STIFFNESS METHOD — 1

Member coordinate system (local)

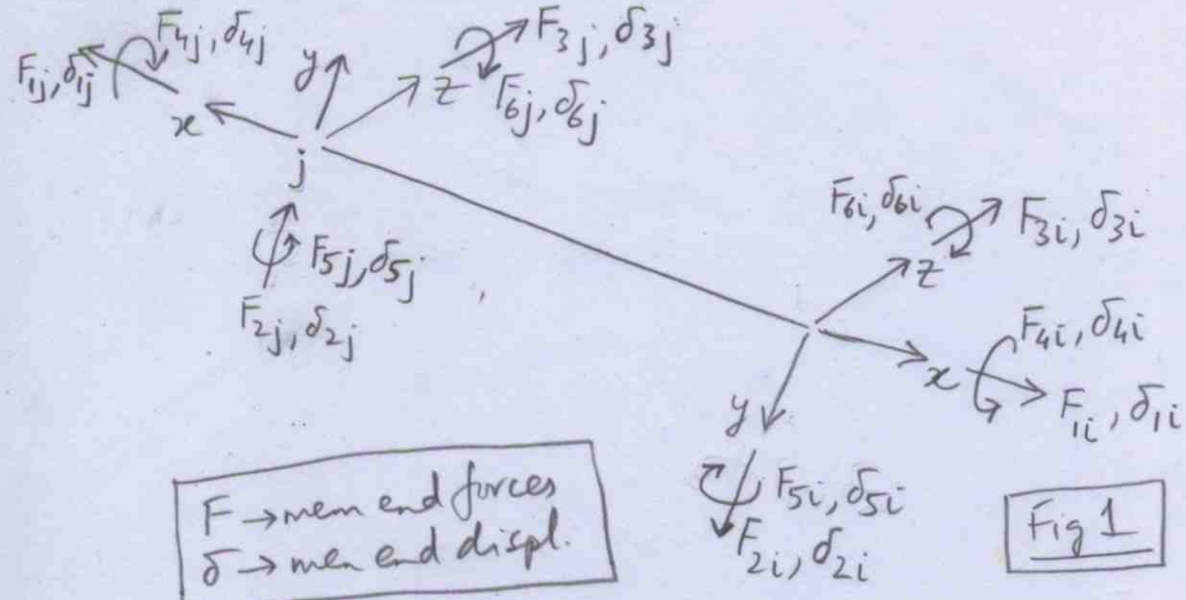


Fig 1

$x \to$ in dir. of tension

$y \to$ in dir. of positive shear for xy plane bending.

$z \to x \times y = z$ (right handed).

F_{4q}, δ_{4q} denote torsion ($q = i$ or j)

F_{6q}, δ_{6q} denote xy plane bending

F_{2q}, δ_{2q} denote xy plane shear

F_{5q}, δ_{5q} denote xz plane bending

F_{3q}, δ_{3q} denote xz plane shear.

Structure coordinate system. (global)

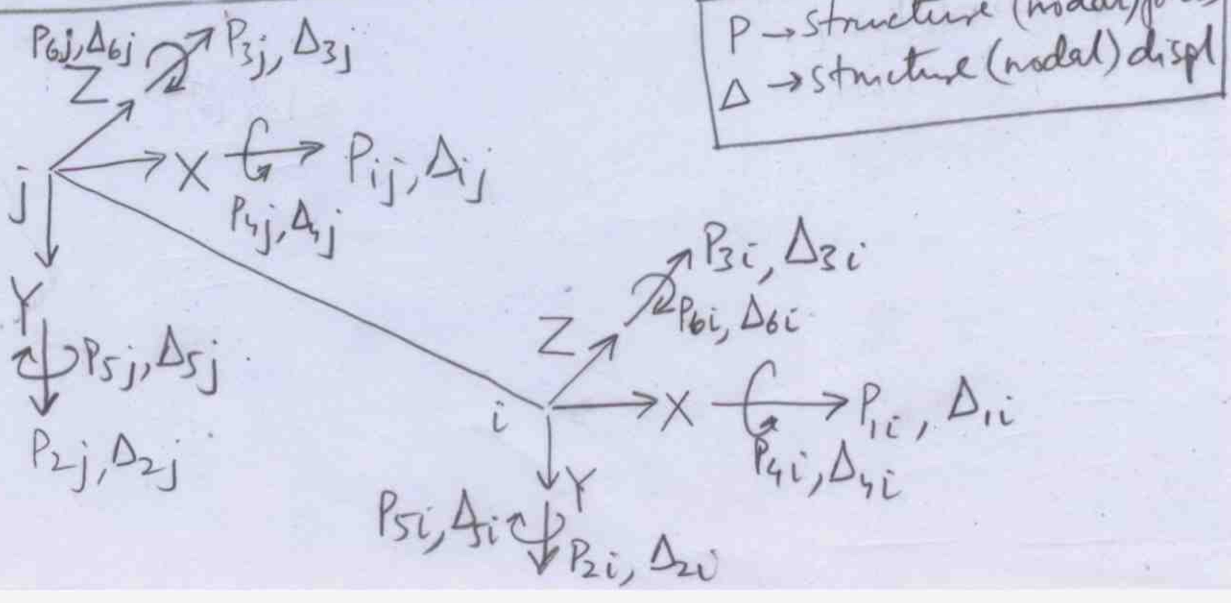


Fig 2

So member forces & displacements are

$\{F\}_{ij} = \{F_1 \; F_2 \; F_3 \; F_4 \; F_5 \; F_6\}^T_{ij}$ → forces at i end of mem ij

$\{\delta\}_{ij} = \{\Delta_1 \; \Delta_2 \; \Delta_3 \; \Delta_4 \; \Delta_5 \; \Delta_6\}^T_{ij}$ → displacements at i end of mem ij

Likewise $\{F\}_{ji}$ & $\{\delta\}_{ji}$ denote member forces and member displacements at j end of member ij

The joint (nodal, structure) forces and displs are

$\{P\}_i = \{P_1 \; P_2 \; P_3 \; P_4 \; P_5 \; P_6\}^T$ → forces at node i

$\{\Delta\}_i = \{\Delta_1 \; \Delta_2 \; \Delta_3 \; \Delta_4 \; \Delta_5 \; \Delta_6\}^T$ → displs at node i

Member force/displs measured in local xyz system, joint (nodal, structure) forces/displs measured in XYZ global system.

Flexibility Matrix

Consider the member constrained as shown, to prevent rigid body displacements.

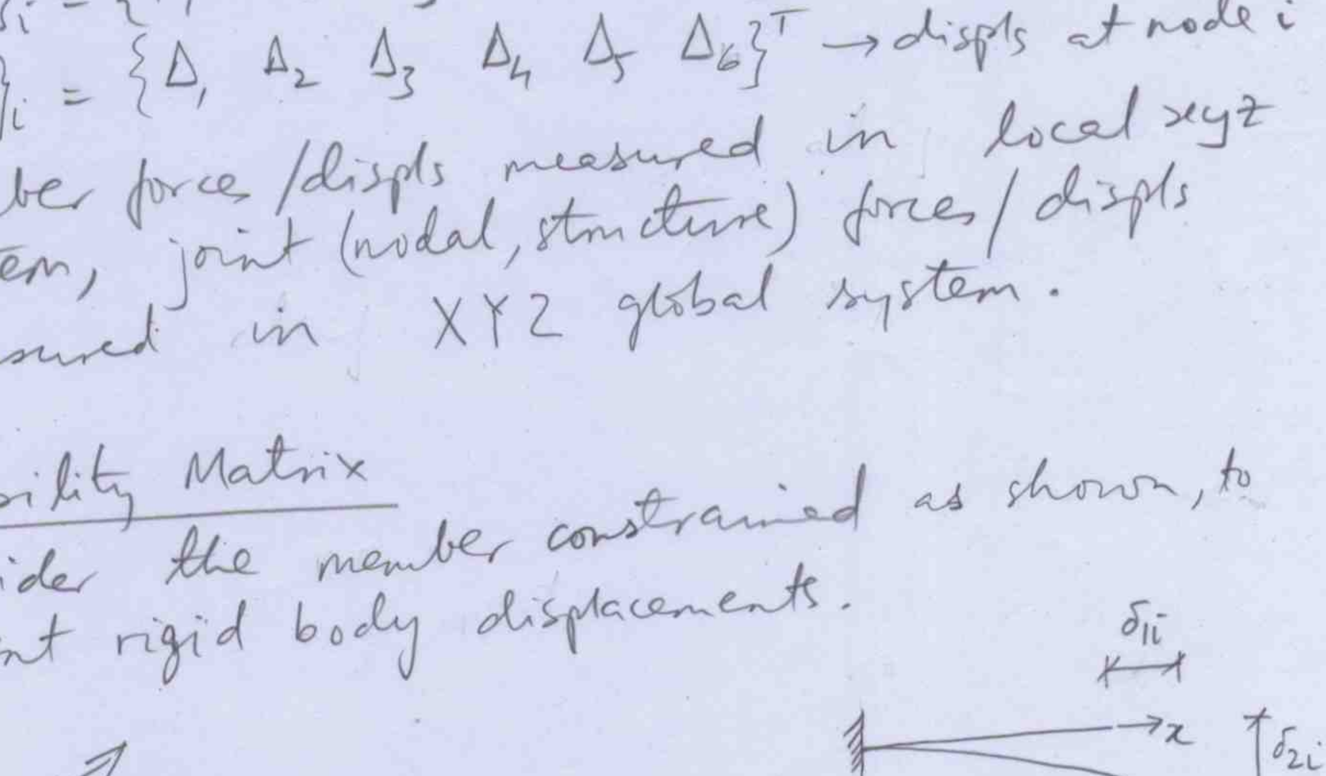


Fig 3

Use Castigliano's second theorem to find flexibility coefficients.

$$U = \frac{1}{2}\int_0^L \left(\frac{F_{1i}^2}{EA} + \frac{F_{4i}^2}{GJ_x} + \frac{M_y^2}{EI_y} + \frac{M_z^2}{EI_z} + \frac{\gamma_y F_{2i}^2}{GA} + \frac{\gamma_z F_{3i}^2}{GA} \right) dx$$

$$M_y = F_{5i} - F_{3i}x \ , \quad M_z = F_{6i} + F_{2i}x$$

$$\delta_{1i} = \frac{\partial U}{\partial F_{1i}} = \frac{F_{1i}L}{EA} \ ; \quad \delta_{4i} = \frac{F_{4i}L}{GJ_x}$$

$$\delta_{2i} = \frac{\partial U}{\partial F_{2i}} = \frac{1}{2}\int_0^L \left(\frac{(F_{6i}+F_{2i}x)^2}{EI_z} + \frac{\gamma_y F_{2i}^2}{GA} \right) dx$$

$$= F_{6i}\frac{L^2}{2EI_z} + F_{2i}\frac{L^3}{3EI_z} + F_{2i}\frac{\gamma_y L}{GA}$$

$$\delta_{6i} = \frac{\partial U}{\partial F_{6i}} = F_{6i}\frac{L}{EI_z} + F_{2i}\frac{L^2}{2EI_z}$$

$$\delta_{5i} = \frac{\partial U}{\partial F_{5i}} = F_{5i}\frac{L}{EI_y} - F_{3i}\frac{L^2}{2EI_y}$$

$$\delta_{3i} = \frac{\partial U}{\partial F_{3i}} = -F_{5i}\frac{L^2}{2EI_y} + F_{3i}\frac{L^3}{3EI_y} + F_{3i}\frac{\gamma_z L}{GA}$$

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3EI_z}+\frac{\gamma_y L}{GA} & 0 & 0 & 0 & \frac{L^2}{2EI_z} \\ 0 & 0 & \frac{L^3}{3EI_y}+\frac{\gamma_z L}{GA} & 0 & -\frac{L^2}{2EI_y} & 0 \\ 0 & 0 & 0 & \frac{L}{GJ_x} & 0 & 0 \\ 0 & 0 & -\frac{L^2}{2EI_y} & 0 & \frac{L}{EI_y} & 0 \\ 0 & \frac{L^2}{2EI_z} & 0 & 0 & 0 & \frac{L}{EI_z} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{ij} \quad (1)$$

far end of member.

$$(1) \leftarrow \{\delta\}_{ij} = [d]^{j}_{ii} \{F\}_{ij}$$

end at which forces applied

far end of mem.

end at which measured

far end of mem

end at which displs are measured

end at which forces applied

Stiffness matrix.

Inverting ①,

$$F_{1i} = \frac{EA}{L}\delta_{1i} \quad ; \quad F_{4i} = \frac{GJ_x}{L}\delta_{4i} \; ;$$

$$\delta_{2i} - \frac{L}{2}\delta_{6i} = F_{2i}\left(\frac{L^3}{3EI_z} + \frac{\gamma_y L}{GA} - \frac{L^3}{4EI_z}\right).$$

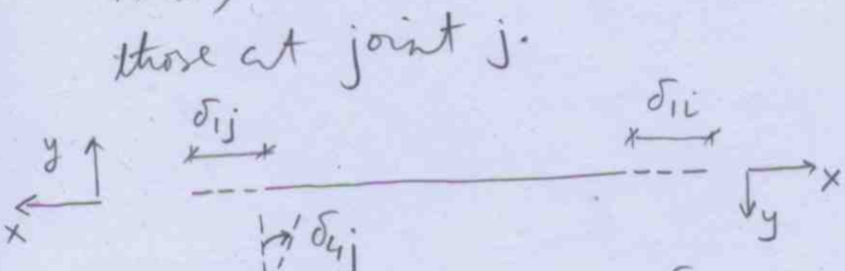
$$F_{2i} = \left(\frac{L^3}{12EI_z} + \frac{\gamma_y L}{GA}\right)^{-1}\left[\delta_{2i} - \frac{L}{2}\delta_{6i}\right] \quad , \text{ let } \frac{L^3}{12EI_z} + \frac{\gamma_y L}{GA} = R_z .$$

$$F_{6i} = \frac{EI_z}{L}\left[\delta_{6i} - \frac{L^2}{2EI_z}\left(\frac{L^3}{12EI_z} + \frac{\gamma_y L}{GA}\right)^{-1}\left[\delta_{2i} - \frac{L}{2}\delta_{6i}\right]\right]$$

$$F_{3i} = \left(\frac{L^3}{12EI_y} + \frac{\gamma_z L}{GA}\right)^{-1}\left[\delta_{3i} + \frac{L}{2}\delta_{5i}\right] \quad , \text{ let } \frac{L^3}{12EI_y} + \frac{\gamma_z L}{GA} = R_y$$

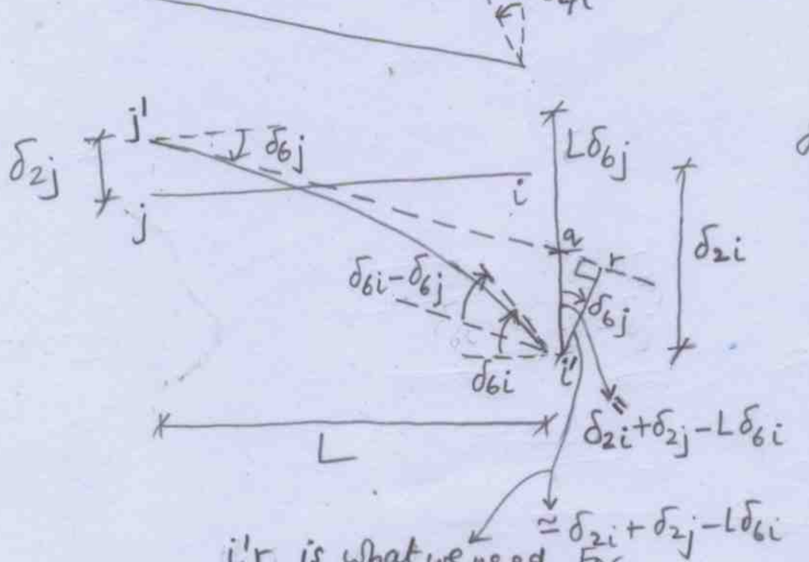
$$F_{5i} = \frac{EI_y}{L}\left[\delta_{5i} + \frac{L^2}{2EI_y}\left(\frac{L^3}{12EI_y} + \frac{\gamma_z L}{GA}\right)^{-1}\left[\delta_{3i} + \frac{L}{2}\delta_{5i}\right]\right]$$

Now permit d.o.f's at joint j, as in Fig 1. Hence, all displs at joint i will be relative to those at joint j.



$\delta_{1i} \rightarrow \delta_{1i} + \delta_{1j}$

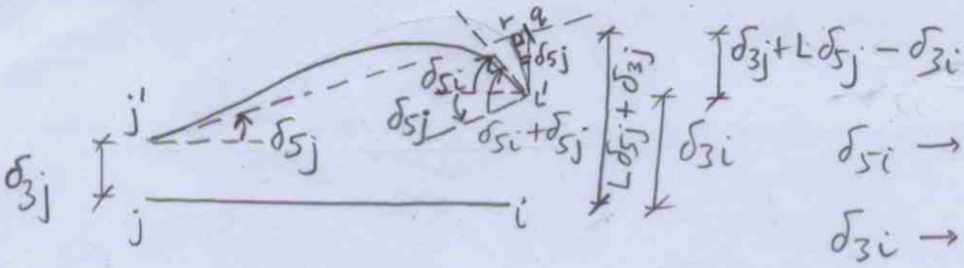
$\delta_{4i} \rightarrow \delta_{4i} + \delta_{4j}$



$\delta_{6i} \rightarrow \delta_{6i} - \delta_{6j}$

$\delta_{2i} \rightarrow \delta_{2i} + \delta_{2j} - L\delta_{6j}$

i.e, consider relative rotation at joint i and relative displ (in y-direction) w.r.t. tangent (at joint j) at joint i. We do this since only deformation component gives rise to mem end forces.

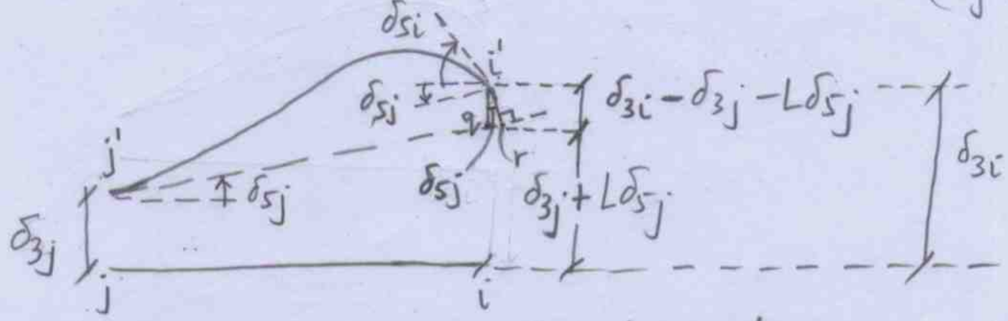


$$\delta_{5i} \to \delta_{5i} + \delta_{5j}$$

$$\delta_{3i} \to -\left(\delta_{3j} + L\delta_{5j} - \delta_{3i}\right)$$

(minus sign ∵ in opp dir of +ve convention for δ_{3i})

or



substitute relative displs into mem-end forces, and use equilibrium of member, get

$$F_{1i} = F_{1j} = \frac{EA}{L}\left(\delta_{1i} + \delta_{1j}\right) \quad ; \quad F_{4i} = F_{4j} = \frac{GJ_x}{L}\left(\delta_{4i} + \delta_{4j}\right)$$

$$F_{2i} = F_{2j} = k_z^{-1}\left[\delta_{2i} + \delta_{2j} - \frac{L}{2}\delta_{6i} - \frac{L}{2}\delta_{6j}\right]$$

$$F_{6i} = \frac{EI_z}{L}\left[-\frac{L^2}{2EI_z}k_z^{-1}\delta_{2i} - \frac{L^2}{2EI_z}k_z^{-1}\delta_{2j} + \left(1 + \frac{L^3}{4EI_z}k_z^{-1}\right)\delta_{6i} + \left(-1 + \frac{L^3}{4EI_z}k_z^{-1}\right)\delta_{6j}\right]$$

$$F_{3i} = -F_{3j} = k_y^{-1}\left[\delta_{3i} - \delta_{3j} + \frac{L}{2}\delta_{5i} - \frac{L}{2}\delta_{5j}\right]$$

$$F_{5i} = \frac{EI_y}{L}\left[\frac{L^2}{2EI_y}k_y^{-1}\delta_{3i} - \frac{L^2}{2EI_y}k_y^{-1}\delta_{3j} + \left(1 + \frac{L^3}{4EI_y}k_y^{-1}\right)\delta_{5i} + \left(1 - \frac{L^3}{4EI_y}k_y^{-1}\right)\delta_{5j}\right]$$

$$F_{6j} = -F_{6i} - F_{2i}L$$

$$F_{5j} = F_{5i} - F_{3i}L$$

$$
\begin{Bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{ij} \\ \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{ji} \end{Bmatrix}
=
\left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & k_z^{-1} & 0 & 0 & 0 & -\frac{L}{2}k_z^{-1} & 0 & k_z^{-1} & 0 & 0 & 0 & -\frac{L}{2}k_z^{-1} \\
0 & 0 & k_y^{-1} & 0 & \frac{L}{2}k_y^{-1} & 0 & 0 & 0 & -k_y^{-1} & 0 & -\frac{L}{2}k_y^{-1} & 0 \\
0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\
0 & 0 & \frac{L}{2}k_y^{-1} & 0 & \left(\frac{EI_y}{L}+\frac{L^2}{4}k_y^{-1}\right) & 0 & 0 & 0 & -\frac{L}{2}k_y^{-1} & 0 & \left(\frac{EI_y}{L}-\frac{L^2}{4}k_y^{-1}\right) & 0 \\
0 & -\frac{L}{2}k_z^{-1} & 0 & 0 & 0 & \left(\frac{EI_z}{L}+\frac{L^2}{4}k_z^{-1}\right) & 0 & -\frac{L}{2}k_z^{-1} & 0 & 0 & 0 & \left(-\frac{EI_z}{L}+\frac{L^2}{4}k_z^{-1}\right) \\
\hline
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & k_z^{-1} & 0 & 0 & 0 & -\frac{L}{2}k_z^{-1} & 0 & k_z^{-1} & 0 & 0 & 0 & -\frac{L}{2}k_z^{-1} \\
0 & 0 & -k_y^{-1} & 0 & -\frac{L}{2}k_y^{-1} & 0 & 0 & 0 & k_y^{-1} & 0 & \frac{L}{2}k_y^{-1} & 0 \\
0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\
0 & 0 & -\frac{L}{2}k_y^{-1} & 0 & \left(\frac{EI_y}{L}-\frac{L^2}{4}k_y^{-1}\right) & 0 & 0 & 0 & \frac{L}{2}k_y^{-1} & 0 & \left(\frac{EI_y}{L}+\frac{L^2}{4}k_y^{-1}\right) & 0 \\
0 & -\frac{L}{2}k_z^{-1} & 0 & 0 & 0 & \left(-\frac{EI_z}{L}+\frac{L^2}{4}k_z^{-1}\right) & 0 & -\frac{L}{2}k_z^{-1} & 0 & 0 & 0 & \left(\frac{EI_z}{L}+\frac{L^2}{4}k_z^{-1}\right)
\end{array}\right]
\begin{Bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}_{ij} \\ \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}_{ji} \end{Bmatrix}
$$

i.e, $\boxed{\underline{F}_{12\times 1} = \underline{\underline{k}}_{12\times 12}\, \underline{\delta}_{12\times 1}}$ $\longrightarrow$ 2

$\underline{F}$ = member end forces
$\underline{\delta}$ = member end displacements
$\underline{\underline{k}}$ = member stiffness matrix

in local (member) coordinate system

This member stiffness matrix for the rod (bar) ij includes axial, bending, and shear deformation effects.

Neglecting shear deformations, ie, $\gamma_y = \gamma_z = 0$,

$k_z = \dfrac{L^3}{12EI_z}$, $k_y = \dfrac{L^3}{12EI_y}$, get member stiffness matrix

$$[k] = \left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\
0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & -\frac{6EI_y}{L^2} & 0 & -\frac{2EI_y}{L} & 0 \\
0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\hline
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & -\frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right] \longrightarrow 3$$

The force deformation relation, in partitioned form, is

$$\left\{\begin{array}{c} \{F\}_{ij} \\ \hline \{F\}_{ji} \end{array}\right\} = \left[\begin{array}{c|c} [k]^{j}_{ii} & [k]_{ij} \\ \hline [k]_{ji} & [k]^{i}_{jj} \end{array}\right] \left\{\begin{array}{c} \{\delta\}_{ij} \\ \hline \{\delta_{ji}\} \end{array}\right\} \longrightarrow 4$$

* $[k]^{j}_{ii}$ and $[k]^{i}_{jj}$ are termed direct stiffness sub-matrices since they relate forces at one end with displ's at same end,

while $[k]_{ij}$ & $[k]_{ji}$ are termed cross stiffness submatrices since they relate forces at one end with displ's at the other end.

* It is evident that $[k]$ is singular (ie $[k]^{-1}$ does not exist, see row 1 = row 7, row 2 = row 8, etc). This means that given $\{\delta\}$ we can find $\{F\}$ but not vice-versa. The reason is that $\{\delta\}$ includes displ's due to straining as well as rigid body motion, and since we have allowed 6 kinematic d.o.f's at each end $(i \& j)$, rbm is possible which yields non-unique $\{\delta\}$ for given $\{F\}$, the non-uniqueness being upto the rbm.

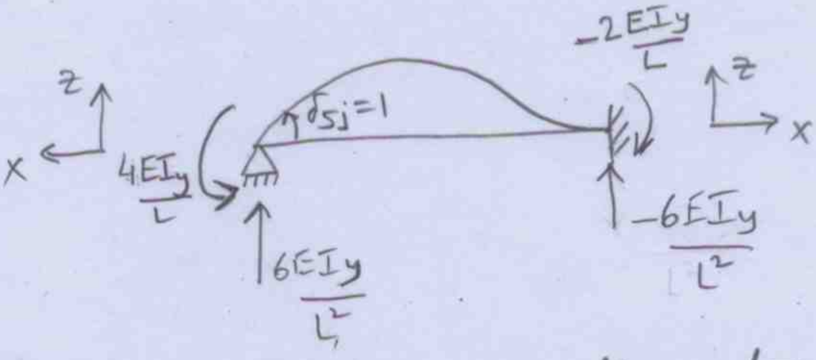
For example, if you restrain all d.o.f's at j, you get $\{F\}_{ij} = [k]^{j}_{ii}\{\delta\}_{ij}$ and $\{F\}_{ji} = [k]_{ji}\{\delta_{ij}\}$ where both $[k]^{j}_{ii}$ and $[k]_{ji}$ are invertible. In fact comparing with (1) you see that $\left[[k]^{j}_{ii}\right]^{-1} = [d]^{j}_{ii}$.

* Note that all sub-matrices shown in partitioned form (4) are invertible. In fact $\det\left([k]^{j}_{ii}\right) = \det\left([k]_{ij}\right) = \frac{EA}{L} \cdot \frac{GJ_x}{L} \cdot \frac{12(EI_z)^2}{L^4} \cdot \frac{12(EI_y)^2}{L^4}$.
Also $[k]^{j}_{ii} = [k]^{i}_{jj}$, $[k]_{ij} = [k]_{ji}$, as expected since ends i, j are interchangeable. Hence all submatrices have same det.

* $[k]$ is termed full stiffness, its invertible submatrices are termed reduced stiffnesses.

* From (2) with $[k]$ as in (3), 1^{st} & 7^{th} row are well known axial force – displ relation; 4^{th} & 10^{th} row are well known torsional moment – angular displ relations; 5^{th} & 11^{th} row are well known slope – deflection relations for xz plane bending; while 6^{th} & 12^{th} rows are slope – defl rel. for xy plane bending.

* For eg. if $\delta_{5j} = 1$ and all other displ's are zero, you get member end forces as the 11^{th} column of $[k]$.

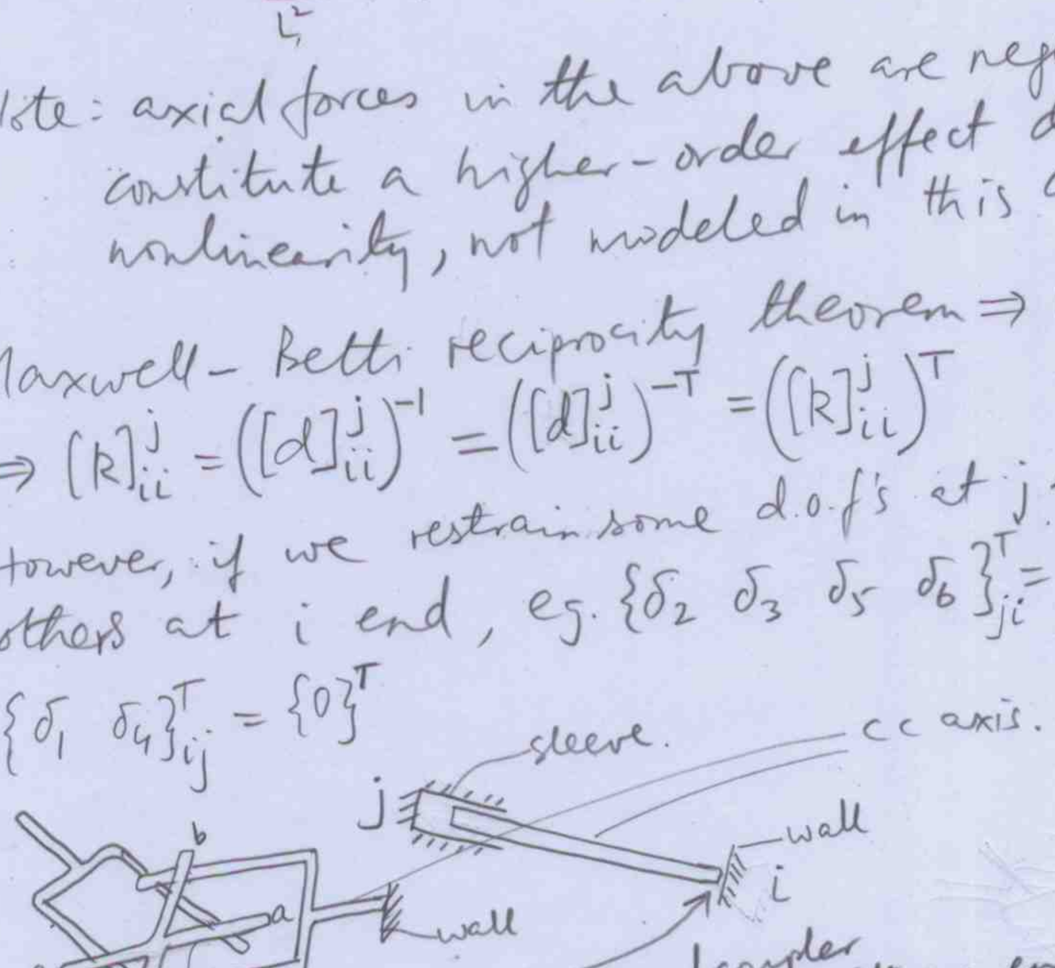


Note: axial forces in the above are negligible & constitute a higher-order effect due to nonlinearity, not modeled in this course.

* Maxwell-Betti reciprocity theorem $\Rightarrow [d]^j_{ii} = \left([d]^j_{ii}\right)^T$

$$\Rightarrow [k]^j_{ii} = \left([d]^j_{ii}\right)^{-1} = \left([d]^j_{ii}\right)^{-T} = \left([k]^j_{ii}\right)^T$$

However, if we restrain some d.o.f's at j end and others at i end, eg. $\{\delta_2\ \delta_3\ \delta_5\ \delta_6\}^T_{ji} = \{0\}^T$ and $\{\delta_1\ \delta_4\}^T_{ij} = \{0\}^T$



universal coupler at this end. with one end "fixed" in wall. Here by "fixed" we mean the end moves in a track in aa direction. This track can move along a perpendicular track in bb direction. Hence at end i translation and rotation along aa & bb axes is free, while at end j translation and rotation along member axis (cc) is free. The aa & bb axes lie in plane of wall.

Note: these are the minimum nos of restraints to render the rod stable, ie no rbm. In this case the structure is statically determinate. But we can have more restraints, in which case it is indeterminate.

The flexibility matrix for bar constrained in this manner would be invertible and yield a corresponding reduced stiffness matrix (6×6) whose elements appear in $[k]_{12\times12}$ when rows & columns corresponding to restrained d.o.f's are eliminated. Now, by similar arguements as for $[k]^j_{ii}$, this reduced stiffness matrix is also symmetric. Thus by changing the constrained d.o.f's we generate different reduced stiffness matrices, corresponding to certain rows and columns of $[k]$, which are symmetric.

Hence entire $[k]$ is symmetric.

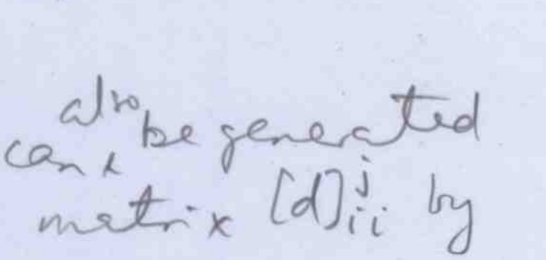
The member stiffness matrix $[k]$ can also be generated directly from the 6x6 flexibility matrix $[d]^j_{ii}$ by using (4), as follows. Let $\{\delta\}_{ji} = \{0\}$ Thus,

$$\{F\}_{ij} = [k]^j_{ii}\{\delta\}_{ij}$$

Comparing with (1) $\Rightarrow$ $[k]^j_{ii} = \left([d]^j_{ii}\right)^{-1} \longrightarrow$ (i)

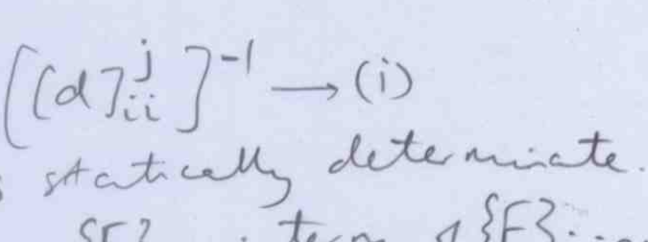
Now with $\{\delta\}_{ij} = \{0\}$ structure is statically determinate. Thus equilibrium equations give $\{F\}_{ji}$ in terms of $\{F\}_{ij}$ as,

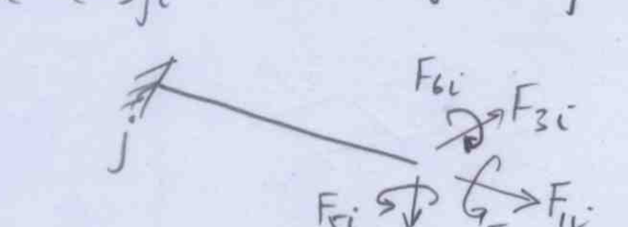
$$\{F\}_{ji} = \overset{(6\times6)}{[b]}\{F_{ij}\} \longrightarrow \text{(a)}.$$



where, $[b] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -L & 0 & 1 & 0 \\ 0 & -L & 0 & 0 & 0 & -1 \end{bmatrix}$

equilibrium matrix





From (4), $\{F\}_{ji} = [k]_{ji}\{\delta\}_{ij}$

From (a), $\{F\}_{ji} = [b]\{F\}_{ij} = [b][k]^j_{ii}\{\delta\}_{ij}$

$\Rightarrow [k]_{ji}\{\delta\}_{ij} = [b][k]^j_{ii}\{\delta\}_{ij}$

$\Rightarrow \left([k]_{ji} - [b][k]^j_{ii}\right)\{\delta\}_{ij} = 0 \longrightarrow$ this represents equil condn (a), which are independent equations.

Thus either $[k]_{ji} = [b][k]^{j}_{ii} = [b]\left[[d]^{j}_{ii}\right]^{-1} \rightarrow$ (ii)

<u>or</u> $\det\left([k]_{ji} - [b][k]^{j}_{ii}\right) = 0 \rightarrow$ not possible $\because$ the equil condts are 6 independent ones.

From symmetry of $[k]$ matrix, $[k]_{ij} = [k]^{T}_{ji}$ — (iii)

Now let $\{\delta\}_{ij} = 0$ instead of $\{\delta\}_{ji}$. Using equilibrium matrix,

$$\{F\}_{ji} = [b]\{F\}_{ij} = [b][k]_{ij}\{\delta\}_{ji} = [b][k]^{T}_{ji}\{\delta\}_{ji}$$

$$= [b]\left[[d_{ii}]^{j}\right]^{-T}[b]^{T}\{\delta\}_{ji} = [b]\left[[d]^{j}_{ii}\right]^{-1}[b]^{T}\{\delta\}_{ji}$$

$$\left(\because [d]^{j}_{ii} = \left([d]^{j}_{ii}\right)^{T}\right)$$

But $\{F\}_{ji} = [k]^{i}_{jj}\{\delta\}_{ji}$.

$$\Rightarrow \left([k]^{i}_{jj} - [b]\left[[d]^{j}_{ii}\right]^{-1}[b]^{T}\right)\{\delta\}_{ji} = 0$$

non singular $\because$ this represents linearly indep equilibrium conditions.

$$\Rightarrow [k]^{i}_{jj} = [b]\left[[d]^{j}_{ii}\right]^{-1}[b]^{T} \rightarrow \text{(iv)}$$

From (4),
$$[k] = \left[\begin{array}{c|c} \left[[d]^{j}_{ii}\right]^{-1} & \left[[d]^{j}_{ii}\right]^{-1}[b]^{T} \\ \hline [b]\left[[d]^{j}_{ii}\right]^{-1} & [b]\left[[d_{ii}]^{j}\right]^{-1}[b]^{T} \end{array}\right] \rightarrow \text{(V)}$$

can be used to generate member stiffness matrix for any statically determinate member.

(<u>Eg</u>) For 3-D bar/rod fixed at end j, from p. (4) you get

$$\left[[d]^{j}_{ii}\right]^{-1} = \begin{bmatrix} EA/L & 0 & 0 & 0 & 0 & 0 \\ 0 & k_z^{-1} & 0 & 0 & 0 & -\frac{L}{2}k_z^{-1} \\ 0 & 0 & k_y^{-1} & 0 & \frac{L}{2}k_y^{-1} & 0 \\ 0 & 0 & 0 & \frac{GJ_x}{L} & 0 & 0 \\ 0 & 0 & \frac{L}{2}k_y^{-1} & 0 & \frac{EI_y}{L}+\frac{L^2}{4}k_y^{-1} & 0 \\ 0 & -\frac{L}{2}k_z^{-1} & 0 & 0 & 0 & \frac{EI_z}{L}+\frac{L^2}{4}k_z^{-1} \end{bmatrix}$$

Using (v) we get $[k]_{12\times 2}$ as in (2), p. 6.

General procedure for Stiffness Method

We seek force displacement relation for the structure in the form

$$\{P\} = [K]\{\Delta\} \longrightarrow (5)$$

$\{P\}^T = \{\{P\}_1^T \cdots \{P\}_i^T \cdots \{P\}_n^T\}^T$ = structure (~~nodal~~ joint) forces in XYZ (global) coords.

$\{\Delta\}^T = \{\{\Delta\}_1^T \cdots \{\Delta\}_i^T \cdots \{\Delta\}_n^T\}^T$ = structure (nodal) displs in XYZ (global) coords.

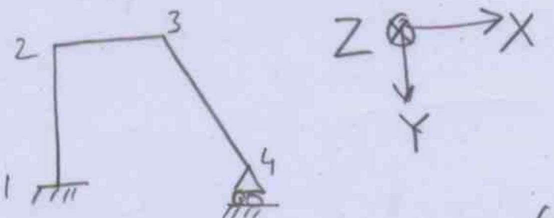
where, for 3-D structure $\{P\}_i^T = \{P_1 \; P_2 \; P_3 \; P_4 \; P_5 \; P_6\}^T$, $\{\Delta\}_i^T = \{\Delta_1 \; \Delta_2 \; \Delta_3 \; \Delta_4 \; \Delta_5 \; \Delta_6\}^T$

$[K]$ = structure total stiffness matrix in XYZ coords.

$\{P\}_i$ and $\{\Delta\}_i$ are structure forces and displ's at node i,

$i = 1, \cdots n$, n = number of ~~nodes~~ joints in structure.

For example consider the planar frame,



$\{P\}_i^T = \{P_1 \; P_2 \; P_6\}^T$, $i = 1, \cdots 4$

$\{\Delta\}_i^T = \{\Delta_1 \; \Delta_2 \; \Delta_6\}^T$

Let $\{\Delta\}_I^T = \{(\Delta_1)_2 \; (\Delta_2)_2 \; (\Delta_6)_2 \; (\Delta_1)_3 \; (\Delta_2)_3 \; (\Delta_6)_3 \; (\Delta_1)_4 \; (\Delta_6)_4\}^T$ = unknown displ vector

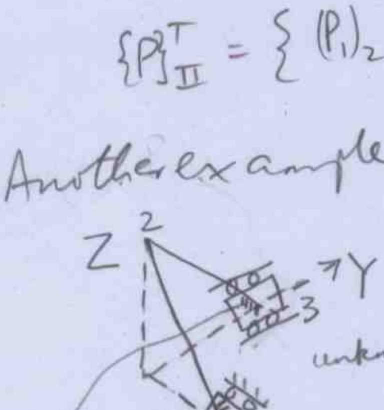
$\{\Delta\}_{II}^T = \{(\Delta_1)_1 \; (\Delta_2)_1 \; (\Delta_6)_1 \; (\Delta_2)_4\}^T$ = prescribed displ vector

$\{P\}_I^T = \{(P_1)_1 \; (P_2)_1 \; (P_6)_1 \; (P_2)_4\}^T$ = unknown (reaction) force vector

$\{P\}_I^T = \{(P_1)_2 \; (P_2)_2 \; (P_6)_2 \; (P_1)_3 \; (P_2)_3 \; (P_6)_3 \; (P_1)_4 \; (P_6)_4\}^T$ = prescribed (applied) forces (loads) at joints only.

$\{P\}_{II}^T = \{(P_1)_2 \; (P_2)_2 \; (P_6)_2 \; (P_1)_3 \; (P_2)_3$

Another example, the space frame



$\{P\}_i^T = \{P_1 \cdots P_6\}_i^T$, $\{\Delta\}_i = \{\Delta_1 \cdots \Delta_6\}_i^T$

unknown $\{\Delta\}_I = \{(\Delta_1)_1 \; (\Delta_1)_2 \; (\Delta_2)_2 \; (\Delta_3)_2 \; (\Delta_4)_2 \; (\Delta_5)_2 \; (\Delta_6)_2 \; (\Delta_2)_3\}^T$

prescribed $\{\Delta\}_{II} = \{(\Delta_2)_1 \; (\Delta_3)_1 \; (\Delta_4)_1 \; (\Delta_5)_1 \; (\Delta_6)_1 \; (\Delta_1)_3 \; (\Delta_3)_3 \; (\Delta_4)_3 \; (\Delta_5)_3 \; (\Delta_6)_3\}^T$

$\{P\}_I = \{(P_2)_1 \; (P_3)_1 \; (P_4)_1 \; (P_5)_1 \; (P_6)_1 \; (P_1)_3 \; (P_3)_3 \; (P_4)_3 \; (P_5)_3 \; (P_6)_3\}^T$

unknown $\{P\}_{II} = \{(P_1)_1 \; (P_1)_2 \; (P_2)_2 \; (P_3)_2 \; (P_4)_2 \; (P_5)_2 \; (P_6)_2 \; (P_2)_3\}^T$

So, in general

$\{\Delta\}_{I}$ = unknown displ vector, $\{\Delta\}_{II}$ = prescribed displ vector

$\{P\}_{I}$ = prescribed force vector, $\{P\}_{II}$ = unknown force vector (ie reactions)

and we partition (5) as

$$\left\{\begin{array}{c}\{P\}_{I}\\ \hline \{P\}_{II}\end{array}\right\} = \left[\begin{array}{c|c}[K]_{I I} & [K]_{I II}\\ \hline [K]_{II I} & [K]_{II II}\end{array}\right]\left\{\begin{array}{c}\{\Delta\}_{I}\\ \hline \{\Delta\}_{II}\end{array}\right\} \longrightarrow (6)$$

which yields

$$\{\Delta\}_{I} = [K]_{I I}^{-1}\left(\underbrace{\{P\}_{I}}_{\text{prescribed forces}} - [K]_{I II}\underbrace{\{\Delta\}_{II}}_{\text{prescribed displs}}\right) \longrightarrow (7)$$

$$\{P\}_{II} = [K]_{II I}[K]_{I I}^{-1}\left(\{P\}_{I} - [K]_{I II}\{\Delta\}_{II}\right) + [K]_{II II}\{\Delta\}_{II} \longrightarrow (8)$$

* If unknown displ's are p in number & unknown forces are q in number, then $[K]_{I I}$ is $(p \times p)$, $[K]_{II II}$ is $(q \times q)$; $[K]_{I II}$ is $(p \times q)$, $[K]_{II I}$ is $(q \times p)$.

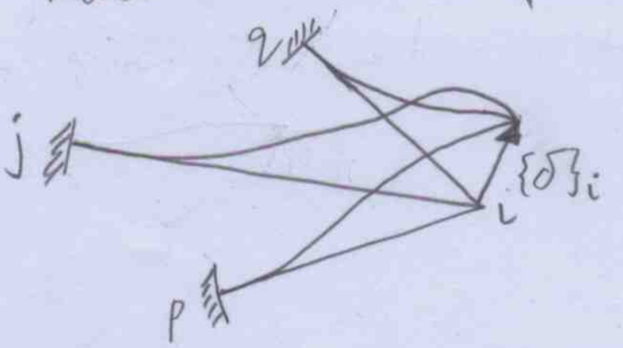
* $(p+q)$ = $6n$ for space frame; $3n$ for plane frame and space truss; $2n$ for plane truss.

All that remains is to find $[K]$.

<u>Generation of [K]</u>

$$\begin{Bmatrix}\{P\}_1\\ \vdots\\ \{P\}_i\\ \vdots\\ \{P\}_j\\ \vdots\\ \{P\}_n\end{Bmatrix} = \begin{bmatrix} \\ \\ [K]_{i1} \cdots [K]_{ii} \cdots [K]_{ij} \cdots [K]_{in} \\ \\ \\ \end{bmatrix}\begin{Bmatrix}\{\Delta\}_1\\ \vdots\\ \{\Delta\}_i\\ \vdots\\ \{\Delta\}_j\\ \vdots\\ \{\Delta\}_n\end{Bmatrix} \longrightarrow (9)$$

$[K]_{ii}$ = stiffness sub-matrix which gives forces at node i in terms of displ's at node i, for structure.

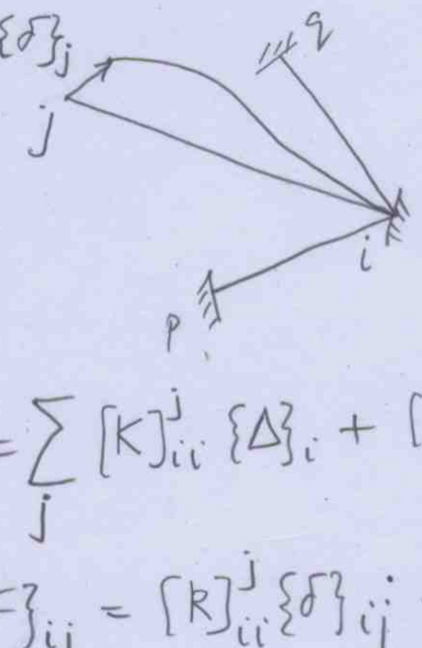


So $[K]_{ii}$ depends on stiffness characteristics of all members framing into i.

$$[K]_{ii} = \sum_{\substack{j = \text{all nodes} \\ \text{framing into } i}} [K]^{j}_{ii} \longrightarrow (10)$$

where $[K]^{j}_{ii}$ = stiffness matrix relating forces at i to displ's at i, for member ij only, in global coordinates.

$[K]_{ij}$ = stiffness sub-matrix which gives forces at i in terms of displ's only at j, for structure. So it depends on stiffness characteristics of member ij only.



So, $$\{P\}_i = \sum_j [K]^{j}_{ii} \{\Delta\}_i + [K]_{ij} \{\Delta\}_j \longrightarrow (11)$$

From (4), $$\{F\}_{ij} = [K]^{j}_{ii} \{\delta\}_{ij} + [K]_{ij} \{\delta\}_{ji} \longrightarrow (4a)$$

We can relate member end displs $\{\delta\}_{ij}$ (j = all node framing into i) to nodal displ $\{\Delta\}_i$ as follows:

$$\{\delta\}_{ij} = [a]_{ij} \{\Delta\}_i \longrightarrow (12)$$

Similarly $\{\delta\}_{ji} = [a]_{ji}\{\Delta\}_j \rightarrow$ (12a)

(4a, 12, 12a) $\rightarrow \{F\}_{ij} = [k]^j_{ii}[a]_{ij}\{\Delta\}_i + [k]_{ij}[a]_{ji}\{\Delta\}_j \rightarrow$ (13)

Details regarding transformation matrix $[a]_{ij}$ are given later on.

Now equilibrium requires that the forces at a node i balances the member end forces at node i, ie,

$$\{P\}_i = \sum_j [a_{ij}]^{-1}\{F\}_{ij} \rightarrow (14)$$

It is shown later that $[a]^{-1}_{ij} = [a]^T_{ij}$. Hence, using (13, 14)

$$\{P\}_i = \sum_j [a]^T_{ij}\{F\}_{ij} = \sum_j \left([a_{ij}]^T[k]^j_{ii}[a]_{ij}\{\Delta\}_i + [a]^T_{ij}[k]_{ij}[a]_{ji}\{\Delta\}_j\right) \rightarrow (15)$$

From (11) & (15),

$$\left.\begin{aligned} [K]^j_{ii} &= [a]^T_{ij}[k]^j_{ii}[a]_{ij} \\ [K]_{ij} &= [a]^T_{ij}[k]_{ij}[a]_{ji} \end{aligned}\right\} \rightarrow (16)$$

$$[K]_{ii} = \sum_j [K]^j_{ii} \rightarrow (10)$$

<u>Transformation matrices.</u>

The table below shows direction cosines (measured CW or CCW) between member (local) axes xyz and structure (global) axes XYZ $\rightarrow$ (17)

	X	Y	Z
x	a_{11}	a_{12}	a_{13}
y	a_{21}	a_{22}	a_{23}
z	a_{31}	a_{32}	a_{33}

these are components of matrix $[a]$.

If xyz is the member coordinate system (ie local) at end i of member ij, then the above matrix of direction cosines is termed $[a]_{ij}$.

Now consider the transformation of a vector whose components in XYZ system are known, ie,

$\underline{V} = V_1 \hat{X} + V_2 \hat{Y} + V_3 \hat{Z}$. , $\hat{X}, \hat{Y}, \hat{Z}$ are unitvectors in XYZ

Here (V_1, V_2, V_3) can be $((P_1)_i, (P_2)_i, (P_3)_i)$ or $((\Delta_1)_i, (\Delta_2)_i, (\Delta_3)_i)$, respectively, ie translational forces/displacements at node i.

Now $\underline{v}$ is invariant (ie its magnitude & direction do not change) irrespective of the coordinate system we use to componentiate it (ie describe it). The above description is in global coordinates. If we chose to describe it in local coordinates, then project each of the global components of $\underline{v}$ above onto the local xyz system and add these projections, ie

$$\left\{\begin{aligned} V_1 \hat{X} &= V_1 (a_{11} \hat{x} + a_{21} \hat{y} + a_{31} \hat{z}) \\ V_2 \hat{Y} &= V_2 (a_{12} \hat{x} + a_{22} \hat{y} + a_{32} \hat{z}) \\ V_3 \hat{Z} &= V_3 (a_{13} \hat{x} + a_{23} \hat{y} + a_{33} \hat{z}) \end{aligned}\right\}$$

here we used, for eg., that projection of $\hat{X}$ on $\hat{y}$ is the direction cosine between $\hat{X}$ & $\hat{y}$, ie a_{21}, an so on.

Here $\hat{x}, \hat{y}, \hat{z}$ are unitvectors in xyz system.

adding these

$$\underline{v} = (a_{11} V_1 + a_{12} V_2 + a_{13} V_3) \hat{x} + (a_{21} V_1 + a_{22} V_2 + a_{23} V_3) \hat{y} + (a_{31} V_1 + a_{32} V_2 + a_{33} V_3) \hat{z}$$

$$= v_1 \hat{x} + v_2 \hat{y} + v_3 \hat{z}$$

where v_1, v_2, v_3 are components of $\underline{v}$ in xyz system.

Thus,

$$\{v\} = [a]\{V\}, \quad \text{where } \{v\} = \{v_1 \; v_2 \; v_3\}^T, \; \{V\} = \{V_1 \; V_2 \; V_3\}^T$$

Now the rotational forces (ie moments) also transform the same way since they can be written as components of a vector in local or global system. However rotational displacements in general cannot be written as components of a vector, i.e. we cannot write a vector $(\Delta_4)_i \hat{X} + (\Delta_5)_i \hat{Y} + (\Delta_6)_i \hat{Z}$ or a vector $(\delta_4)_{ij}\hat{x} + (\delta_5)_{ij}\hat{y} + (\delta_6)_{ij}\hat{z}$, unless these rotational displacements are small. In our case they are small so they too can be transformed in the same manner.

Thus for general 3-D rod/bar structure problems

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}_{ij} = \begin{bmatrix} \overbrace{\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}}^{[a]_{ij}} & 0 \\ 0 & \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \end{bmatrix}_{ij} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}_i$$

for 3-D trusses

$$\{\delta_1\}_{ij} = [a_{11} \quad a_{12} \quad a_{13}]_{ij} \{\Delta_1 \quad \Delta_2 \quad \Delta_3\}_i^T$$

for 2-D trusses

$$\{\delta_1\}_{ij} = [a_{11} \quad a_{12}]_{ij} \{\Delta_1 \quad \Delta_2\}_i^T$$

for 2-D beam/frames

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_6 \end{Bmatrix}_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \to 0 \\ a_{21} & a_{22} & \cancel{a_{23}} \to 0 \\ \cancel{a_{31}} \to 0 & \cancel{a_{32}} \to 0 & \cancel{a_{33}} \to 1 \end{bmatrix}_{ij} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_6 \end{Bmatrix}_i \qquad (18)$$

<u>Algorithm for stiffness method</u>

(1) Identify nodes, and member end (local) coordinate system and associated displs and forces, and structure (global) coordinate system and associated nodal displs & forces

(2) Generate transformation matrices from (17), (18)

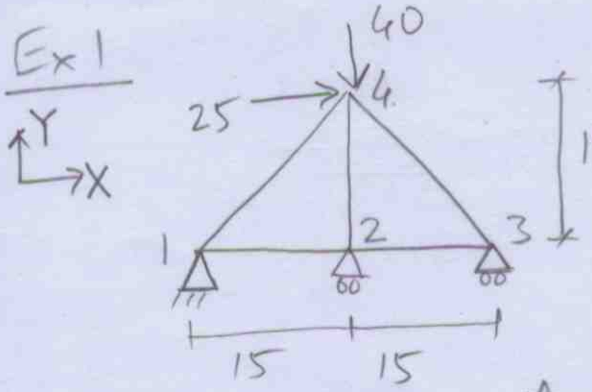
(3) Generate member stiffness matrices based on the identified member-end displs/forces and using 2 or 3 (p. 6,7).

(4) Identify the contraints and d.o.f for the structure, ie $\{\Delta\}_I$ and $\{\Delta\}_{II}$, and corresponding $\{P\}_I$ & $\{P\}_{II}$.
Use these to generate reduced structure stiffness matrices ($[K]_{II}$ etc) of 6 by eliminating suitable rows/cols of total structure stiffness matrix of 9. (p. 13), for which you need to use 16, 10 (p. 15)

(5) Form vector of applied structural loads and displs ($\{P\}_I$ & $\{\Delta\}_{II}$), and use 7 to calculate unknown structural (nodal) displ's & 8 to calculate unknown structural reactions. (p. 13).

(6) Use 13 (p. 15) to get member end forces.

Ex 1



$EA = (29E3)*4.$

$\{P\}_i = \{P_1 \;\; P_2\}^T, \; \{\Delta\}_i = \{\Delta_1 \;\; \Delta_2\}_i^T$

$\{F\}_{ij} = \{F_1\}_{ij}, \; \{\delta\}_{ij} = \{\delta_1\}_{ij}$

Transformation matrices: $[a]_{ij} = [a_{11} \;\; a_{12}]_{ij}, \; [a]_{ji} = -[a]_{ij}$

$[a]_{12} = [-1 \;\; 0] = [a]_{23} \; ; [a]_{14} = \left[-\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right] \; ; [a]_{24} = [0 \;\; -1]$

$[a]_{34} = \left[\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right]$

Member stiffnesses: $[k]_{ii}^{j} = [k]_{jj}^{i}, \; [k]_{ij} = [k]_{ji}$

$[k]_{11}^{2} = [k]_{12} = [k]_{23} = [k]_{22}^{3} = [k]_{24} = [k]_{22}^{4} = \frac{EA}{L}$

$[k]_{14} = [k]_{34} = \frac{EA}{L\sqrt{2}} = [k]_{11}^{4} = [k]_{33}^{4}$

Structure stiffnesses:

$$[K] = \begin{bmatrix} [K]_{11} & [K]_{12} & 0 & [K]_{14} \\ [K]_{21} & [K]_{22} & [K]_{23} & [K]_{24} \\ 0 & [K]_{32} & [K]_{33} & [K]_{34} \\ [K]_{41} & [K]_{42} & [K]_{43} & [K]_{44} \end{bmatrix}$$

$$[K]_{11} = \sum_{j=2,4} [a]_{1j}^T [k]_{11}^{j} [a]_{1j} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{EA}{L} [-1 \;\; 0] + \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \frac{EA}{L\sqrt{2}} \left[-\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right]$$

$$= \frac{EA}{L} \begin{bmatrix} 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$[K]_{22} = \sum_{1,3,4} [a]_{2j}^T [k]_{22}^{j} [a]_{2j} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{EA}{L} [1 \;\; 0] + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{EA}{L} [-1 \;\; 0] + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \frac{EA}{L} [0 \;\; -1]$$

$$= \frac{EA}{L} \begin{bmatrix} 1+1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K]_{33} = \sum_{2,4} [a]^T_{3j} [k]^j_{33} [a]_{3j} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{EA}{L} [1 \quad 0] + \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \frac{EA}{L\sqrt{2}} [1/\sqrt{2} \quad -1/\sqrt{2}]$$

$$= \frac{EA}{L} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$[K]_{44} = \sum_{1,2,3} [a]^T_{4j} [k]^j_{44} [a]_{4j} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \frac{EA}{L\sqrt{2}} [1/\sqrt{2} \quad 1/\sqrt{2}] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \frac{EA}{L} [0 \quad 1]$$

$$+ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \frac{EA}{L\sqrt{2}} [-1/\sqrt{2} \quad 1/\sqrt{2}]$$

$$= \frac{EA}{L} \begin{bmatrix} \frac{1}{2\sqrt{2}} * 2 & \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} * 2 + 1 \end{bmatrix}$$

$$[K]_{ij} = [a]^T_{ij} [k]_{ij} [a]_{ji}$$

$$[K]_{12} = [a]^T_{12} [k]_{12} [a]_{21} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{EA}{L} [1 \quad 0] = \frac{EA}{L} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[K]_{14} = [a]^T_{14} [k]_{14} [a]_{41} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \frac{EA}{L\sqrt{2}} \left[\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right] = \frac{EA}{L} \begin{bmatrix} -1/2\sqrt{2} & -1/2\sqrt{2} \\ -1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix}$$

$$[K]_{23} = [a]^T_{23} [k]_{23} [a]_{32} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{EA}{L} [1 \quad 0] = \frac{EA}{L} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[K]_{24} = [a]^T_{24} [k]_{24} [a]_{42} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \frac{EA}{L} [0 \quad 1] = \frac{EA}{L} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[K]_{34} = [a]^T_{34} [k]_{34} [a]_{43} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \frac{EA}{L\sqrt{2}} [-1/\sqrt{2} \quad 1/\sqrt{2}] = \frac{EA}{L} \begin{bmatrix} -1/2\sqrt{2} & 1/2\sqrt{2} \\ 1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix}$$

$[K]_{ij} = [K]_{ji}$, by definition ($\because [k]_{ij} = [k]_{ji}$ and $[a]_{ij} = -[a]_{ji}$)

<u>Reduced stiffness matrices:</u>

$$\{\Delta\}_{I} = \{(\Delta_1)_2 \quad (\Delta_1)_3 \quad (\Delta_1)_4 \quad (\Delta_2)_4\}^T$$

$$\{\Delta\}_{II} = \{(\Delta_1)_1 \quad (\Delta_2)_1 \quad (\Delta_2)_2 \quad (\Delta_2)_3\}^T = \{0 \quad 0 \quad 0 \quad 0\}^T$$

So delete rows and columns 1, 2, 4, 6 of $[K]$ to get $[K]_{I\,I}$
& delete rows 1, 2, 4, 6 and columns 3, 5, 7, 8 to get $[K]_{I\,II}$.
Note here $[K]_{I\,II}$ not required since $\{\Delta\}_{II} = 0$ (no settlement)

$$[K]_{II} = \frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}+1 \end{bmatrix}$$

Nodal displ's:

$$\Delta_I = [K]_{II}^{-1}\{P\}_I \; ; \; \{P\}_I^T = \{(P_1)_2 \;\; (P_1)_3 \;\; (P_1)_4 \;\; (P_2)_4\}^T$$

$$\{0 \quad 0 \quad 25 \quad -40\}^T$$

$$\Delta_1 = \frac{L}{EA}\{17.2183 \;\; 34.4365 \;\; 52.5736 \;\; -30.5635\}^T$$

$$\Delta_I = \{0.0022 \quad 0.0045 \quad 0.0068 \quad -0.0040\}^T$$

Member forces:

$$\{F\}_{ij} = [K]_{ii}^{j}[a]_{ij}\{\Delta\}_i + [k]_{ij}[a]_{ji}\{\Delta\}_j$$

$$\{F\}_{12} = \frac{EA}{L}[-1 \;\; 0]\begin{Bmatrix}0\\0\end{Bmatrix} + \frac{EA}{L}[1 \;\; 0]\begin{Bmatrix}0.0022\\0\end{Bmatrix} = 17.2183$$

$$\{F\}_{23} = \frac{EA}{L}[-1 \;\; 0]\begin{Bmatrix}0.0022\\0\end{Bmatrix} + \frac{EA}{L}[1 \;\; 0]\begin{Bmatrix}0.0045\\0\end{Bmatrix} = 17.2182$$

$$\{F\}_{14} = \frac{EA}{L\sqrt{2}}\left[\tfrac{-1}{\sqrt{2}} \; \tfrac{-1}{\sqrt{2}}\right]\begin{Bmatrix}0\\0\end{Bmatrix} + \frac{EA}{L\sqrt{2}}\left[\tfrac{1}{\sqrt{2}} \; \tfrac{1}{\sqrt{2}}\right]\begin{Bmatrix}0.0068\\-0.0040\end{Bmatrix} = 11.0051$$

$$\{F\}_{24} = \frac{EA}{L}[0 \;\; -1]\begin{Bmatrix}0.0022\\0\end{Bmatrix} + \frac{EA}{L}[0 \;\; 1]\begin{Bmatrix}0.0068\\-0.0040\end{Bmatrix} = -30.5635$$

$$\{F\}_{34} = \frac{EA}{L\sqrt{2}}\left[\tfrac{1}{\sqrt{2}} \; -\tfrac{1}{\sqrt{2}}\right]\begin{Bmatrix}0.0045\\0\end{Bmatrix} + \frac{EA}{L\sqrt{2}}\left[\tfrac{-1}{\sqrt{2}} \; \tfrac{1}{\sqrt{2}}\right]\begin{Bmatrix}0.0068\\-0.0040\end{Bmatrix} = -24.3503$$

Reactions (unknown forces):

Delete rows & columns 3, 5, 7, 8 in [K] to get $[K]_{II\,II}$.

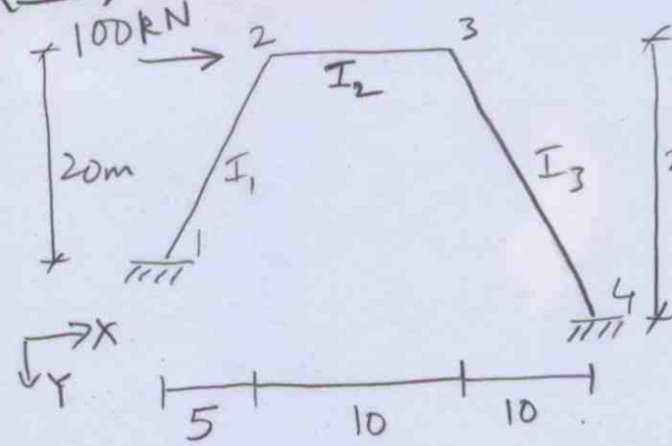
But we dont need it here $\because \{\Delta\}_{II} = \{0\}$

Delete rows 3, 5, 7, 8 & columns 1, 2, 4, 6 to get $[K]_{II\,I}$

$$[K]_{II\,I} = \begin{bmatrix} -1 & 0 & -1/2\sqrt{2} & -1/2\sqrt{2} \\ 0 & 0 & -1/2\sqrt{2} & -1/2\sqrt{2} \\ 0 & 0 & 0 & -1 \\ 0 & -1/2\sqrt{2} & 1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix} ; \{P\}_{II} = [K]_{II\,I}\{\Delta\}_I = \begin{Bmatrix} -25 \\ -7.7817 \\ 30.5635 \\ 17.2183 \end{Bmatrix}$$

<u>Note</u>: More efficient to inspect total [K] matrix and generate only those sub-matrices that are required in computations.

(Ex 2)



$I_1 = 412$, $I_2 = 300$, $I_3 = 807$

$\left(\frac{EI}{EA}\right)_{ij} = \gamma_{ij}^{-1}$ for member ij.

$\underline{\Delta}_I = \{(\Delta_1)_2, (\Delta_2)_2, (\Delta_6)_2, (\Delta_1)_3, (\Delta_2)_3, (\Delta_6)_3\}^T$

$$\underline{\underline{a}}_{12} = \begin{bmatrix} -\frac{5}{\sqrt{425}} & \frac{20}{\sqrt{425}} & 0 \\ -\frac{20}{\sqrt{425}} & -\frac{5}{\sqrt{425}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{a}}_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{a}}_{34} = \begin{bmatrix} \frac{-10}{\sqrt{725}} & -\frac{25}{\sqrt{725}} & 0 \\ \frac{25}{\sqrt{725}} & \frac{-10}{\sqrt{725}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{\underline{a}}_{21}$, $\underline{\underline{a}}_{32}$, $\underline{\underline{a}}_{43}$ obtained from $\underline{\underline{a}}_{12}$, $\underline{\underline{a}}_{23}$, $\underline{\underline{a}}_{34}$, be reversing signs on the upper 2X2 block.

Total stiffness matrix, structure stiffness matrix, member stiffness matrices.

$$\underline{\underline{K}} = \begin{bmatrix} \underline{\underline{K}}_{11} & \underline{\underline{K}}_{12} & 0 & 0 \\ \underline{\underline{K}}_{21} & \underline{\underline{K}}_{22} & \underline{\underline{K}}_{23} & 0 \\ 0 & \underline{\underline{K}}_{32} & \underline{\underline{K}}_{33} & \underline{\underline{K}}_{34} \\ 0 & 0 & \underline{\underline{K}}_{43} & \underline{\underline{K}}_{44} \end{bmatrix}$$

$\underline{\underline{K}}_{II} \rightarrow$ eliminate rows/cols 1, 2, 3, 10, 11, 12

$\underline{\underline{K}}_{III} \rightarrow$ eliminate rows 4, 5, 6, 7, 8, 9, cols 1, 2, 3, 10, 11, 12.

So we require matrices in the dotted box above.

$$\underline{\underline{K}}_{II} = \begin{bmatrix} \underline{\underline{K}}_{22} & \underline{\underline{K}}_{23} \\ \underline{\underline{K}}_{32} & \underline{\underline{K}}_{33} \end{bmatrix}, \quad \underline{\underline{K}}_{III} = \begin{bmatrix} \underline{\underline{K}}_{12} & \underline{0} \\ \underline{0} & \underline{\underline{K}}_{43} \end{bmatrix}$$

we need following member stiffness matrices:

$$k_{11}^2 = k_{22}^1 = \frac{EI_1}{L_1}\begin{bmatrix} \gamma_{12} & 0 & 0 \\ 0 & 12/L_1^2 & -6/L_1 \\ 0 & -6/L_1 & 4 \end{bmatrix}, \quad k_{12} = k_{21} = \frac{EI_1}{L_1}\begin{bmatrix} \gamma_{12} & 0 & 0 \\ 0 & 12/L_1^2 & -6/L_1 \\ 0 & -6/L_1 & 2 \end{bmatrix}$$

For $k_{22}^3 = k_{33}^2$ and k_{23}, replace (I_1, L_1, γ_{12}) by (I_2, L_2, γ_{23}) in k_{11}^2 and k_{12}, respectively

For $k_{33}^4 = k_{44}^3$ and k_{34}, replace (I_1, L_1, γ_{12}) by (I_3, L_3, γ_{34}) in k_{11}^2 & k_{12}, resp'ly.

$$\underline{K}_{22} = \begin{bmatrix} \frac{5}{\sqrt{425}} & \frac{20}{\sqrt{425}} & 0 \\ \frac{-20}{\sqrt{425}} & \frac{5}{\sqrt{425}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{12} & 0 & 0 \\ 0 & 12/L_1^2 & -6/L_1 \\ 0 & -6/L_1 & 4 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{425}} & \frac{-20}{\sqrt{425}} & 0 \\ \frac{20}{\sqrt{425}} & \frac{5}{\sqrt{425}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{EI_1}{L_1}$$

$$+ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{23} & 0 & 0 \\ 0 & 12/L_2^2 & -6/L_2 \\ 0 & -6/L_2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{EI_2}{L_2}$$

$$= \frac{EI_1}{L_1} \begin{bmatrix} \frac{5}{\sqrt{425}}\gamma_{12} & \frac{20}{\sqrt{425}}\cdot\frac{12}{L_1^2} & \frac{-20}{\sqrt{425}}\cdot\frac{6}{L_1} \\ \frac{-20}{\sqrt{425}}\gamma_{12} & \frac{5}{\sqrt{425}}\cdot\frac{12}{L_1^2} & \frac{-5}{\sqrt{425}}\cdot\frac{6}{L_1} \\ 0 & -6/L_1 & 4 \end{bmatrix} \cdot \underline{a}_{21} + \frac{EI_2}{L_2} \begin{bmatrix} \gamma_{23} & 0 & 0 \\ 0 & 12/L_2^2 & 6/L_2 \\ 0 & 6/L_2 & 4 \end{bmatrix}$$

$$\underline{K}_{22} = \frac{EI_1}{L_1} \begin{bmatrix} \left(\frac{25}{425}\gamma_{12} + \frac{400}{425}\cdot\frac{12}{L_1^2}\right) & \left(-\frac{100}{425}\gamma_{12} + \frac{100}{425}\cdot\frac{12}{L_1^2}\right) & \frac{-120}{\sqrt{425}}\cdot\frac{1}{L_1} \\ \left(-\frac{100}{425}\gamma_{12} + \frac{100}{425}\cdot\frac{12}{L_1^2}\right) & \left(\frac{400}{425}\gamma_{12} + \frac{25}{425}\cdot\frac{12}{L_1^2}\right) & \frac{-30}{\sqrt{425}}\cdot\frac{1}{L_1} \\ \frac{-120}{\sqrt{425}}\cdot\frac{1}{L_1} & \frac{-30}{\sqrt{425}}\cdot\frac{1}{L_1} & 4 \end{bmatrix} + \frac{EI_2}{L_2} \begin{bmatrix} \gamma_{23} & 0 & 0 \\ 0 & \frac{12}{L_2^2} & \frac{6}{L_2} \\ 0 & \frac{6}{L_2} & 4 \end{bmatrix}$$

$$\underline{K}_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} k^2_{33} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{EI_3}{L_3} \begin{bmatrix} \frac{-10}{\sqrt{725}} & \frac{25}{\sqrt{725}} & 0 \\ \frac{-25}{\sqrt{725}} & \frac{-10}{\sqrt{725}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{34} & 0 & 0 \\ 0 & 12/L_3^2 & -6/L_3 \\ 0 & -6/L_3 & 4 \end{bmatrix} \begin{bmatrix} \frac{-10}{\sqrt{725}} & \frac{-25}{\sqrt{725}} & 0 \\ \frac{25}{\sqrt{725}} & \frac{-10}{\sqrt{725}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{EI_2}{L_2} \begin{bmatrix} \gamma_{23} & 0 & 0 \\ 0 & 12/L_2^2 & -6/L_2 \\ 0 & -6/L_2 & 4 \end{bmatrix} + \begin{bmatrix} \frac{-10}{\sqrt{725}}\gamma_{34} & \frac{25}{\sqrt{725}}\cdot\frac{12}{L_3^2} & -\frac{25}{\sqrt{725}}\cdot\frac{6}{L_3} \\ \frac{-25}{\sqrt{725}}\gamma_{34} & \frac{-10}{\sqrt{725}}\cdot\frac{12}{L_3^2} & \frac{10}{\sqrt{725}}\cdot\frac{6}{L_3} \\ 0 & -6/L_3 & 4 \end{bmatrix} \cdot \underline{a}_{34} \cdot \frac{EI_3}{L_3}$$

$$= \frac{EI_2}{L_2} \begin{bmatrix} \gamma_{23} & 0 & 0 \\ 0 & 12/L_2^2 & -6/L_2 \\ 0 & -6/L_2 & 4 \end{bmatrix} + \frac{EI_3}{L_3} \begin{bmatrix} \left(\frac{100}{725}\gamma_{34} + \frac{625}{725}\cdot\frac{12}{L_3^2}\right) & \frac{250}{725}\gamma_{34} - \frac{250}{725}\cdot\frac{12}{L_3^2} & \frac{-150}{\sqrt{725}}\cdot\frac{1}{L_3} \\ & \left(\frac{625}{725}\gamma_{34} + \frac{100}{725}\cdot\frac{12}{L_3^2}\right) & \frac{60}{\sqrt{725}}\cdot\frac{1}{L_3} \\ \text{symm} & & 4 \end{bmatrix}$$

$$\underline{K}_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{23} & 0 & 0 \\ 0 & 12/L_2^2 & -6/L_2 \\ 0 & -6/L_2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{EI_2}{L_2} = \begin{bmatrix} -\gamma_{23} & 0 & 0 \\ 0 & -12/L_2^2 & 6/L_2 \\ 0 & -6/L_2 & 2 \end{bmatrix} \frac{EI_2}{L_2}$$

$$\underline{K}_{23}^T = a_{32}^T \cancel{\underline{k}_{23}^T} a_{23} = \underline{K}_{32}$$

$$= \underline{k}_{23} = \underline{k}_{32}$$

$$\underline{\underline{K}}_{12} = \begin{bmatrix} \frac{-5}{\sqrt{425}} & \frac{-20}{\sqrt{425}} & 0 \\ \frac{20}{\sqrt{425}} & \frac{-5}{\sqrt{425}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{12} & 0 & 0 \\ 0 & 12/L_1^2 & -6/L_1 \\ 0 & -6/L_1 & 2 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{425}} & \frac{-20}{\sqrt{425}} & 0 \\ \frac{20}{\sqrt{425}} & \frac{5}{\sqrt{425}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{EI_1}{L_1}$$

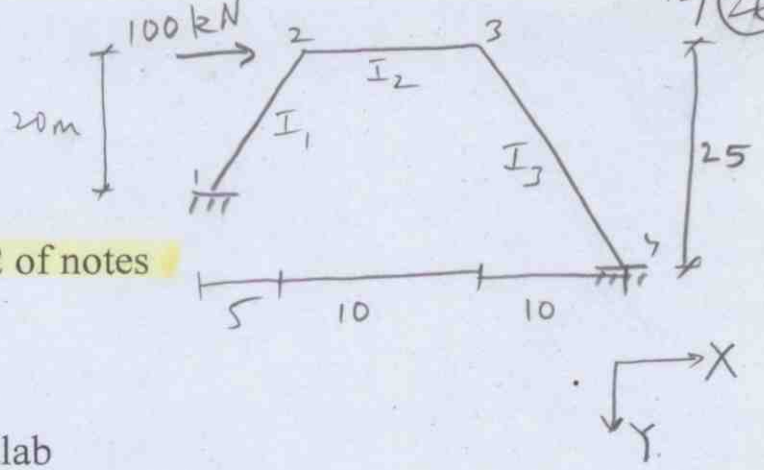
$$= \frac{EI_1}{L_1}\begin{bmatrix} \frac{-5}{\sqrt{425}}\gamma_{12} & \frac{-20}{\sqrt{425}}\cdot\frac{12}{L_1^2} & \frac{20}{\sqrt{425}}\cdot\frac{6}{L_1} \\ \frac{20}{\sqrt{425}}\gamma_{12} & \frac{-5}{\sqrt{425}}\cdot\frac{12}{L_1^2} & \frac{5}{\sqrt{425}}\cdot\frac{6}{L_1} \\ 0 & -\frac{6}{L_1} & 2 \end{bmatrix} \cdot \underline{\underline{a}}_{21} = \frac{EI_1}{L_1}\begin{bmatrix} -\frac{25}{425}\gamma_{12} - \frac{400}{425}\cdot\frac{12}{L_1^2} & \frac{100}{425}\left(\gamma_{12} - \frac{12}{L_1^2}\right) & \frac{120}{\sqrt{425}}\cdot\frac{1}{L_1} \\ \frac{100}{425}\left(\gamma_{12} - \frac{12}{L_1^2}\right) & \left(-\frac{400}{425}\gamma_{12} - \frac{25}{425}\cdot\frac{12}{L_1^2}\right) & \frac{30}{\sqrt{425}}\cdot\frac{1}{L_1} \\ -\frac{120}{\sqrt{425}}\cdot\frac{1}{L_1} & -\frac{30}{\sqrt{425}}\cdot\frac{1}{L_1} & 2 \end{bmatrix}$$

$$\underline{\underline{K}}_{43} = \begin{bmatrix} \frac{10}{\sqrt{725}} & \frac{-25}{\sqrt{725}} & 0 \\ \frac{25}{\sqrt{725}} & \frac{10}{\sqrt{725}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{34} & 0 & 0 \\ 0 & 12/L_3^2 & -6/L_3 \\ 0 & -6/L_3 & 2 \end{bmatrix} \begin{bmatrix} \frac{-10}{\sqrt{725}} & \frac{-25}{\sqrt{725}} & 0 \\ \frac{25}{\sqrt{725}} & \frac{-10}{\sqrt{725}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{EI_3}{L_3}$$

$$= \begin{bmatrix} \frac{10}{\sqrt{725}}\gamma_{34} & \frac{-25}{\sqrt{725}}\cdot\frac{12}{L_3^2} & \frac{25}{\sqrt{725}}\frac{6}{L_3} \\ \frac{25}{\sqrt{725}}\gamma_{34} & \frac{10}{\sqrt{725}}\cdot\frac{12}{L_3^2} & \frac{-10}{\sqrt{725}}\frac{6}{L_3} \\ 0 & -\frac{6}{L_3} & 2 \end{bmatrix} \cdot \underline{\underline{a}}_{34} = \begin{bmatrix} \left(-\frac{100}{725}\gamma_{34} - \frac{625}{725}\cdot\frac{12}{L_3^2}\right) & \frac{250}{725}\left(-\gamma_{34} + \frac{12}{L_3^2}\right) & \frac{150}{\sqrt{725}}\frac{1}{L_3} \\ \frac{250}{725}\left(-\gamma_{34} + \frac{12}{L_3^2}\right) & \left(-\frac{625}{725}\gamma_{34} - \frac{100}{725}\cdot\frac{12}{L_3^2}\right) & \frac{-60}{\sqrt{725}}\frac{1}{L_3} \\ \frac{-150}{\sqrt{725}}\frac{1}{L_3} & \frac{60}{\sqrt{725}}\cdot\frac{1}{L_3} & 2 \end{bmatrix}$$

Use $L_1 = \sqrt{425}$, $L_2 = 10$, $L_3 = \sqrt{725}$, $\gamma_{12} = \gamma_{23} = \gamma_{34} = 1/3000$.

See remainder on MATLAB file. It contains matrix multiplications done in MATLAB as well as done by hand as above.



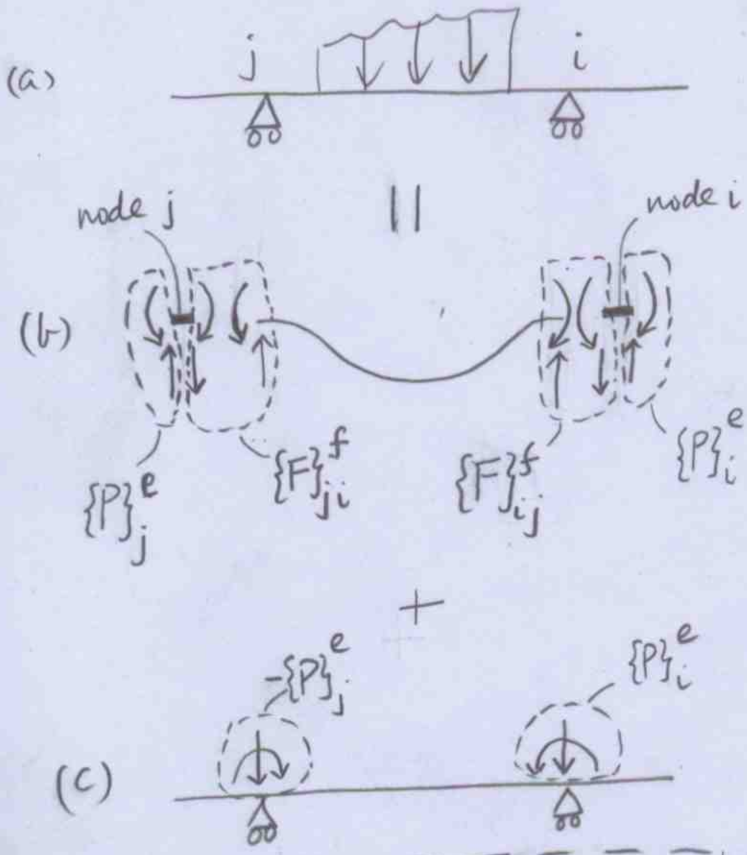

```
%Stiffness matrix method, Ex2 of notes

g=300; %g=A/I in m^2

%matrix multiplications in matlab

a12=[-5/425^.5 20/425^.5 0; -20/425^.5 -5/425^.5 0; 0 0 1];
a21=[5/425^.5 -20/425^.5 0; 20/425^.5 5/425^.5 0; 0 0 1]; a34=[-
10/725^.5 -25/725^.5 0; 25/725^.5 -10/725^.5 0; 0 0 1];
a43=[10/725^.5 25/725^.5 0; -25/725^.5 10/725^.5 0; 0 0 1];
a23=[-1 0 0 ; 0 -1 0; 0 0 1]; a32=[1 0 0; 0 1 0; 0 0 1];

i1=412; i2=300; i3=807; L1=425^.5; L2=10; L3=725^.5;
k112=i1/L1*[g 0 0; 0 12/L1^2 -6/L1; 0 -6/L1 4];  k12=i1/L1*[g 0
0; 0 12/L1^2 -6/L1; 0 -6/L1 2]; k223=i2/L2*[g 0 0; 0 12/L2^2 -
6/L2; 0 -6/L2 4];  k23=i2/L2*[g 0 0; 0 12/L2^2 -6/L2; 0 -6/L2 2];
k334=i3/L3*[g 0 0; 0 12/L3^2 -6/L3; 0 -6/L3 4]; k34=i3/L3*[g 0
0; 0 12/L3^2 -6/L3; 0 -6/L3 2];

K22=a21'*k112*a21 + a23'*k223*a23;
K33=a32'*k223*a32 + a34'*k334*a34;
K23=a23'*k23*a32;
KII=[K22 K23; K23' K33];
p=[100 0 0 0 0 0]';
inv(KII)*p

ans =

$\Delta_b$ ←  40.0518
               9.9999
$\theta_b$ ← -0.9895
$\Delta_c$ ←  40.0459
             -16.0086
$\theta_c$ ← -0.5034
```

horizontal displ of of jts 2 & 3 almost same ∵ $\frac{A}{E}$ lge.

compare with results by SDM (T1 - p. 15). They match very well.

```
%matrix multiplications by hand

K22=[25/425*g+400/425*12/425 -100/425*g+100/425*12/425 -
120/425; -100/425*g+100/425*12/425 25/425*12/425+400/425*g
-30/425; -120/425 -30/425 4]*412/425^.5 + 300/10*[g 0 0 ; 0
12/100 6/10; 0 6/10 4];

K33=300/10*[g 0 0 ; 0 12/100 -6/10; 0 -6/10
4]+807/725^.5*[100/725*g+625/725*12/725 250/725*(g-12/725)
-150/725; 250/725*(g-12/725) 100/725*12/725+625/725*g
60/725; -150/725 60/725 4];

K23=300/10*[-g 0 0 ; 0 -12/100 6/10; 0 -6/10 2];
KII=[K22 K23; K23' K33];
inv(KII)*p

ans =

   40.0518
    9.9999
   -0.9895
   40.0459
  -16.0086
   -0.5034
```

Beams / Frames loaded between node points. – Equivalent Nodal Loads.



(a) Beam/frame loaded between nodes.

(b) Member ends held fixed by applying $\{P_i\}^e$ & $\{P\}^e_j$ joint loads at nodes i, j, respectively. Corresponding fixed end forces for member are $\{F\}^f_{ij}$ & $\{F\}^f_{ji}$, respectively

(c) Now provide release at joints i & j, since they are not fixed-ended in general. This is similar to the MDM. In the present continuously supported beam shown, the release is provided thru $-\{\tilde{P}\}^e_i$ & $-\{\tilde{P}\}^e_j$ which are the moment components of $-\{P\}^e_i$ & $-\{P\}^e_j$, respectively. The corresponding force components, ie $-\{\hat{P}\}^e_i$ & $-\{\hat{P}\}^e_j$ go directly into the continuous supports, ie they are reacted upon by the supports

So $\boxed{\text{Fig(a)} = \text{Fig(b)} + \text{Fig(c)}}$

Note that (b) has zero nodal displs, but non-zero nodal forces & member end forces.

Note that (c) has non-zero nodal displs, nodal forces, & mem-end forces.

In (b) $\rightarrow$ $\{P\}^e_i = [a]^T_{ij}\{F\}^f_{ij}$

For multiple members framing into node i, ie, in general,

$$\{P\}^e_i = \sum_j [a]^T_{ij}\{F\}^f_{ij} \rightarrow (19)$$, Σ taken over all members framing into i.

Let $\{\tilde{P}\}^e_i$ be the part of $\{P\}^e_i$ corresponding to free displacements, ie (true) d.o.f's, ie releases.

Then, nodal loads are

$$\underbrace{\{P\}_i = \{P\}_i^a}_{\text{loads applied directly at node } i} - \{\tilde{P}\}_i^e \longrightarrow (20)$$

→ minus sign ∵ release provided by reversing direction of $\{P\}_i^e$ that we apply to keep end fixed.

Let $\{\hat{P}\}_i$ be the part of $\{P\}_i^e$ corresponding to constrained displacements, ie that part which goes directly into supports, ie that part which gets reacted upon.

Then, reactions are,

→ positive sign ∵ reaction in opp direction of $-\{\hat{P}\}^e$ applied (see Fig (c)).

$$\{P\}_{II} = [K]_{II,I}\{\Delta\}_I + [K]_{II,II}\{\Delta\}_{II} + \{\hat{P}\} \longrightarrow (21)$$

where $\{\hat{P}\}$ is the collection of all $\{\hat{P}\}_i$'s assembled according to node & dof numbering.

Member end forces are,

$$\{F\}_{ij} = \underbrace{[k]_{ii}^j [a]_{ij}\{\Delta\}_i + [k]_{ij}[a]_{ji}\{\Delta\}_j}_{\text{as before (eq 13, p.15), with } \{\Delta\}_i,\ \{\Delta\}_j \text{ computed based on } \{P\}_i \text{ from (21)}} + \underbrace{\{F\}_{ij}^f}_{\text{dont forget this, from fig (b)}} \longrightarrow (22)$$

$$\underbrace{\underbrace{}_{\text{Fig (c)}} + \underbrace{}_{\text{Fig (b)}}}_{\text{superposition.}}$$

Algorithm:

(i) Find fixed end forces (fef's) (see fig (b)), and $\{P\}_i^e$ from (19)

(ii) Find $\{P\}_i$ from (20) and assemble the $\{P\}_i$'s into $\{P\}_I$. Note that in (20) we use only $\{\tilde{P}\}_i^e$ part of $\{P\}_i^e$, ie the part corresponding to free displs (true dof's), ie releases.

(iii) Find $\{\Delta\}_I$ from (7) p. 13

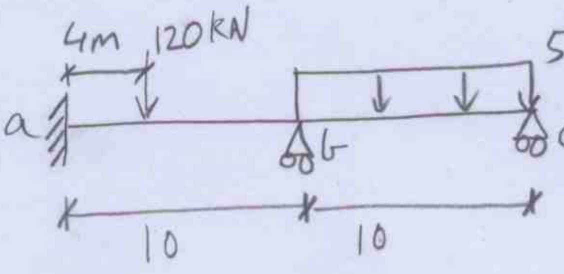
(iv) Find $\{P\}_{II}$, ie reactions from (21). Note that here we use only reactive part of $\{P\}_i^e$, ie $\{\hat{P}\}_i^e$, which corresponds to constrained displs.

(V) Find member end forces from (22). Note that apart from the first two terms (from (13) p15) that correspond to Fig (c), you must add the third term of Fef's to achieve a superposition of Fig (b) with Fig (C).

Q: Do member end forces in Fig(b) & Fig(c) cancel each other

A: Mem-end forces in Fig(b) are just Fef's. The equivalent nodal loads $\{P\}_i^e$ are obtained by adding these Fef's for all members framing into node i. (Thus $\{P\}_i^e$ created by summing effect of applied loads) Then $\{P_i\}^e$ is applied in Fig (c), and it distributes into a new set of member end forces, based on the relative stiffness of members that frame into node i. So mem-end forces in Fig(c) are <u>not</u> equal and opposite to mem-end forces (Fef's) in Fig(b).

(Ex3)



Node b settles 0.03m ↓

(Same example done by SDM, see Ex1, T1/p.7)

See MATLAB file. This is solved by following four variants:

(i) As a plane frame problem (3-dof's per node ie, 1,2,6 dof's)
  (a) By putting settlement as Δ_{II}
  (b) By finding $\{P\}^e$ due to settlement, ie using Fef's, ie as self straining problem (see theory in next section)

(ii) As a plane beam problem (2-dof's per node, ie 2,6 dof's)
  (a)
  (b) } as above

To get Fef's for (Ex3) :

(a) Elegant superposition approach :

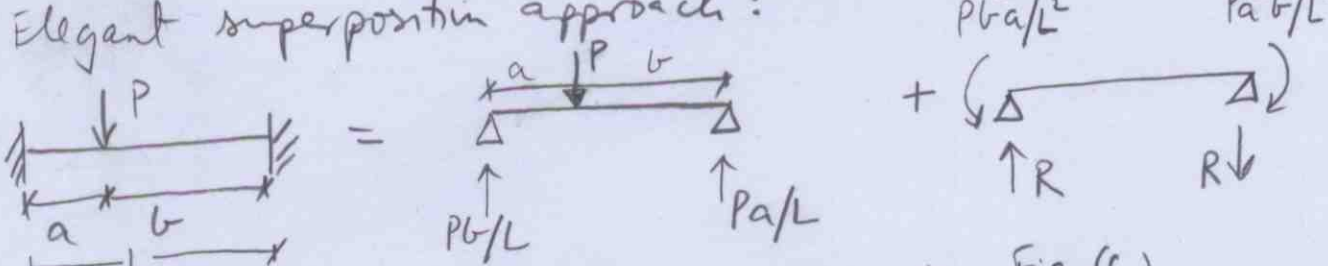


Fig (a) = Fig (b) + Fig (c)

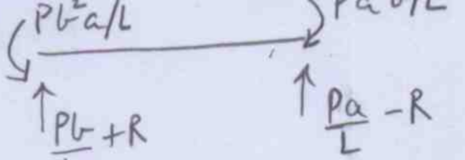
In Fig(b) we have load P that causes rotations at simple supports.
In Fig(c) we cancel these rotations by applying known Fem's, which give rise to reactions R, -R. Then superpose.

From Fig(c) → $R = \frac{P(b^2a - ba^2)}{L^2 \cdot L}$

Thus, $\frac{Pb}{L} + R = Pb\frac{L^2}{L^3} + R = \frac{P}{L^3}\left(b[a+b]^2 + b^2a - ba^2\right) = \frac{P}{L^3}\left(b^3 + 3ab^2\right)$ (*)

Similarly, $\frac{Pa}{L} - R = Pa\frac{L^2}{L^3} - R = \frac{P}{L^3}\left(a[a+b]^2 - b^2a + ba^2\right) = \frac{P}{L^3}\left(a^3 + 3a^2b\right)$ (**)

Superposed Fef's are



(b) Brute force direct integration approach using singularity functions :

$$EI\,w^{IV} = -P\langle x-a\rangle^{-1}$$

$$EI\,w''' = -P\langle x-a\rangle^{0} + C_1$$

$$EI\,w'' = -P\langle x-a\rangle^{1} + C_1x + C_2$$

$$EI\,w' = -P\frac{\langle x-a\rangle^{2}}{2} + C_1\frac{x^2}{2} + C_2x + C_3$$

$$EI\,w = -P\frac{\langle x-a\rangle^{3}}{6} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$$

For BC's, use definition, $\langle x-a\rangle^n = 0$, for $x \le a$
$= (x-a)^n$, for $x > a$

where $n = 0, 1, 2, \ldots$

BC's : $w(0)=0 \Rightarrow c_4=0$, $w'(0)=0 \Rightarrow c_3=0$

$$w(L)=0=-P\frac{(L-a)^3}{6}+c_1\frac{L^3}{6}+c_2\frac{L^2}{2}$$

$$w'(L)=0=-P\frac{(L-a)^3}{2}+c_1\frac{L^2}{2}+c_2 L$$

Solve c_1, c_2 from above. Then,

$EIw'''(0) = c_1 = \frac{P}{L^3}(b^3+3ab^2)$ → same as ⊛ on previous pg.

$EIw''(0) = c_2 = -\frac{Pb^2a}{L^2}$, ie ↺, → same as LHS Fem on prev pg.

$EIw'''(L) = -P+c_1 = -\frac{P}{L^3}(a^3+3a^2b)$, ie ↑, → same as ⊛⊛ on prev pg

$EIw''(L) = -P(L-a)+c_1L+c_2 = -Pb+c_1L+c_2 = -\frac{Pba^2}{L^2}$,

ie ↺, → same as RHS Fem on prev pg

```
%Stiffness matrix method, Ex3 of notes.

%Done as plane frame %(3-dof per node, i.e., 1,2,6 dof's).

g=300; %g=A/I in m^2

%matrix multiplications in matlab

a12=[-1 0 0; 0 -1 0; 0 0 1]; a21=[1 0 0; 0 1 0; 0 0 1]; a23=a12;
a32=a21;

e=200E9; i1=2000e-6; i2=i1; L1=10; L2=L1; ei1=e*i1/1e3;
ei2=e*i2/1e3;
k221=ei1/L1*[g 0 0; 0 12/L1^2 -6/L1; 0 -6/L1 4];  k112=k221;
k12=ei1/L1*[g 0 0; 0 12/L1^2 -6/L1; 0 -6/L1 2]; k21=k12';
k223=ei2/L2*[g 0 0; 0 12/L2^2 -6/L2; 0 -6/L2 4];  k332=k223;
k23=ei2/L2*[g 0 0; 0 12/L2^2 -6/L2; 0 -6/L2 2]; k32=k23';

K11= a12'*k112*a12; K12= a12'*k12*a21;
K13=[0 0 0; 0 0 0; 0 0 0];
K21= K12';
K22=a21'*k221*a21 + a23'*k223*a23;
K23=a23'*k23*a32;
K31= K13'; K32= K23';
K33=a32'*k332*a32;

%remove rows/cols 1,2,3,5,8 from K_Total
KII=[K22(1,1)  K22(1,3) K23(1,1) K23(1,3); K22(3,1)  K22(3,3)
K23(3,1) K23(3,3); K32(1,1) K32(1,3) K33(1,1) K33(1,3);
K32(3,1) K32(3,3) K33(3,1) K33(3,3)];

%remove rows 1,2,3,5,8 cols 4,6,7,9 from K_Total
KI_II=[K21(1,1) K21(1,2) K21(1,3) K22(1,2) K23(1,2); K21(3,1)
K21(3,2) K21(3,3) K22(3,2)  K23(3,2); K31(1,1) K31(1,2)
```

K31(1,3) K32(1,2) K33(1,2); K31(3,1) K31(3,2) K31(3,3)
K32(3,2) K33(3,2)];

%remove rows 4,6,7,9 cols 1,2,3,5,8 from K_Total
KII_I=[K12(1,1) K12(1,3) K13(1,1) K13(1,3); K12(2,1) K12(2,3)
K13(2,1) K13(2,3); K12(3,1) K12(3,3) K13(3,1) K13(3,3);
K22(2,1) K22(2,3) K23(2,1) K23(2,3); K32(2,1) K32(2,3)
K33(2,1) K33(2,3)];

%remove rows/cols 4,6,7,9 from K_Total
KII_II=[K11(1,1) K11(1,2) K11(1,3) K12(1,2) K13(1,2); K11(2,1)
K11(2,2) K11(2,3) K12(2,2) K13(2,2); K11(3,1) K11(3,2)
K11(3,3) K12(3,2) K13(3,2); K21(2,1) K21(2,2) K21(2,3)
K22(2,2) K23(2,2); K31(2,1) K31(2,2) K31(2,3) K32(2,2)
K33(2,2)];

%First method…..putting settlement as Delta_II; done as plane
frame problem (1,2,6, dof's per node).

F12f=[0 ; 120*6/10+120/L1^3*(6^2*4-6*4^2); -120*6^2*4/L1^2];
F21f=[0 ; -120*4/10+120/L1^3*(6^2*4-6*4^2); 120*4^2*6/L1^2];
F23f=[0 ; 50*L2/2 ; -50*L2^2/12];
F32f=[0 ; -50*L2/2 ; 50*L2^2/12];

P1e= a12'*F12f; P2e= a21'*F21f + a23'*F23f; P3e=a32'*F32f;
Pa=[0 0 0 0]';

Pe=[P1e' P2e' P3e']'; Petilde=[Pe(4) Pe(6) Pe(7) Pe(9)]';
PI=Pa-Petilde; Pehat= [Pe(1) Pe(2) Pe(3) Pe(5) Pe(8)]';

DeltaII=[0 0 0 0.03 0]';

DeltaI=inv(KII)*(PI-KI_II*DeltaII)

DeltaI =

```
     0
  0.0031
     0
 -0.0087

%Reactions
P_II=KII_I*DeltaI+KII_II*DeltaII+Pehat

P_II =
      0
 -147.2057
 -644.2857
 -212.0171
 -260.7771


%member end forces
Delta1= [DeltaII(1); DeltaII(2); DeltaII(3)];
Delta2= [DeltaI(1); DeltaII(4); DeltaI(2)];
Delta3= [DeltaI(3); DeltaII(5); DeltaI(4)];
F12=k112*a12*Delta1 + k12*a21*Delta2+F12f
F21=k21*a12*Delta1 + k221*a21*Delta2+F21f
F23=k223*a23*Delta2 + k23*a32*Delta3+F23f
F32=k23*a23*Delta2 + k332*a32*Delta3+F32f

F12 =
      0
  147.2057
 -644.2857


F21 =
      0
   27.2057
 -107.7714
```

F23 =

0
239.2229
107.7714

F32 =

0
-260.7771
-0.0000

%Second method…..finding equivalent nodal loads due to settlement via fixed end forces due to settlement; done as plane frame problem (1,2,6, dof's per node).

```
delta21s=[0 0.03 0]'; delta23s=[0 -0.03 0]';

F12f=[0 ; 120*6/10+120/L1^3*(6^2*4-6*4^2) ; -
120*6^2*4/L1^2]+k12* delta21s;
F21f=[0 ; -120*4/10+120/L1^3*(6^2*4-6*4^2); 120*4^2*6/L1^2]
+k221* delta21s;
F23f=[0 ; 50*L2/2 ; -50*L2^2/12] +k223* delta23s;
F32f=[0 ; -50*L2/2 ; 50*L2^2/12] +k32* delta23s;

P1e= a12'*F12f; P2e= a21'*F21f + a23'*F23f; P3e=a32'*F32f;
Pa=[0 0 0 0]';
Pe=[P1e' P2e' P3e']'; Petilde=[Pe(4) Pe(6) Pe(7) Pe(9)]';
PI=Pa-Petilde; Pehat= [Pe(1) Pe(2) Pe(3) Pe(5) Pe(8)]';

DeltaII=[0 0 0 0 0]';
DeltaI=inv(KII)*(PI-KI_II*DeltaII)
```

DeltaI =

0
0.0031
0

```
  -0.0087

%Reactions
P_II=KII_I*DeltaI+KII_II*DeltaII+Pehat

P_II =
        0
 -147.2057
 -644.2857
 -212.0171
 -260.7771
```

M_{AB} ← −644.2857 → matches with SDM. (T1-p.7). All mem-end forces & reactions match → only M_{AB} chosen for comparison.

```
%member end forces
Delta1= [DeltaII(1); DeltaII(2); DeltaII(3)];
Delta2= [DeltaI(1); DeltaII(4); DeltaI(2)];
Delta3= [DeltaI(3); DeltaII(5); DeltaI(4)];
F12=k112*a12*Delta1 + k12*a21*Delta2+F12f
F21=k21*a12*Delta1 + k221*a21*Delta2+F21f
F23=k223*a23*Delta2 + k23*a32*Delta3+F23f
F32=k23*a23*Delta2 + k332*a32*Delta3+F32f

F12 =
        0
  147.2057
 -644.2857

F21 =
        0
   27.2057
 -107.7714

F23 =
        0
  239.2229
```

```
  107.7714

F32 =
       0
 -260.7771
       0

>> DeltaI(2)*ei1

ans =

  1.2426e+003

>> DeltaI(4)*ei1

ans =

 -3.4630e+003

%results match exactly with SDM
```

$EI\theta_B$

$EI\theta_C$

— (T1 - p. 7).

```
%%Done as plane beam problem (2-dof per node, i.e., 2,6)

%so remove local dof # 1, and also global dof #1 since only
vertical loads, i.e., no inclined loads; correspondingly remove axial
stiffness terms.

a12=[-1 0; 0 1]; a21=[1 0; 0 1]; a23=a12; a32=a21;

e=200E9; i1=2000e-6; i2=i1; L1=10; L2=L1; ei1=e*i1/1e3;
ei2=e*i2/1e3;
k221=ei1/L1*[12/L1^2 -6/L1; -6/L1 4];  k112=k221;
k12=ei1/L1*[12/L1^2 -6/L1; -6/L1 2]; k21=k12';
k223=ei2/L2*[12/L2^2 -6/L2; -6/L2 4];  k332=k223;
k23=ei2/L2*[12/L2^2 -6/L2; -6/L2 2]; k32=k23';

K11= a12'*k112*a12; K12= a12'*k12*a21;
K13=[0 0; 0 0];
K21= K12';
K22=a21'*k221*a21 + a23'*k223*a23;
K23=a23'*k23*a32;
K31= K13'; K32= K23';
K33=a32'*k332*a32;

%remove rows/cols 1,2,3,5 from K_Total
KII=[K22(2,2) K23(2,2); K32(2,2) K33(2,2)];

%remove rows 1,2,3,5 cols 4,6 from K_Total
KI_II=[K21(2,1) K21(2,2) K22(2,1) K23(2,1); K31(2,1) K31(2,2)
K32(2,1) K33(2,1)];

%First method…..putting settlement as Delta_II; done as plane
beam problem (2,6, dof's per node).
```

```
F12f=[120*6/10+120/L1^3*(6^2*4-6*4^2) ; -120*6^2*4/L1^2];
F21f=[-120*4/10+120/L1^3*(6^2*4-6*4^2) ; 120*4^2*6/L1^2];
F23f=[50*L2/2 ; -50*L2^2/12]; F32f=[-50*L2/2 ; 50*L2^2/12];

P1e= a12'*F12f; P2e= a21'*F21f + a23'*F23f; P3e=a32'*F32f;
Pa=[0 0]';

Pe=[P1e' P2e' P3e']'; Petilde=[Pe(4) Pe(6)]';
PI=Pa-Petilde; Pehat= [Pe(1) Pe(2) Pe(3) Pe(5)]';

DeltaII=[0 0 0.03 0]';

DeltaI=inv(KII)*(PI-KI_II*DeltaII)

DeltaI =
   0.0031
  -0.0087

%results match plane frame approach

%Second method…..finding equivalent nodal loads due to
settlement via fixed end forces due to settlement; done as plane
beam problem (2,6, dof's per node).

delta21s=[0.03 0]'; delta23s=[-0.03 0]';

F12f=[120*6/10+120/L1^3*(6^2*4-6*4^2) ; -120*6^2*4/L1^2]
+k12* delta21s;
F21f=[-120*4/10+120/L1^3*(6^2*4-6*4^2) ; 120*4^2*6/L1^2]
+k221* delta21s;
F23f=[50*L2/2 ; -50*L2^2/12] +k223* delta23s; F32f=[-50*L2/2
; 50*L2^2/12] +k32* delta23s;

P1e= a12'*F12f; P2e= a21'*F21f + a23'*F23f; P3e=a32'*F32f;
Pa=[0 0]';
```

```
Pe=[P1e' P2e' P3e']'; Petilde=[Pe(4) Pe(6)]';
PI=Pa-Petilde; Pehat= [Pe(1) Pe(2) Pe(3) Pe(5)]';

DeltaII=[0 0 0 0]';

DeltaI=inv(KII)*(PI-KI_II*DeltaII)
DeltaI =

   0.0031
  -0.0087
```

Self straining problems

We consider loading due to support movement (i.e settlement), temperature variations, and misfit (i.e lack of alignment or improper lengths during fabrication). Note that in case of statically indeterminate structures, these effects (or loads) induce stresses (internal forces, moments). But this is not the case for statically determinate structures.

The analysis is similar to case when mechanical loads applied between loads.

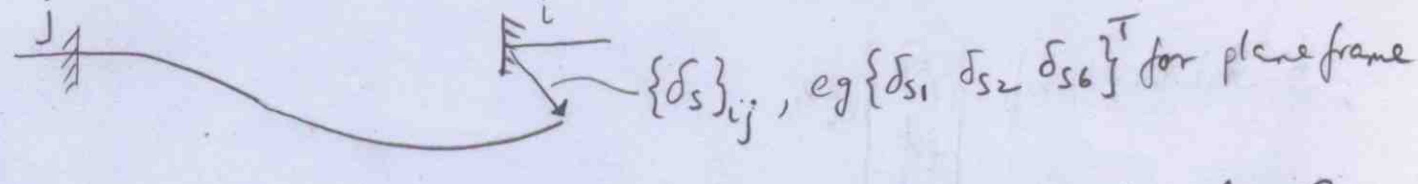


Fig (a) : Member cut at one end (eg. end i). Cut end undergoes free displs (i.e stress free) $\{\delta_s\}_{ij}$ due to self straining

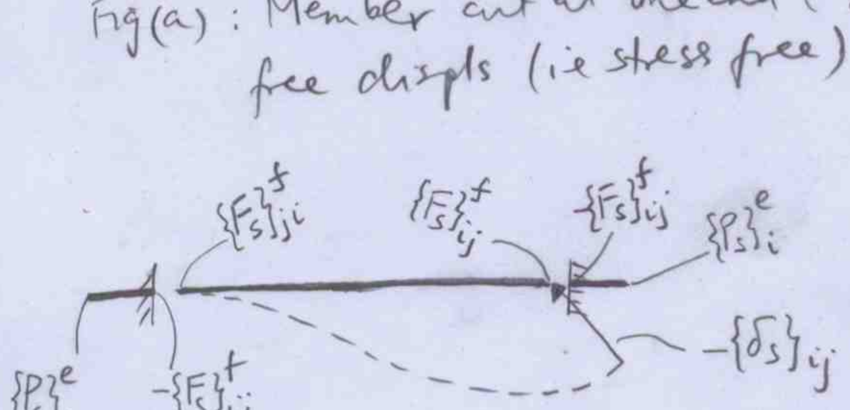


Fig (b) : Displ. $-\{\delta_s\}_{ij}$, i.e opposite to self straining free-displ's, are applied to achieve fixed ended configuration. This is done by means of applied nodal forces $\{P_s\}^e_i$ & $\{P_s\}^e_j$, and it induces Fef's $\{F_s\}^f_{ij}$ and $\{F_s\}^f_{ji}$. This is similar to case of mech loads applied between loads. Thus,

$$\{P_s\}^e_i = [a]^T_{ij}\{F_s\}^f_{ij} \quad ; \quad \{P_s\}^e_j = [a]^T_{ji}\{F_s\}^f_{ji}$$

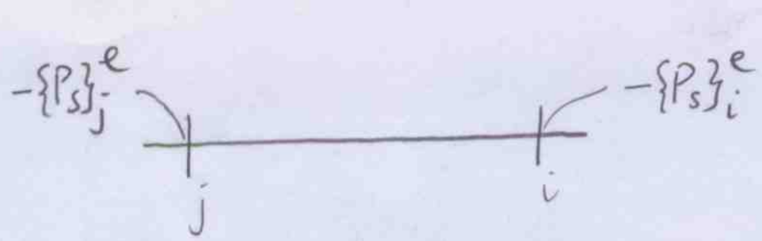


Fig (C): Now the fixity of joints i, j, is released by applying $-\{P_s\}_i^e$ & $-\{P_s\}_j^e$, since joints i, j are not fixed, in general, in actual structure. This is similar, to case of mech loads between nodes. Thus $-\{P_s\}_i^e$ split into $-\{\tilde{P}_s\}_i^e$ which corresponds to (true) dof's, ie, it releases fixity; and $-\{\hat{P}_s\}_i^e$ which transmits directly into supports, ie it gets reacted upon.

Thus you see that analysis closely imitates that of case when mech loads applied between nodes.

Thus, Fef's (in Fig(b) are),

$$\{F_s\}_{ij}^f = [k]_{ii}^j\{-\delta_s\}_{ij} + [k]_{ij}\{-\delta_s\}_{ji} \longrightarrow (23a)$$

one of these usually zero, but both may be present in settlement or misfit problems.

$$\{F_s\}_{ji}^f = [k]_{ji}\{-\delta_s\}_{ij} + [k]_{jj}^i\{-\delta_s\}_{ji} \longrightarrow (23b)$$

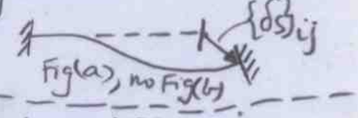
For multiple members framing into node i, as before,

$$\{P_s\}_i^e = \sum_j [a]_{ij}^T\{F_s\}_{ij}^f \longrightarrow (24)$$

↳ split into $\{\tilde{P}_s\}_i^e$ & $\{\hat{P}_s\}_i^e$

So for mech loads directly applied at nodes and also between nodes, and also self straining loads, the nodal loads are obtained from Eq (20) p. 29 with additional term $-\{\tilde{P}_s\}_i^e$; the reactions are obt from Eq (21) with extra term $+\{\hat{P}_s\}_i^e$; the member end forces are obtained from Eq 22 with extra term $+\{F_s\}^f_{ij}$.

For settlement problems, we dont need to reverse the stress free self straining displs in order to achieve end fixity, ie, settlement of a node maintains fixity of that node. Hence the (−) sign is removed from Eq 23 & the $\{\delta_s\}_i$ & $\{\delta_s\}_j$ are the given settlements at nodes i & j, respectively. Given settlements will usually be in global coords, ie $\{\Delta_s\}_i$ & $\{\Delta_s\}_j$ so you need to convert to local member-end coords.

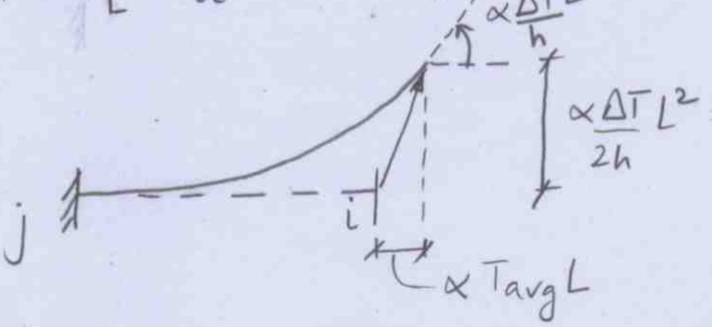


For misfit and temperature effects, the (−) sign in 23 is retained. For misfit $\{\delta_s\}_{ij}$ & $\{\delta_s\}_{ji}$ are the given misfits.

For temp effects, refer T1 p.20. We had,

$$\frac{d\theta}{dx} = \alpha\frac{\Delta T}{h} \Rightarrow \theta = w' = \alpha\frac{\Delta T}{h}x + C_1 \; ; \; W = \frac{\alpha\Delta T}{2h}x^2 + C_1 x + C_2$$

$W(0)=0$, $W'(0)=0 \Rightarrow C_1 = C_2 = 0$. Thus $w'(L) = \theta(L) = \alpha\frac{\Delta T}{h}L$ and $W(L) = \alpha\frac{\Delta T}{2h}L^2$. Here w +ve ↑, w' +ve ↺. Also, $\Delta T = T_L - T_u$. Thus,



ie, $\{\delta_{s2}\}_i = -\alpha\frac{\Delta T}{2h}L^2$

$\{\delta_{s6}\}_i = -\alpha\frac{\Delta T}{h}L$

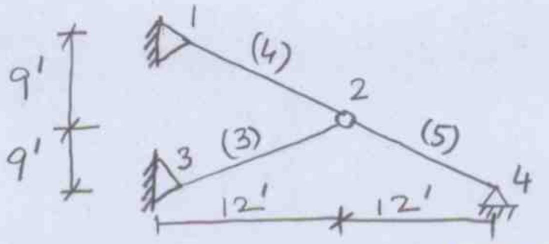
$\{\delta_{s1}\}_i = \alpha T_{avg} \cdot L$

where $T_{avg} = \frac{T_L + T_u}{2}$

∵ temp variation is linear, the extensional strain is average of top & bottom strains.

Algorithm: Find $\{\delta_s\}_{ij}$ as above. Then use 23 to find Fef's, and 24 to find equivalent nodal loads due to self straining. Then proceed as in case of mech loads between nodes, ie use modified Eq 20 − 22.

(Ex4)



$E = 29 \times 10^6$ ksi

Members 12 & 24 undergo $\Delta T = 40°F$

$\alpha = 0.0000065/°F$

Area of members denoted in (in^2)

T3/43

```
%Stiffness matrix method, Ex4 of notes.
```

(West, prob 14.32, p.517)

```
%matrix multiplications in matlab

a21=[0.8 0.6]; a12=-a21; a23=[0.8 -0.6]; a32=-a23; a24=-a21;
a42=-a24;

e=29E3; ar21=4; ar24=5; ar23=3; L21=15; L23=L21; L24=L21;
alpha=0.0000065; DeltaT21=40;  DeltaT24=40; DeltaT23=0;

k221=ar21*e/L21*[1]; k12=k221; k223=ar23*e/L23*[1];
k32=k223; k224=ar24*e/L24*[1]; k42=k224;

K22=a21'*k221*a21 + a23'*k223*a23 + a24'*k224*a24;

KII=K22;

delta21s=-[alpha* DeltaT21* L21]'; delta23s=-[ alpha* DeltaT23*
L23]'; delta24s=-[alpha* DeltaT24* L24]';

F21f=k221* delta21s; F23f=k223* delta23s; F24f=k224* delta24s;
F12f=k12* delta21s; F32f=k32* delta23s; F42f=k42* delta24s;

P2e= a21'*F21f + a23'*F23f + a24'*F24f;
P1e=a12'*F12f; P3e=a32'*F32f; P4e=a42'*F42f;

Pa=[0 0]';

Pe=[P1e' P2e' P3e' P4e']'; Petilde=[ P2e']';
PI=Pa-Petilde; Pehat= [P1e' P3e' P4e']';

DeltaI=inv(KII)*PI

F21=k221*a21* DeltaI+ F21f;
F23=k223*a23* DeltaI + F23f;
```

F24=k224*a24* DeltaI + F24f;

F21 =

-33.5111

>> F23

F23 =

results match with West. 3.5527e-015 → correct. West has incorrect result.

∵ no mech load at jt 2, then jt equil ⇒ $F_{23} = 0$.

>> F24

F24 =

-33.5111

Internal hinge

Two independent rotational dof's at i, ie one on either side of hinge. In general many members can frame into hinge i.

$$\begin{Bmatrix} \vdots \\ P_r \\ \vdots \\ P_i \\ \vdots \\ P_j \\ \vdots \end{Bmatrix} = \begin{bmatrix} & & & & \\ \cdots & K_{ir} \cdots & K_{ii} \cdots & K_{ij} & \cdots \\ & & & & \end{bmatrix} \begin{Bmatrix} \vdots \\ \Delta_r \\ \vdots \\ \Delta_i \\ \vdots \\ \Delta_j \\ \vdots \end{Bmatrix}$$

$\Delta_i = \{\Delta_{1i} \; \Delta_{2i} \; \Delta_{6L,i} \; \Delta_{6R,i}\}^T$ → frame formulation.

or $= \{\Delta_{2i} \; \Delta_{6L,i} \; \Delta_{6R,i}\}^T$ → Beam formulation

$\underline{\underline{K}} = (2n+1)\times(2n+1) \to$ beam formulation
$(3n+1)\times(3n+1) \to$ frame "

} Here +1 is due to extra dof of hinge.

$\underline{\underline{K}}_{ir} \to$ gives nodal loads required at i in order to produce kinematic condition $\underline{\Delta}_r \neq 0, \underline{\Delta}_i = 0, \underline{\Delta}_j = 0, \forall\, j \neq r$. So it is of size (3×2) for beam and (4×3) for frame formulation. Since hinge does not transmit moment, we need a force and a lhs moment at i to achive above kinematic condition, and hence no rhs moment at i is required to achieve this condition.

$$\Rightarrow \underline{\underline{K}}_{ir} = \left[\begin{array}{c} \underline{\underline{a}}_{ir}^T \, \underline{\underline{k}}_{ir} \, \underline{\underline{a}}_{ri} \\ \hline \underline{\underline{O}} \end{array}\right]$$

→ (2×2)-beam, (3×3)-frame

→ zero row vector, size (1×2) for beam & (1×3) for frame formulation.

$\underline{\underline{K}}_{ri} \to$ directly can write as K_{ir}^T. To see physically, it gives nodal loads required at r in order to produce kinematic condt $\Delta_i \neq 0$, $\Delta_r = 0$, $\forall\, r \neq i$. So its size is (2×3)-beam or (3×4)-frame. Now ∵ rhs rotation at i cannot affect the required load at r, since hinge does not transmit moment/rot, we get

$$\underline{\underline{K}}_{ri} = \left[\begin{array}{c|c} \underline{\underline{a}}_{ri}^T \, \underline{\underline{k}}_{ri} \, \underline{\underline{a}}_{ir} & \underline{\underline{O}} \end{array}\right]$$

→ zero column vector, size (2×1)-beam, (3×1)-frame.

(2×2)-beam, (3×3)-frame

Similarly,

$$\underline{\underline{K}}_{iq} = \left[\frac{\underline{\underline{a}}_{iq}^T \underline{\underline{k}}_{iq} \underline{\underline{a}}_{qi}}{\underline{\underline{0}}} \right]$$

here the zero row-vector is not at the bottom row but inserted in the 2nd last row (ie in the 3rd [2nd] col of $\underline{\underline{K}}_{iq}$ ie after 2nd [1st] row of $\underline{\underline{a}}_{iq}^T \underline{\underline{k}}_{iq} \underline{\underline{a}}_{qi}$ in case of frame [beam]).

$$K_{qi} = \left[\underline{\underline{a}}_{qi}^T \underline{\underline{k}}_{qi} \underline{\underline{a}}_{iq} \middle| \underline{\underline{0}} \right]$$

similarly here it is inserted in 2nd last column of $\underline{\underline{K}}_{qi}$ (ie in 3rd [2nd] col of $\underline{\underline{K}}_{qi}$ ie after 2nd [1st] col of $\underline{\underline{a}}_{qi}^T \underline{\underline{k}}_{qi} \underline{\underline{a}}_{iq}$ in case of frame [beam]).

$K_{ii} \to (3\times3)$ – beam, (4×4) – frame.

For beam formulation, 1st col of K_{ii} represents nodal forces at i in $(2, 6L, 6R)$ directions to produce kinematic condt $(\Delta_2)_i \neq 0$, $(\Delta_{6L})_i = (\Delta_{6R})_i = 0$, for which we need all three nodal forces as non-zero in general. In 2nd col, in order to produce $(\Delta_{6L})_i \neq 0$, $(\Delta_2)_i = (\Delta_{6R})_i = 0$ we need forces in $(2, 6L)$ directions but no force in $(6R)$ direction, ie no rhs moment at i, since hinge wont transfer lhs moment $(6L)$. Thus 3rd entry of 2nd col is zero. Similarly 2nd entry of 3rd col will be zero.

$$K_{ii} = \begin{bmatrix} K_{ii}^r(1,1) + K_{ii}^q(1,1) & K_{ii}^r(1,2) & K_{ii}^q(1,2) \\ K_{ii}^r(2,1) & K_{ii}^r(2,2) & 0 \\ K_{ii}^q(2,1) & 0 & K_{ii}^q(2,2) \end{bmatrix} \to \text{beam}.$$

$$K_{ii} = \begin{bmatrix} K_{ii}^r(1,1) + K_{ii}^q(1,1) & K_{ii}^r(1,2) + K_{ii}^q(1,2) & K_{ii}^r(1,3) & K_{ii}^q(1,3) \\ K_{ii}(2,1) + K_{ii}(2,1) & K_{ii}(2,2) + K_{ii}(2,2) & K_{ii}^r(2,3) & K_{ii}^q(2,3) \\ K_{ii}^r(3,1) & K_{ii}^r(3,2) & K_{ii}^r(3,3) & 0 \\ K_{ii}^q(3,1) & K_{ii}^q(3,2) & 0 & K_{ii}^q(3,3) \end{bmatrix}$$

where $\underline{\underline{K}}_{ii}^{r} = \underline{\underline{a}}_{ir}^{T}\, \underline{\underline{k}}_{ii}^{r}\, \underline{\underline{a}}_{ir}$ and $\underline{\underline{K}}_{ii}^{q} = \underline{\underline{a}}_{iq}^{T}\, \underline{\underline{k}}_{ii}^{q}\, \underline{\underline{a}}_{iq}$

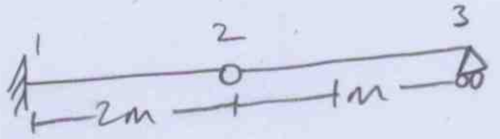
In compact form,

$$K_{ii} = \left[\begin{array}{c|c} \underline{\underline{a}}_{ir}^{T}\, \underline{\underline{k}}_{ii}^{r}\, \underline{\underline{a}}_{ir} & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0 & 0\end{matrix} & 0 \end{array}\right] + \left[\begin{array}{c|c} \underline{\underline{a}}_{iq}^{T}\, \underline{\underline{k}}_{ii}^{q} & \underline{\underline{a}}_{iq} \\ \hline & \end{array}\right]$$

2nd last row & col in zeros is inserted.

\# This problem is re-visited when doing element-wise assembly
(ie Direct Stiffness method).

(Ex 5)



Done as beam elements.

$$\underline{\underline{a}}_{21}^{T}\, \underline{\underline{k}}_{21}\, \underline{\underline{a}}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{12}{2^3} & -\frac{6}{2^2} \\ -\frac{6}{2^2} & \frac{2}{2} \end{pmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{12}{2^3} & -\frac{6}{2^2} \\ \frac{6}{2^2} & \frac{2}{2} \end{bmatrix}$$

$$(a_{21}^{T} k_{21} a_{12})^{T} = a_{12}^{T} k_{21}^{T} a_{21} = a_{12}^{T} k_{12} a_{21}$$

$$a_{32}^{T} k_{32} a_{23} = \begin{bmatrix} -\frac{12}{1^3} & -\frac{6}{1^2} \\ \frac{6}{1^2} & \frac{2}{1} \end{bmatrix} \;;\; (a_{32}^{T} k_{32} a_{23})^{T} = a_{23}^{T} k_{23} a_{32}$$

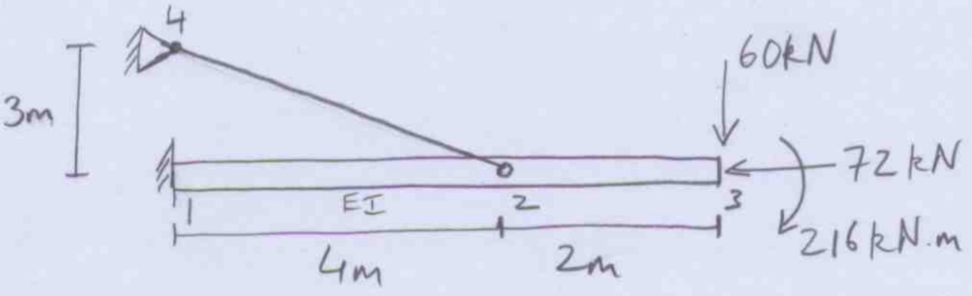
$$;\; a_{21}^{T} k_{22}^{1} a_{21} = k_{22}^{1}$$

$$a_{23}^{T} k_{22}^{3} a_{23} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 12/1^3 & -6/1^2 \\ -6/1^2 & 4/1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12/1^3 & 6/1^2 \\ 6/1^2 & 4/1 \end{bmatrix}$$

$$a_{12}^{T} k_{11}^{2} a_{12} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 12/2^3 & -6/2^2 \\ -6/2^2 & 4/2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12/2^3 & 6/2^2 \\ 6/2^2 & 4/2 \end{bmatrix}$$

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} = \left[\begin{array}{cc|ccc|cc} 12/2^3 & 6/2^2 & -12/2^3 & 6/2^2 & 0 & 0 & 0 \\ 6/2^2 & 4/2 & -6/2^2 & 2/2 & 0 & 0 & 0 \\ \hline -12/2^3 & -6/2^2 & 12/2^3+12/1^3 & -6/2^2 & 6/1^2 & -12/1^3 & 6/1^2 \\ 6/2^2 & 2/2 & -6/2^2 & 4/2 & 0 & 0 & 0 \\ 0 & 0 & 6/1^2 & 0 & 4/1 & -6/1^2 & 2/1 \\ \hline 0 & 0 & -12/1^3 & 0 & -6/1^2 & 12/1^3 & -6/1^2 \\ 0 & 0 & 6/1^2 & 0 & 2/1 & -6/1^2 & 4/2 \end{array}\right]$$

(Ex6) Frame – truss problem. (from CE222 Tute 10, Prob 1)



For beam: $\frac{I}{A} = 3000\,mm^2$

$\Rightarrow \frac{A}{I} = \frac{1000}{3} m^2$

$\left(\frac{AE}{EI}\right)_{beam} = \gamma = \frac{1000}{3}$

$$a_{12} = a_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad a_{21} = a_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad a_{24} = [0.8 \; 0.6] = -a_{42}$$

$$\underline{\underline{K}}_{13} = \underline{\underline{0}}_{3\times3} ; \; \underline{\underline{K}}_{14} = \underline{\underline{0}}_{3\times2} ; \; \underline{\underline{K}}_{14} = \underline{\underline{K}}_{41}^T ; \; \underline{\underline{K}}_{31} = \underline{\underline{K}}_{13}^T ; \; \underline{\underline{K}}_{34} = \underline{\underline{0}}_{3\times2} ; \; \underline{\underline{K}}_{43} = \underline{\underline{K}}_{34}^T$$

$$k_{11}^2 = k_{221} = \frac{EI}{4} \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 12/4^2 & -6/4 \\ 0 & -6/4 & 4 \end{bmatrix} ; \quad k_{22}^3 = k_{33}^2 = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 12/2^2 & -6/2 \\ 0 & -6/2 & 2 \end{bmatrix} \frac{EI}{2}$$

$$k_{12} = k_{21} = \frac{EI}{4} \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 12/4^2 & -6/4 \\ 0 & -6/4 & 2 \end{bmatrix} ; \quad k_{23} = k_{32} = \frac{EI}{2} \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 12/2^2 & -6/2 \\ 0 & -6/2 & 2 \end{bmatrix}$$

$$k_{24} = \frac{EI}{5}[0.2] = k_{22}^4 = k_{442} = k_{42}$$

$$K_{11} = a_{12}^T k_{11}^2 a_{12} ; \quad K_{33} = a_{32}^T k_{33}^2 a_{32}$$

$$K_{22} = a_{21}^T k_{22}^1 a_{21} + a_{23}^T k_{22}^3 a_{23} + \left[\begin{array}{cc|c} \multicolumn{2}{c|}{a_{24}^T k_{22}^4 a_{24}} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$K_{44} = a_{42}^T k_{44}^2 a_{42} ; \quad K_{12} = a_{12}^T k_{12} a_{21} ;$$

$$K_{23} = a_{23}^T k_{23} a_{32} ; \quad K_{24} = \left[\begin{array}{c} a_{24}^T k_{24} a_{42} \\ \hline 0 \quad 0 \end{array}\right]$$

See MATLAB file for details. Two versions are given in MATLAB file, ie,

(a) Done as frame + truss elements, as above.

(b) Done as frame elements only (not recommended).

T3/49

%Stiffness matrix method, Ex6 of notes……Frame+Truss combination

%matrix multiplications in matlab

%Done using frame and truss elements.

```
L1=4; L2=2; L3=5; g1=1000/3; g2=g1; g3=0.2;

a12=[-1 0 0 ; 0 -1 0 ; 0 0 1]; a23=a12; a21=eye(3,3); a32=a21;
a24=[0.8 0.6]; a42=-a24;

k112=1/L1*[g1 0 0 ; 0 12/L1^2 -6/L1 ; 0 -6/L1 4]; k221=k112 ;
k223=1/L2*[g2 0 0 ; 0 12/L2^2 -6/L2 ; 0 -6/L2 4]; k332=k223 ;
k12=1/L1*[g1 0 0 ; 0 12/L1^2 -6/L1 ; 0 -6/L1 2]; k21=k12 ;
k23=1/L2*[g2 0 0 ; 0 12/L2^2 -6/L2 ; 0 -6/L2 2]; k32=k23 ;

k224=1/L3*[g3]; k442=k224; k42=k224; k24=k224;

K13=zeros(3,3); K31=K13';

K14=zeros(3,2); K41=K14'; K34=zeros(3,2); K43=K34';

K11=a12'*k112*a12;

K22=a21'*k221*a21+a23'*k223*a23+[a24'*k224*a24 zeros(2,1);
zeros(1,3)];

K33=a32'*k332*a32;

K44=a42'*k442*a42;

K12=a12'*k12*k21; K21=K12'; K23=a23'*k23*a32; K32=K23';

K24=[a24'*k24*a42; zeros(1,2)]; K42=K24';
```

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```
KII=[K22 K23; K32 K33]
PI=[0 0 0 -72 60 216]' ;
DeltaI=inv(KII)*PI
Delta1=[0 0 0]'; Delta2=[DeltaI(1) DeltaI(2) DeltaI(3)]';
Delta3=[DeltaI(4) DeltaI(5) DeltaI(6)]'; Delta4=[0 0]';

Delta2truss=[Delta2(1) Delta2(2)]';

F12=k112*a12*Delta1+k12*a21*Delta2
F21=k21*a12*Delta1+k221*a21*Delta2
F23=k223*a23*Delta2+k23*a32*Delta3
F32=k32*a23*Delta2+k332*a32*Delta3
F24=k224*a24*Delta2truss+k24*a42*Delta4
F42=k42*a24*Delta2truss+k442*a42*Delta4

KII =
  250.0256    0.0192         0 -166.6667         0         0
    0.0192    1.7019    1.1250         0   -1.5000    1.5000
         0    1.1250    3.0000         0   -1.5000    1.0000
 -166.6667         0         0  166.6667         0         0
         0   -1.5000   -1.5000         0    1.5000   -1.5000
         0    1.5000    1.0000         0   -1.5000    2.0000

DeltaI =
  1.0e+003 *
   -0.0016
    3.0360
    1.4745
   -0.0020
    6.5770
    2.0265

F12 =
 -130.2509
```

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```
  16.3118
-401.2473

F21 =
-130.2509
  16.3118
 336.0000

F23 =
 -72.0000
  60.0000
-336.0000

F32 =
 -72.0000
  60.0000
 216.0000

F24 =
  72.8136

F42 =
  72.8136
```

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```
%Done using frame elements only.

L1=4; L2=2; L3=5; g1=1000/3; g2=g1; g3=0.2;

a12=[-1 0 0 ; 0 -1 0 ; 0 0 1]; a23=a12; a21=eye(3,3); a32=a21;
a24=[0.8 0.6 0 ; -0.6 0.8 0 ; 0 0 1]; a42=[-0.8 -0.6 0; 0.6 -0.8 0 ; 0 0
1];

k112=1/L1*[g1 0 0 ; 0 12/L1^2 -6/L1 ; 0 -6/L1 4]; k221=k112 ;
k223=1/L2*[g2 0 0 ; 0 12/L2^2 -6/L2 ; 0 -6/L2 4]; k332=k223 ;
k12=1/L1*[g1 0 0 ; 0 12/L1^2 -6/L1 ; 0 -6/L1 2]; k21=k12 ;
k23=1/L2*[g2 0 0 ; 0 12/L2^2 -6/L2 ; 0 -6/L2 2]; k32=k23 ;

k224=1/L3*[g3 0 0 ; zeros(2,3)]; k442=k224; k42=k224;
k24=k224;

K13=zeros(3,3); K31=K13';

K14=zeros(3,3); K41=K14'; K34=zeros(3,3); K43=K34';

K11=a12'*k112*a12;

K22=a21'*k221*a21+a23'*k223*a23+[a24'*k224*a24];

K33=a32'*k332*a32;

K44=a42'*k442*a42;

K12=a12'*k12*k21; K21=K12'; K23=a23'*k23*a32; K32=K23';

K24=[a24'*k24*a42]; K42=K24';

K24col3=[K24(1,3) K24(2,3) K24(3,3)]'; K42row3= K24col3'
K34col3=[K34(1,3) K34(2,3) K34(3,3)]'; K43row3= K34col3'
```

KII=[K22 K23 K24col3; K32 K33 K34col3; K42row3 K43row3 K44(3,3)]

```
KII =
 250.0256    0.0192         0 -166.6667         0         0         0
   0.0192    1.7019    1.1250         0   -1.5000    1.5000         0
        0    1.1250    3.0000         0   -1.5000    1.0000         0
-166.6667         0         0  166.6667         0         0         0
        0   -1.5000   -1.5000         0    1.5000   -1.5000         0
        0    1.5000    1.0000         0   -1.5000    2.0000         0
        0         0         0         0         0         0         0
```

%Note that KII singular, as expected. This is since dof 12, i.e., displacement Δ_{12} and nodal force P_{12} (rotational displacement/force at node 4) are spuriously taken for truss element 2-4 when formulating KII. They should actually be zero. Thus we remove this row and col (i.e., 7th row/col) and form the KII. Hence:

KII=[K22 K23 ; K32 K33]

```
KII =
 250.0256    0.0192         0 -166.6667         0         0
   0.0192    1.7019    1.1250         0   -1.5000    1.5000
        0    1.1250    3.0000         0   -1.5000    1.0000
-166.6667         0         0  166.6667         0         0
        0   -1.5000   -1.5000         0    1.5000   -1.5000
        0    1.5000    1.0000         0   -1.5000    2.0000
```

%rest same as frame-truss formulation.

Grids

A planar frame structure subjected to loading normal to the plane of the structure is termed a grid structure. So d.o.f's at end of each element are out of plane displacement, out of plane bending rotation, and torsion, ie, (3, 4, 5) are the element dof's. Thus inplane displacements (axial & bending) & inplane rotation due to bending are absent (ie dof's 1, 2, 6). Thus only $\{\delta_3, \delta_4, \delta_5\}_{ij}^T$ present.

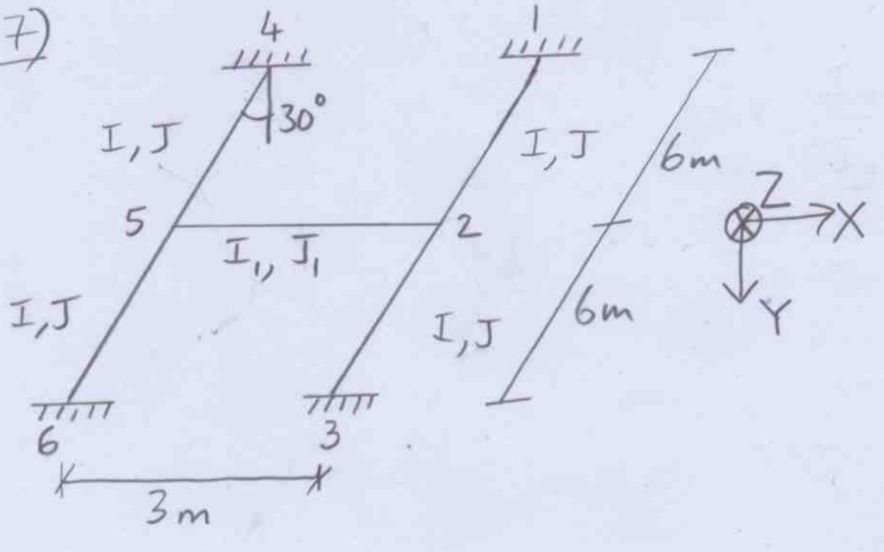
Element stiffness matrix becomes,

$$\frac{EI_y}{L}\left[\begin{array}{ccc|ccc} 12/L^2 & 0 & 6/L & -12/L^2 & 0 & -6/L \\ 0 & GJ/EI_y & 0 & 0 & GJ/EI_y & 0 \\ 6/L & 0 & 4 & -6/L & 0 & -2 \\ \hline & \text{same here} & & & \text{same here} & \end{array}\right]$$

Recall $\{\delta\}_{ij} = \left[\begin{array}{c|c} \underline{\underline{a}}_{ij} & \underline{\underline{0}} \\ \hline \underline{\underline{0}} & \underline{\underline{a}}_{ij} \end{array}\right]\{\Delta\}_i$ for 3-D case

$$\Rightarrow \begin{Bmatrix} \delta_3 \\ \delta_4 \\ \delta_5 \end{Bmatrix}_{ij} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix}}_{\text{Transf matrix for grid.}} \begin{Bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \end{Bmatrix}_i$$

(Ex 7)



For members 12, 23, 45, 56, $b \times d = 450\,mm \times 900\,mm$

For mem 25, $b \times d = 300 \times 900$

$E = 12\,kN/mm^2$, $G = 5\,kN/mm^2$

For mem's 12, 23, 45, 56,

$$I_y = I = \frac{1}{12} * 450 * 900^3 = 27.34 * 10^9 \, mm^4$$

$$J = 900 * 450^3 \left[\frac{1}{3} - 0.21 * \frac{450}{900} \left(1 - \frac{450^4}{12 * 900^4} \right) \right] = 18.77 * 10^9 \, mm^4$$

→ formula from Advanced Solid Mech for torsion of rectangular section. Not reqd to remember.

For mem 25,

$$I_y = I_1 = \frac{1}{12} * 300 * 900^3 = 18.23 * 10^9 \, mm^4$$

$$J_1 = \text{replace } 450 \to 300 \text{ in } J = 6.401 * 10^9 \, mm^4$$

$$a_{21} = a_{54} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -0.5 \end{bmatrix} ; \quad a_{25} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ;$$

$$a_{23} = a_{56} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 0.5 \end{bmatrix} ; \quad a_{52} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$K_{II} = \begin{bmatrix} K_{22} & K_{25} \\ K_{52} & K_{55} \end{bmatrix}$$

let $\alpha_1 = \frac{GJ}{EI}$, $\alpha_2 = \frac{EI}{6}$, $\alpha_3 = \frac{GJ_1}{EI_1}$, $\alpha_4 = \frac{EI_1}{3}$

$$K_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -0.5 \end{bmatrix} \begin{bmatrix} 12/6^2 & 0 & 6/6 \\ 0 & \alpha_1 & 0 \\ 6/6 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -0.5 \end{bmatrix} \alpha_2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{-ve} & \\ 0 & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{-ve} & \\ 0 & & \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 12/3^2 & 0 & 6/3 \\ 0 & \alpha_3 & 0 \\ 6/3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \alpha_4$$

$$K_{55} = a_{54}^T k_{55}^4 a_{54} + a_{56}^T k_{55}^6 a_{56} + a_{52}^T k_{55}^2 a_{52}$$

$$K_{25} = a_{25}^T k_{25} a_{52} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -12/3^2 & 0 & -6/3 \\ 0 & \alpha_3 & 0 \\ -6/3 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \alpha_4 = \begin{bmatrix} -\frac{12}{3^2} & 0 & \frac{6}{3} \\ 0 & -\alpha_3 & 0 \\ -\frac{6}{3} & 0 & 2 \end{bmatrix} * \alpha_4$$

See MATLAB file. Note that twist and out-of-plane bending are coupled in case of skew ≠ 0 (eg skewed bridge, skewed plates)

T3/ 56

```
%Stiffness matrix method, Ex7 of notes....Grid problem

%matrix multiplications in matlab

%skew = 30 degrees
a21=[1 0 0 ; 0 -0.5 3^.5/2 ; 0 -3^.5/2 -0.5]; a23=[1 0 0 ; 0 0.5 -
3^.5/2 ; 0 3^.5/2 0.5]; a25= eye(3,3); a52=[1 0 0 ; 0 -1 0 ; 0 0 -1];
a54=a21; a56=a23;

%skew = 0 degrees
a21=[1 0 0 ; 0 0 1 ; 0 -1 0]; a23=[1 0 0 ; 0 0 -1 ; 0 1 0]; a25=
eye(3,3); a52=[1 0 0 ; 0 -1 0 ; 0 0 -1]; a54=a21; a56=a23;

L1=6; L2=3; i1=27.34e9; i2=18.23e9; j1=18.77e9; j2=6.401e9;
e=12; g=5; alpha1=g*j1/e/i1; alpha2=e*i1/L1; alpha3=g*j2/e/i2;
alpha4=e*i2/L2;

k221=[12/36 0 1 ; 0 alpha1 0 ; 1 0 4]*alpha2*1e-6; k223=k221;
k225=[12/9 0 2 ; 0 alpha3 0 ; 2 0 4]*alpha4*1e-6;
k25=[-12/9 0 -6/3 ; 0 alpha3 0 ; -6/3 0 -2]*alpha4*1e-6;
k554=k221; k556=k221; k552=k225; k52=k25;

K22=a21'*k221*a21+a23'*k223*a23+a25'*k225*a25
K25=a25'*k25*a52
K55=a54'*k554*a54+a56'*k556*a56+a52'*k552*a52
K52=a52'*k52*a25

K=[K22 K25 ; K52 K55]
```

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```
%skew = 30 degrees
K22 =
  1.0e+005 *
   1.3368         0    1.4584
        0    3.4657    1.7587
   1.4584    1.7587    4.2450

K25 =
  1.0e+005 *
  -0.9723         0    1.4584
        0   -0.1067         0
  -1.4584         0    1.4584

K55 =
  1.0e+005 *
   1.3368         0   -1.4584
        0    3.4657    1.7587
  -1.4584    1.7587    4.2450

K52 =
  1.0e+005 *
  -0.9723         0   -1.4584
        0   -0.1067         0
   1.4584         0    1.4584

K =
  1.0e+005 *
   1.3368         0    1.4584   -0.9723         0    1.4584
        0    3.4657    1.7587         0   -0.1067         0
   1.4584    1.7587    4.2450   -1.4584         0    1.4584
  -0.9723         0   -1.4584    1.3368         0   -1.4584
        0   -0.1067         0         0    3.4657    1.7587
   1.4584         0    1.4584   -1.4584    1.7587    4.2450
```

See that twist and out-of-plane bending are coupled.

```
%skew = 0 degrees
K22 =
  1.0e+005 *
    1.3368         0    1.4584
         0    4.4811         0
    1.4584         0    3.2296

K25 =
  1.0e+005 *
   -0.9723         0    1.4584
         0   -0.1067         0
   -1.4584         0    1.4584

K55 =
  1.0e+005 *
    1.3368         0   -1.4584
         0    4.4811         0
   -1.4584         0    3.2296

K52 =
  1.0e+005 *
   -0.9723         0   -1.4584
         0   -0.1067         0
    1.4584         0    1.4584

K =
  1.0e+005 *
    1.3368         0    1.4584   -0.9723         0    1.4584
         0    4.4811         0         0   -0.1067         0
    1.4584         0    3.2296   -1.4584         0    1.4584
   -0.9723         0   -1.4584    1.3368         0   -1.4584
         0   -0.1067         0         0    4.4811         0
    1.4584         0    1.4584   -1.4584         0    3.2296
```

See that twist and out-of-plane bending are uncoupled.