# Skewed Supports.

## Method-I : Transforming global coords ie $K^* \Delta^* = P^*$ $\boxed{1}$

(This is just a formalization of the method used in solving the midsem 2011 problem)

$\Delta \rightarrow$ structural displs in single global coord system

$\Delta^* \rightarrow$ struct displs in multiple global systems (ie diff global coord at each skewed support).

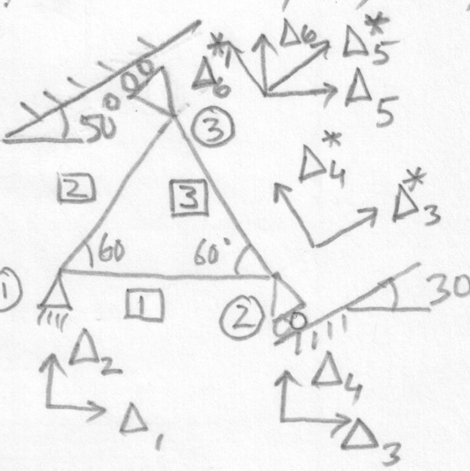
$\Delta = T \Delta^*$, $T \rightarrow$ transf matrix

$K\Delta = P \longrightarrow KT\Delta^* = P \longrightarrow T^T K T \Delta^* = T^T P$

$K^* \Delta^* = P^*$

(note $T^T = T^{-1}$ used)

Here $K, K^*, \Delta, \Delta^*, P, P^*$, of size $n$




$$T = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cos 30 & -\sin 30 & & \\ & & \sin 30 & \cos 30 & & \\ & & & & \cos 50 & -\sin 50 \\ & & & & \sin 30 & \cos 50 \end{bmatrix}$$

Instead of transforming full assembled $K$ to $K^*$, can do transformation during assembly, i.e.,

$$\Delta_e = \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} = \underbrace{T_{ij}}_{T_e} \begin{Bmatrix} \Delta_i^* \\ \Delta_j^* \end{Bmatrix} \longrightarrow K_e \Delta_e = P_e \longrightarrow K_e T_e \Delta_e^* = P_e$$

$\rightarrow T_e^T K_e T_e \Delta_e^* = T_e^T P_e$

$K_e^* \Delta_e^* = P_e^*$

Then do assembly.

$K_e$: element stiffness matrix in global coords

$P_e$: element end forces in global coords

$$K_e = \begin{bmatrix} a_{ij}^T k_{ii}^j a_{ij} & a_{ij}^T k_{ij} a_{ji} \\ a_{ji}^T k_{ji} a_{ij} & a_{ji}^T k_{jj}^i a_{ji} \end{bmatrix}$$

For given example,

$$T_{①③} = T_{\boxed{2}} = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \cos 50 & -\sin 50 \\ & & \sin 50 & \cos 50 \end{bmatrix} \equiv \begin{bmatrix} T_① & 0 \\ 0 & T_③ \end{bmatrix}$$

$$a_{①③} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$K_{\boxed{2}R} = \begin{array}{cccc} \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 \end{array}$$
$$\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ & & & \end{bmatrix}$$

Let $K_{\boxed{2}} = \left[\begin{array}{c|c} A & B \\ \hline B & C \end{array}\right]$, here $C = A$.

$$\begin{array}{cccc} \Delta_1^* & \Delta_2^* & \Delta_3^* & \Delta_4^* \end{array}$$

$$K^*_{\boxed{2}} = \begin{bmatrix} T_{1}^T A & T_{1}^T B \\ T_{3}^T B & T_{3}^T C \end{bmatrix} \begin{bmatrix} T_{1} & 0 \\ 0 & T_{3} \end{bmatrix} = \begin{bmatrix} T_{1}^T A T_{1} & T_{1}^T B T_{3} \\ T_{3}^T B T_{1} & T_{3}^T C T_{3} \end{bmatrix}$$

Note $\Delta_1 = \Delta_1^*$, $\Delta_2 = \Delta_2^*$ in this example.

<u>Method-II</u> : <u>Using Constraint Equations</u> (Multi-point Constraints MPCs)

Split $\Delta = \left\{ \begin{array}{c} \Delta_F \\ \hline \Delta_D \end{array} \right\}$ $\begin{array}{l} \rightarrow \text{Free, ie independent, size } f \\ \rightarrow \text{dependent on } U_F \text{ through } d \text{ constraint eqns, size } d. \end{array}$

$$\left\{ \begin{array}{c} \Delta_F \\ \hline \Delta_D \end{array} \right\} = A \{ \Delta_F \}$$

$K \rightarrow n \times n$, $\Delta_F \rightarrow f \times 1$, $\Delta_D \rightarrow d \times 1$, $A \rightarrow (f+d) \times f$, $n = (f+d)$

$$K\Delta = P \xrightarrow[\text{to bring } \Delta_F \text{ to top.}]{\text{rearrange}} K^{\#} \left\{ \begin{array}{c} \Delta_F \\ \hline \Delta_D \end{array} \right\} = P^{\#} \longrightarrow K^{\#} A \Delta_F = P^{\#}$$

$$\longrightarrow A^T K^{\#} A \Delta_F = A^T P^{\#}$$

$$\longrightarrow K^* \Delta_F = P^*$$

For given example,
First assemble structural $K$ in single $\overset{Y}{\uparrow}\!\rightarrow_X$ global coords.

$$\Delta = \left\{ \begin{array}{c} \Delta_F \\ \hline \Delta_D \end{array} \right\} = \{ \underbrace{\Delta_1 \quad \Delta_2 \quad \Delta_3 \quad \Delta_5}_{\Delta_F^T} \mid \underbrace{\Delta_4 \quad \Delta_6}_{\Delta_D^T} \}^T$$

constraint eqs $\rightarrow \Delta_4 = \Delta_3 \tan 30$ ; $\Delta_6 = \Delta_5 \tan 50$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tan 30 & 0 \\ 0 & 0 & 0 & \tan 50 \end{bmatrix} \leftarrow (6 \times 4)$$

$$K \xrightarrow[\substack{4^{th} \,\&\, 5^{th} \text{ rows} \\ \text{and cols}}]{\text{interchange}} K^{\#} ; \quad P \xrightarrow[\substack{4^{th} \,\&\, 5^{th} \\ \text{rows}}]{\text{interchange}} P^{\#}$$

Generalization to inhomogenous constraints (Cook p. 489-492): ③

$$C\Delta - Q = 0 \longrightarrow \left[C_F \mid C_D\right]\left\{\frac{\Delta_F}{\Delta_D}\right\} - \{Q\} = \{0\}$$

$C$: $c \times n$

$Q$: inhomogenous part, $c \times 1$

$c$ = nos of constraints

$n$ = nos of d.o.f. in $K\Delta = P$

$C_D$ is $c \times c$ (square) and invertible

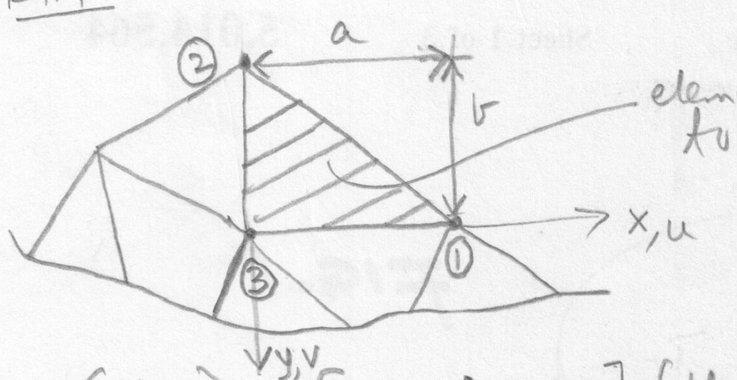
$$\Rightarrow \Delta_D = C_D^{-1}\left[Q - C_F\Delta_F\right] \Rightarrow \left\{\frac{\Delta_F}{\Delta_D}\right\} = \begin{Bmatrix} I \\ -C_D^{-1}C_F \end{Bmatrix}\{\Delta_F\} + \begin{Bmatrix} 0 \\ C_D^{-1}Q \end{Bmatrix}$$

$$\Delta = A\Delta_F + Q_0$$

can form this directly also as done in Ex 1, 2, 3

$$K^{\#}\Delta = P^{\#} \Rightarrow \underbrace{A^T K^{\#} A}_{K^*}\Delta_F = \underbrace{A^T\left(P^{\#} - K^{\#}Q_0\right)}_{P^*} \Rightarrow K^*\Delta_F = P^*.$$

Ex 1




elements with high stiffness compared to neighboring elements, ie tending to rigid. So introduce rigidity via MPC's, as follows:

$$\underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}}_{\text{slave}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a/b & 1 & -a/b \\ 1 & 0 & 0 \\ a/b & 1 & -a/b \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \end{Bmatrix}}_{\text{master}}$$

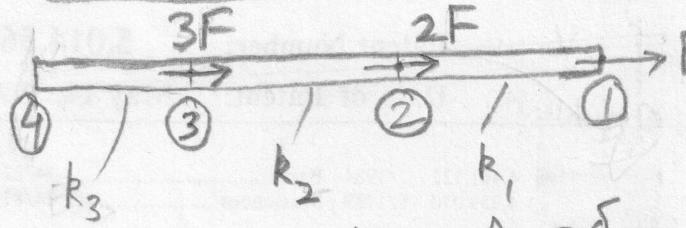
Note: rigid part has 3-d.o.f's, 2 translations and 1 rotation. Hence we choose $\{u_1, v_1, u_2\}^T$ as master d.o.fs. Thus, constraint kinematics is

$$\begin{cases} v_2 = v_1 - a\theta = v_1 - a\left(\dfrac{u_2 - u_1}{b}\right) \\ u_3 = u_1 \\ v_3 = v_1 - a\theta = v_1 - a\left(\dfrac{u_2 - u_1}{b}\right) \end{cases}$$

Note: when finding $K^*$, the elastic properties of rigid part (ie element 1-2-3 above) dont affect $K^*$ (ie they dont matter), since the constraint implying rigidity overides these elastic properties. So you find K structure assuming some elastic properties for ALL its elements, then apply constraint $\Delta = A\,\Delta_F$, and you will find that elastic properties of rigid elements dont appear in $K^* = A^T K^\# A$, as it ought to be, since constraint overides assumed elastic properties of rigid elements. THIS IS DEMONSTRATED IN THE EXAMPLES THAT FOLLOW.

NOTE: We could also have used $\{u_1, v_1, v_3\}^T$ etc as the master d.o.f's and gotten a different $\underline{A}$. But obviously cant use $\{u_1, u_2, u_3\}^T$ etc as that wont yield

## Ex2 Axial bar




$$k_i = \frac{EA_i}{L_i}$$

MPC : $\Delta_3 = 2\Delta_1 , \ \Delta_4 = \delta$

$$\underbrace{\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix}}_{\Delta} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix}}_{\Delta_F} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix}}_{Q_0} ; \quad K\Delta = P , \quad K = K^{\#} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}$$

$$A^T K^{\#} = \begin{bmatrix} k_1 & -k_1-2k_2 & 2(k_2+k_3) & -2k_3 \\ -k_1 & (k_1+k_2) & -k_2 & 0 \end{bmatrix}$$

$$A^T K^{\#} A = K^* = \begin{bmatrix} k_1+4(k_2+k_3) & -k_1-2k_2 \\ -k_1-2k_2 & k_1+k_2 \end{bmatrix}$$

$$P = P^{\#} = \{F \ \ 2F \ \ 3F \ \ 0\}^T$$

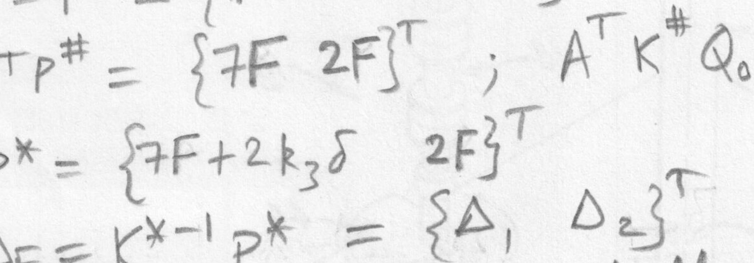
$$A^T P^{\#} = \{7F \ \ 2F\}^T ; \quad A^T K^{\#} Q_0 = \{-2k_3\delta \ \ 0\}^T$$

$$P^* = \{7F + 2k_3\delta \quad 2F\}^T$$

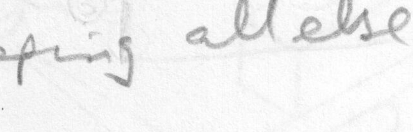
$$\Delta_F = K^{*-1} P^* = \{\Delta_1 \quad \Delta_2\}^T$$ (see MATLAB results)

If we change one of the constraints to $\Delta_3 = \Delta_1$ instead of $\Delta_3 = 2\Delta_1$, ~~keeping~~ all else same, we get $\Delta_1 = \dfrac{6F}{k_3} + \delta$

Note, when $\Delta_3 = \Delta_1$ the physical problem is




for which physically you can see that




$\Delta_3 = \Delta_1 = \dfrac{6F}{k_3} + \delta$

```
% MPC Ex2

syms k1 k2 k3 p d f;

A=[1 0; 0 1; 2 0; 0 0];

Q0=[0; 0; 0; d];

K=[k1 -k1 0 0; -k1 k1+k2 -k2 0; 0 -k2 k2+k3 -k3; 0 0 -k3 k3];

Kpound=K;

Kstar=A'*Kpound*A;

P = [f; 2*f; 3*f; 0];

Ppound=P;

Q0=[0; 0; 0; d];

Pstar=A'*(Ppound-Kpound*Q0);

        Pstar =

         7*f + 2*d*k3

              2*f

Delta=Kstar^-1*Pstar;

        Delta =

        ((k1 + k2)*(7*f + 2*d*k3))/(k1*k2 + 4*k1*k3 + 4*k2*k3) + (2*f*(k1 + 2*k2))/(k1*k2 +
        4*k1*k3 + 4*k2*k3)

        (2*f*(k1 + 4*k2 + 4*k3))/(k1*k2 + 4*k1*k3 + 4*k2*k3) + ((7*f + 2*d*k3)*(k1 + 2*k2))/(k1*k2
        + 4*k1*k3 + 4*k2*k3)
```

If we take MPC with A=[1 0; 0 1; 1 0; 0 0], i.e., rigid link between nodes 1 and 3, we get

```
        Delta =

         (4*f + d*k3)/k3 + (2*f)/k3

        (4*f + d*k3)/k3 + (2*f*(k1 + k2 + k3))/(k1*k3 + k2*k3)
```

As it should be physically.

Alternatively do elementwise transformation and then assemble. $[\Delta_1, \Delta_2]^T$ are master d.o.f's and $[\Delta_3 \;\; \Delta_4]^T$ are slaves. MPC's are:

MPC's: $$\begin{Bmatrix} \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} \; ; \quad \begin{Bmatrix} \Delta_3 \\ \Delta_4 \end{Bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \delta \end{Bmatrix}$$

(inhomogenous constraint)

Element equations in global coords are:

El $\boxed{1}$:
$$K_{\boxed{1}} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} \cancel{P_{12}}\, F \\ P_{21} \end{Bmatrix} = \begin{Bmatrix} F \\ P_{21} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} F \\ P_{21} \end{Bmatrix}$$

El $\boxed{2}$:
$$K_{\boxed{2}} \begin{Bmatrix} \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{Bmatrix} P_{23} \\ P_{32} \end{Bmatrix} \Rightarrow \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} P_{23} \\ P_{32} \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2k_2 & 2k_2 \\ k_2 & -k_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 2P_{32} \\ P_{23} \end{Bmatrix} \Rightarrow \begin{bmatrix} 4k_2 & -2k_2 \\ -2k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 2P_{32} \\ P_{23} \end{Bmatrix}$$

El $\boxed{3}$:
$$K_{\boxed{3}} \begin{Bmatrix} \Delta_3 \\ \Delta_4 \end{Bmatrix} = \begin{Bmatrix} P_{34} \\ P_{43} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \delta \end{Bmatrix} \right\} = \begin{Bmatrix} P_{34} \\ P_{43} \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} P_{34} \\ \underset{\to 0}{P_{43}} \end{Bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ \delta \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4k_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 2P_{34} + 2k_3\delta \\ 0 \end{Bmatrix}$$

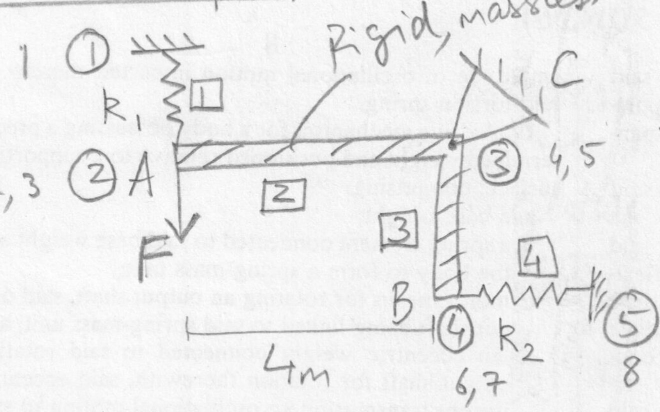
Assembly:
$$\underbrace{\begin{bmatrix} k_1 + 4k_3 + 4k_2 & -k_1 - 2k_2 \\ -k_1 - 2k_2 & k_1 + k_2 \end{bmatrix}}_{K^*} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 2P_{34} + 2P_{32} + F + 2k_3\delta \\ P_{23} + P_{21} \end{Bmatrix}$$

$$P_{23} + P_{21} = 2F$$
$$P_{34} + P_{32} = 3F$$
$$\Rightarrow K^* \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 7F + 2k_3\delta \\ 2F \end{Bmatrix}$$

as before when done by transforming assembled system

# MPC examples.

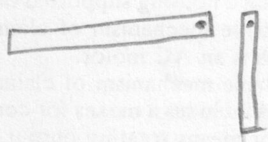
**Ex3**




1, 3, 5, 7 are ↑ d.o.f

2, 4, 6, 8 are → d.o.f.

## Done as Truss

So elements $\boxed{2}$, $\boxed{3}$ considered as  i.e

internal hinge at node 3.

Element stiffness matrices are, $\left\{ \text{use } a_{(2)(3)} = [-1 \;\; 0],\; a_{(3)(4)} = [0 \;\; -1] \right\}$

$$K_{\boxed{1}} = \begin{array}{c} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \\ \begin{matrix} 1 & \quad 3 \end{matrix} \end{array} ; \quad K_{\boxed{2}} = \begin{array}{c} \begin{bmatrix} k_4 & 0 & -k_4 & 0 \\ 0 & 0 & 0 & 0 \\ -k_4 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 2 & \quad 3 & \quad 4 & \quad 5 \end{matrix} \end{array} ; \quad K_{\boxed{3}} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \\ \begin{matrix} 4 & \quad 5 & \quad 6 & \quad 7 \end{matrix} \end{array}$$

$$K_{\boxed{4}} = \begin{array}{c} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \\ \begin{matrix} 6 & \quad 8 \end{matrix} \end{array}$$

Here $k_4 = \dfrac{EA}{4}$, $k_3 = \dfrac{EA}{3}$.

Assembly:

$\left( \Delta_1 = \Delta_2 = \Delta_4 = \Delta_5 = \Delta_7 = \Delta_8 = 0 \right)$

$$K = \begin{bmatrix} k_1 & & -k_1 & & & & & \\ & k_4 & 0 & -k_4 & 0 & & & \\ -k_1 & 0 & k_1+0 & 0 & 0 & & & \\ & -k_4 & 0 & k_4+0 & 0+0 & 0 & 0 & \\ & 0 & 0 & 0+0 & 0+k_3 & 0 & -k_3 & \\ & & & 0 & 0 & 0+k_2 & 0 & -k_2 \\ & & & 0 & -k_3 & 0 & k_3 & \\ & & & & & -k_2 & & k_2 \end{bmatrix}$$

$$K_{II} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}; \quad \text{MPC}: \begin{Bmatrix} \Delta_3 \\ \Delta_6 \end{Bmatrix} = \underbrace{\begin{Bmatrix} 1 \\ \frac{3}{4} \end{Bmatrix}}_{A} \{\Delta_1\}.$$

9

$$K_{II}\begin{Bmatrix}\Delta_3\\ \Delta_6\end{Bmatrix}=\begin{Bmatrix}F\\ 0\end{Bmatrix} \Rightarrow A^T K_{II} A\{\Delta_3\}=A^T\begin{Bmatrix}F\\ 0\end{Bmatrix}$$

$$\Rightarrow \left[k_1 \;\; \tfrac{3}{4}k_2\right]\begin{Bmatrix}1\\ 3/4\end{Bmatrix}\{\Delta_3\}=\{F\} \Rightarrow \Delta_3=\frac{F}{k_1+\frac{9}{16}k_2}$$

$$\Delta_6=\frac{3F}{4k_1+\frac{9}{4}k_2}$$

$$F_{(2)(1)}=k_1(\Delta_3-\underbrace{\Delta_1}_{0})=\frac{F}{1+\frac{9}{16}\frac{k_2}{k_1}}, \quad F_{(4)(5)}=k_2(\underbrace{\Delta_8}_{0}-\Delta_6)=\frac{3F}{4\frac{k_1}{k_2}+\frac{9}{4}}$$

Alternatively, MPC is

$$\Delta^\# = \begin{Bmatrix}\Delta_3\\ \Delta_1\\ \Delta_2\\ \Delta_4\\ \Delta_5\\ \Delta_6\\ \Delta_7\\ \Delta_8\end{Bmatrix}=\underbrace{\begin{bmatrix}1\\ 0\\ 0\\ 0\\ 0\\ 3/4\\ 0\\ 0\end{bmatrix}}_{A}\{\Delta_3\}, \quad K^\#\Delta^\#=P^\#, \quad P=\begin{Bmatrix}P_1\\ P_2\\ P_3=F\\ P_4\\ P_5\\ P_6=0\\ P_7\\ P_8\end{Bmatrix}$$

$$K \xrightarrow[\text{1st, 3rd rows/cols}]{\text{interchange}} K^\#$$

$$P \xrightarrow[\text{1st, 3rd rows}]{\text{interchange}} P^\#$$

$$A^T K^\# A\{\Delta_3\}=A^T P^\#$$

$$\Rightarrow \left[k_1 \;\; 0 \;\; -k_1 \;\; 0 \;\; 0 \;\; \tfrac{3}{4}k_2 \;\; 0 \;\; -\tfrac{3}{4}k_2\right]A=\{F\} \Rightarrow \left(k_1+\frac{9}{16}k_2\right)\Delta_3=F$$

So we get same result.

Alternatively do MPC elementwise transformation & then assembly.

<u>Element $\boxed{1}$</u>: MPC $\rightarrow \begin{Bmatrix}\Delta_1\\ \Delta_3\end{Bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}\{\Delta_3\} = A_{\boxed{1}}\{\Delta_3\}$ ; $K_{\boxed{1}}\begin{Bmatrix}\Delta_1\\ \Delta_3\end{Bmatrix} = \begin{Bmatrix}P^2_{12}\\ P^2_{21}\end{Bmatrix}$

$$\Rightarrow A^T_{\boxed{1}} K_{\boxed{1}} A_{\boxed{1}}\{\Delta_3\} = A^T_{\boxed{1}}\begin{Bmatrix}P^2_{12}\\ P^2_{21}\end{Bmatrix} \Rightarrow \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}k_1 & -k_1\\ -k_1 & k_1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\{\Delta_3\} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{Bmatrix}P^2_{12}\\ P^2_{21}\end{Bmatrix}$$

$$\Rightarrow k_1\Delta_3 = P^2_{21} \rightarrow \underline{El\,\boxed{1}}$$

<u>Element $\boxed{2}$</u>: MPC $\rightarrow \begin{Bmatrix}\Delta_2\\ \Delta_3\\ \Delta_4\\ \Delta_5\end{Bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\{\Delta_3\} = A_{\boxed{2}}\{\Delta_3\}$ ; $K_{\boxed{2}}\begin{Bmatrix}\Delta_2\\ \Delta_3\\ \Delta_4\\ \Delta_5\end{Bmatrix} = \begin{Bmatrix}P^1_{23}\\ P^2_{23}\\ P^1_{32}\\ P^2_{32}\end{Bmatrix}$

$$\Rightarrow A^T_{\boxed{2}} K_{\boxed{2}} A_{\boxed{2}}\{\Delta_3\} = A^T_{\boxed{2}}\begin{Bmatrix}P^1_{23}\\ P^2_{23}\\ P^1_{32}\\ P^2_{32}\end{Bmatrix} \Rightarrow \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}k_4 & 0 & -k_4 & 0\\ 0 & 0 & 0 & 0\\ -k_4 & 0 & k_4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\{\Delta_3\} = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}P^1_{23}\\ P^2_{23}\\ P^1_{32}\\ P^2_{32}\end{Bmatrix}$$

$$\Rightarrow 0.\Delta_3 = P^2_{23} \longrightarrow El\,\boxed{2}$$

<u>Element $\boxed{3}$</u>: MPC $\rightarrow \begin{Bmatrix}\Delta_4\\ \Delta_5\\ \Delta_6\\ \Delta_7\end{Bmatrix} = \begin{bmatrix}0\\0\\3/4\\0\end{bmatrix}\{\Delta_3\} = A_{\boxed{3}}\{\Delta_3\}$ ; $K_{\boxed{3}}\begin{Bmatrix}\Delta_2\\ \Delta_3\\ \Delta_4\\ \Delta_5\end{Bmatrix} = \begin{Bmatrix}P^1_{34}\\ P^2_{34}\\ P^1_{43}\\ P^2_{43}\end{Bmatrix}$

$$\Rightarrow A^T_{\boxed{3}} K_{\boxed{3}} A_{\boxed{3}}\{\Delta_3\} = A^T_{\boxed{3}}\begin{Bmatrix}P^1_{34}\\ P^2_{34}\\ P^1_{43}\\ P^2_{43}\end{Bmatrix} \Rightarrow \begin{bmatrix}0 & 0 & \frac{3}{4} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & k_3 & 0 & -k_3\\ 0 & 0 & 0 & 0\\ 0 & -k_3 & 0 & k_3\end{bmatrix}\begin{bmatrix}0\\0\\3/4\\0\end{bmatrix}\{\Delta_3\} = \begin{bmatrix}0 & 0 & \frac{3}{4} & 0\end{bmatrix}\begin{Bmatrix}P^1_{34}\\ P^2_{34}\\ P^1_{43}\\ P^2_{43}\end{Bmatrix}$$

$$\Rightarrow 0.\Delta_3 = \frac{3}{4}P^1_{43} \longrightarrow El\,\boxed{3}$$

<u>Element $\boxed{4}$</u>: MPC $\rightarrow \begin{Bmatrix}\Delta_6\\ \Delta_8\end{Bmatrix} = \begin{bmatrix}3/4\\0\end{bmatrix}\{\Delta_3\} = A_{\boxed{4}}\{\Delta_3\}$; $K_{\boxed{4}}\begin{Bmatrix}\Delta_6\\ \Delta_8\end{Bmatrix} = \begin{Bmatrix}P^1_{45}\\ P^1_{54}\end{Bmatrix}$

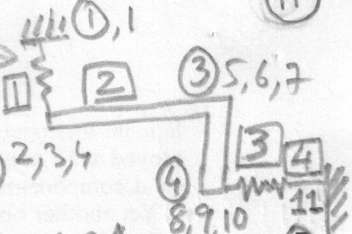
$$\Rightarrow A^T_{\boxed{4}} K_{\boxed{4}} A_{\boxed{4}}\{\Delta_3\} = A^T_{\boxed{4}}\begin{Bmatrix}P^1_{45}\\ P^1_{54}\end{Bmatrix} \Rightarrow \begin{bmatrix}\frac{3}{4} & 0\end{bmatrix}\begin{bmatrix}k_2 & -k_2\\ -k_2 & k_2\end{bmatrix}\begin{bmatrix}3/4\\0\end{bmatrix}\{\Delta_3\} = \begin{bmatrix}\frac{3}{4} & 0\end{bmatrix}\begin{Bmatrix}P^1_{45}\\ P^1_{54}\end{Bmatrix}$$

$$\Rightarrow \left(\frac{3}{4}\right)^2 k_2\,\Delta_3 = \frac{3}{4}P^1_{45} \longrightarrow El\,\boxed{4}$$

<u>Assembly</u>: $\left[k_1 + 0 + 0 + \left(\frac{3}{4}\right)^2 k_2\right]\{\Delta_3\} = \left\{ P^2_{21} + P^2_{23} + \frac{3}{4}P^1_{43} + \frac{3}{4}P^1_{45} \right\} = \{F\}.$

So same result as when assembly done before MPC application   $P_2 \to F$   $\frac{3}{4}.P_4 \to 0$

Done as Frame :




Dof's : 1, 3, 6, 9 ↑
2, 5, 8, 11 → ; 4, 7, 10 ↻

$K_{\boxed{1}}, K_{\boxed{4}}$ as before (for spring element).

$$K_{\boxed{2}} = \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} a & 0 & 0 & -a & 0 & 0 \\ 0 & 12/4^3 & 6/4^2 & 0 & -12/4^3 & 6/4^2 \\ 0 & 6/4^2 & 4/4 & 0 & -6/4^2 & 2/4 \\ -a & 0 & 0 & a & 0 & 0 \\ 0 & -12/4^3 & -6/4^2 & 0 & 12/4^3 & -6/4^2 \\ 0 & 6/4^2 & 2/4 & 0 & -6/4^2 & 4/4 \end{bmatrix} \end{array} EI, \quad a = \frac{EA}{4}\frac{1}{EI}$$

where we used

$$a_{②③} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{\boxed{3}} = \begin{array}{c} \begin{bmatrix} 12/3^3c & 0 & -6/3^2c & -12/3^3c & 0 & -6/3^2c \\ 0 & b & 0 & 0 & -b & 0 \\ -6/3^2c & 0 & 4/3c & 6/3^2c & 0 & 2/3c \\ -12/3^3c & 0 & 6/3^2c & 12/3^3c & 0 & 6/3^2c \\ 0 & -b & 0 & 0 & b & 0 \\ -6/3^2c & 0 & 2/3c & 6/3^2c & 0 & 4/3c \end{bmatrix} \\ \begin{array}{cccccc} 5 & 6 & 7 & 8 & 9 & 10 \end{array} \end{array} EI, \quad b = \frac{EA_1}{3}\frac{1}{EI}$$

$c = EI_1/EI$

where we used

$$a_{③④} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{\boxed{1}} = \begin{array}{c} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \\ \begin{array}{cc} 1 & 3 \end{array} \end{array}; \quad K_{\boxed{4}} = \begin{array}{c} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \\ \begin{array}{cc} 8 & 11 \end{array} \end{array}$$

$$K = \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} k_1 & \cdot & -k_1 & & & & & & & & \\ \vdots & a & 0 & 0 & -a & 0 & 0 & & & & \\ -k_1 & 0 & k_1+12/4^3 & 6/4^2 & 0 & -12/4^3 & 6/4^2 & & & & \\ & 0 & 6/4^2 & 4/4 & 0 & -6/4^2 & 2/4 & & & & \\ & -a & 0 & 0 & (a+\frac{12}{3^3}c) & 0+0 & (0-\frac{6}{3^2}c) & -\frac{12}{3^3}c & 0 & -\frac{6}{3^2}c & \\ & 0 & -12/4^3 & -6/4^2 & 0+0 & (12/4^3+b) & (-6/4^2+0) & 0 & -b & 0 & \\ & 0 & 6/4^2 & 2/4 & (0-\frac{6}{3^2}c) & (-6/4^2+0) & (4/4+\frac{4}{3}c) & \frac{6}{3^2}c & 0 & \frac{2}{3}c & \\ & & & & -12/3^3c & 0 & 6/3^2c & (12/3^3c+k_2) & 0 & 6/3^2c & -k_2 \\ & & & & 0 & -b & 0 & 0 & b & 0 & \\ & & & & -6/3^2c & 0 & 2/3c & 6/3^2c & 0 & 4/3c & \\ & & & & & & & -k_2 & & & k_2 \end{bmatrix} \end{array}$$

MPC : $\{\Delta_3 | \Delta_1, \Delta_2, \Delta_4, \Delta_5, \Delta_6\ \Delta_7\ \Delta_8\ \Delta_9\ \Delta_{10}\ \Delta_{11}\}^T = A\{\Delta_3\}$

$= \Delta^{\#}$

$$A = \left\{1, 0, 0, \frac{-1}{4}, 0, 0, \frac{-1}{4}, \frac{3}{4}, 0, \frac{-1}{4}, 0\right\}^T$$

$P = \{0, 0, F, 0, 0, 0, 0, 0, 0, 0, 0\}^T$ , $P \xrightarrow[\text{1st, 3rd row}]{\text{interchange}} P^{\#}$

$K \xrightarrow[\text{1st, 3rd rows/cols}]{\text{interchange}} K^{\#}$

$$K^{\#}\Delta^{\#} = P^{\#} \longrightarrow A^T K^{\#} A\{\Delta_3\} = A^T P^{\#}$$

From MATLAB, $A^T K^{\#} A = k_1 + \frac{9}{16} k_2$ , $A^T P^{\#} = F$

$$\Rightarrow \Delta_3 = \frac{F}{k_1 + \frac{9}{16} k_2} \longrightarrow \text{same as by Truss approach.}$$

Hence spring forces will also be same as from Truss approach.

Note: Could also delete rows/cols 1, 2, 5, 6, 9, 11 from K and proceed. In that case $\Delta = \Delta^{\#} = \{\Delta_3 | \Delta_4, \Delta_7, \Delta_8, \Delta_{10}\}^T$,

$A = \left\{1, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}\right\}^T$ , $P = P^{\#} = \{F, 0, 0, 0, 0\}^T$.

This entails easier matrix multiplications.

```
% MPC Ex3
```
Done as frame.

```
syms a b c d k1 k2;


ke1=[k1 0 -k1 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0; -k1 0 k1 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0;0 0 0 0
0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0
0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0];


ke4=[0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0
0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 k2 0 0 -k2;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0
0 0 0 0;0 0 0 0 0 0 0 -k2 0 0 k2];


ke2=[0 0 0 0 0 0 0 0 0 0 0; 0 a 0 0 -a 0 0 0 0 0 0; 0 0 12/4^3 6/4^2 0 -12/4^3 6/4^2 0 0 0 0; 0 0 6/4^2
4/4 0 -6/4^2 2/4 0 0 0 0;0 -a 0 0 a 0 0 0 0 0 0;0 0 -12/4^3 -6/4^2 0 12/4^3 -6/4^2 0 0 0 0;0 0 6/4^2
2/4 0 -6/4^2 4/4 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0
0 0];


ke3=[0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0;0 0 0 0
12/3^3*c 0 -6/3^2*c -12/3^3*c 0 -6/3^2*c 0;0 0 0 0 0 b 0 0 -b 0 0;0 0 0 0 -6/3^2*c 0 4/3*c 6/3^2*c 0
2/3*c 0;0 0 0 0 -12/3^3*c 0 6/3^2*c 12/3^3*c 0 6/3^2*c 0;0 0 0 0 0 -b 0 0 b 0 0;0 0 0 0 -6/3^2*c 0
2/3*c 6/3^2*c 0 4/3*c 0;0 0 0 0 0 0 0 0 0 0 0];


kstruct=ke1+ke2+ke3+ke4;

kstruct([1,3],:)=kstruct([3,1],:);

kstruct(:,[1,3])=kstruct(:,[3,1]);

Amat=[1; 0; 0; -1/4; 0; 0; -1/4; 3/4; 0; -1/4; 0] ;

Amat'*kstruct*Amat

load=[0;0;f;0;0;0;0;0;0;0;0]

load([1,3],:)=load([3,1],:)

(Amat'*kstruct*Amat)^-1*(Amat'*load)

        ans =f/(k1 + (9*k2)/16)
```

Alternatively, do MPC elementwise transformation and then assembly.

Element $\boxed{1}$ : MPC $\rightarrow \begin{Bmatrix} \Delta_1 \\ \Delta_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{A_{\boxed{1}}} \{\Delta_3\} \rightarrow$ so same result as in Truss method, ie, $k_1 \Delta_3 = P^2_{21} \rightarrow$ El $\boxed{1}$

Element $\boxed{2}$ : MPC $\rightarrow \begin{Bmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1/4 \\ 0 \\ 0 \\ -1/4 \end{bmatrix}}_{A_{\boxed{2}}} \{\Delta_3\}$ ; $K_{\boxed{2}} \begin{Bmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \end{Bmatrix} = \begin{Bmatrix} P^1_{23} \\ P^2_{23} \\ P^6_{23} \\ P^1_{32} \\ P^2_{32} \\ P^6_{32} \end{Bmatrix}$

$$\Rightarrow A^T_{\boxed{2}} K_{\boxed{2}} A_{\boxed{2}} \{\Delta_3\} = A^T_{\boxed{2}} \begin{Bmatrix} P^1_{23} \\ \vdots \\ P^6_{32} \end{Bmatrix} \Rightarrow \left[ 0 \quad \underbrace{\frac{12 \cancel{-6-6}}{4^2}}_{0} \quad \underbrace{\frac{6 \cancel{-4-2}}{4}}_{0} \quad 0 \quad \underbrace{\frac{-12 \cancel{+6+6}}{4^2}}_{0} \quad \underbrace{\frac{6 \cancel{-2-4}}{4}}_{0} \right] A_{\boxed{2}} \{\Delta_3\} = P^2_{23} - \frac{P^6_{23}}{4} - \frac{P^6_{32}}{4}$$

$$\Rightarrow 0 \cdot \Delta_3 = P^2_{23} - \frac{P^6_{23}}{4} - \frac{P^6_{32}}{4} \longrightarrow \text{El } \boxed{2}$$

Element $\boxed{3}$ : MPC $\rightarrow \begin{Bmatrix} \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \\ \Delta_9 \\ \Delta_{10} \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ 0 \\ -1/4 \\ 3/4 \\ 0 \\ -1/4 \end{Bmatrix}}_{A_{\boxed{3}}} \{\Delta_3\}$ ; $K_{\boxed{3}} \begin{Bmatrix} \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \\ \Delta_9 \\ \Delta_{10} \end{Bmatrix} = \begin{Bmatrix} P^1_{34} \\ P^2_{34} \\ P^6_{34} \\ P^1_{43} \\ P^2_{43} \\ P^6_{43} \end{Bmatrix}$

$$\Rightarrow A^T_{\boxed{3}} K_{\boxed{3}} A_{\boxed{3}} \{\Delta_3\} = A^T_{\boxed{3}} \begin{Bmatrix} P^1_{34} \\ \vdots \\ P^6_{43} \end{Bmatrix} \Rightarrow \left[ \underbrace{\frac{6 \cancel{-12+6}}{4 \cdot 3^2}}_{0} \quad 0 \quad \underbrace{\frac{-4 \cancel{+6-2}}{4.3}}_{0} \quad \underbrace{\frac{-6 \cancel{+12-6}}{4.3^2}}_{0} \quad 0 \quad \underbrace{\frac{-2 \cancel{+6-4}}{4.3}}_{0} \right]$$

$$* \; A_{\boxed{3}} \{\Delta_3\} = -\frac{1}{4} P^6_{34} + \frac{3}{4} P^1_{43} - \frac{1}{4} P^6_{43}$$

$$\Rightarrow 0 . \Delta_3 = -\frac{1}{4} P^6_{34} + \frac{3}{4} P^1_{43} - \frac{1}{4} P^6_{43} \rightarrow \text{El } \boxed{3}$$

Element $\boxed{4}$ : MPC $\rightarrow \begin{Bmatrix} \Delta_8 \\ \Delta_{11} \end{Bmatrix} = \underbrace{\begin{bmatrix} 3/4 \\ 0 \end{bmatrix}}_{A_{\boxed{4}}} \{\Delta_3\}$ ; $K_{\boxed{4}} \begin{Bmatrix} \Delta_8 \\ \Delta_{11} \end{Bmatrix} = \begin{Bmatrix} P^1_{45} \\ P^1_{54} \end{Bmatrix}$

$$\Rightarrow A^T_{\boxed{4}} K_{\boxed{4}} A_{\boxed{4}} \{\Delta_3\} = A^T_{\boxed{4}} \begin{Bmatrix} P^1_{45} \\ P^1_{54} \end{Bmatrix} \Rightarrow \left(\frac{3}{4}\right)^2 k_2 \Delta_3 = \frac{3}{4} P^1_{45} \rightarrow \text{El } \boxed{4}$$

So El. $\boxed{4}$ also same as Truss formulation except dof numbering changes ($\Delta_6 \rightarrow \Delta_8$, $\Delta_8 \rightarrow \Delta_{11}$) Truss → frame

Assembly: $\left[ k_1 + 0 + 0 + \left(\frac{3}{4}\right)^2 k_2 \right] \{\Delta_3\} = P^2_{21} + P^2_{23} - \frac{P^6_{23}}{4} - \frac{P^6_{32}}{4} - \frac{1}{4} P^6_{34} + \frac{3}{4} P^1_{43} - \frac{1}{4} P^6_{43} + \frac{3}{4} P^1_{45}$

Assembly:

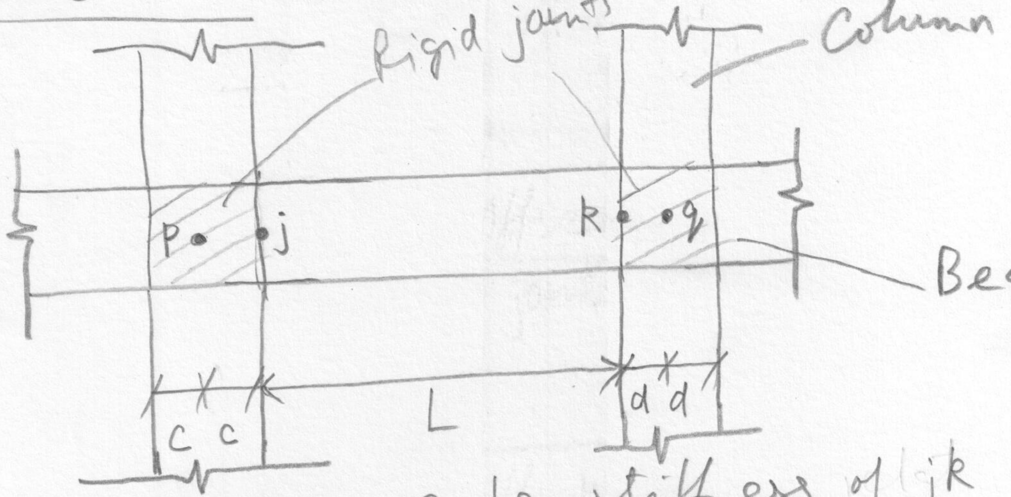
$$\left[k_1 + 0 + 0 + \left(\frac{3}{4}\right)^2 k_2\right]\{\Delta_3\} = P_{21}^2 + P_{23}^2 - \frac{P_{23}^6}{4} - \frac{P_{32}^6}{4} - \frac{1}{4}P_{34}^6 + \frac{3}{4}P_{43}^1 - \frac{1}{4}P_{43}^6 + \frac{3}{4}P_{45}^1$$

$$P_{23}^6 = P_2^6 = 0\ ; \quad P_{32}^6 + P_{34}^6 = P_3^6 = 0\ ; \quad P_{43}^6 = P_4^6 = 0\ ; \quad P_{43}^1 + P_{45}^1 = P_4^1 = 0$$

$$P_{21}^2 + P_{23}^2 = P_2^2 = F$$

$\Rightarrow \left(k_1 + \frac{9}{16}k_2\right)\Delta_3 = F$ $\longrightarrow$ same result as before.

# Finite Joints. (Joint Offsets).




Idea is to express member stiffness of jk in terms of force-displ relations of pq (ie equivalent stiffness of pq).

Consider equivalent force/displ system between general points p and j, where line pj is rigid, i.e,

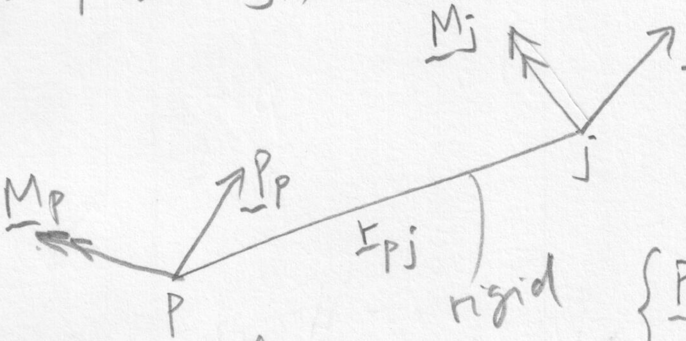



Fig: Translation of forces

forces (N)

$$\underline{P}_p = \underline{P}_j$$

Moments (Nm)

$$\underline{M}_p = \underline{r}_{pj} \times \underline{P}_j + \underline{M}_j$$

$$\begin{Bmatrix} \underline{P}_p \\ \underline{M}_p \end{Bmatrix} = \left[\begin{array}{c|c} \underline{\underline{I}} & \underline{\underline{0}} \\ \hline \underline{\underline{C}}_{pj} & \underline{\underline{I}} \end{array}\right] \begin{Bmatrix} \underline{P}_j \\ \underline{M}_j \end{Bmatrix} = \underline{\underline{A}}_{pj}^T \begin{Bmatrix} \underline{P}_j \\ \underline{M}_j \end{Bmatrix}$$

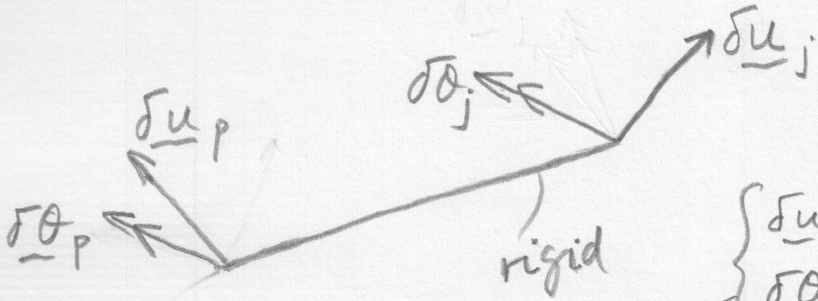
where $\underline{\underline{C}}_{pj} = \begin{bmatrix} 0 & -z_{pj} & y_{pj} \\ z_{pj} & 0 & -x_{pj} \\ -y_{pj} & x_{pj} & 0 \end{bmatrix}$ → arising from cross product $\underline{r}_{pj} \times \underline{P}_j$

so $\underline{\underline{C}}_{pj}^T = -\underline{\underline{C}}_{pj}$

where $z_{pj} = z_j - z_p$, $y_{pj} = y_j - y_p$
$x_{pj} = x_j - x_p$

and inverse transformation is

$$\begin{Bmatrix} \underline{P}_j \\ \underline{M}_j \end{Bmatrix} = A_{pj}^{-T} \begin{Bmatrix} \underline{P}_p \\ \underline{M}_p \end{Bmatrix} = \begin{bmatrix} I & 0 \\ -C_{pj} & I \end{bmatrix} \begin{Bmatrix} \underline{P}_p \\ \underline{M}_p \end{Bmatrix} =$$




$$\delta\theta_j = \delta\theta_p$$

$$\delta\underline{u}_j = \delta\underline{u}_p + \delta\underline{\theta}_p \times \underline{r}_{pj}$$

$$\underbrace{\begin{Bmatrix} \delta\underline{u}_j \\ \delta\underline{\theta}_j \end{Bmatrix}}_{\Delta_j} = \left[\begin{array}{c|c} \underline{\underline{I}} & -\underline{\underline{C}}_{pj} \\ \hline \underline{\underline{0}} & \underline{\underline{I}} \end{array}\right] \begin{Bmatrix} \delta\underline{u}_p \\ \delta\underline{\theta}_p \end{Bmatrix} = \underline{\underline{A}}_{pj} \underbrace{\begin{Bmatrix} \delta\underline{u}_p \\ \delta\underline{\theta}_p \end{Bmatrix}}_{\Delta_p}$$

and inverse transf is

$$\Delta_p \;\; \begin{Bmatrix}\delta \underset{\sim}{u}_p\\ \delta\underset{\sim}{\theta}_p\end{Bmatrix} = \left[\begin{array}{c|c} \underline{\underline{I}} & \underline{\underline{c}}_{pj}\\ \hline \underline{\underline{0}} & \underline{\underline{I}}\end{array}\right]\begin{Bmatrix}\delta \underset{\sim}{u}_j\\ \delta\underset{\sim}{\theta}_j\end{Bmatrix} = A_{pj}^{-1}\begin{Bmatrix}\delta \underset{\sim}{u}_j\\ \delta\underset{\sim}{\theta}_j\end{Bmatrix} \;\; \Delta_j$$

Now $\begin{Bmatrix}\underline{P}_j\\ \underline{M}_j\end{Bmatrix}$ represent joint load at j. * → see detail/explanation below But same transformation would hold for member end forces $\begin{Bmatrix}\underline{P}_{jk}\\ \underline{M}_{jk}\end{Bmatrix}$, ie,

forces, moments

$$\begin{Bmatrix}\underline{P}_{pq}\\ \underline{M}_{pq}\end{Bmatrix} = A_{pj}^T\begin{Bmatrix}\underline{P}_{jk}\\ \underline{M}_{jk}\end{Bmatrix} \longrightarrow P_{pq} = A_{pj}^T\, P_{jk}$$

forces, moments

$P_{jk}$ — forces & moments

$P_{pq}$ → as per Striffness matrix notation.

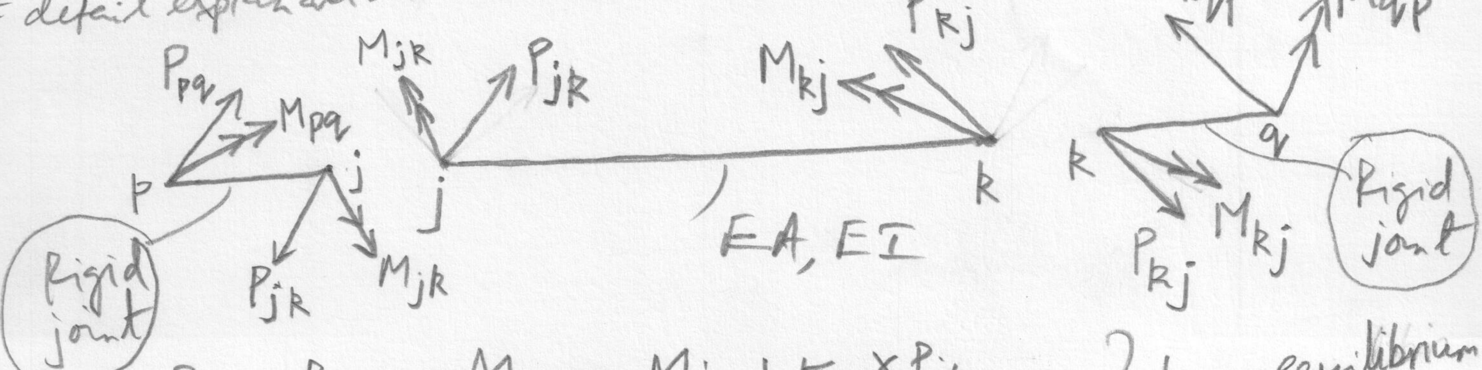
forces & moments.

$$\text{and similarly } \begin{Bmatrix}\underline{P}_{qp}\\ \underline{M}_{qp}\end{Bmatrix} = A_{qk}^T\begin{Bmatrix}\underline{P}_{kj}\\ \underline{M}_{kj}\end{Bmatrix} \longrightarrow P_{qp} = A_{qk}^T\, P_{kj}$$

$P_{qp}$ $\qquad$ $P_{kj}$

---

* detail explanation




$$\underline{P}_{pq} = \underline{P}_{jk}\; ; \;\; \underline{M}_{pq} = \underline{M}_{jk} + \underline{r}_{pj}\times\underline{P}_{jk}$$
$$P_{qp} = P_{kj}\; ; \;\; \underline{M}_{qp} = \underline{M}_{kj} + \underline{r}_{qk}\times\underline{P}_{kj}$$

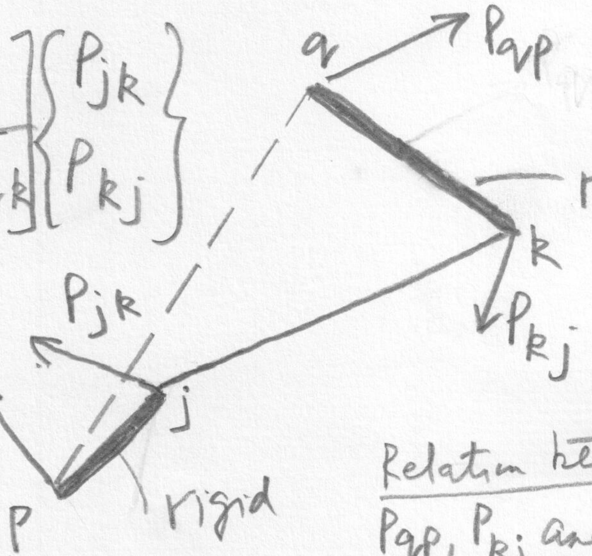
} from equilibrium of rigid joints p and q

---

(contd)

Thus,
$$\begin{Bmatrix}P_{pq}\\ P_{qp}\end{Bmatrix} = \left[\begin{array}{c|c} A_{pj}^T & 0\\ \hline 0 & A_{qk}^T\end{array}\right]\begin{Bmatrix}P_{jk}\\ P_{kj}\end{Bmatrix}$$

and
$$\begin{Bmatrix}\Delta_j\\ \Delta_k\end{Bmatrix} = \left[\begin{array}{c|c} A_{pj} & 0\\ \hline 0 & A_{qk}\end{array}\right]\begin{Bmatrix}\Delta_p\\ \Delta_q\end{Bmatrix}$$




<u>Relation between</u>

$P_{qp}$, $P_{kj}$ and $P_{pq}$, $P_{jk}$

Now $\begin{Bmatrix} P_{jk} \\ P_{kj} \end{Bmatrix} = K_e^{jk} \begin{Bmatrix} \Delta_j \\ \Delta_k \end{Bmatrix}$

$$\Rightarrow \underbrace{\begin{Bmatrix} P_{pq} \\ P_{qp} \end{Bmatrix}}_{\bar{P}} = \underbrace{\left[\begin{array}{c|c} A_{pj}^T & \\ \hline & A_{qk}^T \end{array}\right]}_{A_{pq}^T} K_e^{jk} \underbrace{\left[\begin{array}{c|c} A_{pj} & \\ \hline & A_{qk} \end{array}\right]}_{A_{pq}} \underbrace{\begin{Bmatrix} \Delta_p \\ \Delta_q \end{Bmatrix}}_{\bar{\Delta}}$$

$$\Rightarrow \bar{P} = \underbrace{A_{pq}^T K_e^{jk} A_{pq}}_{K_e^{pq}} \bar{\Delta}$$

→ element force-displ eqns for element pq in global coords.

$\bar{P}$: Mem. end Forces/moments at p, q

$\bar{\Delta}$: displ at p, q

This is the element equations for element pq in terms of stiffness of element jk and offsets c and d.

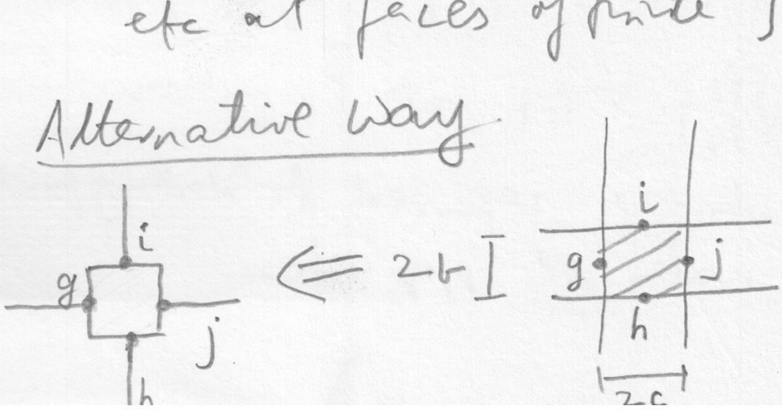
-X- Loads applied at joints j and k need to be transferred to joints p, q in the same manner, ie using

$P_p = A_{pj} P_j$, $P_q = A_{qk} P_k$ where $\{P_p\} = \begin{Bmatrix} \underline{P}_p \\ \underline{M}_p \end{Bmatrix}$, $\{P_j\} = \begin{Bmatrix} \underline{P}_j \\ \underline{M}_j \end{Bmatrix}$

(see over) $\{P_q\} = \begin{Bmatrix} \underline{P}_q \\ \underline{M}_q \end{Bmatrix}$, $\{P_k\} = \begin{Bmatrix} \underline{P}_k \\ \underline{M}_k \end{Bmatrix}$ → directly applied ($P_a$) or equivalent ($P_e$) due to loads between nodes or self-straining

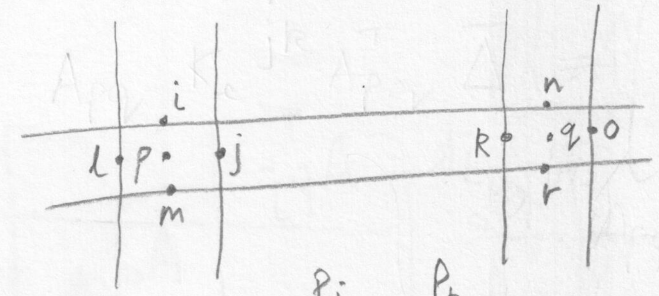
Thus you assemble structure considering the "working points" p, q, etc at ℄ of finite joints instead of joints j, k, etc at faces of finite joints.

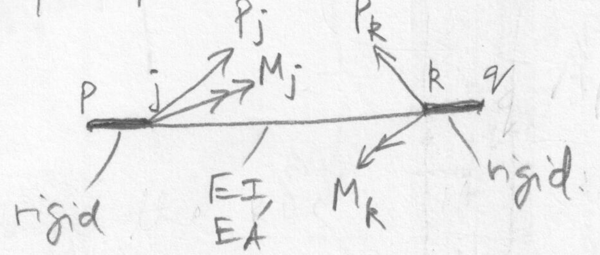
Alternative way




Using MPC's express dof's of jts i, g, h, in terms of displs of jt j since ▨ finite joint is rigid. Assume stiffnesses

# Explanation of Assembly process




Joint loads at j, k determined from loading of member jk (due to mechanical and self-straining).

Now dul find Fef's due to $P_j = \begin{Bmatrix} P_j \\ M_j \end{Bmatrix}$, $P_k = \begin{Bmatrix} P_k \\ M_k \end{Bmatrix}$.

$\because$ pj and qk are rigid, under fixed end conditions the whole (ie rigid & flexible parts) of pq stays undeformed. Thus jk (flexible) part doesent carry and BM/SF/AF. Thus Fef's at p [and q] are equal but opposite to equivalent force-couple system at p [and q] due to joint loads at j [and k], respectively. Thus jt loads at p [and q] due to jt loads at j [and k] are same as eqvt force-couple system at p [and q] due to jt loads at j [and k], resply.

So assemble $A_{pq}^T K_e^{jk} A_{pq}$ for all elements pq to get $\overline{K}$

and assemble $\sum A_{pq}^T \begin{Bmatrix} P_j \\ P_k \end{Bmatrix} = \begin{Bmatrix} P_p \\ P_q \end{Bmatrix}$

j,k framing into p, q, resply, ie j, l, i, m framing into p and k, n, o, r framing into q in above Fig.

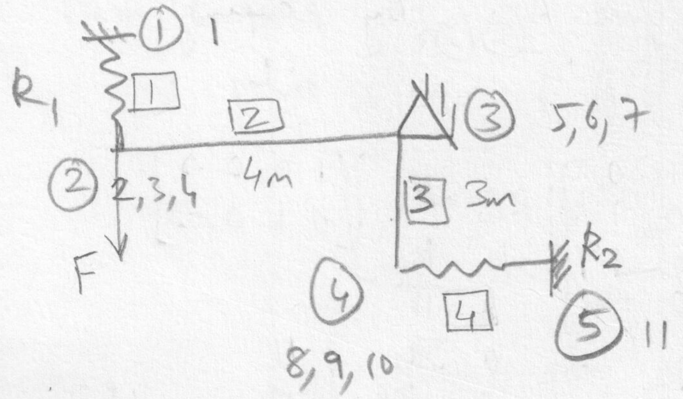
and then assemble $\begin{Bmatrix} P_p \\ P_q \end{Bmatrix}$ $\forall$ p, q into $\overline{P}$. Thus

$$\overline{K}\,\overline{\Delta} = \overline{P}$$

which is referred to "working point" nodes, ie p, q, etc.

for rigid finite joint $\square$, but these will get overridden by MPC's when doing $A^T K A$ etc. Thus final $\underline{\underline{K}}^*$ and $\{\Delta_f\}$ will involve only d.o.f's of jt. 'j' and not of jts i, g, h.

Back to Ex3 → Done by Joint offset method




So now we deal with flexible elements $\boxed{1}$, $\boxed{4}$ only. We write their element equations in terms of d.o.f's at offset joint 3, 1, 5 (which correspond to joints p, q, etc in theory).

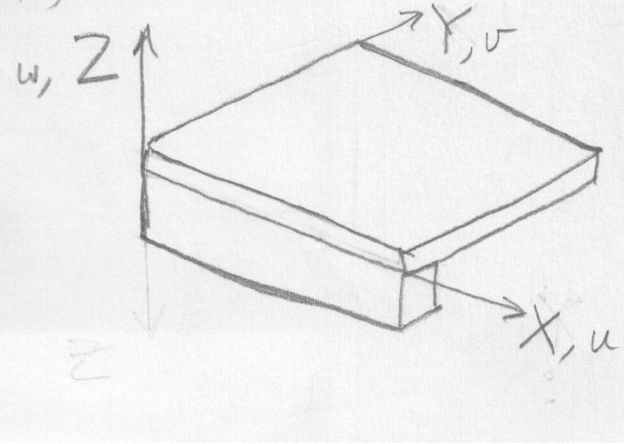
Note that jts 1, 5 also correspond to jts j, k, in the theory, ie there is a rigid element of length zero connecting 1 — 1, and connecting 5 — 5

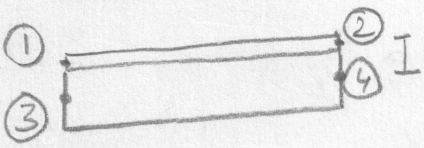
$$\begin{Bmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \Delta_4\end{Bmatrix} = A_{13}\begin{Bmatrix}\Delta_1\\ \Delta_5\\ \Delta_6\\ \Delta_7\end{Bmatrix};\quad A_{13} = \left[\begin{array}{c|c} A_{11} & 0\\ \hline 0 & A_{32}\end{array}\right] = \begin{bmatrix}\underbrace{[1]}_{A_{11}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -4\\ 0 & 0 & 1\end{bmatrix}}_{A_{32}}\end{bmatrix}$$

$$\begin{Bmatrix}\Delta_{11}\\ \Delta_8\\ \Delta_9\\ \Delta_{10}\end{Bmatrix} = A_{53}\begin{Bmatrix}\Delta_{11}\\ \Delta_5\\ \Delta_6\\ \Delta_7\end{Bmatrix};\quad A_{53} = \left[\begin{array}{c|c} A_{55} & 0\\ \hline 0 & A_{34}\end{array}\right] = \begin{bmatrix}\underbrace{[1]}_{A_{55}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \underbrace{\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{A_{34}}\end{bmatrix}$$

(see over)

(Ex) Eccentric stiffener in plate




$$\begin{Bmatrix}u_3\\ w_3\\ \theta_{y3}\end{Bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & -b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{A_{13}^T}\begin{Bmatrix}u_1\\ w_1\\ \theta_{y1}\end{Bmatrix}$$

$K_e^{34}$ → element stiff of 3 4

$K_e^{12} = A_{12}^T K_e^{34} A_{12}$, $\begin{Bmatrix}P_1\\ P_2\end{Bmatrix} = A_{12}^T\begin{Bmatrix}P_3\\ P_4\end{Bmatrix}$

where $A_{12} = \left[\begin{array}{c|c} A_{13} & 0\\ \hline 0 & A_{24}\end{array}\right]$ Here $A_{13} = A_{24}$.

El $\boxed{1}$ → transforming from ①-② to ①-③:

Required $\bar{A}_{13}$ is obtained from $A_{13}$ keeping rows corresponding to dof $\Delta_1, \Delta_3$, ie ~~rows~~ 1, 3 only.

$$K_e^{13} = \bar{A}^T_{13} K_e^{12} \bar{A}_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -4 \end{bmatrix} \begin{array}{c} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \\ \begin{matrix} 1 & \quad 3 \end{matrix} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} = \begin{array}{c|c} \begin{bmatrix} k_1 & 0 & -k_1 & 4k_1 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & -4k_1 \\ 4k_1 & 0 & -4k_1 & 16k_1 \end{bmatrix} & \begin{matrix} 1 \\ 5 \\ 6 \\ 7 \end{matrix} \end{array}$$

El $\boxed{4}$ → transforming from ④-⑤ to ③-⑤:

Required $\bar{A}_{53}$ is obtained from $A_{53}$ by keeping rows corresponding to d.o.f $\Delta_8, \Delta_{11}$, ie rows 1, 2 only.

$$K_e^{53} = \bar{A}^T_{53} K_e^{45} \bar{A}_{53} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{array}{c} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \\ \begin{matrix} 8 & \quad 11 \end{matrix} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

$$= \begin{array}{c|c} \begin{bmatrix} k_2 & -k_2 & 0 & 3k_2 \\ -k_2 & k_2 & 0 & -3k_2 \\ 0 & 0 & 0 & 0 \\ 3k_2 & -3k_2 & 0 & 9k_2 \end{bmatrix} & \begin{matrix} 11 \\ 5 \\ 6 \\ 7 \end{matrix} \end{array}$$

Transforming loads on ①-② to ①-③:

$$\bar{P}_{13} = \bar{A}^T_{13} P_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{array}{c|c} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ -4\underbrace{P_3}_{=F\text{(given)}} + P_4 \end{Bmatrix} & \begin{matrix} 1 \\ 5 \\ 6 \\ 7 \end{matrix} \end{array}$$

overbar to denote tranformed loads

or if we assume only $P_1$ and $P_3 = F$ applied, as in given problem

$$\bar{P}_{13} = \bar{A}^T_{13} P_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -4 \end{bmatrix} \begin{Bmatrix} P_1 \\ \underbrace{P_3}_{F} \end{Bmatrix} = \begin{array}{c|c} \begin{Bmatrix} P_1 \\ 0 \\ P_3 \\ -4P_3 \end{Bmatrix} & \begin{matrix} 1 \\ 5 \\ 6 \\ 7 \end{matrix} \end{array}$$

Transforming loads on ④-⑤ to ③-⑤:

$$\bar{P}_{35} = A^T_{53} P_{45} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P_{11} \\ P_8 \\ P_9 \\ P_{10} \end{Bmatrix} = \begin{array}{c|c} \begin{Bmatrix} P_{11} \\ P_8 \\ P_9 \\ -3P_8 + P_{10} \end{Bmatrix} & \begin{matrix} 11 \\ 5 \\ 6 \\ 7 \end{matrix} \end{array}$$

or if we assume only $P_8, P_{11}$ applied, then

$$\bar{P}_{35} = \bar{A}^T_{53} P_{45} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{Bmatrix} P_{11} \\ P_8 \end{Bmatrix} = \begin{array}{c|c} \begin{Bmatrix} P_{11} \\ P_8 \\ 0 \\ -3P_8 \end{Bmatrix} & \begin{matrix} 11 \\ 5 \\ 6 \\ 7 \end{matrix} \end{array}$$

Assembly:

$$
\begin{array}{c}
1 \\ 5 \\ 6 \\ 7 \\ 11 \\ \\
\end{array}
\left[
\begin{array}{ccccc}
k_1 & 0 & -k_1 & 4k_1 & 0 \\
0 & 0+k_2 & 0+0 & 0-3k_2 & -k_2 \\
-k_1 & 0+0 & k_1+0 & -4k_1+0 & 0 \\
4k_1 & 0-3k_2 & -4k_1+0 & 16k_1+9k_2 & 3k_2 \\
0 & -k_2 & 0 & 3k_2 & k_2 \\ \hline
1 & 5 & 6 & 7 & 11
\end{array}
\right]
\begin{Bmatrix} \Delta_1 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_{11} \end{Bmatrix}
=
\begin{Bmatrix} P_1 \\ P_2 + P_8 \\ P_3 + P_9 \\ -4P_3 + P_4 - 3P_8 + P_{10} \\ P_{11} \end{Bmatrix}
$$

since dofs 1, 5, 6, 11 restrained, these jt. loads, if applied, would go into reactions.

Dof's 1, 5, 6, 11 restrained (so cancel 1,5,6,11 rows/cols)

$$\Rightarrow (16k_1 + 9k_2)\Delta_7 = -4\underbrace{P_3}_{F} + \underbrace{P_4 - 3P_8 + P_{10}}_{\text{not given in problem but this is the generalization if they are given.}}$$