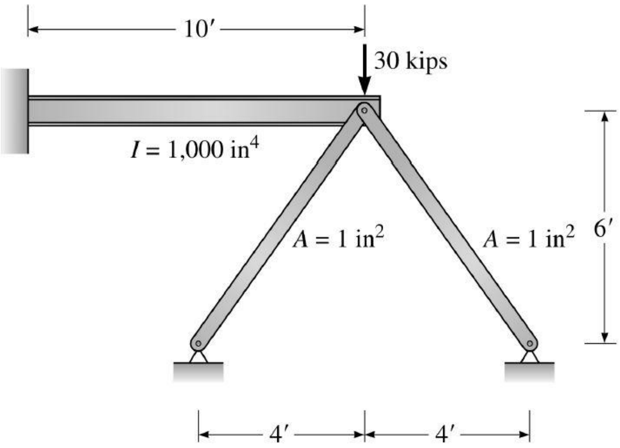
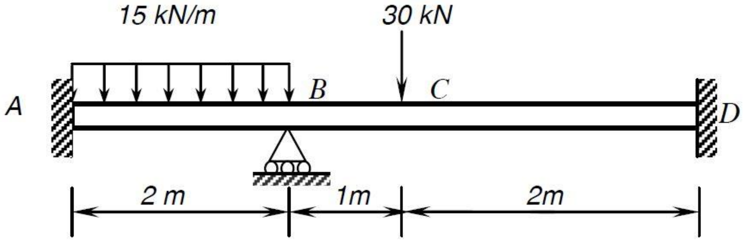
CE317 Tutorial 6 (problems on Beams, Frames, and combination with Truss)

Analyze the following structures using direct stiffness method:

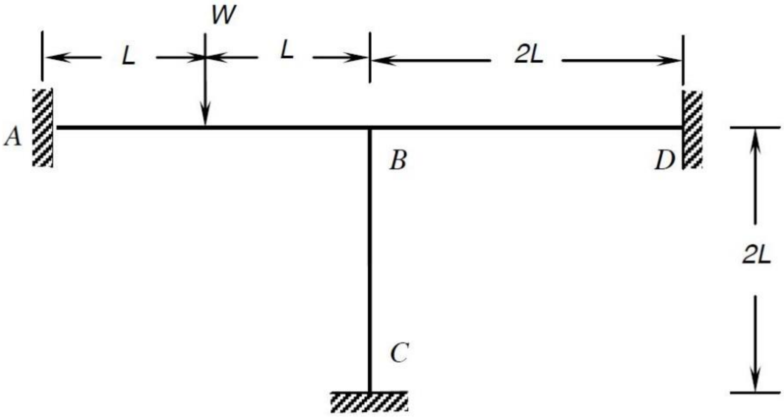
1.



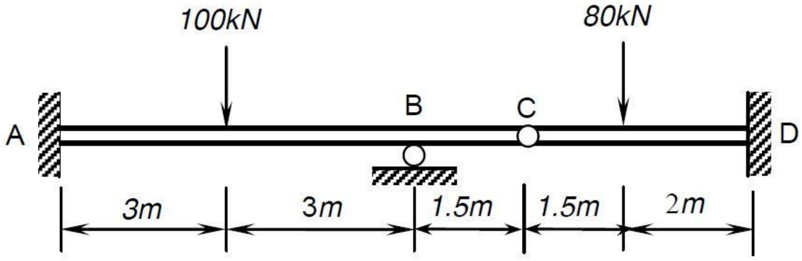
2.



3.

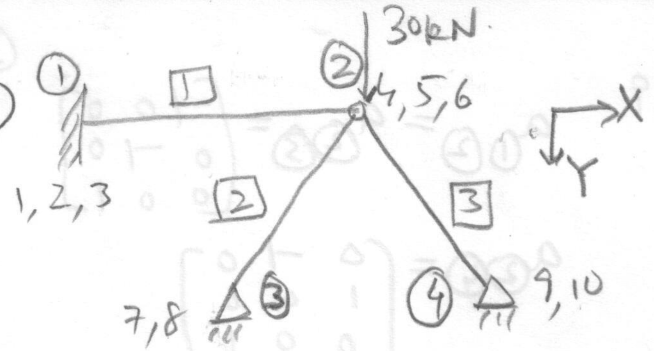


4.



①

1)



$a_{②③} = \left(\frac{4}{\sqrt{52}} \quad \frac{-6}{\sqrt{52}}\right)$, $a_{②④} = \left(\frac{-4}{\sqrt{52}} \quad \frac{-6}{\sqrt{52}}\right)$

$$a_{①②} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$I = 1e9\ mm^4$

$A = 100\ mm^2$

$E = 1E5\ MPa$

$\frac{A}{E} = 1e{-1}\ m^{-2}$

$$K_{II} = EI \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 4 & \frac{1}{10}\frac{1}{10} + \frac{1}{10}\frac{1}{\sqrt{52}}\cdot\frac{16}{52} * 2 & \overbrace{\frac{1}{10}\frac{1}{\sqrt{52}}\left(\frac{-24}{52}\right) + \frac{1}{10}\frac{1}{\sqrt{52}}\left(\frac{24}{52}\right)}^{0\text{, so beam formulation for } \boxed{1} \text{ will also work}} & \\ 5 & & \frac{12}{10^3} + \frac{1}{10}\frac{1}{\sqrt{52}}\frac{36}{52} * 2 & \frac{-6}{10^2} \\ 6 & \text{symm} & & \frac{4}{10} \end{array}$$

$$K_{II\,I} = \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & -\frac{1}{10}\frac{1}{10} & & \\ 2 & & \frac{-12}{10^3} & \frac{6}{10^2} \\ 3 & & -\frac{6}{10^2} & \frac{2}{10} \\ 7 & \frac{1}{10}\frac{1}{\sqrt{52}}\left(\frac{-16}{52}\right) & \frac{1}{10}\frac{1}{\sqrt{52}}\frac{24}{52} & 0 \\ 8 & \frac{1}{10}\frac{1}{\sqrt{52}}\frac{24}{52} & \frac{1}{10}\frac{1}{\sqrt{52}}\left(\frac{-36}{52}\right) & 0 \\ 9 & \frac{1}{10}\frac{1}{\sqrt{52}}\left(-\frac{16}{52}\right) & \frac{1}{10}\frac{1}{\sqrt{52}}\left(\frac{-24}{52}\right) & 0 \\ 10 & \frac{1}{10}\frac{1}{\sqrt{52}}\left(\frac{-24}{52}\right) & \frac{1}{10}\frac{1}{52}\left(\frac{-36}{52}\right) & 0 \end{array} EI$$

$$P_I = \begin{pmatrix} 0 \\ 30E3 \\ 0 \end{pmatrix} \qquad \Delta_{II} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Delta_I = K_{II}^{-1} P_I = \left[0\ ,\ \underset{m}{0.0135128}\ ,\ \underset{rad}{0.0020269}\right]^T$$

$$P_{II} = K_{II\,I}\Delta_I = \left(\left[\underset{kN}{0}\ \ \underset{kN}{-0.4054}\ \ \underset{kNm}{-4.054}\ \ \underset{kN}{0.8649}\ \ \underset{kN}{-1.2973}\ \ -0.8649\ \ -1.2973\right]\right) * 10$$

By beam formulation for $\boxed{1}$, remove d.o.f's 1, 4:

$$K_{II} = \begin{array}{c|cc} & 5 & 6 \\ \hline 5 & & \\ 6 & & \end{array}, \quad K_{II\,I} = \begin{array}{c|cc} & 5 & 6 \\ \hline 2 & & \\ 3 & & \\ 7 & & \\ 8 & & \\ 9 & & \\ 10 & & \end{array} \quad \text{where } K^{\boxed{1}} = \begin{array}{c|cccc} & 2 & 3 & 5 & 6 \\ \hline 2 & & & & \\ 3 & & & & \\ 5 & & & & \\ 6 & & & & \end{array}$$

$$K^{\boxed{2}} = \begin{array}{c|cccc} & 4 & 5 & 7 & 8 \\ \hline 4 & & & & \\ 5 & & & & \\ 7 & & & & \\ 8 & & & & \end{array}; \quad K^{\boxed{3}} = \begin{array}{c|cccc} & 4 & 5 & 9 & 10 \\ \hline 4 & & & & \\ 5 & & & & \\ 9 & & & & \\ 10 & & & & \end{array}$$

$F_{(12)} = k_{11}^{2} a_{12} \Delta_1 + k_{12} a_{21} \Delta_2$, $\Delta_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\Delta_2 = \Delta_{I}$,

$$F_{12} = EI \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ & \frac{12}{10^3} & \frac{-6}{10^2} \\ & -\frac{6}{10^2} & \frac{2}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.0135128 \\ 0.0020269 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4054 \\ -4.054 \end{bmatrix} * 10^4$$

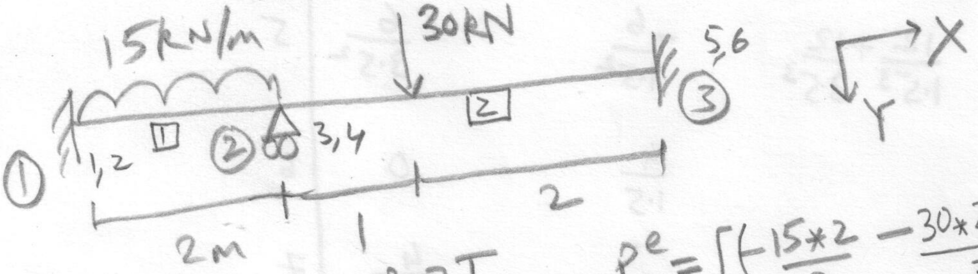
$$F_{21} = EI \begin{bmatrix} 1/10 & 0 & 0 \\ 0 & 12/10^3 & -6/10^2 \\ 0 & -6/10^2 & 4/10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \swarrow \end{bmatrix} = \begin{pmatrix} 0 \\ 4.054 \\ 0 \end{pmatrix} * 10^4$$

$$F_{23} = F_{32} = \frac{AE}{\sqrt{52}} \begin{bmatrix} \frac{4}{\sqrt{52}} & \frac{-6}{\sqrt{52}} \end{bmatrix} \begin{bmatrix} 0 \\ 0.0135128 \end{bmatrix} = -1.5592 * 10^4$$

$$F_{24} = F_{42} = \frac{AE}{\sqrt{52}} \begin{bmatrix} \frac{-4}{\sqrt{52}} & \frac{-6}{\sqrt{52}} \end{bmatrix} \begin{bmatrix} \swarrow \end{bmatrix} = \swarrow \text{same}$$

use E, I as in P.1

2)



$$p^e_{①} = -\left[\frac{15*2}{2}, -\frac{15*2^2}{12}\right]^T, \quad p^e_{②} = \left[\left(-\frac{15*2}{2} - \frac{30*2^2*(3*1+2)}{3^3}\right), \left(\frac{15*2^2}{12} - \frac{30*2^2*1}{3^2}\right)\right]^T$$

$$p^e_{③} = \left[-\frac{30*1^2*(3*2+1)}{3^3}, \frac{30*1^2*2}{3^2}\right]^T$$

$$p^e_{①} = [-15, -5]^T, \quad p^e_{②} = \left(-\frac{335}{9}, \frac{-25}{3}\right)^T, \quad p^e_{③} = \left[-\frac{70}{9}, \frac{20}{3}\right]^T$$

$$\tilde{p}^e = \left[\underset{4}{\frac{-25}{3}}\right], \quad \hat{p}^e = \left[\underset{1}{-15}, \underset{2}{-5}, \underset{3}{-\frac{335}{9}}, \underset{5}{-\frac{70}{9}}, \underset{6}{\frac{20}{3}}\right]^T$$

$$P_I = [0] - \left(-\frac{25}{3}\right] = \left[\frac{25}{3}\right]$$

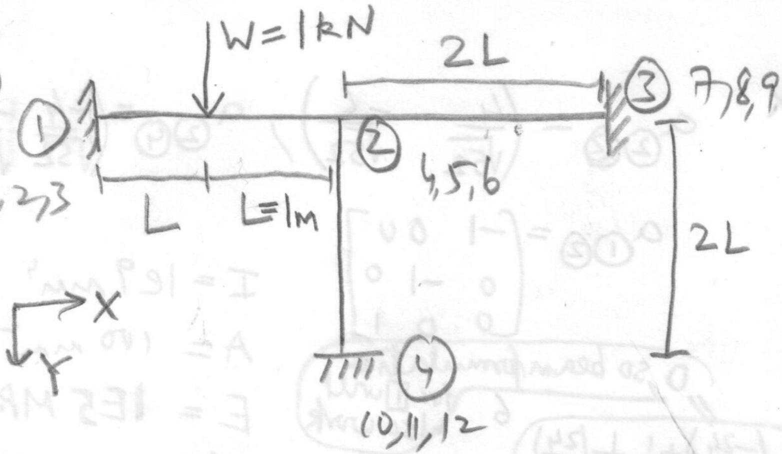
$$K_{II} = EI \begin{array}{c|c} & 4 \\ \hline & \\ 4 & \left[\frac{4}{2} + \frac{4}{3}\right] \end{array} = \left[\frac{10}{3}\right] EI$$

$$K_{III} = \begin{array}{cccccc} & 1 & 2 & 3 & 5 & 6 \\ 4 & \frac{6}{2^2}, & \frac{2}{2}, & \left(-\frac{6}{2^2} + \frac{6}{3^2}\right) & \frac{-6}{3^2} & \frac{2}{3} \end{array}^T$$

$$\Delta_I = K_{II}^{-1} P_I = \frac{3}{10} * \frac{25}{3} \text{kN} \frac{1}{EI} = \frac{25}{10} \frac{1}{10^8 . 10^{-3}} = 2.5\text{E-}5 \text{ rad}$$

$$P_{II} = K_{III} \Delta_I + \hat{p}^e = \left[\underset{\text{kN}}{-11.25}, \underset{\text{kNm}}{-2.5}, \underset{\text{kN}}{-39.3056}, \underset{\text{kN}}{-9.444}, \underset{\text{kNm}}{8.333}\right]^T,$$

3)



$$a_{①②} = a_{②③} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{②④} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E, I, A, as in P.1

$$\tilde{p}^e = \begin{bmatrix} \underset{4}{0}, & \underset{5}{-\frac{W}{2}}, & \underset{6}{\frac{W(2L)}{8}} \end{bmatrix}^T, \quad \hat{p}^e = \begin{bmatrix} \underset{1}{0}, & \underset{2}{-\frac{W}{2}}, & \underset{3}{-\frac{W(2L)}{8}}, & \underset{7}{0}, & \underset{8}{0}, & \underset{9}{0}, & \underset{10}{0}, & \underset{11}{0}, & \underset{12}{0} \end{bmatrix}^T$$

$$K_{II} = \frac{EI}{2L} \left[\begin{array}{ccc|c} 4 & 5 & 6 & \\ \frac{1}{10}+\frac{1}{10}+\frac{12}{4L^2} & & -\frac{6}{2L} & 4 \\ & \frac{12}{4L^2}+\frac{12}{4L^2}+\frac{1}{10} & -\frac{6}{2L}+\frac{6}{2L} & 5 \\ \text{symm} & & 4+4+4 & 6 \end{array}\right]$$

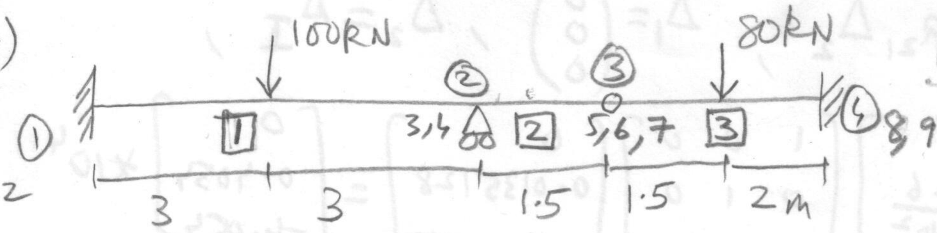
$$K_{III} = \left[\begin{array}{c|ccc} & 4 & 5 & 6 \\ 1 & -\frac{1}{10} & 0 & 0 \\ 2 & 0 & -\frac{12}{4L^2} & \frac{6}{2L} \\ 3 & 0 & -\frac{6}{2L} & 2 \\ 7 & -\frac{1}{10} & 0 & 0 \\ 8 & 0 & -\frac{12}{4L^2} & -\frac{6}{2L} \\ 9 & 0 & \frac{6}{2L} & 2 \\ 10 & -\frac{12}{4L^2} & 0 & \frac{6}{2L} \\ 11 & 0 & -\frac{1}{10} & 0 \\ 12 & -\frac{6}{2L} & 0 & 2 \end{array}\right] \frac{EI}{2L}$$

$$P_I = [0,0,0]^T - \tilde{p}^e = -\tilde{p}^e$$

$$D_I = K_{II}^{-1} P_I = \begin{bmatrix} \underset{m}{\overset{\xrightarrow{\text{nonzero}}}{-0.05102}}, & \underset{m}{0.16393}, & \underset{rad}{-0.054422} \end{bmatrix} * 10^{-5}$$

$$P_{II} = \begin{bmatrix} \underset{kN}{0.002551}, & \underset{kN}{-0.8275}, & \underset{kNm}{-0.5503}, & \underset{kN}{0.002551}, & \underset{kN}{-0.1643}, & \underset{kNm}{0.1915}, & \underset{kN}{-0.005102}, & \overset{kN}{-0.008197}, & 0.02211\,kNm \end{bmatrix}^T$$

4)



E, I, as in P.1

$$\tilde{p}^e = \left[\underset{4}{\frac{100*6}{8}}, \underset{5}{\frac{-80*2^2(3*1.5+2)}{3.5^3}}, \underset{6}{0}, \underset{7}{\frac{-80*1.5*2^2}{3.5^2}}\right]^T, \quad P_I = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} - \tilde{p}^e = -\tilde{p}^e$$

$$\hat{p}^e = \left[\underset{1}{-\frac{100}{2}}, \underset{2}{-\frac{100*6}{8}}, \underset{3}{-\frac{100}{2}}, \underset{78}{-\frac{80*1.5^2(3*2+1.5)}{3.5^3}}, \underset{9}{\frac{80*1.5^2*2}{3.5^2}}\right]$$

$$K_{II} = EI \begin{array}{c} \\ \left[\begin{array}{cccc} 4 & 5 & 6 & 7 \\ \frac{4}{6}+\frac{4}{1.5} & \frac{-6}{1.5^2} & \frac{2}{1.5} & 0 \\ & \frac{12}{1.5^3}+\frac{12}{3.5^3} & -\frac{6}{1.5^2} & \frac{6}{3.5^2} \\ & & \frac{4}{1.5} & 0 \\ \text{symm} & & & \frac{4}{3.5} \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

$$K_{III} = \begin{array}{c|cccc} & 4 & 5 & 6 & 7 \\ \hline 1 & 6/6^2 & 0 & 0 & 0 \\ 2 & 2/6 & 0 & 0 & 0 \\ 3 & (-6/6^2+6/1.5^2) & -12/1.5^3 & 6/1.5^2 & 0 \\ 8 & 0 & -12/3.5^3 & 0 & -6/3.5^2 \\ 9 & 0 & 6/3.5^2 & 0 & 2/3.5 \end{array} \; EI$$

$$\Delta_I = K_{II}^{-1} P_I = [\underset{\text{rad}}{-0.3801}, \underset{\text{m}}{-0.1978}, \underset{\text{rad}}{-0.007733}, \underset{\text{rad}}{0.4276}] * 10^{-3}$$

$$P_{II} = [\underset{\text{kN}}{-56.336}, \underset{\text{kNm}}{-87.672}, \underset{\text{kN}}{-76.768}, \underset{\text{kN}}{-46.896}, \underset{\text{kNm}}{44.135}]$$