INTRODUCTION TO FEM.

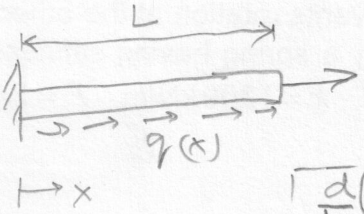
2nd order BVP

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0 \quad , \quad 0<x<L, a=a(x), c=c(x), q=q(x)$$

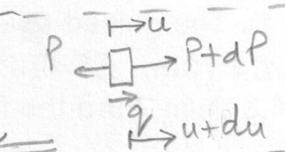
$$u(0)=u_0 \quad , \quad a\left(\frac{du}{dx}\right)\Bigg|_{x=L} = Q_L$$

Examples ($c(x)=0$):

Axial bar : $a = EA$, q = distributed traction, u = longitudinal displ , Q_L = axial force at end



$$\rightarrow Q_L = \left(a\frac{du}{dx}\right)\Bigg|_{x=L}$$



$$\frac{dP}{dx} + q = 0$$

$$du = \frac{(P+\alpha dP)dx}{AE} = \frac{Pdx}{AE}$$

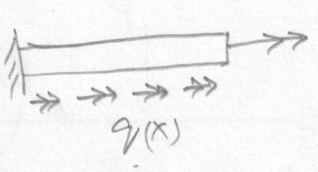
$$\boxed{\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q = 0} \Longleftarrow$$

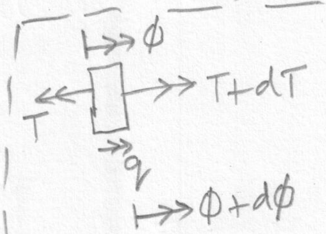
1-D Conduction : a = thermal conductivity, q = distr heat source

u = Temp , Q_L = Heat out at $x=L$.

Torsion : $a = GJ(x)$, q = distributed torque,

u = rotation $\phi(x)$, Q_L = end torque T_L.





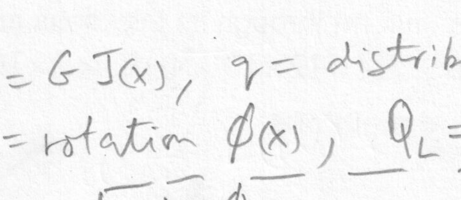
$$\frac{dT}{dx} + q = 0 \;;$$

$$\frac{(T+\alpha dT)dx}{GJ} = d\phi \Rightarrow T = GJ\frac{d\phi}{dx}$$

$$\Rightarrow \frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) + q = 0$$

Cable : $a = T$ (cable tension), q = distributed load,

u = vertical displ of cable $w(x)$, Q_L=



$$d(T\sin\theta) + q\,dx = 0$$

$$\sin\theta \cong \tan\theta = \frac{dw}{dx}$$

$$\frac{d}{dx}T\left(\frac{dw}{dx}\right) + q(x) = 0$$

Flow thru pipes (Darcy) $a = \frac{\pi D^4}{128\mu}$, $u \equiv$ hydrostatic pressure

q = flow source, Q_L = flow rate.

Laminar flow thru channel, : a = viscosity, u = velocity, q = pressure gradient

incompressible under constant pressure gradient. Q_L = axial stress

Flow thru porous media: a = coeff of permeability; u = fluid head, q = fluid flux, Q_L = flow seepage.

Electrostatics: a = dielectric const, u = electrostatic potential, q = charge density, Q_L = electric flux.

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Approx solution

$$u \simeq U_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)$$

$\phi_0(0) = u_0$, $\phi_i(0) = 0$, ie ϕ_0 used to satisfy non-homogenous essential BC's.

$\phi_0(x)$, $\phi_i(x)$ chosen, usually polynomials.

In general, if you chose U_N to satisfy all BC's (say for $u_0 = 1$, $Q_L = 0$, $a(x) = x$, $c(x) = 1$, $q(x) = 0$, $L = 1$, choose $\phi_0(x) = 1$, $\phi_1(x) = x^2 - 2x$, $\phi_2(x) = x^3 - 3x$, $N = 2$, $U_N = 1 + c_1(x^2 - 2x) + c_2(x^3 - 3x)$)

subst U_N in BVP, get,

$$(2c_1 + 3c_2 + 1) + (-4c_1 - 2c_1 - 3c_2)x + (-9c_2 + c_1)x^2 + (c_2)x^3 = 0$$

$\Rightarrow$ each ( ) = 0 $\Rightarrow$ inconsistent set of equations for c_1, c_2.

So we need a weighted-integral or weak formulation of the BVP.

Weak Form (Weighted Integral/Residual)

$$\int_0^L W\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-q\right)dx=0 \rightarrow ① , \quad W=W(x)$$

$$\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wq\right)dx - w\,a\frac{du}{dx}\Big|_0^L = 0 \rightarrow ②$$

wt-Residual:
Galerkin → $w_i = \phi_i$
Collocation → $w_i = \delta(x-x_i)$

→ Eq ① (yields non-symmetric form.) requires u to be twice differentiable, but not so for weight fn. w. So choosing $u \simeq U_N$, (that satisfies all BC's) and writing ① N times using $w = w_1, \dots w_N(x)$ we can get N eqns for $c_1, \dots c_N$. This is weighted-residual method.

→ Eq ② requires u to be once differentiable, ie weaker continuity requirement on u. Also it often results in a symmetric form ($w \equiv u$ in first two terms under integral), yielding symmetric algebraic eqns. Also natural BC (ie $a\frac{du}{dx}$) included in ② so u need only satisfy geometric BC's. ② is the WEAK FORM. Note that such trading of differentiation from u to w should be avoided if it yields physically meaningless boundary terms. Weak form imposes continuity/differentiability requirements on w.

→ Primary variables (PV) → w and its derivatives in bndry terms, written in terms of u (ie w → u), i.e., u in present problem. Specification of primary variables at bndry's constitute Essential/Kinematic/Geometric BC's (EBC), i.e $U(0) = u_0$ in present problem.

→ Secondary variables (SV) → coeffs on w and its derivatives in bndry term, ie $a\frac{du}{dx}$ in present case. Specification of SV at bndry's constitutes Natural/Force BC's (NBC), ie, $a\frac{du}{dx}\Big|_{x=L} = Q_L$ in present case.

→ $2m^{th}$ order BVP will have m pairs of (PV, SV). Each PV has associated SV.

→ Apart form differentiability/continuity of w as dictated by weak form ②, w is required to satisfy homogenous form of EBC's, ie $w(0)=0$ in present case.

Thus weak form ② becomes,

$$\int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx - w(L)\,Q_L = 0$$

↪ ②*

WEAK FORM of 2nd order BVP

→ Weak form exists for all linear/nonlinear BVP's of order ≥ 2

When BVP is linear and even ordered, weak form is symmetric and Bilinear in w, u.

$$B(w,u) \triangleq \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right)dx$$ — Bilinear, symmetric in w, u.

$$L(w) \triangleq \int_0^L wq\,dx + w(L)\,Q_L$$ — linear in w.

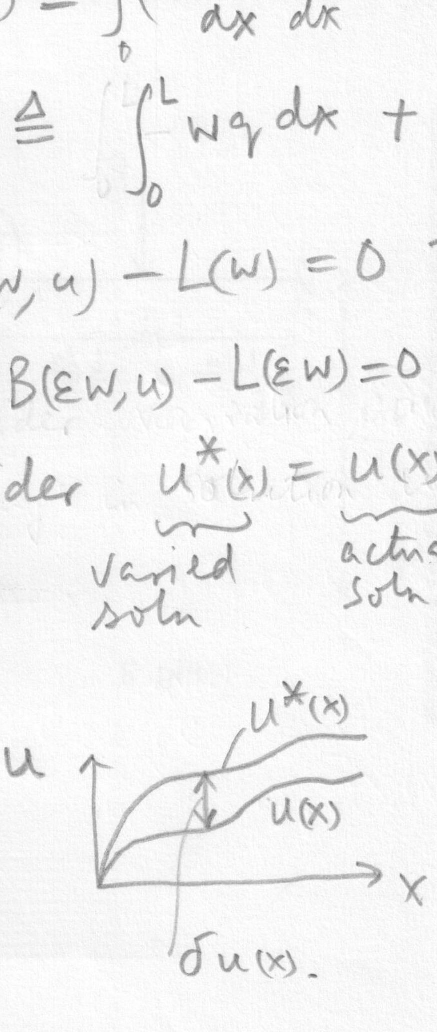
$$B(w,u) - L(w) = 0$$ — ③ — WEAK FORM. where $B(w,u)$ is symm & bilinear, $L(w)$ is linear.

$$\Rightarrow B(\varepsilon w,u) - L(\varepsilon w) = 0$$

$$\text{Consider } \underbrace{u^*(x)}_{\text{varied soln}} = \underbrace{u(x)}_{\text{actual soln}} + \underbrace{\cancel{\delta u(x)}}\rightarrow \varepsilon w(x)$$

variation or small arbitrary change in $u(x)$, where $\delta u(x)$ satisfies homogenous EBC's of u. Thus we can write

$$\delta u(x) \equiv \varepsilon w(x)$$



$$\rightarrow \underbrace{\delta u(x)}_{\substack{\text{virtual change}\\ \text{in actual soln } u(x)\\ \text{at a station } x}} \neq \underbrace{du(x)}_{\substack{\text{actual/real change in soln } u(x)\\ \text{across stations } dx \text{ apart.}}}$$

→ However, laws of differential calculus apply to $\delta u(x)$. Note $\delta x = 0$ (no variation of indep variable).

Symmetric Bilinear WEAK FORM ③ becomes,

$$B(\delta u, u) - L(\delta u) = 0$$

$$B(\delta u, u) = \int_0^L \left(a \frac{d\delta u}{dx}\frac{du}{dx} + c\,\delta u . u\right) dx = \frac{1}{2}\delta\int\left[a\left(\frac{du}{dx}\right)^2 + cu^2\right]dx$$

$$= \frac{1}{2}\delta B(u,u)$$

$$\Rightarrow \delta \underbrace{\left[\frac{1}{2}B(u,u) - L(u)\right]}_{I(u)} = \delta I(u) = 0$$

(used $L(\delta u) = \delta L(u)$ ∵ L is linear.

$$L(\delta u) = \int_0^L \delta u . q\, dx + \delta u(L) Q_L$$

$$= \delta\left[\int_0^L u . q\, dx + u(L)\, Q_L\right]$$

$$= \delta[L(u)]$$

where we used

$\delta(uq) = \delta u . q$, ∵ $\delta q = 0$

$\delta(u(L) Q_L) = \delta u(L)\, Q_L$ ∵ $\delta Q_L = 0$

∵ $q(x)$, Q_L are prescribed.)

Thus for symmetric bilinear $B(w,u)$ and linear $L(w)$, WEAK FORM becomes

WEAK FORM ← $\boxed{\delta I(u) = 0}$ → Extremum condt. In SolMech ≡ Min PE, I ≡ PE.

$\boxed{I(u) = \frac{1}{2}B(u,u) - L(u)}$ → Quadratic functional (in u).

Thus in Sol Mech, WEAK FORM ≡ MIN. PE.

→ FEM doesent require quad functional $I(u)$. Can directly work with WEAK FORM which is more general than $I(u)$.

$$I(u) = \underbrace{\frac{1}{2}\int_0^L\left[a\left(\frac{du}{dx}\right)^2\right.}_{S.E} + \underbrace{cu^2}_{\substack{\equiv SE \text{ due}\\ \text{to discrete}\\ \text{springs}}}\left.\right]dx - \left[\underbrace{\int_0^L u . q\, dx}_{\substack{\text{Work done}\\ \text{by } q}} + \underbrace{u(L)\, Q_L}_{\substack{\text{work done}\\ \text{by } Q_L}}\right]$$

S.E →

$$\frac{1}{2}\int_V \sigma_x \varepsilon_x\, dV$$

$$= \frac{1}{2}\int_0^L E \varepsilon_x^2 A\, dx$$

$$= \frac{1}{2}\int_0^L a\left(\frac{du}{dx}\right)^2 dx$$

(Variational) Methods of Approximation

(I) Rayleigh - Ritz method — uses WEAK FORM.

$$u_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0 \quad , \quad \underline{w = \phi_i} \; , \; i = 1, \cdots N.$$

$$\underbrace{\underline{B(w,u)}}_{\text{bilinear}} = \underbrace{L(w)}_{\text{linear.}}$$

$$\left.\begin{aligned} B\Big(\phi_i, \sum_{j=1}^{N} c_j \phi_j + \phi_0\Big) &= L(\phi_i) \\ \because B \text{ bilinear} \Rightarrow \sum_{j=1}^{N} B(\phi_i, \phi_j)\, c_j &= \underbrace{L(\phi_i) - B(\phi_i, \phi_0)}_{F_i} \\ \sum_{j=1}^{N} B_{ij}\, c_j &= F_i \longrightarrow \textcircled{1} \end{aligned}\right\} i = 1, \cdots N \rightarrow \text{Ritz Eqns for } c_j$$

Additionally, if B is also symmetric then $\exists\, I(u)$ and WEAK FORM is $\delta I(u) = 0$

$$\Rightarrow \frac{\partial I}{\partial c_i} = 0 , \quad i = 1, \cdots N \quad \rightarrow \text{Ritz Eqns for } c_j \quad \rightarrow \textcircled{1}^*$$

① = ①* but ① more general as it requires only bilinearity of $B(w, u)$

→ $\phi_0(x)$ required only to satisfy non-homo EBC's, i.e,

$$\underset{u_0}{\underset{\parallel}{u(x_0)}} = U_N(x_0) = \sum c_j \underbrace{\phi_j(x_0)}_{=0} + \underbrace{\phi_0(x_0)}_{=u_0} .$$

→ ϕ_0 should be sufficiently differentiable as required by $B(\phi_i, \phi_0)$

→ ϕ_i
- should be sufficiently differentiable as required $\boxed{\text{by } B(\phi_i, \phi_j)}$
- should satisfy homogenous EBC's.
- should be linearly independent and should yield indep rows/cols in $[B_{ij}]$
- should form complete set.

Ex

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \;, \quad 0<x<1$$

$$u(0) = u(1) = 0$$

$$\phi_0 = 0, \quad \phi_i = x^i(1-x)$$

$$B_{ij} = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i \phi_j \right) dx$$

$$= \int_0^1 \left\{ \left[i\,x^{i-1}(1-x) - x^i \right]\left[j\,x^{j-1}(1-x) - x^j \right] - x^{i+j}(1-x)^2 \right\} dx$$

$$= \frac{2ij}{(i+j)\left[(i+j)^2-1\right]} - \frac{2}{(i+j+1)(i+j+2)(i+j+3)}$$

$$F_i = \int_0^1 -\phi_i\, x^2\, dx = -\int_0^1 x^2 x^i (1-x)\, dx = \frac{1}{(3+i)(4+i)}$$

$$N=2 \longrightarrow \frac{1}{420}\begin{bmatrix} 126 & 63 \\ 63 & 52 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = -\frac{1}{60}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

$$c_1 = \frac{-10}{123}, \quad c_2 = \frac{-21}{123}, \quad U_{N=2} = \frac{-1}{123}\left(10x + 11x^2 - 21x^3\right)$$

Exact soln: $u(x) = \dfrac{\sin x + 2\sin(1-x)}{\sin 1} + x^2 - 2$

$U_{N=2} \cong u, \quad \underset{\longrightarrow}{U_{N=3} = u}$ ie overlaps in plot, but not exactly equal

If BC is $u(0)=0, \quad \left.\dfrac{du}{dx}\right|_{x=1} = 1$

$$\phi(0) = 0, \quad \phi_i = x^i$$

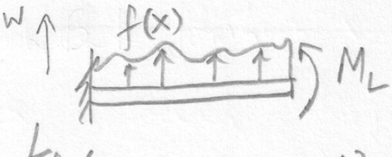
$$B_{ij} = \int_0^1 \left(ij\, x^{i+j-2} - x^{i+j} \right) dx$$

$F_i = -\int_0^1 x^{i+2}\, dx + 1^i \cdot 1$ and proceed.

Here too $U_{N=2} \cong u, \quad U_{N=3} \overset{\rightarrow \text{almost same}}{=} u, \quad u = \dfrac{2\cos(1-x) - \sin x}{\sin 1} + x^2 - 2$

Ex $$\frac{d^2}{dx^2}\left[b(x)\frac{d^2w}{dx^2}\right] - f(x) = 0, \quad 0<x<L$$

$$w(0) = \frac{dw}{dx}\Big|_{x=0} = 0, \quad \left(b\frac{d^2w}{dx^2}\right)_{x=L} = M_L, \quad \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x=L} = V_L = 0$$



Euler Bernoulli cantilever beam with $f(x)$, M_L applied

$b(x) = EI(x)$

$$\int_0^L \left(b(x)\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - fv\right)dx + v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\Big|_0^L - \frac{dv}{dx}\,b\frac{d^2w}{dx^2}\Big|_0^L = 0$$

$\left[w, \underbrace{\frac{d}{dx}\underbrace{\left(b\frac{d^2w}{dx^2}\right)}_{M}}_{V\equiv SF}\right]$ and $\left[\frac{dw}{dx}, \underbrace{b\frac{d^2w}{dx^2}}_{M}\right]$ are the two (PV, SV) pairs.

$$B(v,w) = \int_0^L b(x)\, v'' w''\, dx, \quad L(v) = \int_0^L f(x)\, v\, dx + v'(L) M_L$$

where we used $v(0) = v'(0) = V_L = 0$.

$B(v,w) - L(v) = 0 \rightarrow$ WEAK FORM

$$I(w) = \underbrace{\frac{1}{2}\int_0^L b\,(w'')^2 dx}_{SE} - \Bigg(\underbrace{\int_0^L f\,w\,dx}_{\text{work done by } f(x)} + \underbrace{w'(L) M_L}_{\text{work done by } M_L}\Bigg)$$

$$\boxed{\begin{aligned}&\frac{1}{2}\int \sigma_x \varepsilon_x\, dV \\ &= \frac{1}{2}\int E\varepsilon_x^2\, dV = \frac{1}{2}\int_0^L\int_A E(y\,w'')^2\, dA\, dx = \frac{1}{2}\int_0^L EI(w'')^2 dx\end{aligned}}$$

$\delta I(w) = 0 \rightarrow$ Also WEAK FORM

Rayleigh Ritz solution: for $b(x) = EI = \text{const}$, $f(x) = f_0 = \text{const}$.

$\phi_i(x) = x^{i+1}$ (twice differentiable, satisfies homogenous form of EBC's)

$\phi_0(x) = 0$ (EBC's homogenous, ie no inhomogenous term).

$$B_{ij} = \int_0^L EI(i+1)(i)(j+1)(j)\, x^{i+j-2}\, dx = EI\, ij\, \frac{(i+1)(j+1)\, L^{i+j-1}}{i+j-1} = B_{ji}$$

$$F_i = \int_0^L f_0\, x^{i+1}\, dx + (i+1)L^i M_L = f_0\frac{L^{i+2}}{i+2} + (i+1)L^i M_L$$

For N=2 (two term soln), $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

$$\text{gives } W_{N=2} = \underbrace{\frac{5f_0L^2+12M_L}{24EI}}_{c_1} \underset{\phi_1}{x^2} + \underbrace{\left(\frac{-f_0L}{12EI}\right)}_{c_2} \underset{\phi_2}{x^3} \simeq W_{exact}$$

For N=3, $\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B & B_{22} & B_{23} \\ \text{symm} & & B_{33} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$

$$\text{gives } W_{N=3} = \sum_{i=1}^{3} c_i \phi_i = \frac{f_0 x^2}{24EI}\left(6L^2-4Lx+x^2\right) + \frac{M_0 x^2}{2EI}$$

$$= W_{exact}$$

↳(exactly true).

⇒ For N=4, we will get $c_i = 0,\ i > 3$

(II) <u>Weighted Residual Methods.</u>

$\phi_j \rightarrow$ differentiable as much as (original) BVP requires
satisfies homogenous form of ALL BC's.

$\phi_0 \longrightarrow$ satisfies ALL BC's.

Galerkin Method : $w_j = \phi_j$

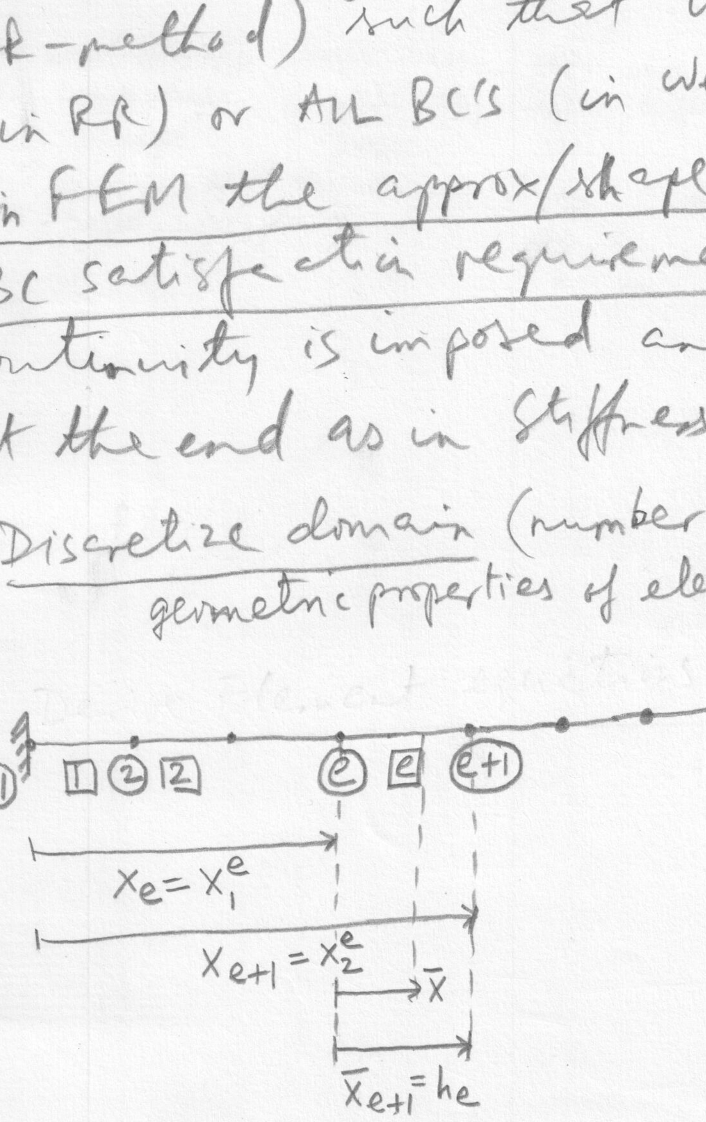
Collocation method: $w_j = \delta(x-x_j)$

FINITE ELEMENT METHOD for 2nd order BVP

Approx fns ϕ_i, ϕ_0, difficult to generate in a systematic way for complex geometries. Only continuity, completeness, EBC satisfaction, linear independence requirements prescribed but their systematic generation for complex geometries is not prescribed. Hence, traditional variational methods such as RR, Galerkin, etc methods, where ϕ_i, ϕ_0 prescribed over entire domain, is not competitive. In FEM the complex geometry doesnt alter the approx fns.

→ • Unlike traditional variational methods where ϕ_0, ϕ_i must satisfy certain BC's (eg ϕ_0 satisfies all EBC's, ϕ_i satisfy all homogenous form of EBC's in RR-method) such that U_N satisfies EBC's (in RR) or ALL BC's (in weighted residual), in FEM the approx/shape fns have no BC satisfaction requirements, since interelement continuity is imposed and BC's are imposed at the end as in Stiffness method.

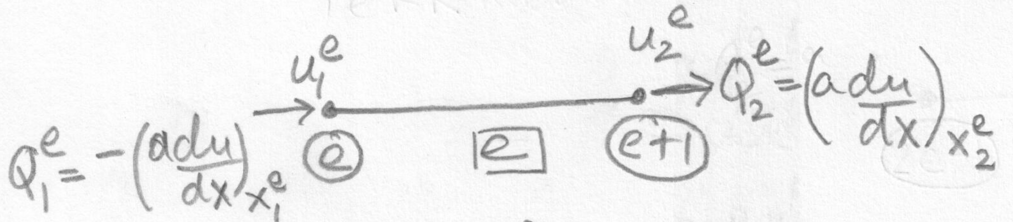
Step 1 Discretize domain (number nodes, elements, generate geometric properties of elements and coords of nodes).



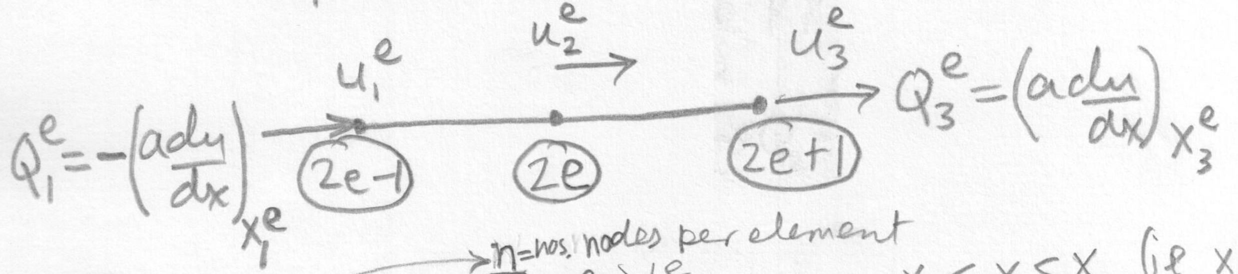
Linear 2-noded elements.

Step 2 Derive Element Equations → relation between PV's & SV's

Substep (i) Weak form or Weighted Residual
↳ (we do this, ie RR-method)



2-noded Linear Lagrange, C^0



3-noded ~~Quadrat~~ Quadratic Lagrange, C^0

Here $n \equiv N$ in traditional Variational Methods. We changed notation ∵ later we encounter N_i shape function

n = nos. nodes per element

$$U^e = \sum_{j=1}^{n} u_j^e \, \overset{N_j^e(x)}{\cancel{\phi_j^e(x)}} \quad , \quad x_1 < x < x_n \text{ (ie } x \text{ lies within element } e)$$

- weak form requires $\overset{N_j^e}{\cancel{\phi_j^e}}$ at least linear. Weighted residual requires $\cancel{\phi_j^e}$ at least quadratic.

Weak form is,

$$\int_{x_1^e}^{x_2^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx - w(x_1^e)\,Q_1^e - w(x_n^e)\,Q_n^e$$

$$Q_1^e = -\left(a\frac{du}{dx}\right)_{x_1^e} \quad , \quad Q_n^e = \left(a\frac{du}{dx}\right)_{x_n^e}$$

, i-e, both → +ve ie Q_1^e compr +ve Q_n^e tensile +ve.

or alternatively $\delta I(u) = \delta\left[\frac{1}{2}B(u,u) - L(u)\right] = 0$

$$0 = \delta I(u) = \delta\Bigg[\underbrace{\frac{1}{2}\int_{x_1^e}^{x_n^e}\left(a\left(\frac{du}{dx}\right)^2 + cu^2\right)dx}_{\text{strain energy due to deformation}} - \underbrace{\int_{x_1^e}^{x_n^e} uq\,dx - Q_1^e u(x_1^e) - Q_n^e u(x_n^e)}_{\text{work done by loads}}\Bigg]$$

- In deriving weak form / I(u), all BC's for element are "considered/treated/assumed" as NBC's, and thus retained in the weak form. In the next step, while deriving shape/approximating/interpolation f$\underline{^{ns}}$ ∴ ϕ_i (aka N_i) we "consider/assume/treat" all BCs as EBC's by considering coeffs to be determined, ie u_i^e, as $u_1^e = U^e(x_1^e)$, $u_i^e = U^e(x_i^e)$

- Note that this does not imply that we are applying both NBC's & EBC's which would be fundamentally wrong. In fact in entire step 2 we dont apply any BC's. BC application happens only after assembly, whence we identify prescribed displ's (EBCs) & prescribed forces (NBCs), and then solve for unknown displs, u_j^e, and forces Q_j^e (i-e, reactions)

substep (ii) <u>Approximating the solution</u> (deriving approx/shape/interpolation fns)

We assume/consider/treat all BCs as EBCs.

{ Thus, we require $U^e(x_i) = u_i^e$, $i = 1, \cdots n$, (1,..,n are node numbers of element e). Thus, $\underset{\searrow N_i^e}{\cancel{\phi_i^e}}$ are interpolating fns.

- $\overset{N_i}{\cancel{\phi_i}}$ usually chosen as polynomials, ∵ easy to differentiate and integrate systematically, and interpolation theory using poly's well established.

|||

- Differentiability, completeness, independence, requirements of $\overset{N_i}{\cancel{\phi_i}}$ same as for RR (or Weighted Residual methods), as the case may be.
- $\overset{N_i}{\cancel{\phi_i}}$ should be interpolant of PV's (u in present case)

<u>Linear $\phi_i \equiv N_i$</u> $U^e(x) = a + bx$, $\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$

$$U^e(x) = \frac{x_2^e - x}{\underset{h^e \swarrow}{\cancel{x_2^e - x_1^e}}} u_1^e + \frac{x - x_1^e}{x_2^e - x_1^e} u_2^e = N_1^e(x) u_1^e + N_2^e(x) u_2^e$$

x → global coord

$x_1^e \leq x \leq x_2^e$

$$= \frac{x_{e+1} - x}{h^e} u_1^e + \frac{x - x_e}{h^e} u_2^e$$

$$U^e(\bar{x}) = \frac{h^e - \bar{x}}{h^e} u_1^e + \frac{\bar{x}}{h^e} u_2^e = N_1^e(\bar{x}) u_1^e + N_2^e(\bar{x}) u_2^e$$

$\bar{x}$ → local coord (element)

$0 \leq \bar{x} \leq h_e$

Quadratic $\phi_i \equiv N_i$: $U^e_{(x)} = a+bx+cx^2$, $\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e & (x_1^e)^2 \\ 1 & x_2^e & (x_2^e)^2 \\ 1 & x_3^e & (x_3^e)^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$

Tedious to solve a, b, c in terms of u_1^e, u_2^e, u_3^e. So use n^{th} order Lagrangian Polynomial (which gives same result), given as

$$N_i(x) = \prod_{j=1, j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$$

Here onwards superscript 'e' dropped in x_j^e for convenience in writing.

$$N_1(x) = \left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right), \quad N_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$$x_3 - x_1 = h^e, \quad x_2 - x_1 = \alpha h^e, \quad x_1 - x = \bar{x}$$

$$N_1(\bar{x}) = \frac{(x-x_1-\alpha h^e)(x-x_1-h^e)}{(-\alpha h^e)(-h^e)} = \left(1-\frac{\bar{x}}{\alpha h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right) \overset{\text{when ② at midpoint } \alpha = 1/2}{=} \left(1-\frac{\bar{x}}{h^e}\right)\left(1-\frac{2\bar{x}}{h^e}\right)$$

$$N_2(\bar{x}) = \frac{(\bar{x})(\bar{x}-h^e)}{(\alpha h^e)(\alpha-1)h^e} = \frac{1}{\alpha(1-\alpha)}\frac{\bar{x}}{h^e}\left(1-\frac{\bar{x}}{h^e}\right) = 4\left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right)$$

$$N_3(\bar{x}) = \frac{(\bar{x})(\bar{x}-\alpha h^e)}{(h^e)(1-\alpha)h^e} = -\frac{\alpha}{(1-\alpha)}\left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{\bar{x}}{\alpha h^e}\right) = -\left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{2\bar{x}}{h^e}\right)$$

Cubic $\phi_i \equiv N_i$ $\quad U^e_x = a+bx+cx^2+dx^3$, 4x4 system to solve for a, b, c, d. TOO tedious!!

so instead, $N_1(x) = \dfrac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$, $N_2(x) = \dfrac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$

$$N_3(x) = \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}, \quad N_4(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$x_4 - x_1 = h^e$, $x_2 - x_1 = \alpha h^e$, $x_3 - x_1 = \beta h^e$, $x_1 - x = \bar{x}$

$$N_1(\bar{x}) = \frac{(x-x_1-\alpha h^e)(x-x_1-\beta h^e)(x-x_1-h^e)}{(-\alpha h^e)(-\beta h^e)(-h^e)} = \left(1-\frac{\bar{x}}{\alpha h^e}\right)\left(1-\frac{\bar{x}}{\beta h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right)$$

$$= \left(1-\frac{\bar{x}}{h^e}\right)\left(1-\frac{3}{2}\frac{\bar{x}}{h^e}\right)\left(1-3\frac{\bar{x}}{h^e}\right) \quad \text{for } \alpha=\frac{1}{3}, \beta=\frac{2}{3}$$

$$N_2(\bar{x}) = \frac{(\bar{x})(\bar{x}-\beta h^e)(\bar{x}-h^e)}{(\alpha h^e)(\alpha-\beta)h^e(\alpha-1)h^e} = \left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right)\left(\beta-\frac{\bar{x}}{h^e}\right)\frac{1}{\alpha(1-\alpha)(\beta-\alpha)}$$

$$= 9\left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right)\left(1-\frac{3}{2}\frac{\bar{x}}{h^e}\right) \quad \text{for } \alpha=\frac{1}{3}, \beta=\frac{2}{3}$$

$$N_3(\bar{x}) = \frac{(\bar{x})(\bar{x}-\alpha h^e)(\bar{x}-h^e)}{(\beta h^e)(\beta-\alpha)h^e(\beta-1)h^e} = -\left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right)\left(\alpha-\frac{\bar{x}}{h^e}\right)\frac{1}{\beta(\beta-\alpha)(1-\beta)}$$

$$= -\frac{9}{2}\left(\frac{\bar{x}}{h^e}\right)\left(1-\frac{\bar{x}}{h^e}\right)\left(1-3\frac{\bar{x}}{h^e}\right) \quad \text{for } \alpha=\frac{1}{3}, \beta=\frac{2}{3}$$

$$N_4(x) = \frac{(\bar{x})(\bar{x}-\alpha h^e)(\bar{x}-\beta h^e)}{h^e(1-\alpha)h^e(1-\beta)h^e} = \frac{\bar{x}}{h^e}\left(\alpha-\frac{\bar{x}}{h^e}\right)\left(\beta-\frac{\bar{x}}{h^e}\right)\frac{1}{(1-\alpha)(1-\beta)}$$

$$= \left(\frac{\bar{x}}{h^e}\right)\left(1-3\frac{\bar{x}}{h^e}\right)\left(1-\frac{3}{2}\frac{\bar{x}}{h^e}\right) \quad \text{for } \alpha=\frac{1}{3}, \beta=\frac{2}{3}$$

$$\boxed{\begin{array}{ll} \text{(i)} & N_i^e(x_j^e) = N_i(\bar{x}_j) = \delta_{ij} \\ \text{(ii)} & \sum_{i=1}^{n} N_i^e(x) = \sum_{i=1}^{n} N_i^e(\bar{x}) = 1 \Rightarrow \sum_{i=1}^{n} \frac{d}{dx} N_i(x) = \sum_{i=1}^{n} \frac{d}{d\bar{x}} N_i(\bar{x}) = 0 \end{array}} \rightarrow \text{Property of } N_i$$

(i) ∵ N_i is an interpolation fn, ie $U^e(x_i) = u_i^e$, $i=1, \cdots n$

(ii) ∵ $U^e(x) = U^e(\bar{x})$ must allow for the possibility of a constant solution, $U^e(x) = U^e(\bar{x}) = \text{const} = u_1^e = u_2^e \cdots = u_n^e$, which implies property (ii).

Note: In beams (u→w) the d.o.f at each node are w, $\frac{dw}{dx}$ so we interpolate PV's w and $\frac{dw}{dx}$ and hence a constant w or $\frac{dw}{dx}$ solution wont imply property (ii)

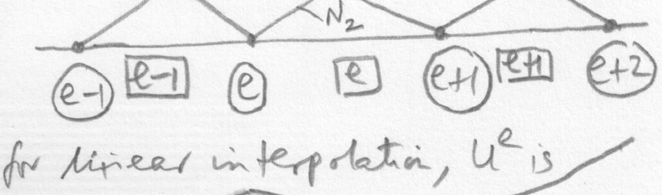
→ • In axial rod, the interpolation of PV's (only u) involve only dependent unknown u. These are Lagrange family of interpolation f^n^s and possess C^0 continuity ∵ only displacements (u) and not its derivatives are used to derive the interpolation f^n^s and only displs and not its derivatives are interpolated by these f^n^s.

→ • When interpolation f^n^s are derived using dependent variable & its derivatives, they are Hermite family of interpolation f^n^s and possess C^1 continuity. Example beam, where PV's are w, w', both of which are used to derive interpolation f^n^s and both of which are interpolated by these f^n^s and their 1st derivative.

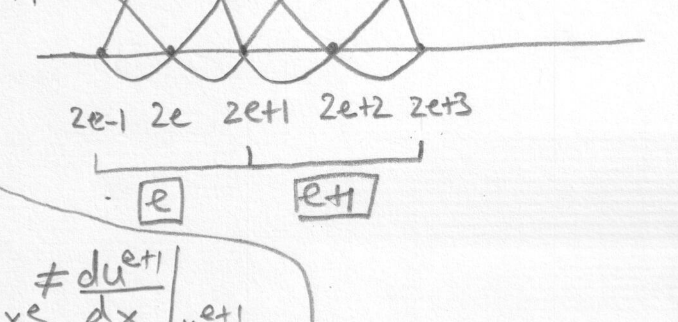
→ • $C^0 \Rightarrow$ the approximation using interpolation f^n^s yields continuous dependent unknown (eg. u) but not continuous derivatives (eg. u') as in axial rod using Lagrangian

$C^1 \Rightarrow$ approx yields continuous dep. unknown (eg. w) and its derivatives (eg. w') as in beam using Hermite.

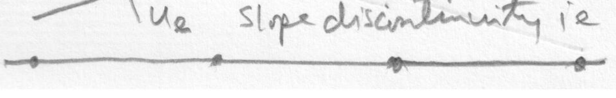
N_i for linear Lagrange element



N_i for quadratic Lagrange element



so for linear interpolation, U^e is



slope discontinuity ie $\left.\frac{du^e}{dx}\right|_{x_2^e} \neq \left.\frac{du^{e+1}}{dx}\right|_{x_1^{e+1}}$

Can derive interpolation functions using property (i)

Quadratic Lagrange

$$N_1 = C\underbrace{\left(\bar{x}-\tfrac{1}{2}h_e\right)\left(\bar{x}-h_e\right)}_{\because N_1=0 \text{ at } \bar{x}=\frac{1}{2}h_e \text{ and } h_e} \; ; \; N_1\Big|_{\bar{x}=0} = 1 \Rightarrow C = \frac{2}{h_e^2}$$

$$\Rightarrow N_1 = \left(1-\frac{\bar{x}}{h_e}\right)\left(1-2\frac{\bar{x}}{h_e}\right)$$

$$N_2 = C(\bar{x})(\bar{x}-h_e) \; ; \; N_2\Big|_{\bar{x}=\frac{h_e}{2}} = 1 \Rightarrow C = -\frac{4}{h_e^2} \Rightarrow N_2 = 4\left(\frac{\bar{x}}{h_e}\right)\left(1-\frac{\bar{x}}{h_e}\right)$$

$$N_3 = C(\bar{x})\left(\bar{x}-\frac{h_e}{2}\right) \; ; \; N_3\Big|_{\bar{x}=h_e} = 1 \Rightarrow C = \frac{2}{h_e^2} \Rightarrow N_3 = -\left(\frac{\bar{x}}{h_e}\right)\left(1-2\frac{\bar{x}}{h_e}\right)$$

Cubic Lagrange

$$N_1 = C\left(\bar{x}-\frac{h_e}{3}\right)\left(\bar{x}-\frac{2}{3}h_e\right)\left(\bar{x}-h_e\right) \; ; \; N_1\Big|_{\bar{x}=0} = 1 \Rightarrow C = -\frac{9}{2}h_e^3$$

$$\Rightarrow N_1 = \left(1-\frac{\bar{x}}{h_e}\right)\left(1-\frac{3}{2}\frac{\bar{x}}{h_e}\right)\left(1-3\frac{\bar{x}}{h_e}\right)$$

$$N_2 = C(\bar{x})\left(\bar{x}-\frac{2}{3}h_e\right)\left(\bar{x}-h_e\right) \; ; \; N_2\Big|_{\bar{x}=\frac{h_e}{3}} = 1 \Rightarrow C = \frac{27}{2}\frac{1}{h^3}$$

$$\Rightarrow N_2 = 9\left(\frac{\bar{x}}{h_e}\right)\left(1-\frac{3}{2}\frac{\bar{x}}{h_e}\right)\left(1-\frac{\bar{x}}{h_e}\right)$$

$$N_3 = C_1(\bar{x})\left(\bar{x}-\frac{1}{3}h_e\right)\left(\bar{x}-h_e\right) \; ; \; N_3\Big|_{\bar{x}=\frac{2}{3}h_e} = 1 \Rightarrow C = -\frac{27}{2}\frac{1}{h^3}$$

$$\Rightarrow N_3 = -\frac{9}{2}\left(\frac{\bar{x}}{h_e}\right)\left(1-\frac{\bar{x}}{h_e}\right)\left(1-3\frac{\bar{x}}{h_e}\right)$$

$$N_4 = C(\bar{x})\left(\bar{x}-\frac{h_e}{3}\right)\left(\bar{x}-\frac{2}{3}h_e\right) \; ; \; N_4\Big|_{\bar{x}=h_e} = 1 \Rightarrow C = \frac{9}{2}\frac{1}{h_e^3}$$

$$\Rightarrow N_4 = \left(\frac{\bar{x}}{h_e}\right)\left(1-\frac{3}{2}\frac{\bar{x}}{h_e}\right)\left(1-3\frac{\bar{x}}{h_e}\right)$$

- Lagrange interpolation fn^{s} readily available in books. They depend only on geometry (shape) of element, & nos and location of nodes in element.

Substep (iii) Finite Element Model (relation between PV & SV for element)

Recall $U^e = \sum_{j=1}^{n} u_j^e \underset{\equiv \phi_j^e}{\boxed{N_j^e}}$. Subst in Weak form of Element

Recall W.F → $\int_{x_1^e}^{x_n^e} \left(a \frac{dw}{dx} \frac{du}{dx} + c \overset{N_i}{w} \overset{U^e}{u} - wq \right) dx - w(x_1^e) Q_1^e - w(x_n^e) Q_n^e$

n = nos of nodes in element e

where, $Q_1^e = -a \frac{du}{dx} \Big|_{x=x_1^e}$, $Q_n^e = a \frac{du}{dx} \Big|_{x=x_n^e}$ defined +ve convention

$$B_{ij}^e \equiv K_{ij}^e = \int_0^{he} \left[a^e(\bar{x}) \frac{dN_i^e(\bar{x})}{d\bar{x}} \frac{dN_j^e(\bar{x})}{d\bar{x}} + c^e(\bar{x}) N_i^e(\bar{x}) N_j^e(\bar{x}) \right] d\bar{x}$$

$$F_i^e = \int_0^{he} N_i^e(\bar{x}) q^e(\bar{x}) d\bar{x} + \sum_{j=1}^{n} N_i(\bar{x}_j^e) Q_j^e$$

Note: In F_i^e we added applied point sources (ie applied loads) at nodes 2, … (n−1), i.e, $Q_2^e, \dots Q_{n-1}^e$, by representing them as $Q_j^e \delta(\bar{x} - \bar{x}_j)$ in the $q(\bar{x})$ term and using

$$\int_0^{he} N_i^e(\bar{x}) Q_j^e \delta(\bar{x} - \bar{x}_j) d\bar{x} = N_i^e(\bar{x}_j) Q_j^e$$

Also $B_{i0}^e = B(N_i^e, \underset{\psi_0}{N_0^e})$ not present (ie zero) ∵ $N_0^e = 0$, ie all BC's considered as NBC's for element.

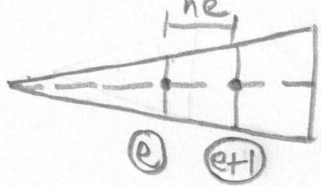
Linear N_i : $a^e(\bar{x}) = a_0$ $c^e(\bar{x}) = c_0$, $q^e(\bar{x}) = q_0$, ie consts

$$K^e = \int_0^{he} \left\{ a_0 \begin{bmatrix} -1/he \\ 1/he \end{bmatrix} \begin{bmatrix} -1/he & 1/he \end{bmatrix} + c_0 \begin{bmatrix} 1 - \bar{x}/he \\ \bar{x}/he \end{bmatrix} \begin{bmatrix} 1 - \frac{\bar{x}}{he} & \frac{\bar{x}}{he} \end{bmatrix} \right\} d\bar{x}$$

$$= \frac{a_0}{he} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \boxed{c_0 he \begin{bmatrix} 1 - 1 + \frac{1}{3} & \frac{1}{2} - \frac{1}{3} \\ \frac{1}{2} - \frac{1}{3} & \frac{1}{3} \end{bmatrix}} = \frac{c_0 he}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$F^e = \int_0^{h_e}\begin{bmatrix}1-\bar{x}/h_e\\ \bar{x}/h_e\end{bmatrix}q_0 + \begin{bmatrix}\sum_{j=1}^{2}(1-\bar{x}/h_e)\big|_{x_j}Q_j\\ \sum_{j=1}^{2}(\bar{x}/h_e)\big|_{x_j}Q_j\end{bmatrix} = q_0h_e\begin{bmatrix}1-\frac{1}{2}=\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}+\begin{bmatrix}Q_1\\ Q_2\end{bmatrix}$$

If $a(x)=a_0x$ (tapered section), $a^e(\bar{x})=a_0(x_e+\bar{x})$ then first term in K^e becomes,



Linearly varying $a(x)$

$$\int_0^{h_e}\frac{a_0(x_e+\bar{x})}{h_e^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}d\bar{x} = \underbrace{\frac{a_0}{h_e}\left(x_e+\frac{h_e}{2}\right)}_{\text{average of areas at } x_e,\ x_{e+1}}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

$$\boxed{K^e\begin{Bmatrix}u_1^e\\u_2^e\end{Bmatrix}=F^e} \rightarrow \text{Linear element Force-displ (SV-PV) relation}$$

<u>Quadratic N_i</u> : $a(x)=a_0$, $c(x)=c_0$, $q(x)=q_0$

$$K^e=\int_0^{h_e}\left\{\begin{array}{l}\frac{a_0}{h_e^2}\begin{bmatrix}-1+\frac{2\bar{x}}{h_e}-2+\frac{2\bar{x}}{h_e}\\ 4-\frac{4\bar{x}}{h_e}-\frac{4\bar{x}}{h_e}\\ -1+\frac{2\bar{x}}{h_e}+\frac{2\bar{x}}{h_e}\end{bmatrix}\left[-3+\frac{4\bar{x}}{h_e},\ 4-\frac{8\bar{x}}{h_e},\ -1+\frac{4\bar{x}}{h_e}\right] + \\ c_0\begin{bmatrix}1-\frac{3\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2\\ \frac{4\bar{x}}{h_e}-4\left(\frac{\bar{x}}{h_e}\right)^2\\ -\frac{\bar{x}}{h_e}+2\left(\frac{\bar{x}}{h_e}\right)^2\end{bmatrix}\left[\text{transpose}\right]\end{array}\right\}d\bar{x}$$

$$-\frac{1}{2}+\frac{5}{3}-\frac{8}{4}+\frac{4}{5}=-1/30$$

$$=\frac{a_0}{h_e}\begin{bmatrix}\overset{=7/3}{\left(9-\frac{24}{2}+\frac{16}{3}\right)} & \overset{=-\frac{8}{3}}{\left(-12+\frac{40}{2}-\frac{32}{3}\right)} & \overset{=1/3}{\left(3-\frac{16}{2}+\frac{16}{3}\right)}\\ & 16/3=\left(16-\frac{64}{2}+\frac{64}{3}\right) & \left(-4+\frac{24}{2}-\frac{32}{3}\right)=-\frac{8}{3}\\ \text{symm} & & 7/3=\left(1-\frac{8}{2}+\frac{16}{3}\right)\end{bmatrix}+c_0h_e\begin{bmatrix}\overset{2/15}{\left(1+\frac{9}{3}+\frac{4}{5}-\frac{6}{2}+\frac{4}{3}-\frac{12}{4}\right)} & \overset{1/15}{\left(\frac{4}{2}-\frac{16}{3}+\frac{20}{4}-\frac{8}{5}\right)} & (\quad)\\ & \frac{8}{15}=\left(\frac{16}{3}+\frac{16}{5}-\frac{32}{4}\right) & \underset{=1/15}{\left(-\frac{4}{3}+\frac{12}{4}-\frac{8}{5}\right)}\\ & & \frac{2}{}=\left(\frac{1}{3}+\frac{4}{}-\frac{4}{}\right)\end{bmatrix}$$

$$F^e = \int_0^{he} \begin{bmatrix} 1-3\frac{\bar{x}}{he}+2\left(\frac{\bar{x}}{he}\right)^2 \\ 4\frac{\bar{x}}{he}-4\left(\frac{\bar{x}}{he}\right)^2 \\ -\frac{\bar{x}}{he}+2\left(\frac{\bar{x}}{he}\right)^2 \end{bmatrix} q_0\, d\bar{x} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} = \underbrace{\begin{Bmatrix} 1-\frac{3}{2}+\frac{2}{3} = 1/6 \\ \frac{4}{2}-\frac{4}{3} = 2/3 \\ -\frac{1}{2}+\frac{2}{3} = 1/6 \end{Bmatrix} q_0 he}_{f^e} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix}$$

$$\boxed{K^e \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = F^e}$$

Quadratic element Force-Displ (SV-PV) relation

Note: $\sum_{i=1}^{n=3} f_i^e = \left(\frac{1}{6}+\frac{2}{3}+\frac{1}{6}\right) q_0 he = q_0 he$

also evident from $f_i = \int_0^{he} N_i q\, d\bar{x}$

and $\sum_{i=1}^{n} N_i^e = 1$. This means that source when distributed to element nodes should add up to source integrated over element.

Note: if you distribute q_0 as per linear element of length $he/2$ you get

$$\begin{Bmatrix} \frac{1}{2}\cdot\frac{1}{2} \\ \frac{1}{2}\cdot\frac{1}{2}\times 2 \\ \frac{1}{2}\cdot\frac{1}{2} \end{Bmatrix} q_0 he$$

which is wrong!!

Step 3 Assembly (Structure SV-PV relation)

Continuity of PV's: eg., $u_n^e = u_1^{e+1} = u_N$

Balance of SV's: eg. $Q_n^e + Q_1^{e+1} = Q_N$

for sequentially arranged axial elements.

$N =$ global number of n^{th} (last) node of e^{th} element

$= ne - (e-1) = (n-1)e + 1$

don't confuse with N-term approximation in traditional variational methods

Q_N = applied point source (load) at global node N.

Actually balance of SV is equivalent to continuity of SV.

i.e., $\left(a\frac{du}{dx}\right)_n^e = \left(a\frac{du}{dx}\right)_1^{e+1} \Rightarrow Q_n^e = -Q_1^{e+1}$, $Q_n^e + Q_1^{e+1} = 0$

if $Q_N = 0$, i.e no point source applied

Note: $\left(a\frac{dU^e}{dx}\right)_n \neq \left(a\frac{dU^{e+1}}{dx}\right)_1$, i.e., gradient of approx soln is not continuous across element end nodes

(obvious ∵ N_i's

Adding n^{th} equil eqn of e^{th} element and 1^{st} equil eqn of $(e+1)^{th}$ element (assuming sequential global node numbering and both elements n-noded)

$$K^e_{n1}u^e_1+\ldots K^e_{nn}u^e_n+K^{e+1}_{11}u^{e+1}_1+\ldots K^{e+1}_{1n} = f^e_n+f^{e+1}_1+Q^e_n+Q^{e+1}_1$$

and introducing interelement continuity of PV's & equilibrium of SV's,

$$\begin{bmatrix} \\ \\ 0\ldots\ldots0\ K^e_{n1}\ \ K^e_{n2}\ldots K^e_{nj}\ldots\left(K^e_{nn}+K^{e+1}_{11}\right)\ K^{e+1}_{12}\ldots K^{e+1}_{1j}\ldots K^{e+1}_{1n}\ 0\ldots0 \\ \\ \\ \\ \\ \\ \end{bmatrix} \begin{array}{l} u_1 \\ \vdots \\ u_{N-(n-1)} \\ u_{N-n+2} \\ \vdots \\ u_N \\ u_{N+1} \\ \vdots \\ u_{N+(n-1)} \\ \vdots \\ u_{E(n-1)+1} \end{array}$$

▭ are applied point sources or reactions at element end nodes

⬭ applied point sources at element intermediate nodes

$$\begin{Bmatrix} \cancel{f^1_1}\to f_1 \\ \vdots \\ \cancel{f^1_j}\to f_j \\ \vdots \\ \cancel{f^1_n+f^2_1}\to f_{(n-1).\overset{1}{\cancel{e}}+1} \\ \vdots \\ \cancel{f^2_j}\to f_{(n-1).(\overset{2}{\cancel{e}}-1)+\cancel{1}+j-\cancel{1}} \\ \vdots \\ \cancel{f^2_n+f^3_1}\to f_{(n-1)\overset{2}{\cancel{e}}+1} \\ \vdots \\ \cancel{f^{e-1}_n+f^e_1}\to f_{(n-1)(\overset{e}{\cancel{e}}-1)+1}=N-(n-1) \\ \cancel{f^e_2}\to f_{N-n+2} \\ \vdots \\ \cancel{f^e_n+f^{e+1}_1}\to f_{(n-1)e+1}=N \\ \cancel{f^{e+1}_2}\to f_{N+1} \\ \vdots \end{Bmatrix} + \begin{Bmatrix} \boxed{Q^1_1} \\ \vdots \\ Q^1_j \\ \vdots \\ \boxed{\cancel{Q^1_n+Q^2_1}\to Q_{(n-1)\overset{1}{\cancel{e}}+1}} \\ \vdots \\ \cancel{Q^2_j}\to Q_{(n-1)(\overset{2}{\cancel{e}}-1)+j} \\ \vdots \\ \boxed{\cancel{Q^2_n+Q^3_1}\to Q_{(n-1)\overset{2}{\cancel{e}}+1}} \\ \vdots \\ \boxed{\cancel{Q^{e-1}_n+Q^e_1}\to Q_{N-(n-1)}} \\ \cancel{Q^e_2}\to Q_{N-n+2} \\ \vdots \\ \boxed{\cancel{Q^e_n+Q^{e+1}_1}\to Q_N} \\ \cancel{Q^{e+1}_2}\to Q_{N+1} \\ \vdots \end{Bmatrix}$$

equivalent nodal loads due to $q(x)$.

=

$q(x)$ can also include applied point loads at nodes, ie, $\delta(x-x^*)$ in which case the corresponding Q's should be made zero.

So for mesh with 'E' linear (2-noded) elements,

$$\left[\begin{array}{cccccccc}
K^1_{11} & K^1_{12} & & & & & & \\
K^1_{21} & +(K^1_{22}+K^2_{11}) & K^2_{12} & & & & & \\
 & K^2_{21} & (K^2_{22}+K^3_{11}) & K^3_{12} & & & & \\
\hline
\hline
 & & K^{e-1}_{21} & (K^{e-1}_{22}+K^e_{11}) & K^e_{12} & & & \\
 & & & K^e_{21} & (K^e_{22}+K^{e+1}_{11}) & K^{e+1}_{12} & & \\
\hline
\hline
 & & & & & K^{E-1}_{21} & (K^{E-1}_{22}+K^E_{11}) & K^E_{12} \\
 & & & & & & K^E_{21} & K^E_{22}
\end{array}\right]$$

$$= \left\{\begin{array}{c}
\cancel{f^1_1} \rightarrow f_1 \\
f^1_2+f^2_1 \\
f^2_2+f^3_1 \\
\vdots \\
f^{e-1}_2+f^e_1 \\
f^e_2+f^{e+1}_1 \\
\vdots \\
f_E \leftarrow \cancel{f^{E-1}_2+f^E_1} \\
\cancel{f^E_2} \rightarrow f_{E+1}
\end{array}\right\}
+
\left\{\begin{array}{c}
\cancel{Q^1_1} \rightarrow Q_1 \\
Q^1_2+Q^2_1 \\
Q^2_2+Q^3_1 \\
\vdots \\
Q^{e-1}_2+Q^e_1 \\
Q^e_2+Q^{e+1}_1 \\
\vdots \\
\cancel{Q^{E-1}+Q^E} \rightarrow Q_E \\
\cancel{Q^E_2} \rightarrow Q_{E+1}
\end{array}\right\}$$

equivalent nodal loads due to $q(x)$ (which can be due to point sources $q(x)=\delta(x-x^*)$

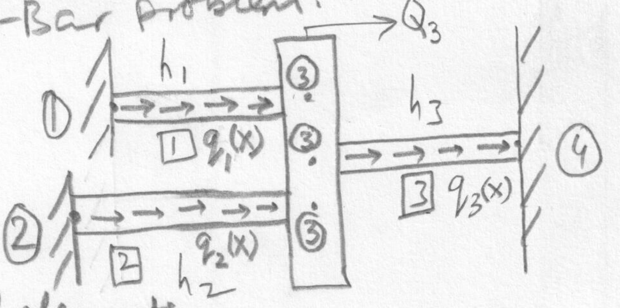
applied point sources or reactions at element end nodes,

i.e, $1, \ldots \underset{E}{e}(\underset{2}{k}-1)+1$

i.e $1, \ldots E+1$

In general (ie when elements are not sequentially arranged), assembly follows same method as for direct Stiffness method, ie as per d.o.f numbering and connectivity.

(Ex) 3-Bar problem. (c=0)



$$G = \begin{array}{c|cc} \text{element} \backslash \text{node} & 1^{st} & 2 \\ \hline \boxed{1} & 1 & 3 \\ \boxed{2} & 2 & 3 \\ \boxed{3} & 3 & 4 \end{array} \rightarrow \text{connectivity matrix}$$

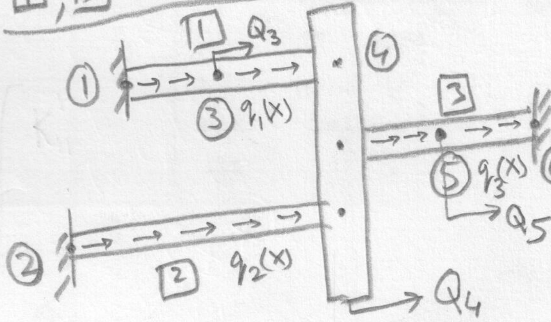
G_{ij} = global node number of j^{th} node of i^{th} element.

All 2-noded elements

$$\begin{bmatrix} K^1_{11} & 0 & K^1_{12} & 0 \\ & K^2_{11} & K^2_{12} & 0 \\ & & K^1_{22}+K^2_{22}+K^3_{11} & K^3_{12} \\ & & & K^3_{22} \\ \text{symm} & & & \end{bmatrix} \begin{Bmatrix} u^1_1 \equiv U_1 \\ u^2_1 \equiv U_2 \\ u^1_2 = u^2_2 = u^3_1 \equiv U_3 \\ u^3_2 \equiv U_4 \end{Bmatrix} = \begin{Bmatrix} f^1_1 \equiv f_1 \\ f^2_1 \equiv f_2 \\ f^1_2 + f^2_2 + f^3_1 \equiv f_3 \\ f^3_2 \equiv f_4 \end{Bmatrix} + \begin{Bmatrix} Q^1_1 \equiv Q_1 \\ Q^2_1 \equiv Q_2 \\ Q^1_2 + Q^2_2 + Q^3_1 \equiv Q_3 \\ Q^3_2 \equiv Q_4 \end{Bmatrix}$$

(✱)

$\boxed{1}$, $\boxed{3}$ 3-noded, $\boxed{2}$ 2-noded:



$$G = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline \boxed{1} & 1 & 3 & 4 \\ \boxed{2} & 2 & - & 4 \\ \boxed{3} & 4 & 5 & 6 \end{array}$$

$$\begin{bmatrix} K^1_{11} & 0 & K^1_{12} & K^1_{13} & 0 & 0 \\ & K^2_{11} & 0 & K^2_{12} & 0 & 0 \\ & & K^1_{22} & K^1_{23} & 0 & 0 \\ & & & K^1_{33}+K^2_{22}+K^3_{11} & K^3_{12} & K^3_{13} \\ & & & & K^3_{22} & K^3_{23} \\ \text{symm} & & & & & K^3_{33} \end{bmatrix} \begin{Bmatrix} u^1_1 \equiv U_1 \\ u^2_1 \equiv U_2 \\ u^1_2 \equiv U_3 \\ u^1_3 = u^2_2 = u^3_1 \equiv U_4 \\ u^3_2 \equiv U_5 \\ u^3_3 \equiv U_6 \end{Bmatrix} = \begin{Bmatrix} f^1_1 \equiv f_1 \\ f^2_1 \equiv f_2 \\ f^1_2 \equiv f_3 \\ f^1_3 + f^2_2 + f^3_1 \equiv f_4 \\ f^3_2 \equiv f_5 \\ f^3_3 \equiv f_6 \end{Bmatrix}$$

$$+ \begin{Bmatrix} Q^1_1 \equiv Q_1 \\ Q^2_1 \equiv Q_2 \\ Q^1_2 \equiv \enclose{circle}{Q_3} \\ Q^1_3 + Q^2_2 + Q^3_1 \equiv Q_4 \\ Q^3_2 \equiv \enclose{circle}{Q_5} \\ Q^3_3 \equiv Q_6 \end{Bmatrix}$$

— here Q_3, Q_5 (circled) are *always* known applied point sources.

Applying BC's

For the 3-bar problem with loads as shown,

All 2-noded elements: $u_1 = u_2 = u_4 = 0$, Q_3 = given; unknowns are u_3, Q_1, Q_2, Q_4

$\boxed{1}, \boxed{3}$ 3-noded, $\boxed{2}$ 2-noded: $u_1 = u_2 = u_6 = 0$, Q_3, Q_4, Q_5 = given; unknowns are $u_3, u_4, u_5, Q_1, Q_2, Q_6$.

$$\begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix} = \int_0^{h_e} \begin{Bmatrix} N_1^e(\bar{x}) \\ N_2^e(\bar{x}) \end{Bmatrix} q_e(\bar{x})\, d\bar{x}$$

for $e = 1, 2, 3$

For $e = 2$.

For $e = 1, 3$

$$\begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \end{Bmatrix} = \int_0^{h_e} \begin{Bmatrix} N_1^e(\bar{x}) \\ N_2^e(\bar{x}) \\ N_3^e(\bar{x}) \end{Bmatrix} q_e(\bar{x})\, d\bar{x}$$

Solution

$$\circledast \leftarrow \begin{bmatrix} K_{I\,I} & K_{I\,II} \\ K_{II\,I} & K_{II\,II} \end{bmatrix} \begin{Bmatrix} u_I \\ u_{II} \end{Bmatrix} = \begin{Bmatrix} f \end{Bmatrix} + \begin{Bmatrix} Q_I \\ Q_{II} \end{Bmatrix}$$

, u_I, Q_{II} are unknown PV's & SV's respectively.

ie, just as in Stiffness Method.

First compute u_I, then Q_{II}, using Equil. eqns $\circledast$. Alteratively

$$\frac{du}{dx} \approx \frac{dU^e}{dx} = \sum_{j=1}^{n} u_j^e \frac{dN_j^e}{dx}.$$

can be used to compute Q_{II}, i-e,

$$Q_1 = Q_1^1 = -\left[a_1(\bar{x}) \sum_{j=1}^{n} u_j^1 \frac{dN_j^1}{d\bar{x}}\right]_{\bar{x}=0}, \quad Q_2 = Q_1^2 = -\left[a_2(\bar{x}) \sum_{j=1}^{n} u_j^2 \frac{dN_j^2}{d\bar{x}}\right]_{\bar{x}=0}$$

(element no)

for 2-noded $\boxed{3}$ ie $n=2$

$$Q_4 = Q_2^3 = \left[a_3(\bar{x}) \sum_{j=1}^{n} u_j^3 \frac{dN_j^3}{d\bar{x}}\right]_{\bar{x}=h_3}$$

or $Q_6 = Q_3^3$ (for 3-noded $\boxed{3}$ ie $n=3$)

The above way of computing SV's by definition, ie $Q = \pm \frac{a\,du}{dx}$ is less accurate that computing thru equilibrium eqn $\circledast$, but it is used in FEM due to ease of computation. Note that $\left.\frac{dU^e}{d\bar{x}}\right|_{x=x_n^e} \neq \left.\frac{dU^{e+1}}{d\bar{x}}\right|_{x=x_1^{e+1}}$ ∵ Lagrange elements are C^0 continuous, ie they interpolate dependent variable but not its derivatives.

This inaccuracy of computing SV's by definition decreases as nos of elements 'E' and/or degree of interpolation 'n' increase.

For (Ex) problem, if $q_1 = q_2 = q_3 = 0$ (no distributed sources/loads), considering <u>all-2-noded</u> elements we have, <u>from definition</u>,

$$Q_1 = Q_1^1 = -\left(EA.\frac{U_3 - \cancelto{0}{U_1}}{h_1}\right)_{\bar{x}=0} = -\frac{EA}{h_1}U_3 = K_{12}^1 U_3$$

$$Q_2 = Q_1^2 = -\left(EA.\frac{U_3 - \cancelto{0}{U_2}}{h_2}\right)_{\bar{x}=0} = -\frac{EA}{h_2}U_3 = K_{12}^2 U_3$$

$$Q_3 = Q_2^3 = \left(EA.\frac{\cancelto{0}{U_4} - U_3}{h_3}\right)_{\bar{x}=0} = -\frac{EA\,U_3}{h_3} = K_{12}^3 U_3$$

which is exactly same as <u>from equilibrium</u> (★ p.22)

However, this match of SV obtained via definition and via equilibrium, is not to be expected, with latter being more accurate. In fact if q_1, q_2, q_3 were non-zero, this match would not have occurred.

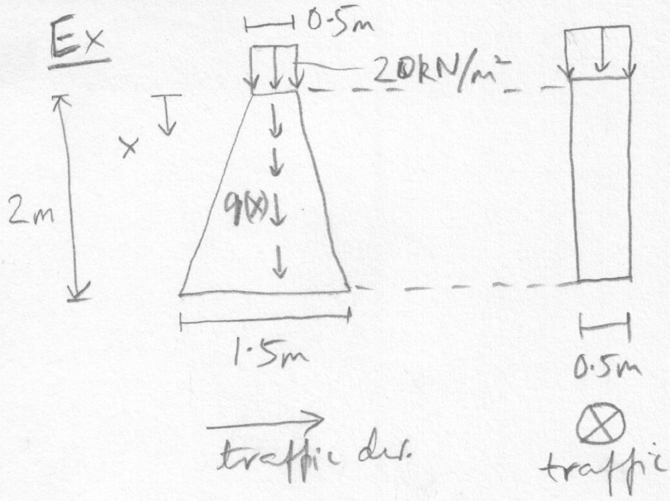
- can repeat these steps using Galerkin or any weighted residual method
- Interpolation f^ns depend on element geometry, number & position of nodes.
- Same formulation holds for class of problems (ie governed by $-(au')' + cu - q = 0$, in present case)
- Integrations for K, f, may need to be done numerically if they are complicated.
- Point sources at intermediate nodes can be handled thru Q terms or $q(\bar{x}) = P\delta(\bar{x}-\bar{x}_p)$ terms, but not both. For point sources not at nodes we handle thru q, ie $\int_0^{he} [N_1\ N_2 \cdots N_n]^T P\delta(\bar{x}-\bar{x}_p)\, d\bar{x}$

$$= P[N_1, \cdots N_n]^T\Big|_{\bar{x}=\bar{x}_p}$$

Thus for linear element, due to point source at $\bar{x}_p = \alpha he$ we have $f = [1-\alpha, \alpha]^T P$

- Errors arise due to domain approximation, computation of K_{ij}, f_i (numerical integration), and solution (numerical soln of simultaneous equations).
- K is sparse due to FE model where non-zero K_{ij} occur only at entries corresponding to an element d.o.f. Further in given problem K is symmetric. Both these result in efficient numerical solution of equil. eqns.

Ex



Concrete Pier of bridge with deck load (DL+LL).

Concrete weighs $25\,kN/m^3 = \rho$

$E = 28 * 10^6 kN/m^2$

Model as 1-D problem

$$Q_0 = 20 \cdot (0.5)^2 = 5\,kN$$

$$q(x) = \underbrace{0.5(1+x)0.5}_{A(x)} \cdot \underbrace{25}_{\rho} = 6.25(1+x) \quad kN/m$$

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + q(x) = 0 \quad , \; EA = E\,0.25(1+x)$$

Use linear elements.

$$\underline{WF}: \; -\int_0^{h_e} EA \frac{d\overset{N_i}{\cancel{w}}}{dx}\frac{d\overset{N_j}{\cancel{u}}}{dx}\,dx + \int_0^{h_e} \overset{N_i}{\cancel{w}} q\,dx + \overset{N_i}{\cancel{w}}(x_2^e)Q_2^e + \overset{N_i}{\cancel{w}}(x_1^e)Q_1^e = 0 \; , \; i = 1,2$$

$$[B_{ij}]^e = [K_{ij}]^e = \int_0^{h_e} \frac{0.25E(1+\overset{x_e+\bar{x}}{\cancel{x}})}{h_e^2}\begin{bmatrix} -1/h_e \\ 1/h_e \end{bmatrix}\begin{bmatrix} -1/h_e & 1/h_e \end{bmatrix} d\bar{x}$$

$$= 0.25E\left[(1+x_e)h_e + \frac{h_e^2}{2}\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.25(x_{e+1}-x_e)\left(1+\frac{x_{e+1}+x_e}{2}\right) * \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F^e\} = \int_0^{h^e} \begin{bmatrix} 1 - \frac{\bar{x}}{h_e} \\ \frac{\bar{x}}{h_e} \end{bmatrix} 6.25(1+x_e+\bar{x})\,d\bar{x} + \begin{bmatrix} Q_1^e \\ Q_2^e \end{bmatrix}$$

$$= 6.25\begin{Bmatrix} (1+x_e)h_e + \frac{h_e^2}{2} - (1+x_e)\frac{h_e}{2} - \frac{h_e^2}{3} \\ (1+x_e)\frac{h_e}{2} + \frac{h_e^2}{3} \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

For 2 elements, $x_1^1 = 0$, $x_2^1 = 1 = x_1^2$, $x_2^2 = 2$,

$$0.25E\begin{bmatrix} 3/2 & -3/2 & 0 & 0 \\ -3/2 & 3/2+5/2 & -5/2 & \\ & -5/2 & 5/2 & \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ \underset{=0}{\cancel{u_3}} \end{Bmatrix} = 6.25\begin{Bmatrix} 2/3 \\ 5/6+7/6 \\ 4/3 \end{Bmatrix} + \begin{Bmatrix} \overset{5}{\cancel{Q_1^1}} \\ \overset{0}{\cancel{Q_2^1 + Q_1^2}} \\ Q_2^2 \end{Bmatrix}$$

$$0.25E\begin{bmatrix}1.5 & -1.5\\ -1.5 & 4\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \frac{6.25}{6}\begin{Bmatrix}4\\ 12\end{Bmatrix} + \begin{Bmatrix}5\\ 0\end{Bmatrix}$$

$u_1 = 2.111 * 10^{-6}m$, $u_2 = 1.238 * 10^{-6}m$, $Q_2^2 = Q_3 = -E.\frac{5}{2}u_2 - 6.25.\frac{4}{3}$

$= -29.998$

ie $\simeq$ 30 kN (C).

Exact solution: $u(x) = \frac{1}{E}\left[56.25 - 6.25(1+x)^2 - 7.5\ln\left(\frac{1+x}{3}\right)\right]$

$u(0) = 2.08 * 10^{-6}m$, $u(1) = 1.225 * 10^{-6}m$.