_y.$

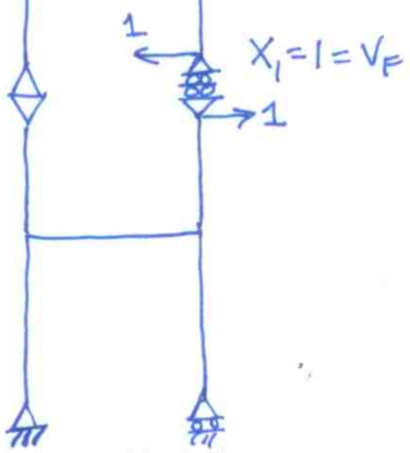


$V_{DE} = -\frac{P}{2}\;;\; V_{BG} = A_y + N_C = -\frac{3P}{2}$



$\Longrightarrow$ BMD for Primary with appl. load; $X_1 = 0$ ie, M diagram.

(c)

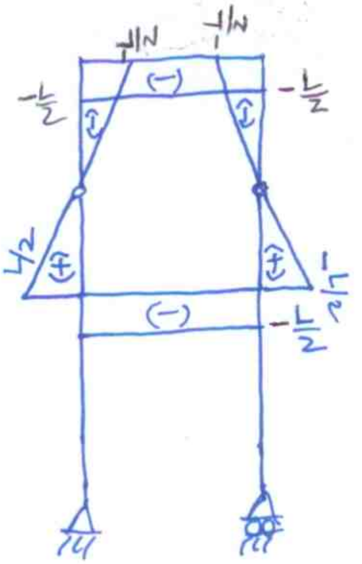


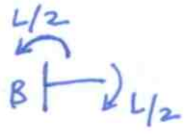
Primary, $X_1 = 1$, no appl. load.

$A_x = A_y = H_y = 0$

$N_F = N_c = 0$; $V_c = -V_F = -1$

(Note: $N_F = N_c = 0$ only because internal hinges at same level, otherwise we would obtain them as $\neq 0$ & proceed in the usual manner).



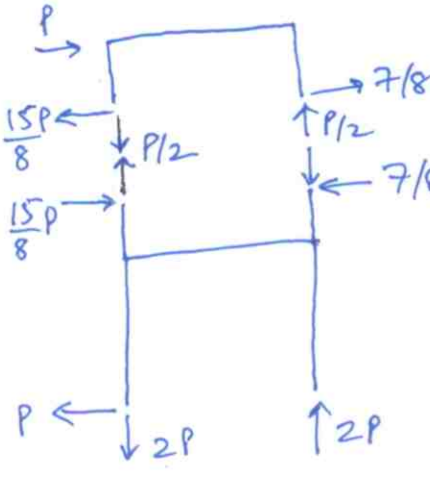


$$\Delta_{10} = \int m \frac{M}{EI} dx = \frac{1}{EI}\left[-\frac{1}{3} * 2 * \left(\frac{PL}{2}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right) - \frac{1}{2}\left(\frac{3PL}{2}\right)\left(\frac{L}{2}\right)(L) - \frac{1}{2}\left(\frac{PL}{2}\right)\left(\frac{L}{2}\right)(L)\right]$$

$$= \frac{7}{12} PL^3 \cdot \frac{1}{EI}$$

$$f_{11} = \int \frac{m^2}{EI} dx = \frac{1}{EI}\left[4 * \left(\frac{1}{3}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right) + 2 * \left(\frac{L}{2}\right)\left(\frac{L}{2}\right)(L)\right] = \frac{2}{3}\frac{L^3}{EI}$$

$$\Delta_{10} + f_{11} X_1 = 0 \Rightarrow \boxed{X_1 = -\frac{7}{8} P = V_F}$$



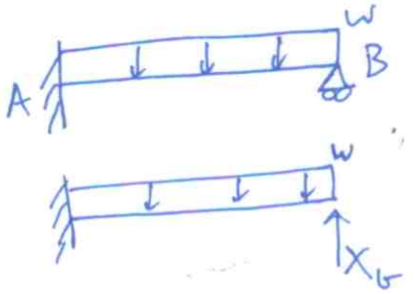
→ Final forces (from (a) = (b) + X_1*(c))

→ From this get BMD, SFD, AFD.

→ Can easily include effect of SF and AF in deformations when finding Δ_{10} & f_{11}.

CASTIGLIANOS SECOND THEOREM.

We can solve indeterminate structures by this method. In the present form it is applicable only to structures with mechanical loads (ie without temperature, settlement, fabrication error).



$$U = \frac{1}{2}\int \frac{M^2}{EI}\,dx$$ due to applied load and redundant X_b.

$$\frac{\partial U}{\partial X_b} = 0 = \int \frac{M}{EI}\frac{\partial M}{\partial X_b}\,dx$$

$M = M_0 + X_b M_b$, $M_0 =$ BM due to applied load only

$M_b =$ BM due to $X_b = 1$ only.

$$\Rightarrow \quad 0 = \int \frac{(M_0 + X_b M_b)}{EI} M_b\,dx = \int \left(\frac{M_0 M_b}{EI} + X_b \frac{M_b^2}{EI}\right) dx$$

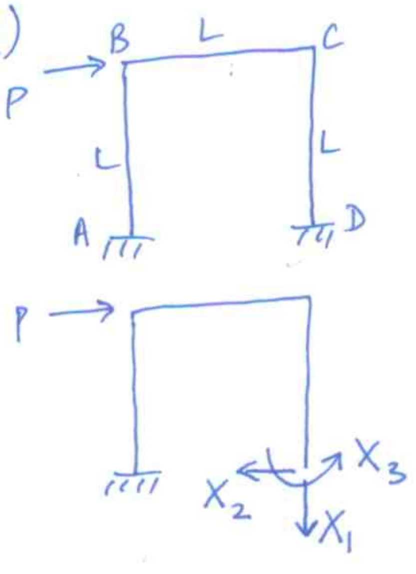
$= \Delta_{b0} + X_b f_{11} \rightarrow$ so same as Unit load method.

For Multi-D.O.I,

$U = U(X_1, \cdots X_n)$, $\dfrac{\partial U}{\partial X_i} = 0$, $i = 1, \cdots n$ gives

n linear equations in the X_i, to solve.

(Ex 1)



3-D.O.I.

$X_1 = N_D$, $X_2 = V_D$, $X_3 = M_D$.

$M = X_3 - X_2 x$, in DC

$= X_3 - X_2 L - X_1 x$, in CB

$= X_3 - X_2 L - X_1 L - (P - X_2) x$, in BA.

$$\frac{\partial M}{\partial X_1} = 0 ,\ DC \qquad = -x ,\ CB \qquad = -L ,\ BA$$

$$\frac{\partial M}{\partial X_2} = -x ,\ DC \qquad = -L ,\ CB \qquad = -L+x ,\ BA$$

$$\frac{\partial M}{\partial X_3} = 1 ,\ DC \qquad = 1 ,\ CB \qquad = 1 ,\ BA .$$

$$\frac{\partial U}{\partial X_1} = 0 = \int_0^L \left[(X_3 - X_2 x)(0) + (X_3 - X_2 L - X_1 x)(-x) + (X_3 - X_2(L-x) - X_1 L - Px)(-L) \right] dx$$

$$0 = X_3\left(-\frac{L^2}{2} - L^2\right) + X_2\left(\frac{L^3}{2} + L^3 - \frac{L^3}{2}\right) + X_1\left(\frac{L^3}{3} + L^3\right) + \frac{PL^3}{2}$$

$$\frac{\partial U}{\partial X_2} = 0 = \int_0^L \left\{ (X_3 - X_2 x)(-x) + (X_3 - X_2 L - X_1 x)(-L) + (X_3 - X_2(L-x) - X_1 L - Px) * (-L+x) \right\} dx$$

$$0 = X_3\left(-\frac{L^2}{2} - L^2 - L^2 + \frac{L^2}{2}\right) + X_2\left(\frac{L^3}{3} + L^3 + L^3 - L^3 + \frac{L^3}{3}\right) + X_1\left(L^3 - \frac{L^3}{2}\right) + P\left(\frac{L^3}{2} - \frac{L^3}{3}\right)$$

$$\frac{\partial U}{\partial X_3} = 0 = \int_0^L \left\{ (X_3 - X_2 x)(1) + (X_3 - X_2 L - X_1 x)(1) + (X_3 - X_2(L-x) - X_1 L - Px)(1) \right\} dx$$

$$0 = X_3(L+L+L) + X_2\left(-\frac{L^2}{2} - L^2 - L^2 + \frac{L^2}{2}\right) + X_1\left(-\frac{L^2}{2} - L^2\right) - \frac{PL^2}{2}$$

$$\Rightarrow \begin{bmatrix} 4/3 & 1 & -1.5 \\ 0.5 & 5/3 & -2 \\ -1.5 & -2 & 3 \end{bmatrix} \begin{Bmatrix} X_1 L \\ X_2 L \\ X_3 \end{Bmatrix} = P \begin{Bmatrix} -L/2 \\ -L/6 \\ 0.5 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 L \\ X_2 L \\ X_3 \end{Bmatrix} = \begin{bmatrix} 1.7143 & 0 & 0.8571 \\ 2.5714 & 3 & 3.2857 \\ 2.5714 & 2 & 2.9524 \end{bmatrix} P \begin{Bmatrix} -0.5L \\ -L/6 \\ 0.5 \end{Bmatrix}$$

$$X_1 = \left(-0.85715 + \frac{0.42855}{L}\right) P$$

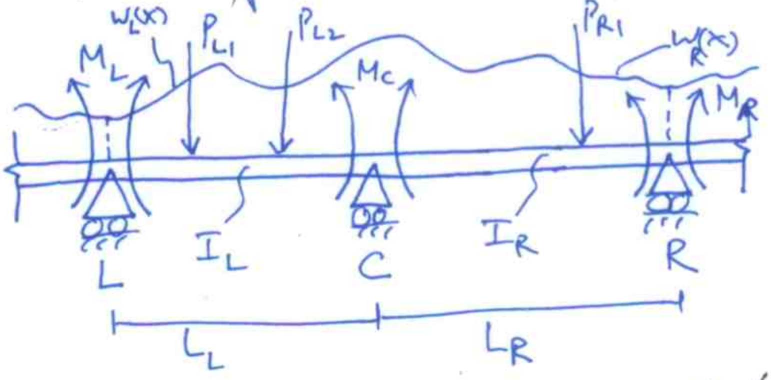
$$X_2 = \left(-1.7857 + \frac{1.64285}{L}\right) P$$

$$X_3 = (-1.619L + 1.4762)\, P$$

THREE MOMENT EQUATION.

This is a convenient formula to find reactions / displ's of continuously supported beams. We will do a simplified version that doesn't include support settlement.

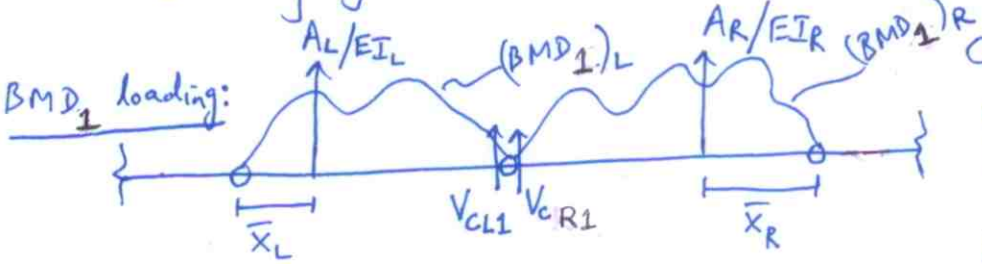
Consider a part of a continuously supported beam with loading as shown, with 3-supports in the part considered (labelled L – left, C – center, R – right).



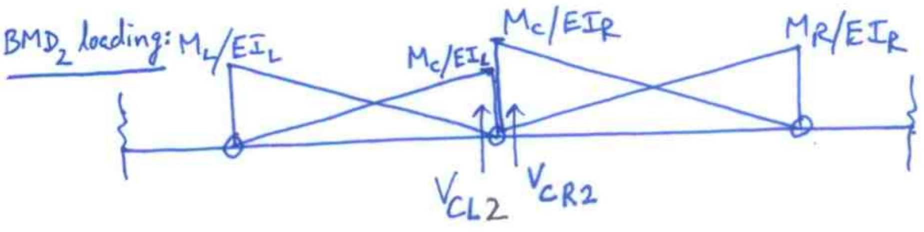
Loads: $P_{L1}, P_{L2}, P_{R1}, w_L(x), w_R(x)$

M_L, M_C, M_R are the bending moments at L, C, R supports.

Use Conjugate beam method. (see p. 29a for more explanation).



Conj beam with loading due to that part of BMD that is due to applied loads with L, C, R as hinges in real beam.



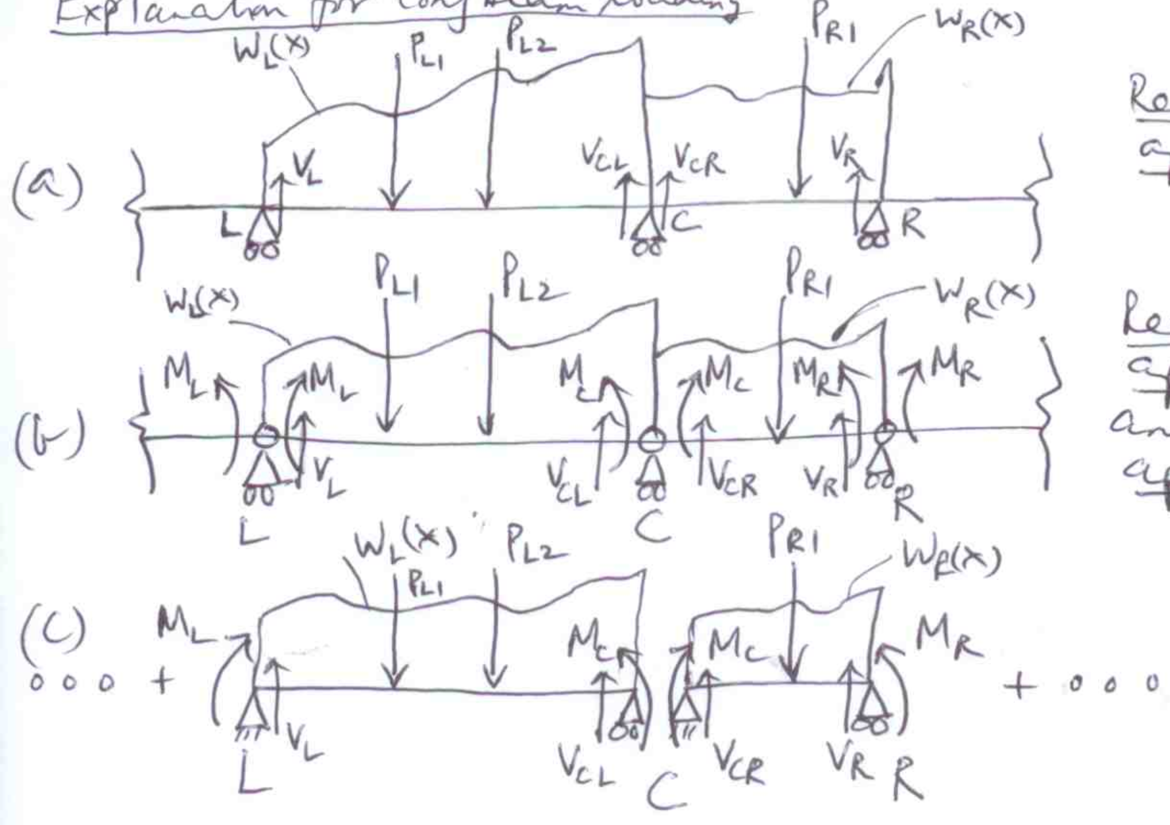
Conj beam with loading due to that part of BMD that is due to internal moments M_L, M_C, M_R applied to internal hinges in real beam.

Total loading on conj beam = $BM_1 + BM_2$.

For $BM_1 + BM_2$ loading, imposing shear continuity across hinge in conjugate beam, i.e. slope continuity across intermediate support in real beam) $\Longrightarrow$ $\boxed{V_{CL1} + V_{CL2} = -(V_{CR1} + V_{CR2})} \rightarrow (1)$

In the above, $\frac{A_L}{EI_L}$ and $\frac{A_R}{EI_R}$ are the load resultants (i.e. areas) due to loading $(BMD_1)_L$ and $(BMD_1)_R$ with line of action at $\bar{x}_L$ and $\bar{x}_R$, respectively, as shown.

Explanation for conj beam loading



Real beam with applied load

Real beam with applied load and moment release applied. (redundant)

Real beams with spans separated out having applied load and applied moment release (redundant)

(a) = (b) = (c).

In (a), (b), (c) the shear forces V_L, V_{CL}, V_{CR}, V_R are real but internal (ie not applied). These satisfy, for example, $R_C + V_{CL} + V_{CR} = 0$ where R_C is ↑ reaction at C in (a), (b). Also, for example, $R_{CL} + V_{CL} = 0$, $R_{CR} + V_{CR} = 0$, $R_L + V_L = 0$, $R_R + V_R = 0$ in (c), where R_{CL}, R_{CR} are ↑ reactions at C for left & right spans, respectively, and R_L, R_R are reactions at L, R, respectively, for left & right spans shown.

Now, loading on conjugate beam is superposition of BMD due to applied loads and applied moment releases (redundants) in fig (c).

Apply eqn 1 :

$$V_{CL1} = -\frac{A_L}{EI_L}\bar{x}_L \cdot \frac{1}{L_L} \quad ; \quad V_{CL2} = -\frac{1}{\cancel{L_L}}\left[\frac{1}{2}\cdot\frac{M_C}{EI_L}\cdot\cancel{L_L}\cdot\frac{2}{3}L_L + \frac{1}{2}\frac{M_L}{EI_L}\cdot\cancel{L_L}\cdot\frac{1}{3}L_L\right]$$

$$V_{CR1} = -\frac{A_R}{EI_R}\bar{x}_R\cdot\frac{1}{L_R} \quad ; \quad V_{CR2} = -\frac{1}{\cancel{L_R}}\left[\frac{1}{2}\frac{M_C}{EI_R}\cdot\cancel{L_R}\cdot\frac{2}{3}L_R + \frac{1}{2}\frac{M_R}{EI_R}\cancel{L_R}\cdot\frac{1}{3}L_R\right]$$

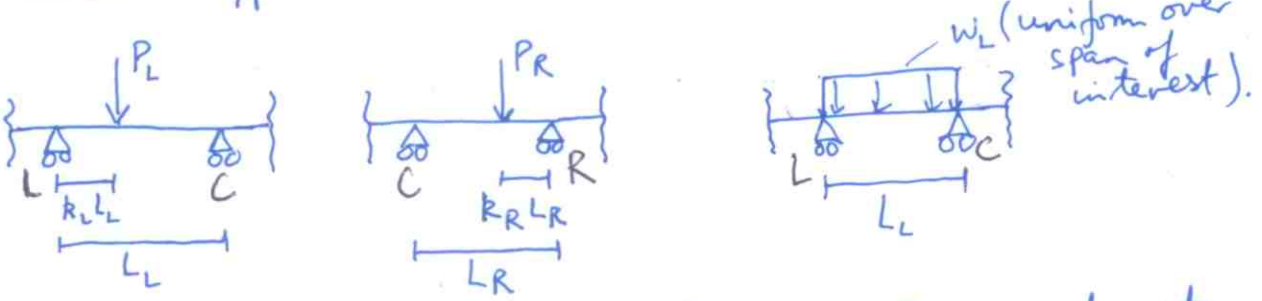
$$\Rightarrow \frac{M_R L_R}{I_R} + \frac{M_L L_L}{I_L} + 2M_C\left(\frac{L_R}{I_R} + \frac{L_L}{I_L}\right) = -6\left(\frac{A_R\bar{x}_R}{I_R L_R} + \frac{A_L\bar{x}_L}{I_L L_L}\right)$$

Generalizing for BMD_1 due to various applied loads (ie., superposing various applied loads),

3 moment Eqn

$$\boxed{\frac{M_R L_R}{I_R} + \frac{M_L L_L}{I_L} + 2M_C\left(\frac{L_R}{I_R} + \frac{L_L}{I_L}\right) = -6\left(\underbrace{\sum \frac{A_R\bar{x}_R}{I_R L_R}}_{\text{Sum for various appl loads in right beam}} + \underbrace{\sum\frac{A_L\bar{x}_L}{I_L L_L}}_{\text{Sum for various appl. loads in left beam.}}\right)}$$

Some common applied loads are,



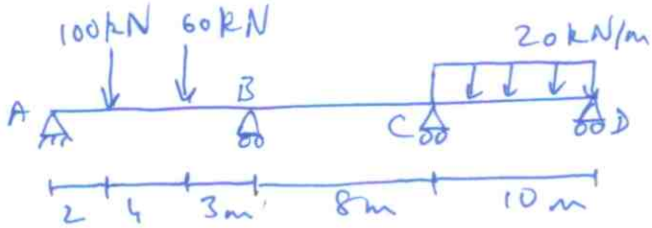
For these cases we can find A_L, A_R, $\bar{x}_L$, $\bar{x}_R$ and get following result:

3-moment equation

$$\boxed{\begin{aligned}\frac{M_R L_R}{I_R} + \frac{M_L L_L}{I_L} + 2M_C\left(\frac{L_R}{I_R} + \frac{L_L}{I_L}\right) &= -\sum\frac{P_L L_L^2}{I_L}(k_L - k_L^3) - \frac{w_L L_L^3}{4I_L} \\ &\quad -\sum\frac{P_R L_R^2}{I_R}(k_R - k_R^3) - \frac{w_R L_R^3}{4I_R}\end{aligned}}$$

Procedure: Choose section of beam having two adjacent spans, ie, supports L_1, C_1, R_1. Write 3-moment equation. It will contain unknowns M_{L1}, M_{C1}, M_{R1} only ($\because$ loading & geometry is known). Then move to next (adjacent) two-span section and repeat process. It will have supports $L_2 \equiv C_1$, $C_2 \equiv R_1$, R_2 and unknowns $M_{L2} = M_{C1}$, $M_{C2} = M_{R1}$, M_{R2}. This way set up simultaneous equations & solve for all support bending moments. Then from statics, solve support reactions.

Ex 1



Straightforward example.

EI = constant.

(ie I_R, I_L cancel out).

Spans ABC: $M_L = 0$, $M_c = M_B$, $M_R = M_C$, $L_L = 9$, $L_R = 8$, $P_{L1} = 100$, $P_{L2} = 60$, $k_{L1} = 2/9$, $k_{L2} = 6/9$,

Spans BCD: $M_L = M_B$, $M_c = M_C$ (center), $M_R = 0$, $L_L = 8$, $L_R = 10$, $w_R = 20$

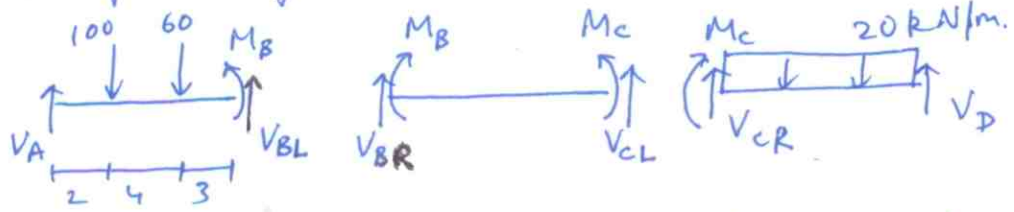
Write 3-moment equations:

$$8M_C + 2M_B(9+8) = -9^2\left(100\left[\tfrac{2}{9} - \left(\tfrac{2}{9}\right)^3\right] + 60\left[\tfrac{2}{3} - \left(\tfrac{2}{3}\right)^3\right]\right) = -3511.11$$

$$8M_B + 2M_C(8+10) = -\frac{(20)(10)^3}{4} = -5000.$$

$\Rightarrow M_C = -122$ kNm, $M_B = -74.5$ kN.m

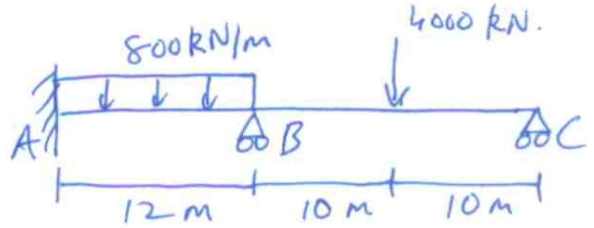
Now use following FBD's



to solve for V_A, V_{BL}, V_{BR}, V_{CL}, V_{CR}, V_D by statics.

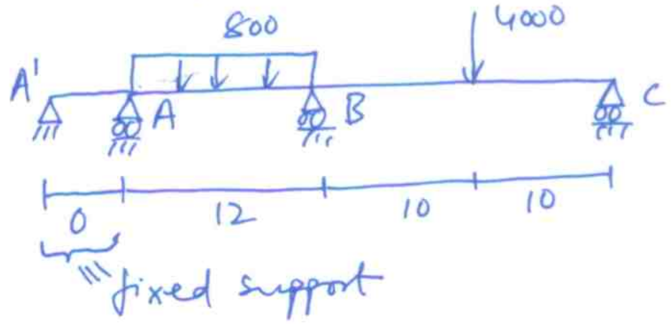
Then $R_A = V_A$, $R_D = V_D$, $R_B = -V_{BL} - V_{BR}$, $R_C = -V_{CL} - V_{CR}$.

Ex 2



Example on how to handle fixed support.
$EI = \text{const}.$

The 3-moment equation applied on above give 1 equation in 2 unknowns (M_A, M_B; $M_C = 0$).
So, we represent fixed support by two infinitesimally close pinned supports, ie,



Spans $A'AB$: $M_L = 0$, $M_c = M_A$, $M_R = M_B$, $L_L = 0$, $L_R = 12$, $w_R = 800$,
Spans ABC : $M_L = M_A$, $M_c = M_B$, $M_R = 0$, $L_L = 12$, $L_R = 20$, $w_L = 800$,
$P_{R1} = 4000$, $k_{R1} = 0.5$

$$12 M_B + 2 M_A (12 + 0) = -\frac{800}{4}(12)^3 = -345600$$

$$12 M_A + 2 M_B (12 + 20) = -4000(20^2)(0.5 - 0.5^3) - \frac{800}{4}(12)^3 = -945600$$

$$\Rightarrow M_A = -7.74 \text{ MN.m}, \quad M_B = -13.3 \text{ MN.m}$$

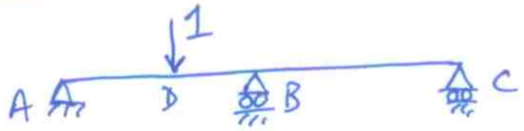
INFLUENCE LINES — STATICALLY INDETERMINATE STRUCTURES.

BEAMS :

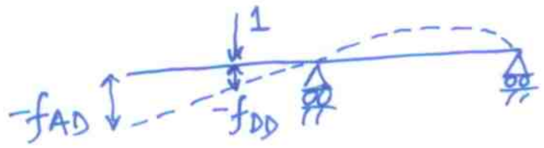
Unlike SD case, the IL will not be piecewise linear for SID case.

Muller-Breslau principle still holds, since in the previous derivation (for SD case) we used VW (unit displacement version) method, wherein SD or SID was not implied. However, SD implied piecewise linear IL.

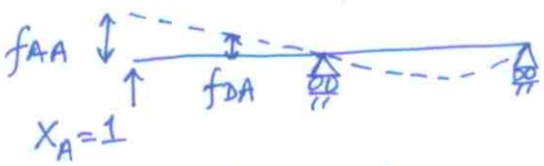
Reaction IL. (R_A)

(a) 

$R_A = X_A$

(b) 

f_{AD} = ↑ displ at A due to ↑ unit load at D.

(c) 

$f_{DA} = f_{AD}$ from Maxwells law.

f_{AA} = ↑ displ at A due to ↑ unit load at A.

f_{DA} = ↑ displ at D due to ↑ unit load at A.

$\Delta_{A0} = -f_{AD} = -f_{DA}$

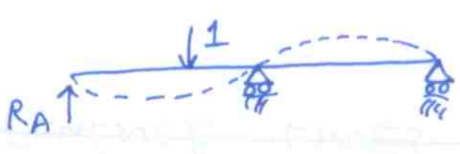
(a)=(b)+X_A(c) ⇒ $0 = \Delta_{A0} + X_A f_{AA}$ ⇒ $\boxed{X_A = R_A = \dfrac{f_{DA}}{f_{AA}}}$ → i.e, R_A = displ at D per unit displ at A, ie Muller–Breslau verified

f_{DA} → represents elastic curve for unit load at A ($X_A = 1$)

f_{AA} → displ at A for $X_A = 1$.

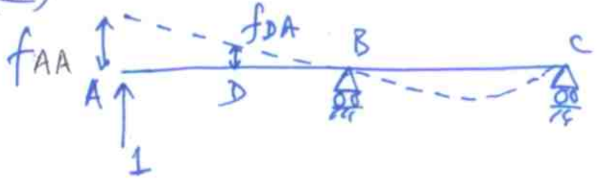
Procedure: Release R_A, apply $X_A = 1$, obtain displacement at various points by VW, Castigliano's, Conjugate beam method (latter most preferred). Then normalize (divide) by displ at A. This gives IL (ie resulting elastic curve [normalized] is IL).

Can also see this thru Betti's Law, as follows.

(I) 

Primary with appl. load (=1) and correct redundant (R_A) so that displ at A is zero.

(II)



Primary, $x_A = 1$.

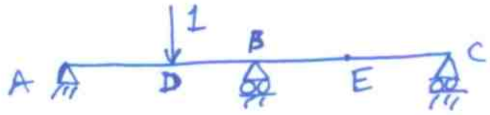
Betti's Law: Ext work done by forces in (I) undergoing displ due to forces in (II) =
Ext work done by forces in (II) undergoing displ due to forces in (I).

$$\Rightarrow \quad R_A f_{AA} + (-1)(f_{DA}) = (1)(0)$$

$$\Rightarrow \quad \boxed{R_A = f_{DA}/f_{AA}} \longrightarrow \text{same as before.}$$

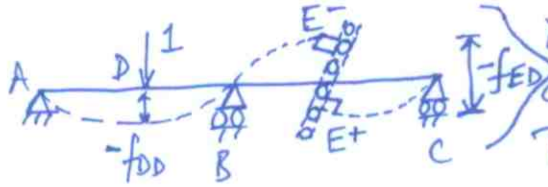
Shear IL. (V_E)

(a)



$X_E = V_E$

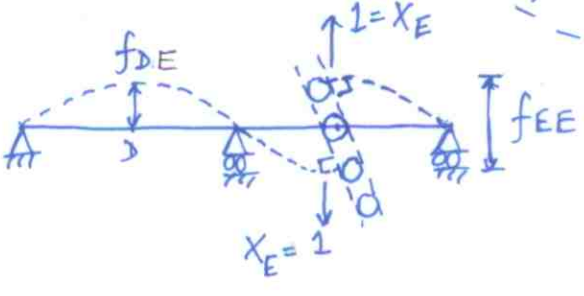
(b)



Note the right angles of elastic curve with roller guide at E^-, E^+. This is consistent with slope continuity at E, ∵ only shear released, moment not released at E.

f_{ED} = ↓↑ relative displ between E^-, E^+, due to ↑ unit load at D.

(c)



f_{DD} = ↑ displ at D due to ↑ unit load at D.

f_{DE} = ↑ displ at D due to ↓↑ unit load (shear) at E

f_{EE} = ↓↑ rel. displ between E^-, E^+ due to ↓↑ unit (shear) at E.

$$\Delta_{E0} = -f_{ED} = -f_{DE}$$

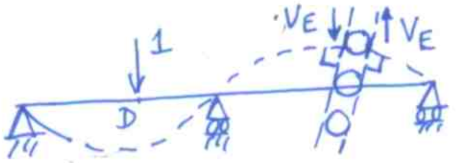
$$(a) = (b) + X_E(c) \Rightarrow 0 = \Delta_{E0} + f_{EE} X_E = -f_{DE} + f_{EE} X_E$$

$$\Rightarrow \boxed{X_E = V_E = \frac{f_{DE}}{f_{EE}}}$$

→ displ at D per unit displ at E (ie rel. displ betwn E^- & E^+) ie Muller-Breslau verified.

Procedure: As before (in R_A IL), ie, release shear at E, apply $X_E = 1$, obtain displ at various points (preferably by Conj. beam method), normalize displ's thus obtained by displ at E. Resulting elastic curve (normalized) is IL of SF at E.

Can also see it by Betti's Law, as follows.

(I) 

Primary with appl load (=1) and correct redundant (V_E) so that displ continuity (in addition to slope continuity, ie right angles) is maintained..

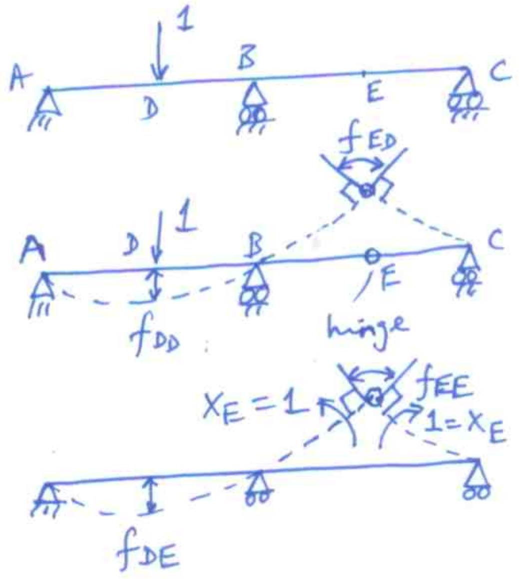
(II) → same as (c). → ie, primary with $X_E = 1$, only.

Betti's Law ⇒ $(-1) f_{DE} + V_E f_{EE} = (1)(0)$

$$\Rightarrow \boxed{V_E = \frac{f_{DE}}{f_{EE}}} \rightarrow \text{as before.}$$

Moment IL (M_E)

$X_E = M_E$

(a)
(b)
(c)


f_{DD} = ↓ displ at D due to ↓ unit load at D.

f_{ED} = ↶↷ displ at E due to ↓ unit load at D.

f_{DE} = ↓ displ at D due to ↶↷ $X_E = 1$.

f_{EE} = ↶↷ displ at E due to ↶↷ $X_E = 1$.

$\Delta_{EO} = f_{ED} = f_{DE}$

(a) $= (b) + X_E (c) \Rightarrow 0 = f_{DE} + f_{EE} X_E \Rightarrow X_E = -\dfrac{f_{DE}}{f_{EE}}$,

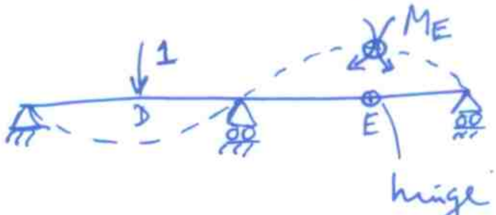
for displ at D measured ↓.

$$\Rightarrow \boxed{X_E = \frac{f_{DE}}{f_{EE}} = M_E}$$

→ Muller Breslau verified.
→ for displ at D measured ↑, and relative rotation at E measured ↺↻

<u>Procedure</u>: As before, ie, release moment at E by inserting hinge, apply $X_E = 1$, obtain displ(↑) at various points (preferably by Conj. beam method), normalize this displ by displ at E (ie rel. rot. ↺↻). Resulting elastic curve (normalized) is IL of BM at E.

Can also see it by Betti's Law, as follows,

(I) 

Primary with appl load (=1) & correct redundant (M_E) so that slope continuity maintained at hinge

(II) → same as (c) → ie, primary with $X_E = 1$, only.

Betti's law $\Rightarrow (1)(f_{DE}) + (-M_E)(f_{EE}) = (\cancel{1}^{X_E})(0) = 0$

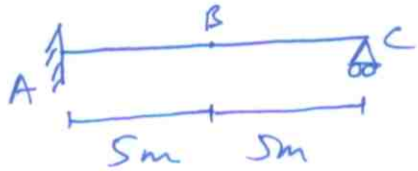
$$\Rightarrow \boxed{M_E = \frac{f_{DE}}{f_{EE}}}$$

→ M_E ↺↻, f_{DE} ↓, f_{EE} ↺↻, ie, same as, M_E ↺↻, f_{DE} ↑, f_{EE} ↺↻ ie same as before.

So procedure (generalized) is: Release the function, apply X_E = function = 1, calculate displacements at various points, normalize by f_{EE}, resulting elastic curve is IL of function.

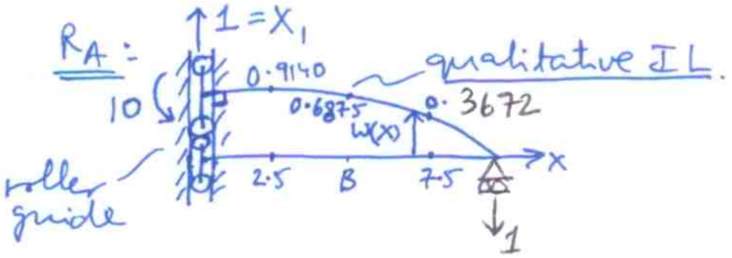
NOTE: If D.O.I. > 1, after we release the function, we (to get "primary" str — indet if DOI > 1 — The term "primary" is used loosely.) still have an indeterminate structure with DOI reduced by 1, so displacement calculations will require solving the indeterminate (ie "primary") structure by methods already discussed in this topic chapter.

Ex 1

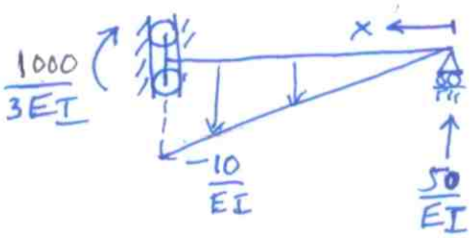


IL for R_A, R_C, V_B, M_B.

$\underline{R_A}$:



Release R_A with roller guide
$R_A = X_1$
Apply $X_1 = 1$, get displ's by Conj-Beam method.

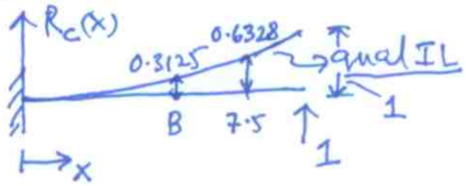


Conjugate beam (same as real), loaded with $\frac{BMD}{EI}$ of real beam.

$$\bar{M} = \frac{50}{EI}x - \frac{x^2}{2}\cdot\frac{x}{3}\cdot\frac{1}{EI} = w(x)$$

$$\boxed{R_A = \frac{\bar{M}(x) \to f_{DA}}{\bar{M}(10) \to f_{AA}} = \frac{3}{1000}\left(50x - \frac{x^3}{6}\right) \longrightarrow \text{IL for } R_A \text{ in CA.}}$$

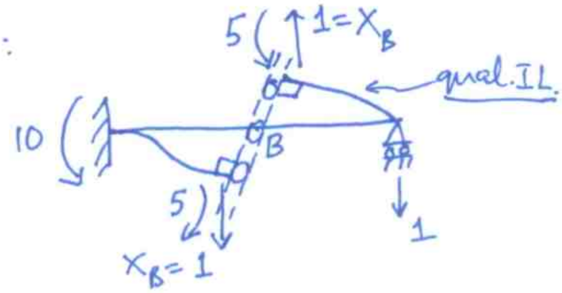
$\underline{R_C}$:



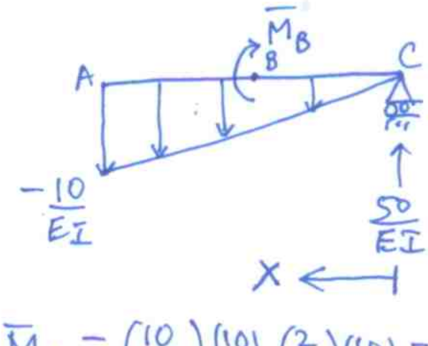
Use tables. Release R_C.

$$\boxed{R_C = \frac{(-x^3 + 3*10*x^2)}{6}\cdot\frac{3}{10^3} = \frac{-x^3+30x^2}{2000} = \text{IL of } R_C}$$

$\underline{V_B}$:



Release $V_B = X_B$.
Apply $X_B = 1$



$$\bar{M}_B = \left(\frac{10}{EI}\right)\frac{(10)}{2}\left(\frac{2}{3}\right)(10) = \frac{1000}{3EI}$$

conjugate beam, loaded with $\frac{M}{EI}$ diag of real beam.
Roller guide: Displ. discontinuity, slope continuity, in real beam. Hence, moment discontinuity — provided by point moment $\bar{M}_B$ applied in conj beam in addition to $\frac{M}{EI}$; but shear continuity — so no support at B in conj. beam.

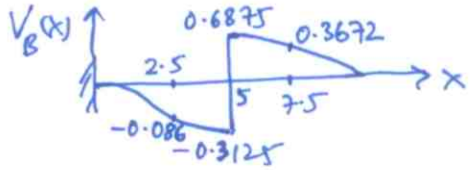
$$\bar{M} = \frac{50}{EI}x - \frac{x^2}{2}\cdot\frac{x}{3}\cdot\frac{1}{EI} = \frac{1}{EI}\left(50x - \frac{x^3}{6}\right), \text{ in } CB,\ 0 \le x \le 5$$

$$= \left(50x - \frac{x^3}{6} - \frac{1000}{3}\right)\cdot\frac{1}{EI}, \text{ in } BA,\ 5 \le x \le 10.$$

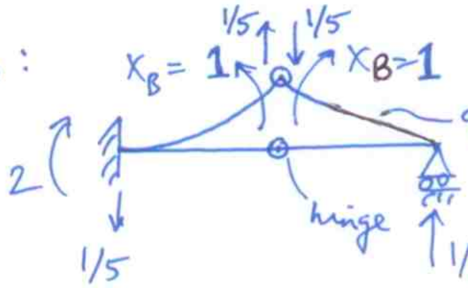
$= w(x) \equiv f_{DB}$

IL for $V_B(x) = \dfrac{w(x)}{\bar{M}_B}$, $\because f_{BB} = \bar{M}_B$ = rel. displ between B^- & B^+ due to unit shear applied at B.

$$\boxed{\begin{aligned} V_B(x) &= \frac{3}{1000}\left(50x - \frac{x^3}{6}\right), \text{ in } CB,\ 0 \le x \le 5 \\ &= \frac{3}{1000}\left(50x - \frac{x^3}{6} - \frac{1000}{3}\right), \text{ in } BA,\ 5 \le x \le 10 \end{aligned}}$$

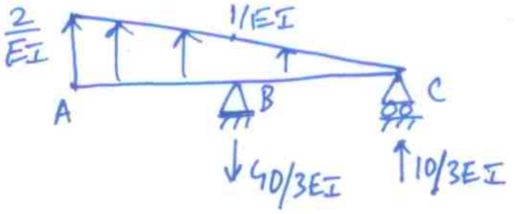


$\underline{M}_B$:



Release $M_B = X_B$

Apply $X_B = 1$



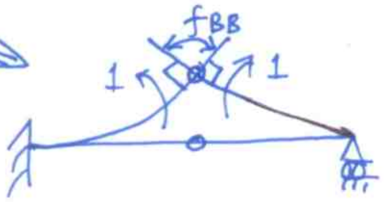
Conj. beam with M/EI load.

$$\bar{M} = \frac{10}{3EI}x + \frac{1}{2}\cdot\frac{x^2}{5}\cdot\frac{x}{3}\cdot\frac{1}{EI}, \text{ in } CB,\ 0 \le x \le 5$$

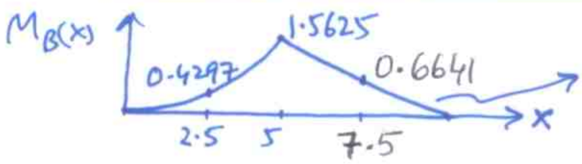
$$= \frac{1}{EI}\left[\left(\frac{10}{3}x + \frac{x^3}{30}\right) - \frac{40}{3}(x-5)\right] \text{ in } BA,\ 5 \le x \le 10$$

$f_{DB} \equiv w(x)$ //

$$M_B(x) = \frac{f_{DB}}{f_{BB}} = \frac{w(x)}{f_{BB}}\ ; \quad f_{BB} = \bar{V}_{B^-} - \bar{V}_{B^+} = \frac{40}{3EI}$$

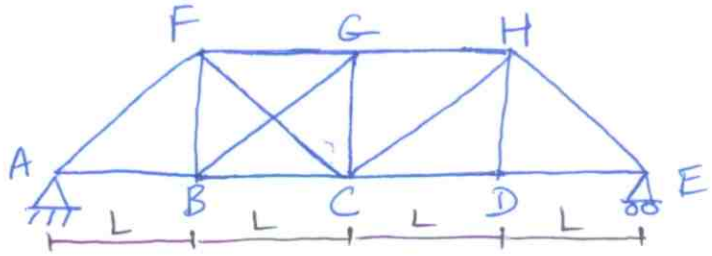


$$\Rightarrow \boxed{\begin{aligned} M_B(x) &= \frac{3}{40}\left[\frac{10}{3}x + \frac{x^3}{30}\right], \text{ in } CB,\ 0 \le x \le 5 \\ &= \frac{3}{40}\left[-10x + \frac{x^3}{30} + \frac{200}{3}\right], \text{ in } BA,\ 5 \le x \le 10 \end{aligned}}$$



IL for TRUSSES.

These will still be piecewise linear.



Draw IL for BG.

Unit load (↓) at B → get BG_1

" " " " C → " BG_2

" " " " D → " BG_3

For unit load(↓) between B & C → $BG = BG_1\left(\frac{x}{L}\right) + BG_2\left(1-\frac{x}{L}\right)$

" " " C & D → $BG = BG_3\left(\frac{x}{L}\right) + BG_2\left(1-\frac{x}{L}\right)$

where x = dist of unit (↓) load from C.

Similar arguements for unit load(↓) between A & B and D & E, with BG = 0 when unit load at A or E. The above discussion assumes that load applied thru floor system that tranfer load at joints only. Thus, you see that BG is a piecewize linear function of x. So to draw IL, find BG for unit load at B, C, D & join by straight lines (B, C, D are key points).