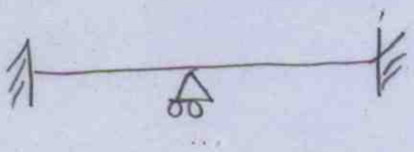
Kinematic indeterminacy — Degrees of freedom

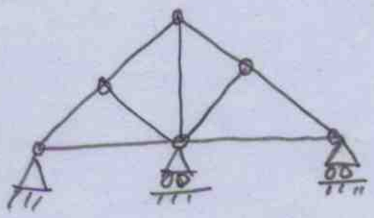
Number of unknown nodal displacements is the kinematic d.o.f's. (just like number of unknown forces, which cannot be obtained by equilibrium is the degree of static indeterminacy.



DoSI = 7 − 3 = 4

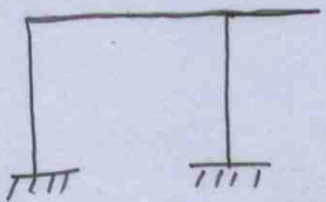
DoKI = 2 (member extension considered)

= 1 (" " neglected).



DoSI = $\overset{ext}{1} + \overset{int}{0} = 1$

DoKI = 2*6 − 4 = 8



DoSI = 3

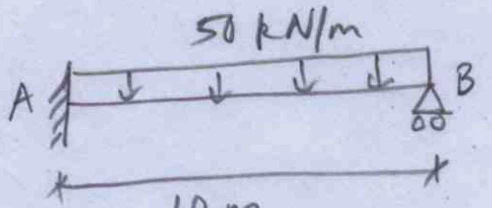
DoKI = 9 (axial def considered)

= 9 − 1 − 2 − 1 = 5 (axial def excluded)

1) Locate the nodes at supports, ends of members, joints, or where member properties (EI, A) change
2) Dof's are (minimum) number of displ's required to define deformed configuration of structure.
3) In Force/Compatibility/Flexibility method, unknowns are redundant forces. <u>Their choice is not unique</u>. Primary structure is statically determinate. Displacements solved for primary structure with loading & redundants applied seperately. Then displacements superposed to satisfy compatibility. The compatibility eqns yield redundant forces.

4) In Displacement/Equilibrium/Stiffness method, unknowns are the kinematic indeterminacies (ie d.o.f's). Their choice is unique. Primary structure is kinematically determinate. Member end forces solved for primary structure with loading and kinematic indeterminacies applied seperately. Then these forces are superposed to satisfy equilibrium. The equilibrium equations yield the kinematic indeterminacies (unknowns).

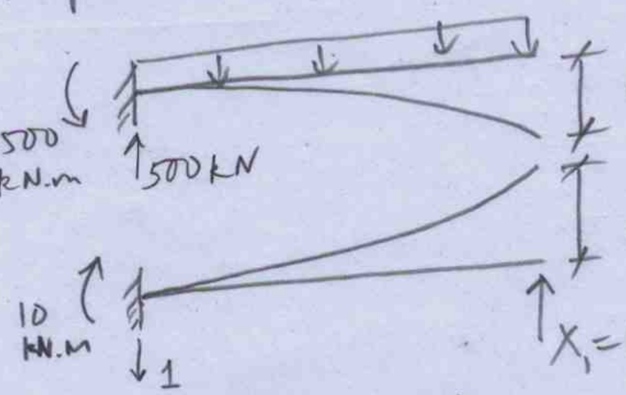
(eg).



DoSI = 1
DoKI = 1

Compatibility method :

equil satisfied here



$\Delta_1 = \frac{50 \cdot 10^4}{8EI}$ Primary w/ loading.

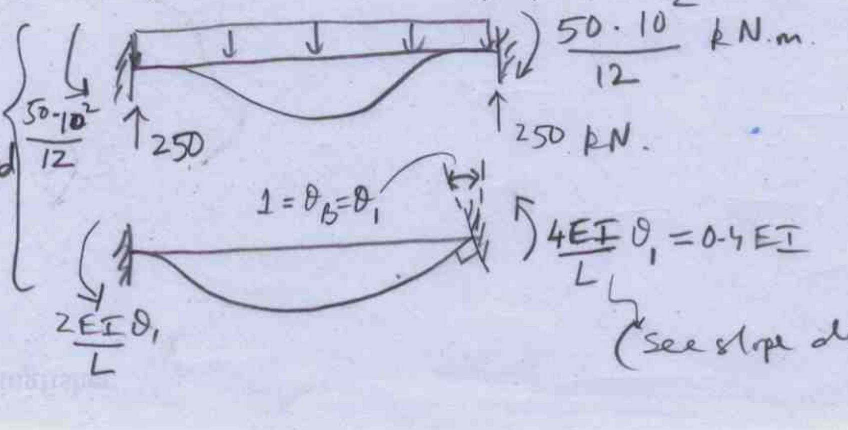
$f_{11} = \frac{10^3}{3EI}$ Primary w/ unit redundant (ie static indeterminacy)

Compat condition is,

$\Delta_1 + f_{11} X_1 = 0 \longrightarrow$ yields X_1.

Equilibrium method :

compat satisfied here



Primary w/ loading

Primary w/ kin. indet. applied.

(see slope defl method later on)

Equil condition is

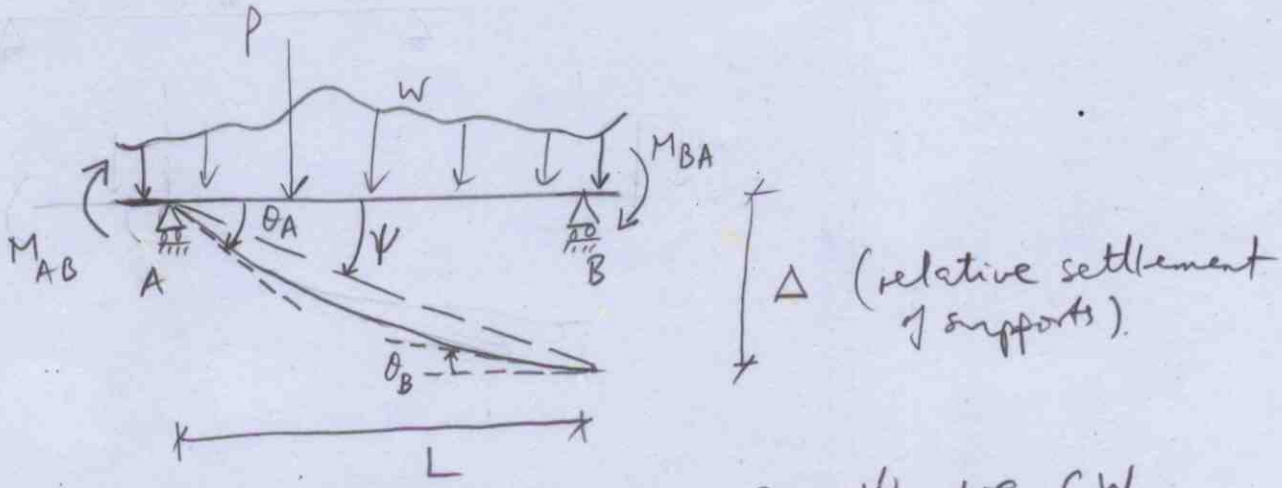
$$\frac{50 \cdot 10^2}{12} - 0.4EI\theta_1 = 0 \rightarrow \text{yields } \theta_1$$

Then by integrating $EIw^{IV} = 50$ four times and using $w(0) = w'(0) = w(L) = 0$ and $w'(L) = \theta_1$ you solve for deflected shape and internal forces.

5) Generally displacement method more attractive since: (i) in large structures kinematic indeterminacies are less than static indeterminacies.
(ii) choice of kinematic indeterminacies is unique, unlike static indeterminacies.

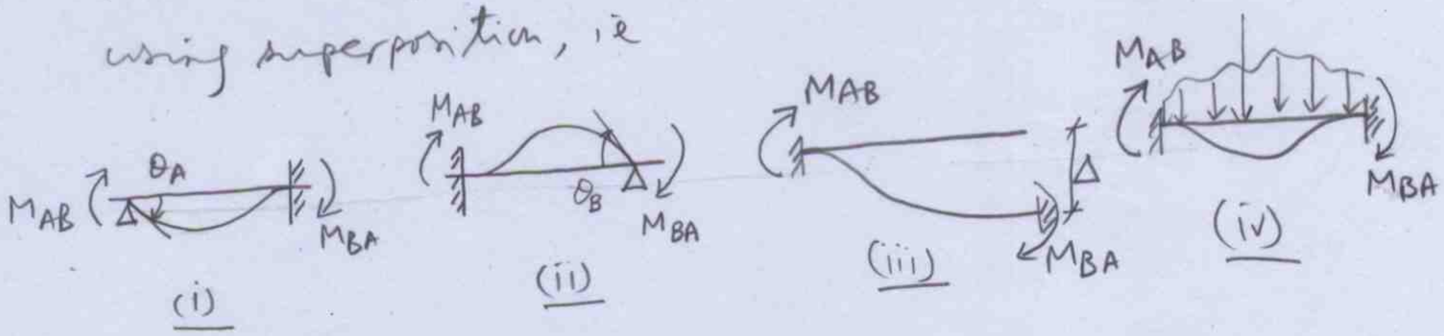
SLOPE DEFLECTION METHOD (Displ/Equil method)

Relates slopes/deflections to applied loads.
Consider a span of continuous beam, EI constant.



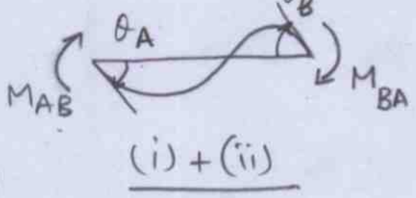
Sign convention: M_{AB}, M_{BA}, θ_A, θ_B, ψ, +ve CW.
(Δ +ve for +ve ψ).

Relate M_{AB}, M_{BA} to θ_A, θ_B, ψ and loading, using superposition, ie



(i) + (ii)

$$EIw^{IV} = 0$$
$$w = c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$



$$\left.\begin{aligned}
w'\big|_{x=0} &= -\theta_A = c_3\\
w'\big|_{x=L} &= -\theta_B = c_1\frac{L^2}{2} + c_2L + c_3\\
w\big|_{x=0} &= 0 = c_4\\
w\big|_{x=L} &= 0 = c_1\frac{L^3}{6} + c_2\frac{L^2}{2} + c_3L
\end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= \frac{6}{L^2}(-\theta_A - \theta_B)\\ c_2 &= \frac{2}{L}(2\theta_A + \theta_B)\\ c_3 &= -\theta_A,\ c_4 = 0.\end{aligned}$$

$$M_{AB} = EIw''\big|_{x=0} = EIc_2 = \frac{EI}{L}(4\theta_A + 2\theta_B)$$

$$-M_{BA} = EIw''\big|_{x=L} = -\frac{EI}{L}(2\theta_A + 4\theta_B)$$

(iii) $w'(0) = 0 = c_3$

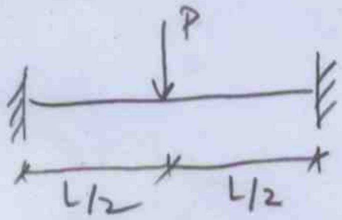
$$\left.\begin{aligned} w'(L) &= 0 = c_1\frac{L^2}{2} + c_2 L \\ w(0) &= 0 = c_4 \\ w(L) &= -\Delta = c_1\frac{L^3}{6} + c_2\frac{L^2}{2} \end{aligned}\right\} \Rightarrow c_1 = \frac{12}{L^3}\Delta\,,\quad c_2 = -\frac{6}{L^2}\Delta$$

$$M_{AB} = EIw''(0) = EIc_2 = -\frac{6}{L^2}EI\Delta$$

$$M_{BA} = -EIw''(L) = -EI(c_1 L + c_2) = -\frac{6}{L^2}EI\Delta$$

(iv) Here M_{AB}, M_{BA} are the end moments due to applied load, with ends fixed, ie they are fixed end moments. These are listed in Tables.

For example,



$$EIw^{IV} = P\langle x - L/2\rangle^{-1}$$

$$w = c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 + \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3$$

$$c_2 = \frac{PL}{8}\,,\quad c_1 = -\frac{P}{2} \quad \Leftarrow \left\{\begin{aligned} w(0) &= 0 = c_4 \\ w(L) &= 0 = c_1\frac{L^3}{6} + c_2\frac{L^2}{2} + c_3 L + \frac{P}{6}\frac{L^3}{8} \\ w'(0) &= 0 = c_3 \\ w'(L) &= 0 = \frac{c_1}{2}L^2 + c_2 L + \frac{P}{2}\frac{L^2}{4} \end{aligned}\right.$$

$$M_{AB} = EIw''(0) = c_1 x + c_2 + P\left\langle x - \frac{L}{2}\right\rangle = 0 + \frac{PL}{8} + 0 = \frac{PL}{8}$$

$$M_{BA} = -EIw''(L) = -\frac{P}{2}L + \frac{PL}{8} + P\frac{L}{2} = \frac{PL}{8}$$

$$\therefore M_{AB} = M_{BA} = \frac{PL}{8} = (FEM)_{AB} = (FEM)_{BA}$$

Superposing, (i) + (ii) + (iii) + (iv),

$$M_{AB} = 2E\frac{I}{L}\left(2\theta_A + \theta_B - 3\frac{\Delta}{L}\right) + (FEM)_{AB}$$

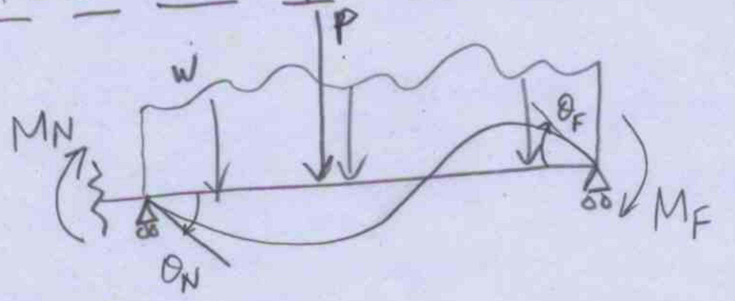
$$M_{BA} = 2E\frac{I}{L}\left(\theta_A + 2\theta_B - 3\frac{\Delta}{L}\right) + (FEM)_{BA}$$

In compact form,

$$\boxed{M_N = 2Ek\,(2\theta_N + \theta_F - 3\psi) + (FEM)_N.}$$

M_N = internal moment on near end of span, CW +ve.
θ_N = rot at near end, CW +ve
θ_F = " " far ", CW +ve
ψ = rot of chord of span, $\frac{\Delta}{L}$, CW +ve
$k = \frac{I}{L}$ = span stiffness
$(FEM)_N$ = fixed-end moment at near end, CW +ve, refer Tables.

Specialization for case where end support is SS (not essential, ie can do without it, but useful to reduce d.o.f's.)



From S-D eqns,

$$M_N = 2Ek\,(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$
$$0 = M_F = 2Ek\,(2\theta_F + \theta_N - 3\psi) + (FEM)_F$$

subtract suitably to eliminate θ_F

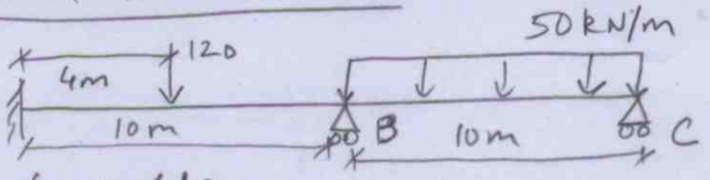
$$\Rightarrow \; M_N = 2Ek\left(\tfrac{3}{2}\theta_N - \tfrac{3}{2}\psi\right) + (FEM)_N - \frac{(FEM)_F}{2}$$

$$\boxed{\begin{aligned} M_N &= 3Ek\,(\theta_N - \psi) + \underbrace{(FEM)_N - \frac{(FEM)_F}{2}}_{\text{from original Table (ie LHS column in Hibbeler)}} \\ &= 3Ek(\theta_N - \psi) + \underbrace{(FEM)_N}_{\text{from revised Table (ie RHS col in Hibbeler)}} \end{aligned}}$$

So we need apply S-D eqns only once, if θ_F is not of interest to us (ie it is eliminated).
THIS PROCEDURE IS TERMED STATIC CONDENSATION IN FEM, ie WE CONDENSE OUT D.O.F'S BY USING STATIC EQUILIBRIUM

Application to beams.

Ex(1)



Support B settles 0.03m

2-dof problem.

SD eqns:

$$M_{AB} = 2E\frac{I}{10}\left(2\cancelto{0}{\theta_A} + \theta_B - 3*\frac{0.03}{10}\right) - \frac{(120)(6)^2(4)}{10^2} \longrightarrow (i)$$

$$M_{BA} = 2E\frac{I}{10}\left(2\theta_B + \cancelto{0}{\theta_A} - 3*\frac{0.03}{10}\right) + \frac{120(4)^2(6)}{10^2} \quad -(ii)$$

$$M_{BC} = 2E\frac{I}{10}\left(2\theta_B + \theta_C + 3*\frac{0.03}{10}\right) - 50\frac{(10)^2}{12} \quad -(iii)$$

$$M_{CB} = 2E\frac{I}{10}\left(2\theta_C + \theta_B + 3*\frac{0.03}{10}\right) + 50\frac{(10)^2}{12} \quad -(iv).$$

Equilibrium:

$$M_{BA} + M_{BC} = 0$$

$$M_{CB} = 0$$

$$EI\begin{bmatrix} 0.4+0.4 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \begin{Bmatrix} -\frac{120(4)^2(6)}{10^2} + 50\frac{(10)^2}{12} \\ -2\frac{EI}{10}*3*\frac{0.03}{10} - 50\frac{(10)^2}{12} \end{Bmatrix}$$

use $E = 200\,E9\ N/m^2$, $I = 2000\,E{-6}\ m^4$

$EI\theta_B = 515014\ kN.m^2$, $EI\theta_C = -2058548.7\ kNm^2$

As 1-dof problem (∵ end support pinned.)

Ref RHS Table of FEM's from Hibbeler.

Eqs (i), (ii) remain same. Eq (iii) becomes,

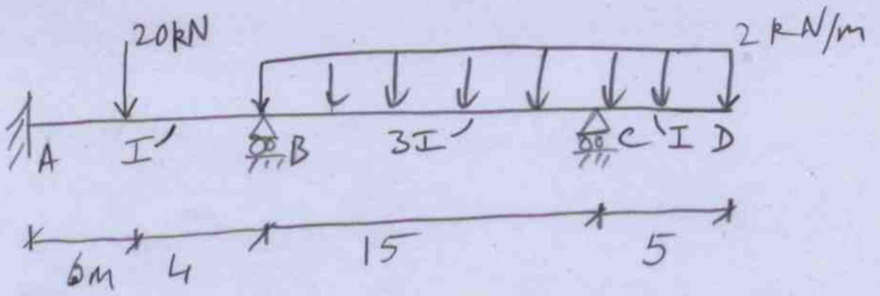
$$M_{BC} = 3E\frac{I}{10}\left(\theta_B + \frac{0.03}{10}\right) - 50\frac{(10)^2}{8}$$

Equil condt is $M_{BA} + M_{BC} = 0$

$$EI[0.7]\,\theta_B = \frac{-120(4)^2(6)}{10^2} + 50\frac{(10)^2}{8} + EI\left(\frac{2}{10}*3*\frac{0.03}{10} - \frac{3}{10}*\frac{0.03}{10}\right)$$

Using same EI values, $EI\theta_B = 515014\ kN.m^2 \rightarrow$ same result.

Ex 2



As 4-dof : $\theta_B,\ \theta_C,\ \theta_D,\ \Delta_D$

$$EI\begin{bmatrix} \frac{4}{10}+\frac{4*3}{15} & \frac{2*3}{15} & 0 & 0 \\ \frac{2*3}{15} & \frac{4*3}{15}+\frac{4}{5} & \frac{2}{5} & -\frac{6}{5^2} \\ 0 & \frac{2}{5} & \frac{4}{5} & -\frac{6}{5^2} \\ 0 & \frac{4}{5} & \frac{2}{5} & -\frac{6}{5^2} \end{bmatrix}\begin{Bmatrix}\theta_B \\ \theta_C \\ \theta_D \\ \Delta_D\end{Bmatrix} = \begin{Bmatrix} -\frac{20(6)^2(4)}{10^2}+\frac{2(15)^2}{12} \\ -\frac{2(15)^2}{12}+\frac{2(5)^2}{12} \\ -\frac{2(5)^2}{12} \\ \frac{2(5)^2}{12}-10(5)+\frac{2(5)^2}{2} \end{Bmatrix} \rightarrow (*)$$

where we have used the following :-

S-D eqns:

$$M_{BA} = \frac{EI}{10}(4\theta_B) + \frac{20(6^2)4}{10^2}, \quad M_{AB} \text{ not reqd.}$$

$$M_{BC} = \frac{E3I}{15}(4\theta_B + 2\theta_C) - \frac{2(15)^2}{12}$$

$$M_{CB} = \frac{E3I}{15}(2\theta_B + 4\theta_C) + \frac{2(15)^2}{12}$$

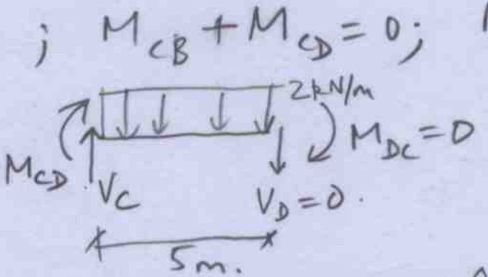
$$M_{CD} = \frac{EI}{5}\left(4\theta_C + 2\theta_D - 6\frac{\Delta_D}{5}\right) - \frac{2(5)^2}{12}$$

$$M_{DC} = \frac{EI}{5}\left(2\theta_C + 4\theta_D - 6\frac{\Delta_D}{5}\right) + \frac{2(5)^2}{12}$$

Equil eqns:

$$M_{BA} + M_{BC} = 0 \quad ; \quad M_{CB} + M_{CD} = 0; \quad M_{DC} = 0;$$ → these 3 eqns directly written above in matrix form.

$$(a) \leftarrow V_C - \cancel{V_D}^{\,0} - 2*5 = 0$$



$$\Sigma M_D = 0 = M_{CD} + V_C(5) - \frac{2(5)^2}{2} + \cancel{M_{DC}}^{\,0} \rightarrow (b)$$

$$(a),(b) \Rightarrow \frac{EI}{5}\left(4\theta_C + 2\theta_D - 6\frac{\Delta_D}{5}\right) - \frac{2(5)^2}{12} + 10(5) - \frac{2(5)^2}{2} = 0$$

This is the 4th eqn in (*)

Soln of (*) is $\boxed{EI\theta_B = 14.95,\ EI\theta_C = -23.1,\ EI\theta_D = 18.5667,\ EI\Delta_D = 40.75}$

As 2-dof : θ_B, θ_C

$$EI\begin{bmatrix} \frac{4}{10}+\frac{4}{15}*3 & \frac{2}{15}*3 \\ \frac{2}{15}*3 & \frac{4}{15}*3 \end{bmatrix}\begin{Bmatrix}\theta_b \\ \theta_c\end{Bmatrix} = \begin{Bmatrix} -\frac{20(6)^2(4)}{10^2}+\frac{2(15)^2}{12} \\ -\frac{2(15)^2}{12}+\frac{2(5)^2}{2} \end{Bmatrix} \rightarrow (**)$$

soln is $\theta_b = 14.95\,EI$, $\theta_C = -23.1\,EI$

where in 2^{nd} equation, ie $M_{CB}+M_{CD}=0$ we used

$M_{CD} = -\frac{2(5)^2}{2}$ = BM at C due to overhang with applied load, obtained directly from statics.

So both ways yield same solution !!

Now consider the same structure w/o loading and with following support movements

support A : $\Delta_A = 0.01m \downarrow$, 0.001 radian CW

B : $\Delta_B = 0.04m \downarrow$

C : $\Delta_C = 0.0175m \downarrow$

2-dof system: — So only RHS in (**) changes.

$\theta_A = 0.001$, $\psi_{AB} = \psi_{BA} = \frac{0.04-0.01}{10}$, $\psi_{BC} = \psi_{CB} = \frac{0.0175-0.04}{15}$

So RHS becomes,

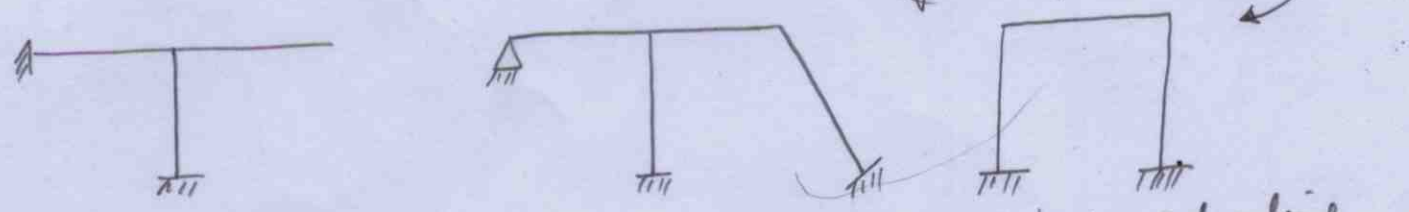
$$EI\begin{Bmatrix} -\frac{2(0.001)}{10}+\frac{6\psi_{BA}}{10}+\frac{6\psi_{BC}}{15}*3 \\ \frac{6\psi_{BC}}{15}*3 \end{Bmatrix}$$

$\theta_B = 0.0007$, $\theta_C = -0.0026$.

Application to Frames.

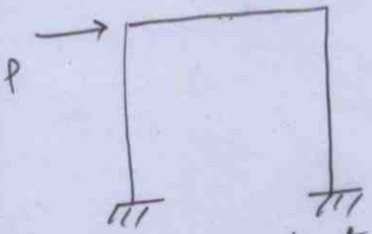
No-sidesway frames: Axial def. neglected. Frame properly restrained

(geom symmetric frame).



↳ here loading should also be symmetric for no sidesway

Sidesway frames:



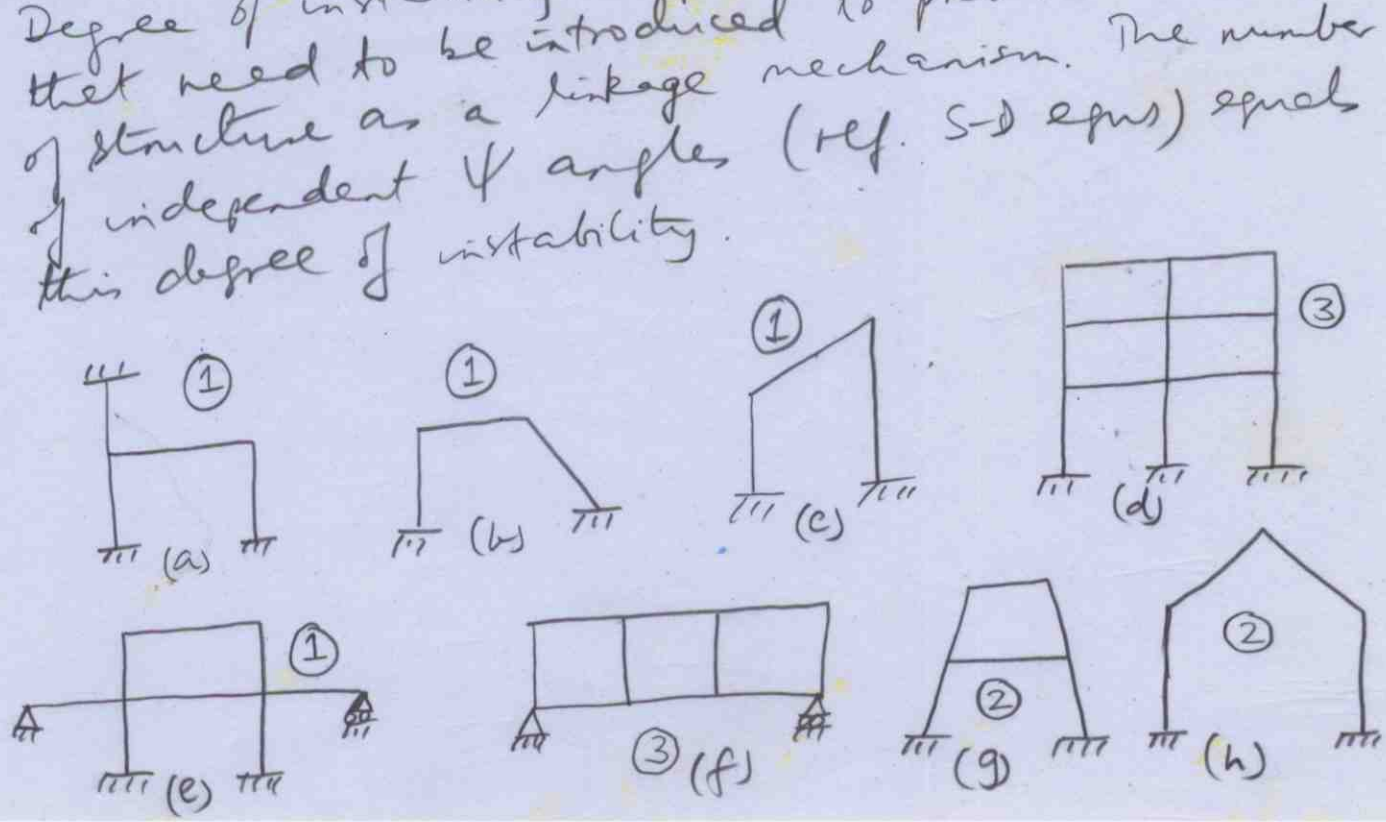
geom symm but loading not symm.

Determining degree of sidesway

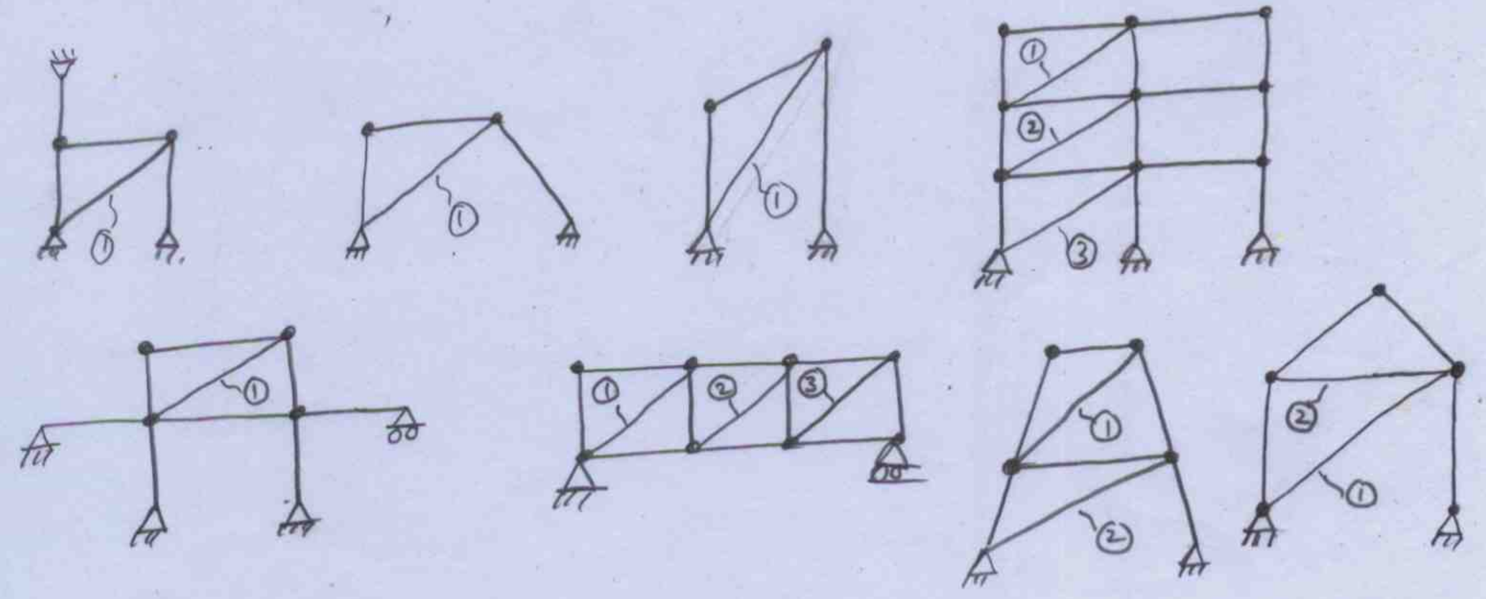
(i) Consider all rigid jts as pin connections and all fixed supports as hinge supports. Now investigate whether modified structure is statically / kinematically stable with respect to sway.

(ii) Degree of instability is equal to number of links that need to be introduced to prevent movement of structure as a linkage mechanism. The number of independent ψ angles (ref. S-D eqns) equals this degree of instability.

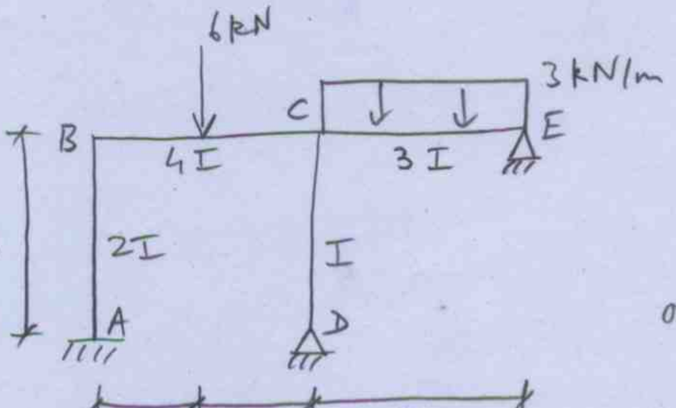
(eg)



Degree of sidesway (instability) is noted against each frame. Frame (a) is statically unstable when converted to pin-joints (apply horizontal load and see it). All other frames are kinematically unstable (ie form mechanism) when converted to pin-joints. The number of links to be introduced to make them stable is shown below:



(Ex 3)



No sway frame.

4-dof. ($\theta_B, \theta_C, \theta_D, \theta_E$)

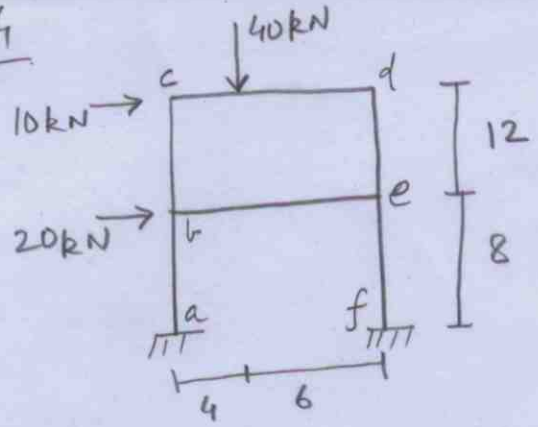
or 2-dof (θ_B, θ_C) — using modified S-D eqns.

Done here as 2-dof.

$$EI\begin{bmatrix} \frac{2}{15}\cdot 4+\frac{4}{16}\cdot 4 & \frac{4}{16}\cdot 2 \\ \frac{4}{16}\cdot 2 & \frac{4}{16}\cdot 4+\frac{1}{15}\cdot 3+\frac{3}{12}\cdot 3 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \begin{Bmatrix} \frac{6\cdot 16}{8} \\ -\frac{6\cdot 16}{8}+\frac{3\cdot 12^2}{8} \end{Bmatrix}$$

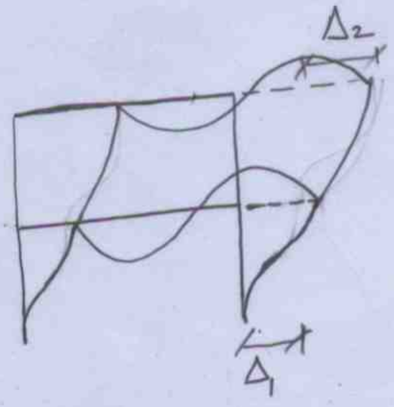
$EI\theta_B = 0.8759, \quad EI\theta_C = 21.31 \ kNm^2$

Ex 4



Sway frame

6 dof $\theta_b, \theta_c, \theta_d, \theta_e, \Delta_1, \Delta_2$



$\psi_1 = \frac{\Delta_1}{8}, \quad \psi_2 = \frac{\Delta_2}{12}$

$$EI\begin{bmatrix} 4\left(\frac{1}{8}+\frac{1}{10}+\frac{1}{12}\right) & 2\left(\frac{1}{12}\right) & 0 & 2\left(\frac{1}{10}\right) & -6\left(\frac{1}{8^2}\right) & -6\left(\frac{1}{12^2}\right) \\ 2\left(\frac{1}{12}\right) & 4\left(\frac{1}{12}+\frac{1}{10}\right) & 2\left(\frac{1}{10}\right) & 0 & 0 & -6\left(\frac{1}{12^2}\right) \\ 0 & 2\left(\frac{1}{10}\right) & 4\left(\frac{1}{10}+\frac{1}{12}\right) & 2\left(\frac{1}{12}\right) & 0 & -6\left(\frac{1}{12^2}\right) \\ 2\left(\frac{1}{10}\right) & 0 & 2\left(\frac{1}{12}\right) & 4\left(\frac{1}{8}+\frac{1}{10}+\frac{1}{12}\right) & -6\left(\frac{1}{8^2}\right) & -6\left(\frac{1}{12^2}\right) \\ 2\left(\frac{1}{8}\right)+4\left(\frac{1}{8}\right) & 0 & 0 & 2\left(\frac{1}{8}\right)+4\left(\frac{1}{8}\right) & -6\left(\frac{1}{8^2}\right)\cdot 2\cdot 2 & 0 \\ 2\left(\frac{1}{12}\right)+4\left(\frac{1}{12}\right) & 2\left(\frac{1}{12}\right)+4\left(\frac{1}{12}\right) & 2\left(\frac{1}{12}\right)+4\left(\frac{1}{12}\right) & 2\left(\frac{1}{12}\right)+4\left(\frac{1}{12}\right) & 0 & -6\left(\frac{1}{12^2}\right)\cdot 2\cdot 2 \end{bmatrix}$$

$$* \begin{Bmatrix} \theta_b \\ \theta_c \\ \theta_d \\ \theta_e \\ \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{40\cdot 6^2\cdot 4}{10^2} \\ -\frac{40\cdot 4^2\cdot 6}{10^2} \\ 0 \\ -8*30 \\ -12*10 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \theta_b \\ \theta_c \\ \theta_d \\ \theta_e \\ \Delta_1 \\ \Delta_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} 0.1035 \\ 0.1672 \\ -0.0223 \\ 0.1341 \\ 1.1152 \\ 1.8675 \end{Bmatrix} * 10^3$$

-ve due to 40kN load

If you remove 40kN load, ie circled FEM's then all displs come +ve (ie θ_d also +ve) where we used

$$M_{ba}+M_{be}+M_{bc}=0 \;;\; M_{cb}+M_{cd}=0 \;;\; M_{dc}+M_{de}=0 \;;$$

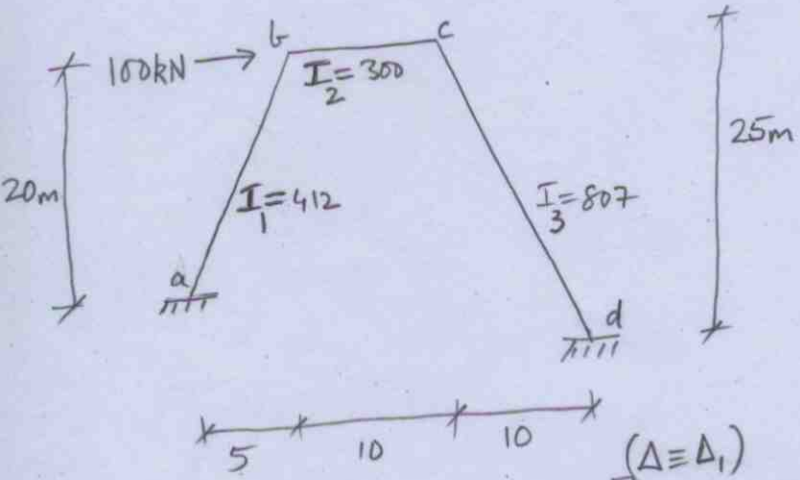
$$M_{ed}+M_{eb}+M_{ef}=0$$

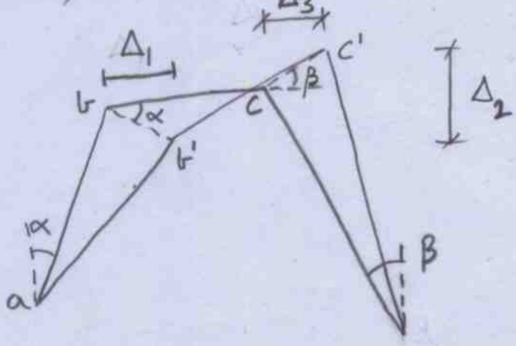
$$V_{ab}+V_{fe}-30=0=-\frac{(M_{ab}+M_{ba})}{8}-\frac{(M_{fe}+M_{ef})}{8}-30$$

$$V_{bc}+V_{ed}-10=0=-\frac{(M_{bc}+M_{cb})}{12}-\frac{(M_{ed}+M_{de})}{12}-10$$

(Ex 5)

$\theta_b,\ \theta_c,\ \Delta$ (3-dof)





$$\psi_{ab}=\psi_{ba}=\frac{bb'}{\sqrt{425}}=\frac{\Delta_1}{\cos\alpha}\cdot\frac{1}{\sqrt{425}}=\Delta\frac{\sqrt{425}}{20}\frac{1}{\sqrt{425}}=\frac{\Delta}{20} \quad (\Delta\equiv\Delta_1)$$

see below

$$\psi_{bc}=\psi_{cb}=-\frac{\Delta_2}{10}=\frac{-1}{10}(\Delta_1\tan\alpha+\Delta_3\tan\beta)=-\Delta\left(\frac{5}{20}+\frac{10}{25}\right)\cdot\frac{1}{10}=\frac{-13}{200}\Delta$$

$$\psi_{cd}=\psi_{dc}=\frac{cc'}{\sqrt{725}}=\frac{\Delta_3}{\cos\beta}\cdot\frac{1}{\sqrt{725}}=\Delta\cdot\frac{\sqrt{725}}{25}\cdot\frac{1}{\sqrt{725}}=\frac{\Delta}{25}$$

$$\Delta_3=\sqrt{(b'c')^2-\Delta_2^2}+\Delta_1-bc\ ,\quad \text{note } b'c'=bc \text{ (no axial ext)}$$

$$=bc\sqrt{1-\left(\frac{\Delta_2}{bc}\right)^2}+\Delta_1-bc=bc\left[1-\frac{1}{2}\underbrace{\left(\frac{\Delta_2}{bc}\right)^2}_{\text{H.O.T (higher order term) quadratic in }\Delta\text{'s.}}\right]+\Delta_1-bc$$

$$\simeq\Delta_1$$

Equilibrium: $M_{ba}+M_{bc}=0 \ \text{(i)}\;;\; M_{cb}+M_{cd}=0 \ \text{(ii)} \rightarrow$ 2 rotational equil. eqns for rot. d.o.f's.

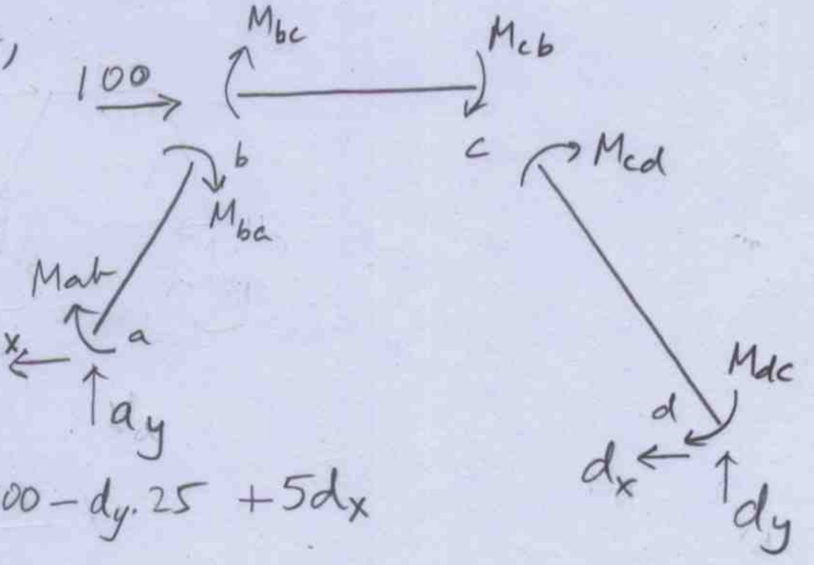
3rd eqn must involve forces for translational d.o.f. Δ.

(see later)

$$\begin{array}{l}\text{(i)}\\ \\ \text{(ii)}\\ \\ \text{(iii)*}\end{array} E\begin{bmatrix} 4\left(\frac{412}{\sqrt{425}}+\frac{300}{10}\right) & 2\left(\frac{300}{10}\right) & 6\left(-\frac{1}{20}\cdot\frac{412}{\sqrt{425}}+\frac{13}{200}\cdot\frac{300}{10}\right) \\ 2\left(\frac{300}{10}\right) & 4\left(\frac{300}{10}+\frac{807}{\sqrt{725}}\right) & 6\left(\frac{13}{200}\cdot\frac{300}{10}-\frac{1}{25}\cdot\frac{807}{\sqrt{725}}\right) \\ (+2*0.7692+4*1.7692)\cdot\frac{412}{\sqrt{425}} & (2*0.6154+4*1.6154)*\frac{807}{\sqrt{725}} & -6\cdot\frac{1}{20}\cdot\frac{412}{\sqrt{425}}(0.7692+1.7692) -6\cdot\frac{1}{25}\cdot\frac{807}{\sqrt{725}}(0.6154+1.6154)\end{bmatrix}\begin{Bmatrix}\theta_b\\ \theta_c\\ \Delta\end{Bmatrix} =$$

$$E\begin{bmatrix} 199.9397 & 60 & 5.7045 \\ 60 & 239.8849 & 4.5069 \\ 172.1742 & 230.5507 & -31.2653 \end{bmatrix} \qquad \begin{Bmatrix} 0 \\ 0 \\ -1538.4615 \end{Bmatrix} \Rightarrow E\begin{Bmatrix}\theta_b\\ \theta_c\\ \Delta\end{Bmatrix} = \begin{Bmatrix}-0.9908\\ -0.5043\\ 40.0318\end{Bmatrix}$$

where 1st, 2nd row correspond to rot equil eqns (i), (ii). Third equil eqn can be, for example,



$a_x + d_x = 100$ — (iii)

$\sum_{\text{for ab}} M_b = 0 = (M_{ab}+M_{ba}) + 20a_x + 5a_y$

$\sum_{\text{for cd}} M_c = 0 = (M_{cd}+M_{dc}) + 25d_x - 10d_y$

$\sum_{\text{whole struct}} M_a = 0 = M_{ab}+M_{dc}+20*100 - d_y.25 + 5d_x$

$a_y = -d_y$

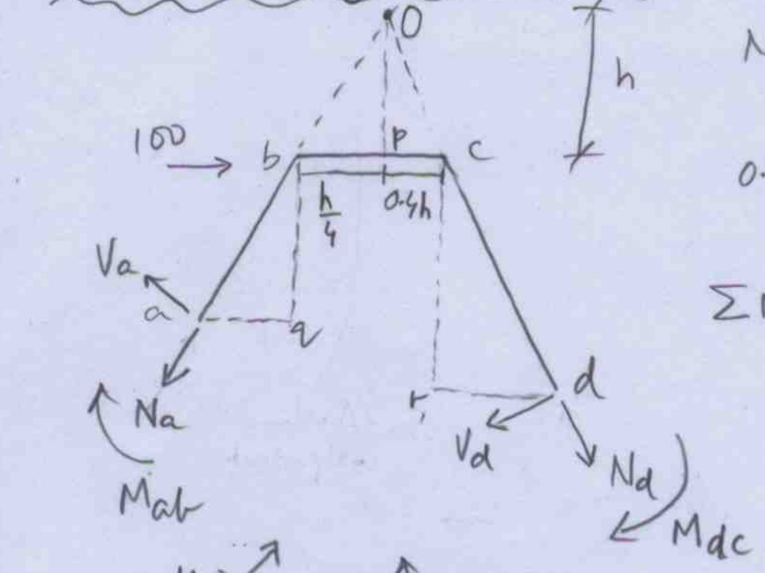
$$\Rightarrow d_x = \frac{-(M_{cd}+M_{dc})}{25} + \frac{10}{25}\left(\frac{M_{ab}+M_{dc}+2000}{25}+\frac{d_x}{5}\right)$$

$$a_x = \frac{-(M_{ab}+M_{ba})}{20} + \frac{5}{20}\left(\frac{M_{ab}+M_{dc}+2000}{25}+\frac{d_x}{5}\right)$$

$$d_x = \frac{-(M_{cd}+M_{dc})}{23}+\frac{10}{23}\left(\frac{M_{ab}+M_{dc}+2000}{25}\right)$$

(iii) $\rightarrow$ $100 = -\frac{1}{46}M_{ab} - \frac{1}{20}M_{ba} - \frac{2}{115}M_{dc} - \frac{21}{460}M_{cd} + \frac{1300}{23}$ $\rightarrow$ too much effort to gether!!

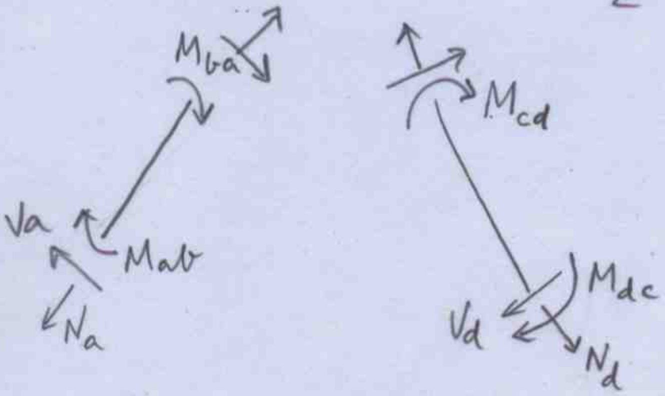
Easier way instead of (iii) is,



Note Δopb and Δbqa are similar
Δopc and Δcrd are similar

$$0.65h = 10 \Rightarrow h = \frac{10}{0.65}$$

$$\Sigma M_o = 0 = V_a\sqrt{(h+20)^2 + (0.25h+5)^2} + V_d\sqrt{(h+25)^2 + (0.4h+10)^2} + M_{ab} + M_{dc} - 100h \rightarrow \text{(iii)}^*$$



$$\Longrightarrow V_a = -\frac{(M_{ab} + M_{ba})}{\sqrt{425}}$$

$$V_d = -\frac{(M_{cd} + M_{dc})}{\sqrt{725}}$$

$$\text{(iii)}^* \rightarrow -1.7692\,(M_{ab} + M_{ba}) - 1.6154\,(M_{cd} + M_{dc}) + M_{ab} + M_{dc} - 1538.4615 = 0$$

$$-0.7692\,M_{ab} - 1.7692\,M_{ba} - 0.6154\,M_{dc} - 1.6154\,M_{cd} - 1538.4615 = 0 \rightarrow \text{(iii)}^*$$

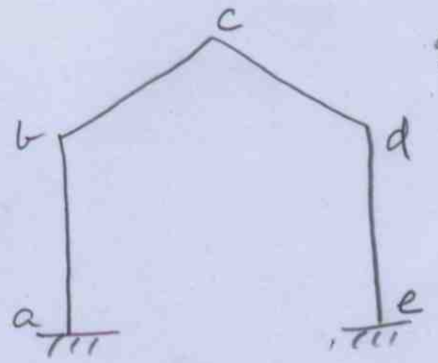
The matrix eqn shown on prev. pg is when we use (iii)*
If we use (iii), last row of matrix eqn becomes,

$$E\left[\left(-\frac{1}{46}\cdot 2 - \frac{1}{20}\cdot 4\right)\left(\frac{412}{\sqrt{425}}\right) \quad \left(-\frac{2}{115}\cdot 2 - \frac{21}{460}\cdot 4\right)\left(\frac{807}{\sqrt{725}}\right) \quad -6\left(\frac{1}{20}\cdot\frac{412}{\sqrt{425}}\cdot\left[-\frac{1}{46} - \frac{1}{20}\right] + \frac{1}{25}\,\frac{807}{\sqrt{725}}\left[-\frac{2}{115} - \frac{21}{460}\right]\right)\right]\begin{Bmatrix}\theta_b \\ \theta_c \\ \Delta\end{Bmatrix}$$

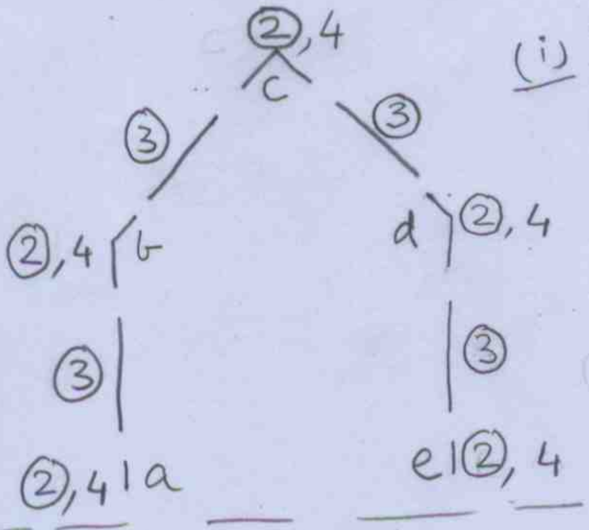
$$= \left\{100 - \frac{1300}{23}\right\}$$

This gives soln, $E\begin{Bmatrix}\theta_b \\ \theta_c \\ \Delta\end{Bmatrix} = \begin{Bmatrix}-0.9908 \\ -0.5043 \\ 40.0316\end{Bmatrix} \rightarrow$ same as when using (iii)*

Obtaining sway (equilibrium) equations — systematic approach:



5 dof (3 rot., 2 sway)



(i) Joints a, b, c, d, e shown with no. of unknowns shown, without encircling, against each joint. Note that only force unknowns (ie, axial, shear or x, y forces) are being accounted for, since BM unknowns are assumed known thru S-D eqns in terms of the displacements. Each joint has two force equilibrium equations (shown as ②). Note that moment equilibrium at a jt. (e.g $M_{cd} + M_{cb} = 0$) is already accounted in S-D method, so not accounted in this discussion.

(ii) Each member has three force/moment equilibrium equations (shown as ③).

Thus; nos of equations of equilibrium $= e = 5 * ② + 4 \times ③ = 22$

nos of unknowns in above FBD's (note again that BM's assumed known in terms of displacements thru S-D equations)

$= u = 5 * 4 = 20$

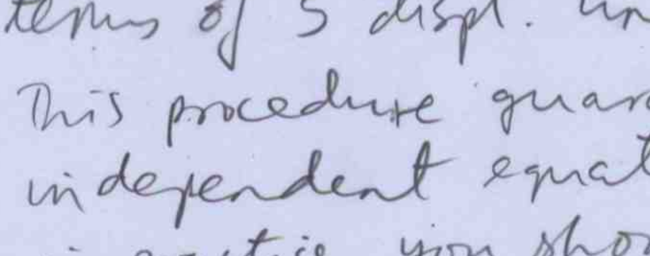
$\Rightarrow 22 - 20 = 2 \rightarrow$ ie these are the required sway equations.

The above 22 equations contain 20 force unknowns and all the BM's assumed known. So solve for these 20 force unknowns in terms of the BM's, using 20 equations, & put the result back in the remaining 2 equations. These 2 equations are now in terms of

BM's, and they represent the two additional equations (ie the sway equations) required. The remaining three equations are the jt. momt. equilibrium equations, as usual. Inserting the BM's from S-D equations in these 2(sway) + 3(jt. momt equil) eqns, we get the 5 equations in terms of 5 displ. unknowns.

This procedure guarantees that we get independent equations of equilibrium. However in practice you should try and exhaust the external equilibrium equations & then use internal equilibrium equations, depending on how many sway equations you require.

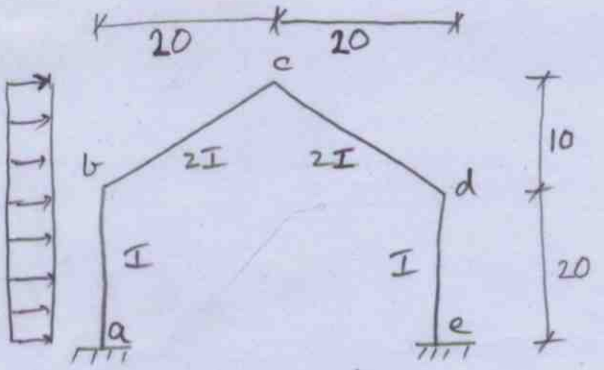
Another example:



3 sway d.o.f's. + 8 rot. d.o.f's.
Require 3 sway eqns.
Force unknowns shown at jts.
$u = 43$.
$e = 8 * 2 + 10 * 3 = 46$.

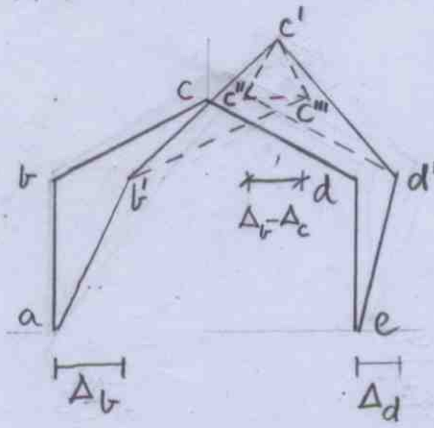
So $46 - 43 = 3 \rightarrow$ these are the "required" sway equations
↳ (ie one possible set of sway eqns)

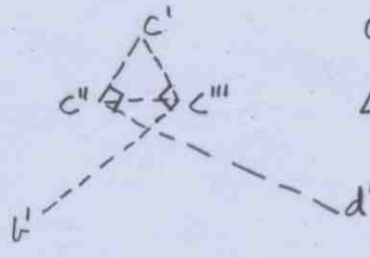
(Ex6)



2 sway d.o.f's.

Total d.o.f's : $\theta_b, \theta_c, \theta_d, \Delta_b, \Delta_d$.





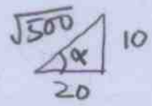
$cc''' = \Delta_b, \; cc'' = \Delta_c$

$\angle c'c''c''' = \angle c'c'''c''$

$= \frac{\pi}{2} - \angle c'''c''d'$

$= \frac{\pi}{2} - \tan^{-1}\frac{10}{20}$

$c'c'' = c'c''' = \frac{c''c'''}{2} \cdot \frac{1}{\cos(\angle c'c''c''')}$



$= \left(\frac{\Delta_b - \Delta_c}{2}\right)\frac{1}{\sin\alpha} = \frac{\sqrt{5}}{2}(\Delta_b - \Delta_c)$

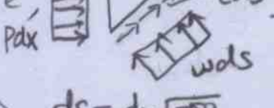
$\psi_{ba} = \frac{\Delta_b}{20}, \; \psi_{bc} = -\frac{\sqrt{5}}{2}(\Delta_b - \Delta_c)\cdot\frac{1}{\sqrt{500}} = -\psi_{cd}, \; \psi_{de} = \frac{\Delta_d}{20}$

$$EI\begin{bmatrix} 4\left(\frac{1}{20}+\frac{2}{\sqrt{500}}\right) & 2\left(\frac{2}{\sqrt{500}}\right) & 0 & -6\left(\frac{1}{20^2}-\frac{\sqrt{5}}{2}\cdot\frac{2}{500}\right) & -6\left(\frac{\sqrt{5}}{2}\cdot\frac{2}{500}\right) \\ 2\left(\frac{2}{\sqrt{500}}\right) & 4\left(\frac{2}{\sqrt{500}}\cdot 2\right) & 2\left(\frac{2}{\sqrt{500}}\right) & 0 & 0 \\ 0 & 2\left(\frac{2}{\sqrt{500}}\right) & 4\left(\frac{1}{20}+\frac{2}{\sqrt{500}}\right) & -6\left(\frac{\sqrt{5}}{2}\cdot\frac{2}{500}\right) & -6\left(\frac{1}{20^2}-\frac{\sqrt{5}}{2}\frac{2}{500}\right) \\ (2+4)\left(\frac{1}{20}\right) & 0 & (2+4)\left(\frac{1}{20}\right) & -6\left(\frac{1}{20^2}\right)\cdot 2 & -6\left(\frac{1}{20^2}\right)\cdot 2 \\ (2\cdot 2+4\cdot 3)\left(\frac{1}{20}\right) + 2\cdot\left(\frac{2}{\sqrt{500}}\right)\cdot[-2] & 4\left(\frac{2}{\sqrt{500}}\right)\cdot[-2] & 2\left(\frac{1}{20}\right) & -6\cdot\left(\frac{1}{20^2}[2+3] - \frac{\sqrt{5}}{2}\cdot\frac{2}{500}\cdot[-2]\right) & -6\cdot\left(\frac{1}{20^2}+\frac{\sqrt{5}}{2}\cdot\frac{2}{500}[-2]\right) \end{bmatrix} *$$

$$\begin{Bmatrix} \theta_b \\ \theta_c \\ \theta_d \\ \Delta_b \\ \Delta_c \end{Bmatrix} = \begin{Bmatrix} -\frac{0.6(20)^2}{12} + \underbrace{\frac{0.6}{5}\frac{(\sqrt{500})^2}{12}}; \\ -\frac{0.6}{5}\frac{(\sqrt{500})^2}{12}; \\ 0; \\ -20(0.6)(30) + 0.6\frac{(20)^2}{2}; \\ -(0.6)(30)^2\frac{3}{2} - [3-2]\cdot(0.6)\frac{(20)^2}{12} - [-2]\cdot\frac{(0.6)}{5}\frac{(\sqrt{500})^2}{12} + 0.6\frac{(20)^2}{2}\cdot 3 \end{Bmatrix}$$

due to load on inclined bc

ie,



$\Rightarrow pdx\frac{10}{\sqrt{500}} = wds$

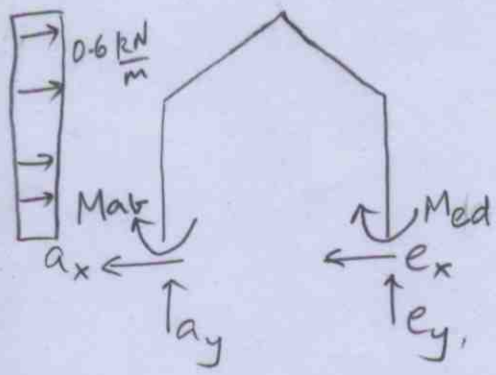
$= wdx\frac{\sqrt{500}}{10}$

$ds = dx\frac{\sqrt{500}}{10};$

$\Rightarrow w = p/5$

where we use $M_{ba} + M_{bc} = 0 \to (i)$; $M_{cb} + M_{cd} = 0 \to (ii)$; $M_{dc} + M_{de} = 0 \to (iii)$

and also 2 equil equations for the 2 sway d.o.f's.



$a_x + e_x = 0.6(30) \to (iv)$

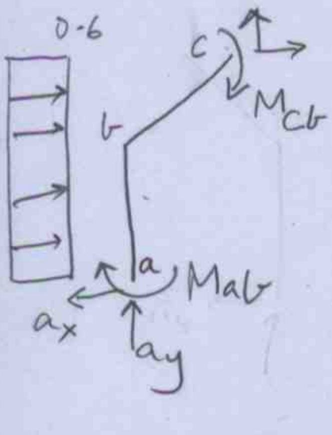
$a_y = -e_y \to (v)$

$a_x = -\left(\dfrac{M_{ab} + M_{ba} - 0.6(20)^2/2}{20}\right) \to (I)$; $e_x = -\left(\dfrac{M_{ed} + M_{de}}{20}\right) \to (II)$

considering legs ab, de and Σ moments appropriately.

$$\Sigma M_a = 0 \Rightarrow e_y = \left(M_{ab} + M_{ed} + \frac{0.6(30)^2}{2}\right)\Big/40 \to (a)$$

Note $\Sigma M_e = 0$ for a_y won't work $\because$ it gives identity when you subst in (v) — obvious $\because$ (iv), (v), (a) are 3 eqns of ext equil, so any other eqn of ext equil will be dependent on them.



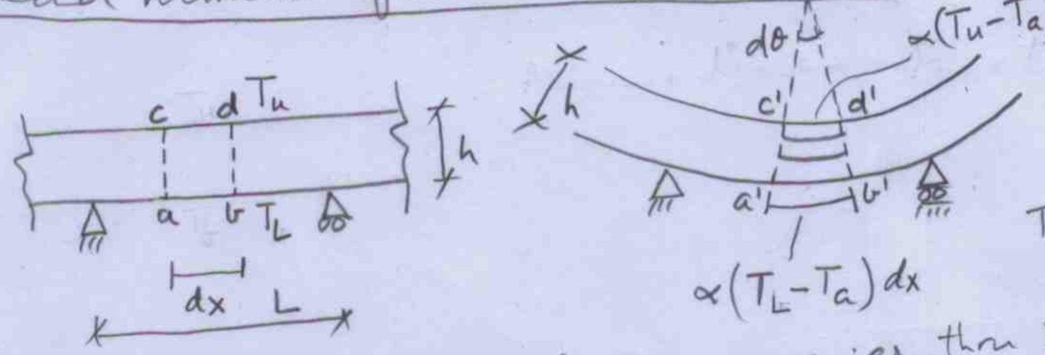
$$\Sigma M_c = 0 \Rightarrow a_y = -\frac{\left(M_{ab} + M_{cb} + a_x \cdot 30 - \frac{0.6(30)^2}{2}\right)}{20}$$

$$(iv), (I), (II) \Rightarrow -(M_{ab} + M_{ba} + M_{de} + M_{ed}) = 20(0.6)(30) - \frac{0.6(20)^2}{2}$$

$$(v), (a), (b) \Rightarrow M_{ab} + M_{ed} - 2M_{ab} - 2M_{cb} + \frac{(M_{ab} + M_{ba} - 0.6(20)^2/2)}{20}.2.30 + 0.6(30)^2 + \frac{0.6(30)^2}{2} = 0$$

$$EI\begin{Bmatrix} \theta_b \\ \theta_c \\ \theta_d \\ \Delta_b \\ \Delta_c \end{Bmatrix} = \begin{Bmatrix} 0.1190 \\ -0.0914 \\ 0.2187 \\ 6.2140 \\ 5.1630 \end{Bmatrix} * 10^3$$

Fixed end moments for thermal load.



T_a = ambient temp

So we have assumed that: (i) Temp varies thru thickness only (ii) Variation is linear (note the fact that a'c' and b'd' are straight lines implies this assumption).

$$d\theta = \frac{a'b' - c'd'}{h} = \frac{\alpha(T_L - T_u)\,dx}{h}$$

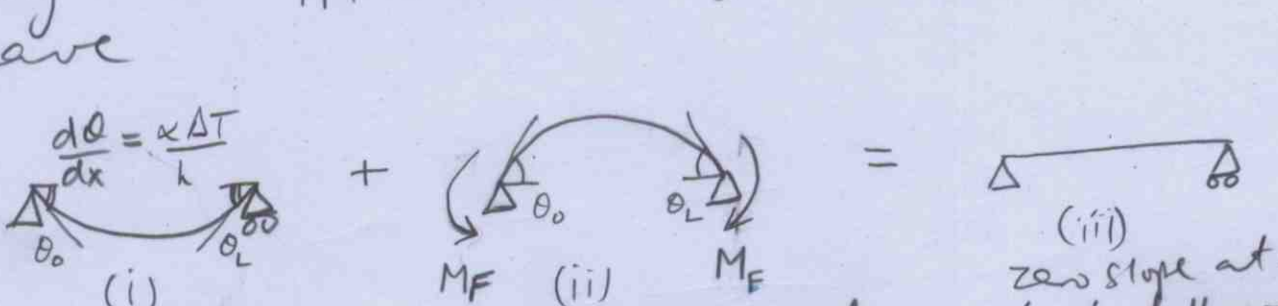
$$\Rightarrow \frac{d\theta}{dx} = \frac{\alpha \Delta T}{h} \longrightarrow \text{this is a constant so } \frac{d\theta}{dx} = w'' = \text{const}$$

ie beam bends into circular arc.

$$\theta = \frac{\alpha\Delta T}{h}x + c_1\,,\quad \theta\left(\tfrac{L}{2}\right) = 0 \Rightarrow c_1 = -\frac{\alpha\Delta T}{h}\frac{L}{2}$$

$$\theta(0) = -\frac{\alpha\Delta T L}{2h}\,,\quad \theta(L) = \frac{\alpha\Delta T L}{2h}$$

Now FEM is the moment reqd at each end to cause zero slope at the ends in a beam that has been deformed into a circular arc by thermal loads. Note that beam subject to ΔT carries no stresses since the simple supports at the two ends cause no BM to be induced when only ΔT is applied. Superposing ΔT and FEM (M_F), we have



(i) ΔT applied

(ii) FEM (M_F) applied

(iii) zero slope at ends (in fact thruout, $\because$ M_F provides curvature opp to ΔT)

<u>Note</u>: from here itself you see that $\because$ M_F provides constant curvature w'' wrt x, it should provide curvature equal in magnitude to $d\theta/dx$
$\Rightarrow M_F = EI\, d\theta/dx = EI\,\alpha\Delta T/h$.

For M_F applied, BM is M_F thruout.

$$\Rightarrow -M_F = EIw''$$

$$-M_F x = EIw' + c_1 \; ; \; -M_F \frac{x^2}{2} = EIw + c_1 x + c_2$$

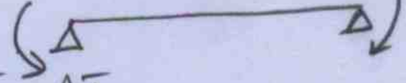
$$w(0) = 0 \Rightarrow c_2 = 0$$

$$w'(0) = \theta_0 = \alpha \frac{\Delta T L}{2h} \Rightarrow c_1 = -EI\alpha \frac{\Delta T L}{2h}$$

$$w'(L) = -\theta_L = -\alpha \frac{\Delta T L}{2h} \Rightarrow -M_F L = -EI\alpha\frac{\Delta T L}{2h} - EI\alpha\frac{\Delta T L}{2h}$$

$$\Rightarrow M_F = +EI\alpha\frac{\Delta T}{h}$$ → can get it by observation, see bot. of prev. pg.

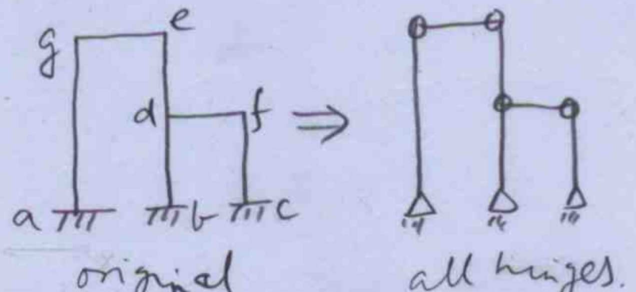
So FEM is,



$$-M_{Fn} = EI\alpha\frac{\Delta T}{h} \qquad EI\alpha\frac{\Delta T}{h} = M_{Ff}$$

(as per FEM sign convention.)

Ex 7

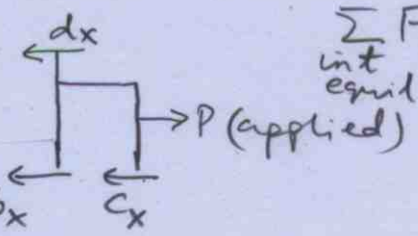


2 sway d.o.f's.

1^{st} sway eqn → $\sum_{\text{ext equil}} F_x = 0 = \underbrace{a_x + b_x + c_x}$ + applied horz load

get in terms of BM's from momt equil of ag, bd, cf, in standard manner.

2^{nd} sway eqn →



$\sum_{\text{int equil}} F_x = 0 = P - \underbrace{b_x - c_x - d_x}$

get all these in terms of BM's from momt equil of bd, cf, de, in standard manner.