BENDING OF BEAMS.

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Coordinate system, sign convention, relation between q, V, M

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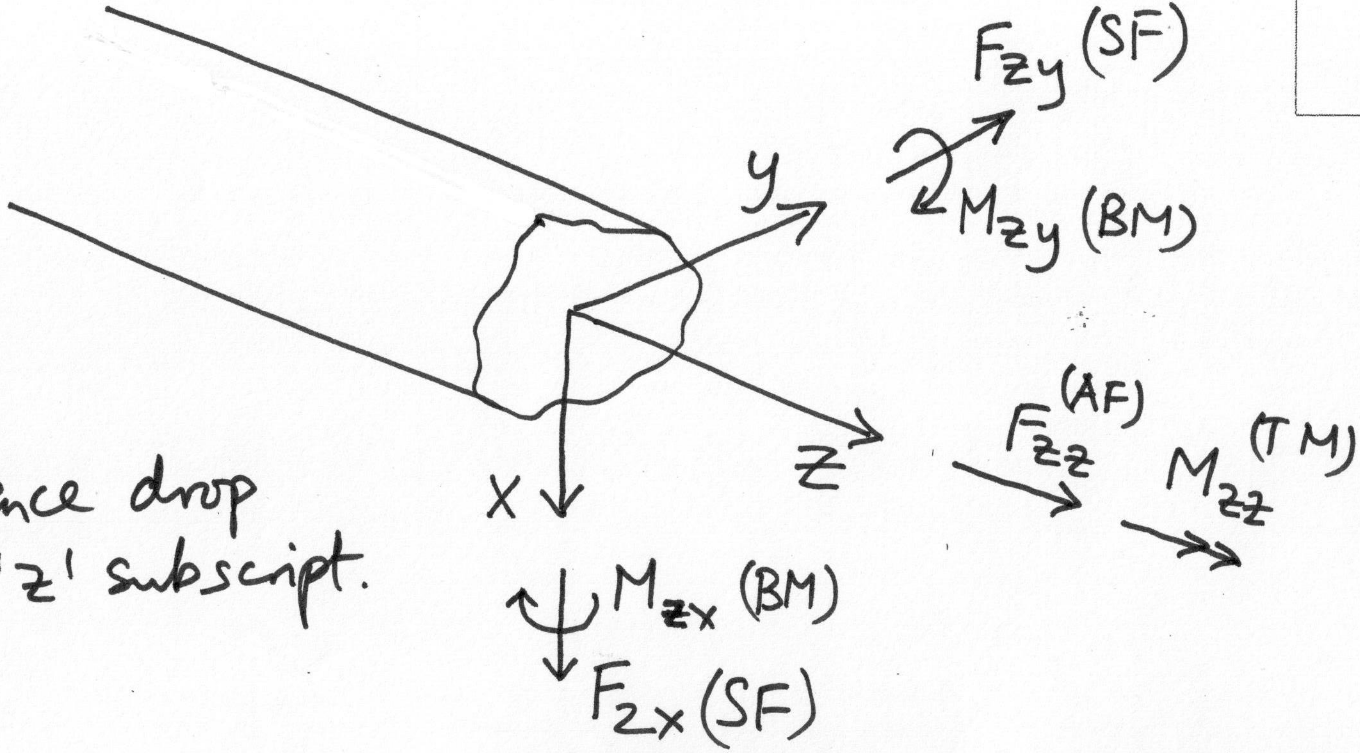
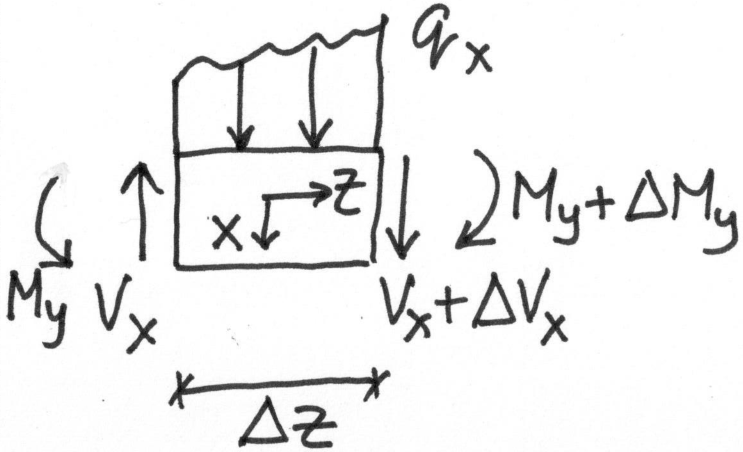


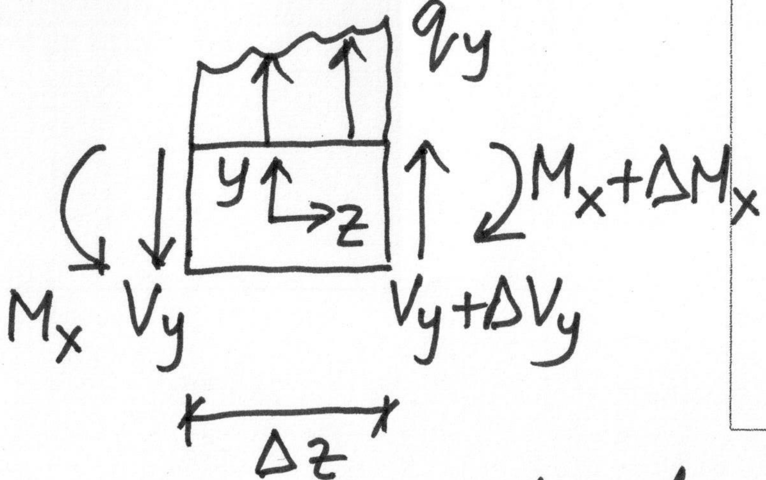
For convenience drop the first 'z' subscript.

$M_{zx} \equiv M_x$, and $F_{zy} \equiv F_y$ correspond to yz-plane bending.

$M_{zy} \equiv M_y$, and $F_{zx} \equiv F_x$ correspond to xz-plane bending.

$F_{zz} \equiv F_z$ is axial force; $M_{zz} \equiv M_z$ is Torsional moment.





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xz-plane bending

Equil →

$$\frac{dV_x}{dz} = -q_x$$

$$\frac{dM_y}{dz} = -V_x$$

$$\frac{d^2 M_y}{dz^2} = q_x$$

yz-plane bending

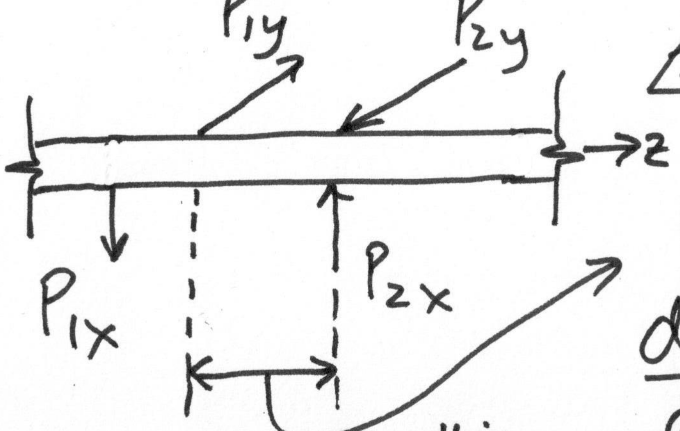
$$\frac{dV_y}{dz} = -q_y$$

$$\frac{dM_x}{dz} = V_y$$

$$\frac{d^2 M_x}{dz^2} = -q_y$$

For distributed load ~~the~~ max BM occurs where corresponding SF is zero.

For point loads the BM varies linearly between load points, & max BM occurs at load point.



Consider this region

M_x, M_y vary linearly between load points.

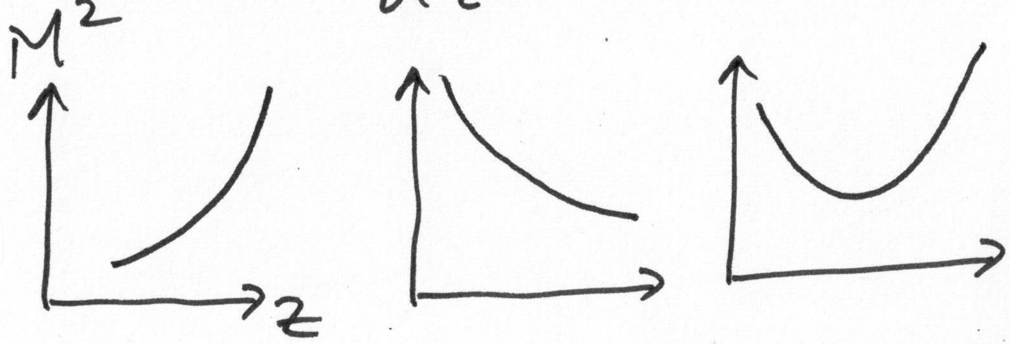
$$M^2 = M_x^2 + M_y^2 = (Az+B)^2 + (Cz+D)^2$$

$$\frac{d(M^2)}{dz} = 2\left[(Az+B)A + (Cz+D)C\right] = 0$$

gives unique z lying inside or outside region.

$$\frac{d^2(M^2)}{dz^2} = 2(A^2+C^2) \geq 0$$

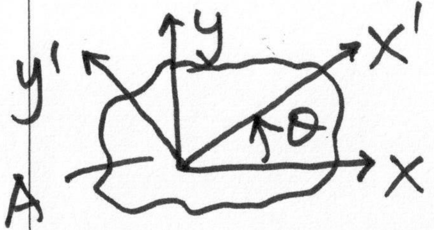
3 possible scenarios for M^2



In all cases M^2 max at one extreme, ie at a load point.

Area Moments of Inertia — Transformations

Rotation of axes.



$$\underline{\underline{a}} = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}, \quad \begin{aligned} x' &= x c\theta + y s\theta \\ y' &= -x s\theta + y c\theta \end{aligned}$$

$$I'_x = \int_A (y')^2 dA = \int \left(x^2 \sin^2\theta + y^2 \cos^2\theta - 2xy \sin\theta\cos\theta\right) dA = I_y s^2\theta + I_x c^2\theta - 2I_{xy} s\theta c\theta$$

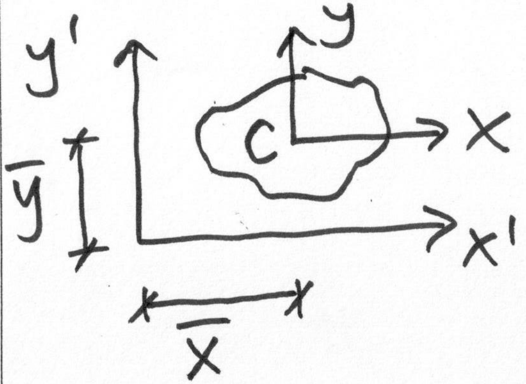
$$I'_y = \int_A (x')^2 dA = I_y c^2\theta + I_x s^2\theta + 2I_{xy} c\theta s\theta$$

$$I'_{xy} = \int_A (x'y') dA = (-I_y + I_x) c\theta s\theta + I_{xy}(c^2\theta - s^2\theta)$$

Now from $\underline{a}\,\underline{\underline{I}}\,\underline{a}^T$, where $\underline{\underline{I}} = \begin{pmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{pmatrix}$ we get same result, ie

$$\boxed{\underline{\underline{I}}' = \underline{\underline{a}}\,\underline{\underline{I}}\,\underline{\underline{a}}^T} = \begin{pmatrix} I'_{xx} & I'_{xy} \\ -I'_{xy} & I'_{yy} \end{pmatrix}$$

Translation of axes. $x' = x + \bar{x}$, $y' = y + \bar{y}$



$$I_{x'} = \int_A (y')^2 dA = I_x + (\bar{y})^2 A$$

$$I_{y'} = \int_A (x')^2 dA = I_y + (\bar{x})^2 A$$

$$I_{x'y'} = \int_A x'y' \, dA = I_{xy} + \bar{x}\bar{y} A$$

where C is centroidal axes (so $\int \bar{y} y \, dA = \bar{y} \int y \, dA = 0$, similarly $\int x \, dA = 0$).

So $\underline{\underline{I}}$ transforms as second order tensor,

So $\underline{\underline{I}}$ is a 2nd order tensor & has similar properties like $\underline{\underline{\sigma}}$, $\underline{\underline{\varepsilon}}$ (ie invariants, principle values, etc).

(eg) $\rightarrow$ $I'_x I'_y - I'_{xy} > 0$ $\longrightarrow$ this is invariant I_3. So in p-system

$I_3 = \det \begin{pmatrix} I_x & 0 \\ 0 & I_y \end{pmatrix} > 0$ $\longrightarrow$ Hence $\det(\underline{\underline{I}}') > 0$.

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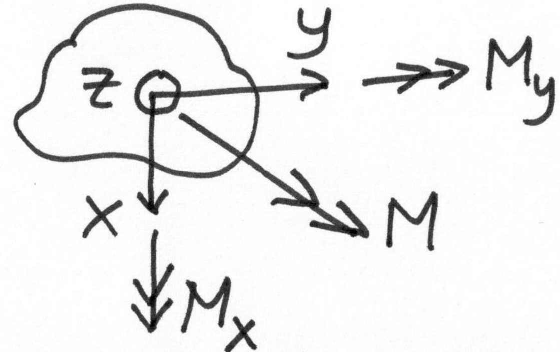
(I) Pure Bending

Consider prismatic bar with ~~terminal~~ end couples.



Choose z-axis as locus of centroids of sections.

Semi-inverse method of solution:
Guided by basic Solid Mech, choose σ_z varying linearly with x, y, and other stresses zero.

$$\sigma_z = -\frac{E}{R_x}x + \frac{E}{R_y}y \quad , \quad \text{other } \sigma_{ij} = 0 \; , \; E = \text{Youngs mod,}$$

R_x, R_y to be determined.

- Above σ_{ij} satisfy equilibrium eqns (for zero body forces).
- Resulting strains linear in x, y, so compatibility satisfied

• Traction free BC's on lateral (longitudinal) faces satisfied $\because n_z = 0$ & $\sigma_x = \sigma_y = \sigma_{xy} = 0$

i.e., $\underline{\underline{\sigma}}\,\underline{n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \end{pmatrix} \begin{Bmatrix} n_x \\ n_y \\ 0 \end{Bmatrix} = 0$, <u>i.s.</u>

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• End face BC'S:

$F_x = \int_A \sigma_{xz}\, dA = 0$, <u>i.s.</u> ; $F_y = \int_A \sigma_{yz}\, dA = 0$, <u>i.s.</u>

$$F_z = \int_A \sigma_z\, dA = -\frac{E}{R_x} \underbrace{\int_A x\, dA}_{0} + \frac{E}{R_y} \underbrace{\int_A y\, dA}_{0} = 0, \text{ i.s.}$$

($\because$ z is line of centroids)

$$M_x = \int_A y \sigma_z\, dA = -\frac{E}{R_x} I_{xy} + \frac{E}{R_y} I_x$$

$$M_y = -\int_A x \sigma_z\, dA = \frac{E}{R_x} I_y - \frac{E}{R_y} I_{xy}$$

$\Rightarrow$

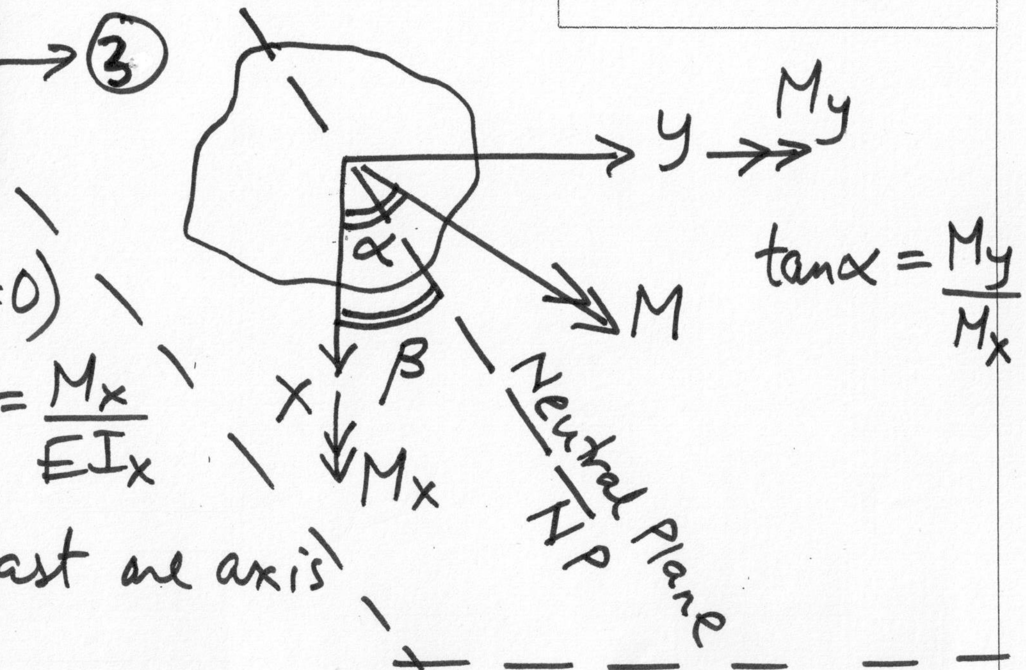
$$\boxed{\frac{1}{R_x} = \frac{M_x I_{xy} + M_y I_x}{E\left(I_x I_y - I_{xy}^2\right)} \qquad \frac{1}{R_y} = \frac{M_x I_y + M_y I_{xy}}{E\left(I_x I_y - I_{xy}^2\right)}} \rightarrow (1')$$

Euler-Bernoulli Law.

$$\boxed{\sigma_z = -\frac{(M_x I_{xy} + M_y I_x)}{I_x I_y - I_{xy}^2}\, x + \frac{(M_x I_y + M_y I_{xy})}{I_x I_y - I_{xy}^2}\, y} \rightarrow (2)$$

Neutral plane is plane on which $\sigma_z = 0$.

$$\Rightarrow \boxed{\tan\beta = \frac{y}{x} = \frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}}} \rightarrow (3)$$



$$\tan\alpha = \frac{M_y}{M_x}$$

If x, y are principal axes (ie $I_{xy} = 0$)

$$\sigma_z = -\frac{M_y}{I_y} x + \frac{M_x}{I_x} y \;;\; \frac{1}{R_x} = \frac{M_y}{EI_y} \;;\; \frac{1}{R_y} = \frac{M_x}{EI_x}$$

eg., $I_{xy} = 0$ for area having at least one axis of symmetry.

Displacements:

$$e_x = -\frac{\nu \sigma_z}{E} = \frac{\nu}{R_x} x - \frac{\nu}{R_y} y = \frac{\partial u}{\partial x} \rightarrow \text{(i)}$$

$$e_y = -\frac{\nu}{E} \sigma_z = \frac{\nu}{R_x} x - \frac{\nu}{R_y} y = \frac{\partial v}{\partial y} \rightarrow \text{(ii)}$$

$$e_z = \frac{\sigma_z}{E} = -\frac{x}{R_x} + \frac{y}{R_y} = \frac{\partial w}{\partial z} \rightarrow \text{(iii)}$$

$$\underbrace{\gamma_{xz} = 0 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}}_{\text{(iv)}} ; \quad \underbrace{\gamma_{xy} = 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}_{\text{(v)}} ; \quad \underbrace{\gamma_{yz} = 0 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}}_{\text{(vi)}}$$

$$\text{(iii)} \Rightarrow w = -\frac{xz}{R_x} + \frac{yz}{R_y} + g(x,y) \rightarrow \text{(vii)}$$

$$\text{(iv),(vii)} \Rightarrow \frac{\partial u}{\partial z} = \frac{z}{R_x} - \frac{\partial g}{\partial x} \Longrightarrow u = \frac{z^2}{2R_x} - z\frac{\partial g}{\partial x} + f(x,y) \rightarrow \text{(viii)}$$

$$\text{(vi),(vii)} \Rightarrow \frac{\partial v}{\partial z} = -\frac{z}{R_y} - \frac{\partial g}{\partial y} \Longrightarrow v = \frac{-z^2}{2R_y} - z\frac{\partial g}{\partial y} + h(x,y) \rightarrow \text{(ix)}$$

$$\text{(v),(viii),(ix)} \Rightarrow -2z\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial f}{\partial y} + \frac{\partial h}{\partial x} = 0 \Rightarrow \frac{\partial^2 g}{\partial x \partial y} = 0 \rightarrow \text{(x)} ; \quad \frac{\partial f}{\partial y} = -\frac{\partial h}{\partial x} \rightarrow \text{(xi)}$$

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(i), (viii) $\Rightarrow \overbrace{-z\frac{\partial^2 g}{\partial x^2}}^{=0} + \overbrace{\frac{\partial f}{\partial x} = \frac{\nu}{R_x}x - \frac{\nu}{R_y}y}$

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} = 0 \longrightarrow \text{(xii)} \; ; \quad f = \frac{\nu}{2R_x}x^2 - \frac{\nu}{R_y}xy + m(y) \longrightarrow \text{(xiii)}$$

(ii), (ix) $\Rightarrow \underbrace{-z\frac{\partial^2 g}{\partial y^2}}_{=0} + \underbrace{\frac{\partial h}{\partial y} = \frac{\nu}{R_x}x - \frac{\nu}{R_y}y}$

$$\Rightarrow \frac{\partial^2 g}{\partial y^2} = 0 \longrightarrow \text{(xiv)} \; ; \quad h = \frac{\nu}{R_x}xy - \frac{\nu}{2R_y}y^2 + n(x) \longrightarrow \text{(xv)}$$

(xi), (xiii), (xv) $\Rightarrow -\frac{\nu}{R_y}x + \underbrace{\frac{dm}{dy} = -\frac{\nu}{R_x}y}_{f^{n} \text{ of } y} - \frac{dn}{dx} \Rightarrow m = \frac{-\nu}{2R_x}y^2 + C_1 y + C_2$

$$n = \frac{\nu}{2R_y}x^2 - C_1 x + C_3$$

fn of x

(x), (xii), (xiv) $\Rightarrow$ $g = c_4 x + c_5 y$

$$\Rightarrow u = \frac{z^2}{2R_x} - C_4 z + \frac{\nu}{2R_x}x^2 - \frac{\nu}{R_y}xy - \frac{\nu}{2R_x}y^2 + C_1 y + C_2$$

$$v = -\frac{z^2}{2R_y} - C_5 z + \frac{\nu}{R_x}xy - \frac{\nu}{2R_y}y^2 + \frac{\nu}{2R_y}x^2 - C_1 x + C_3$$

$$w = -\frac{xz}{R_x} + \frac{yz}{R_y} + C_4 x + C_5 y$$

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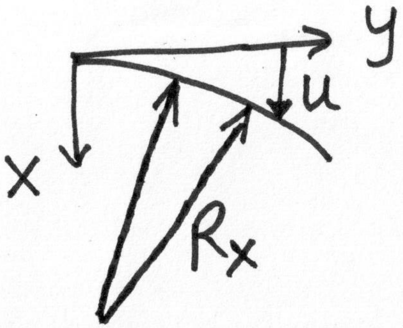
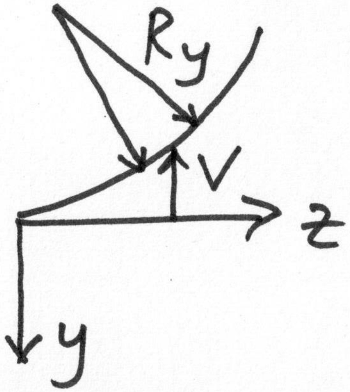
To get RB motions, apply $u = v = w = \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$ at $(x, y, z) = (0, 0, 0)$. Or simply discard constant & linear terms (i.e., get $C_1 = C_2 = C_3 = C_4 = C_5 = 0$).

$$\Rightarrow \boxed{\begin{aligned} u &= \frac{1}{2R_x}\left[z^2 + \nu(x^2 - y^2)\right] - \frac{\nu}{R_y}xy \\ v &= \frac{1}{2R_y}\left[-z^2 + \nu(x^2 - y^2)\right] + \frac{\nu}{R_x}xy \\ w &= -\frac{xz}{R_x} + \frac{yz}{R_y} \end{aligned}} \longrightarrow (4) \quad \Rightarrow \quad \boxed{\begin{aligned} \frac{\partial^2 u}{\partial z^2} &= \frac{1}{R_x} \\ \frac{\partial^2 v}{\partial z^2} &= -\frac{1}{R_y} \end{aligned}}$$

Deflection of centroidal line ($x=y=0$) is,

$$u=\frac{z^2}{2R_x}, \quad v=\frac{-z^2}{2R_y}, \quad w=0.$$





Radius of curvature in xz plane $= \frac{1}{\kappa_x} = \frac{\left[1+(\partial u/\partial z)^2\right]^{3/2}}{\partial^2 u/\partial z^2} \simeq \frac{1}{\partial^2 u/\partial z^2}$

$= R_x$ $\quad (\because \partial u/\partial z \ll 1)$

So physical interpretation of R_x, R_y are that they are radii of curvature of centroidal line, in xz yz plane, respectively (note R_y is -ve of the rad. of curvature in yz plane).

Surface $\phi(x,y,w) = w + x\frac{z}{R_x} - y\frac{z}{R_y} = 0$ defines a plane in (x,y,w) space (for $z=$ const), i.e, $\nabla\phi = \frac{z}{R_x}\underline{i} - \frac{z}{R_y}\underline{j} + \underline{k}$ $\Rightarrow$ Plane sections remain plane after deformation

On neutral plane,

$$\sigma_{zz} = e_{zz} = 0 \Leftrightarrow \frac{y}{x} = \frac{R_y}{R_x} \Leftrightarrow W = 0$$

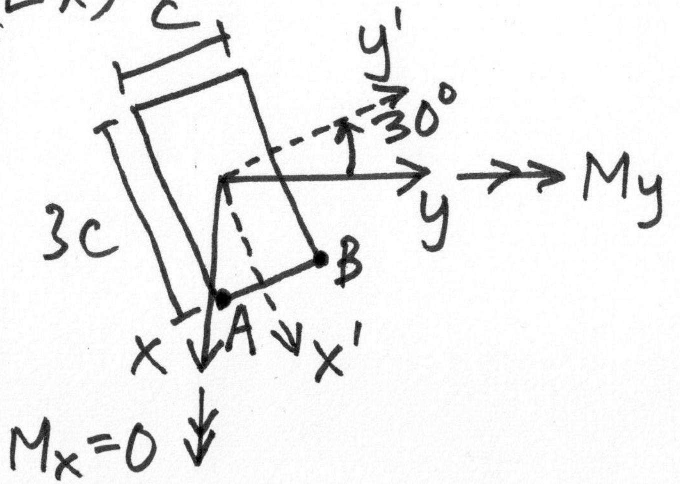
If $I_{xy} = 0$ (i.e, p-axes system),

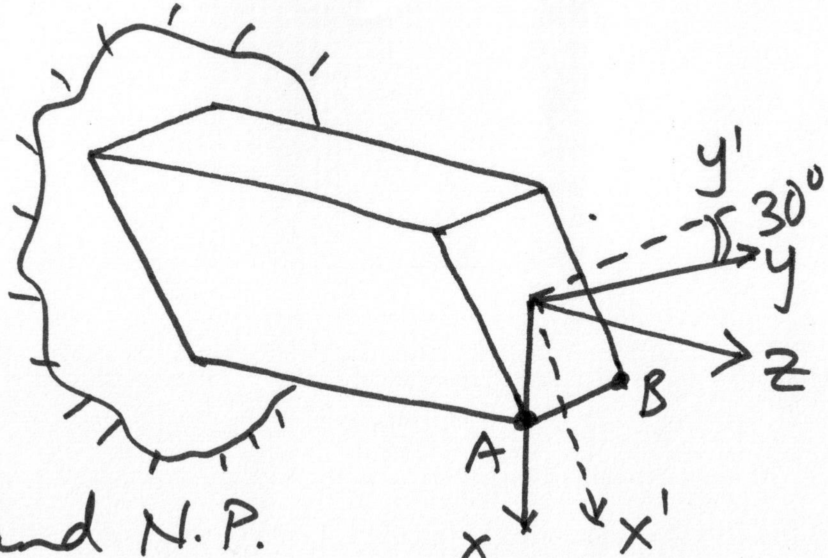
Euler-Bernoulli Law

$$\boxed{\frac{\partial^2 u}{\partial z^2} = \frac{1}{R_x} = \frac{M_y}{EI_y} \; ; \; \frac{\partial^2 v}{\partial z^2} = -\frac{1}{R_y} = -\frac{M_x}{EI_x}} \rightarrow (5)$$

Eqn (5) strictly valid for finding deflections of ℄ (ie elastic curve) of beam with end couples applied. However it is used, approximately, for entire cross section and even when transverse loads (that cause shear forces/stresses) are applied. This approximation good only for thin metallic beams; not for thick (deep) and/or non-metallic beams for which shear affects deformations significantly (we already saw this during plane stress solution of s.s. deep beam with u.d.l.).

(Ex)





Note: x-axis not necessarily passing thru A.

Find: σ_z at A, B, and N.P.

$$\begin{bmatrix} I_x & -I_{xy} \\ -I_{xy} & I_y \end{bmatrix} = \underline{\underline{a}}\,\underline{\underline{I'}}\,\underline{\underline{a}}^T = \begin{bmatrix} C(-30) & S(-30) \\ -S(-30) & C(-30) \end{bmatrix}\begin{bmatrix} c^4/4 & 0 \\ 0 & 9c^4/4 \end{bmatrix}\begin{bmatrix} C(-30) & -S(-30) \\ S(-30) & C(-30) \end{bmatrix}$$

$$= c^4 \begin{bmatrix} 0.75 & 0.87 \\ 0.87 & 1.75 \end{bmatrix}$$

$$② \rightarrow \sigma_z = -1.35\frac{M_y}{c^4}x + 1.566\frac{M_y}{c^4}y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{a}}\begin{pmatrix} x' \\ y' \end{pmatrix} \longrightarrow x'_A = \frac{3c}{2},\ y'_A = -\frac{c}{2},\ x'_B = \frac{3c}{2},\ y'_B = \frac{c}{2} \Rightarrow x_A = 1.55c,$$

$y_A = 0.317c,$

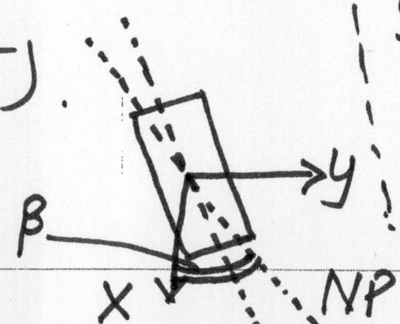
$x_B = 1.05c,$

$y_B = 1.18c$

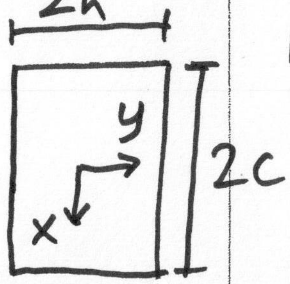
$$\Rightarrow (\sigma_z)_A = -\frac{1.6}{c^3}M_y\ (C)\ ;\ (\sigma_z)_B = \frac{0.43}{c^3}M_y\ (T).$$

$$\tan\beta = \frac{1.35}{1.566},\ \beta = 40.763°.$$

A, B on opp sides of N.P.



(Ex) Rectangular section beam with $M_y = -M_b$, $M_x = 0$.
Analyze deflections & sketch deformed section

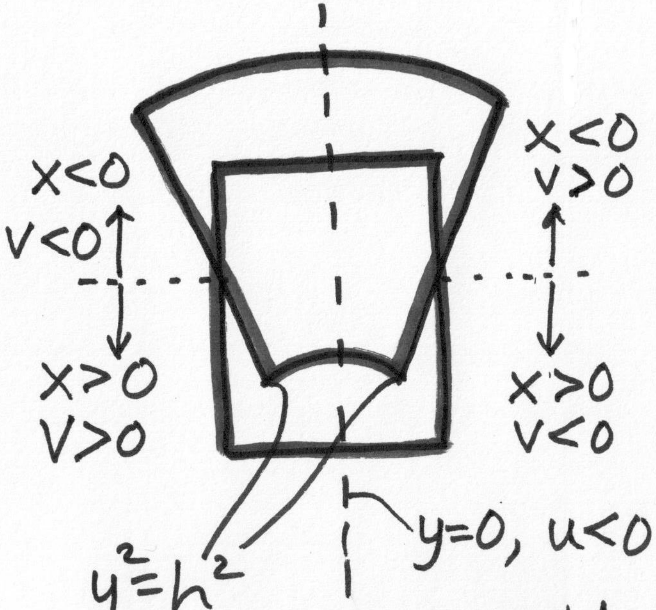


$$\frac{1}{R_x} = \frac{M_y}{EI_y} = -\frac{M_b}{EI_y} \quad ; \quad \frac{1}{R_y} = \frac{M_x}{EI_x} = 0$$

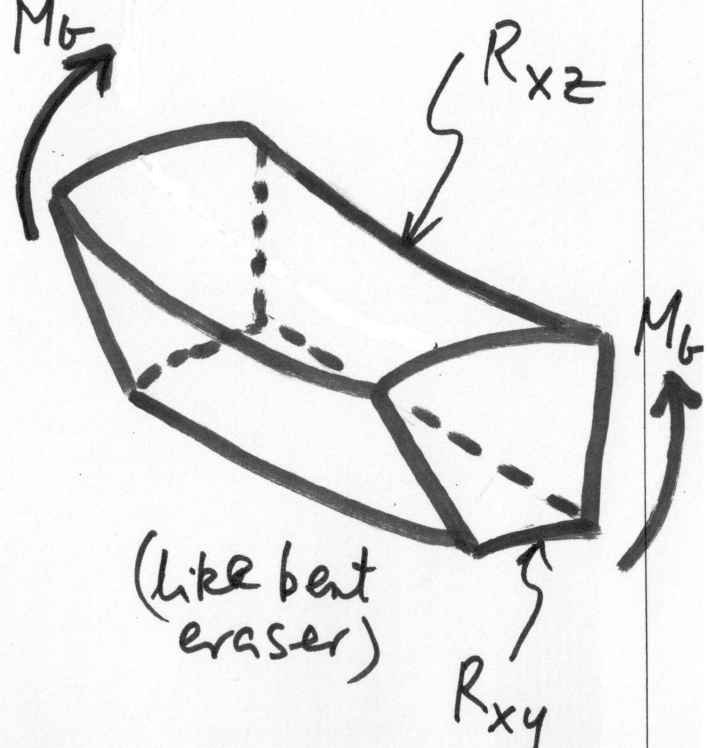
$$u\Big|_{x=\pm c} = \frac{M_b}{2EI_y}\left[-z^2 + \nu\left(-c^2+y^2\right)\right]$$

$$v\Big|_{y=\pm h} = \mp \nu \frac{M_b h}{EI_y} x$$

So surfaces originally parallel to yz plane get deformed into anticlastic (double and opp curvature, ie saddle shaped) surfaces.

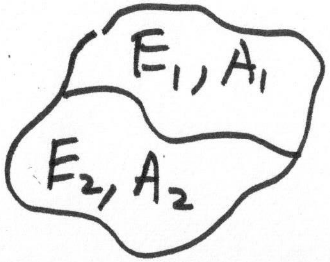


so u > or < 0 possible.



$$\frac{\partial^2 u}{\partial z^2} = -\frac{M_b}{EI_y} = -\frac{1}{R_{xz}} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = \frac{M_b}{EI_y}\nu = \frac{1}{R_{xy}} \Rightarrow R_{xy} = \frac{R_{xz}}{\nu}$$

Composite Beams. (pure bending)



$\underbrace{\nu_1 = \nu_2}$, prismatic beam.
↳ else delamination occurs.

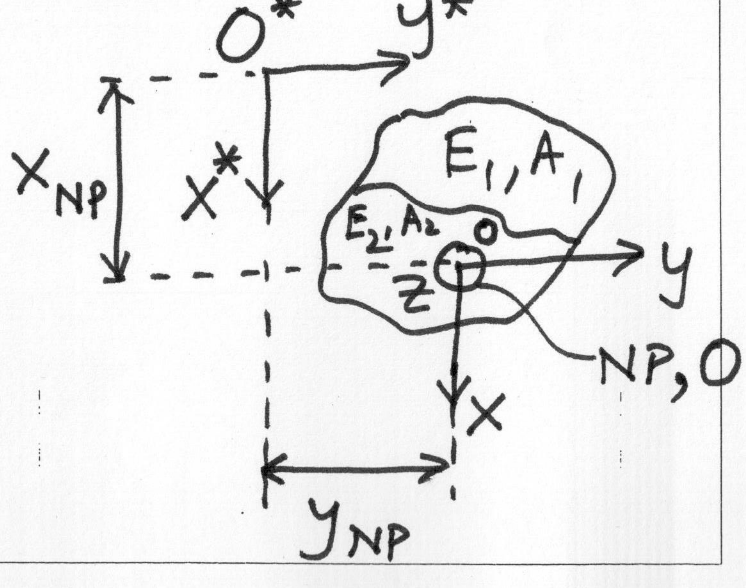
Following semi-inverse procedure, assume

$$\text{⑥} \leftarrow \boxed{\begin{aligned} \sigma_z &= -E_1 K_x x + E_1 K_y y, \ \text{in } A_1 \\ &= -E_2 K_x x + E_2 K_y y, \ \text{in } A_2 \end{aligned}}$$

remaining $\sigma_{ij} = 0$.

Equil & compat satisfied, as before, as are $\sum F_y = 0 = \sum F_x$ on end faces, ie find (x_{NP}, y_{NP})

Here z-axis is not centroidal. We locate z axis by zero axial force condition, ie,



$$\sum F_z = \int_{A_1+A_2} \sigma_z \, dA = -E_1 K_x \int_{A_1} x\,dA + E_1 K_y \int_{A_1} y\,dA$$

$$- E_2 K_x \int_{A_2} x\,dA + E_2 K_y \int_{A_2} y\,dA = 0$$

Put $x = x^* - x_{NP}$, $y = y^* - y_{NP}$,

$$-E_1 K_x \bar{x}_1^* A_1 - E_2 K_x \bar{x}_2^* A_2 + (E_1 A_1 + E_2 A_2) K_x x_{NP}$$
$$+ E_1 K_y \bar{y}_1^* A_1 + E_2 K_y \bar{y}_2^* A_2 - (E_1 A_1 + E_2 A_2) K_y y_{NP} = 0$$

The above should be valid for $K_x \neq 0$, $K_y = 0$ and vice-versa (ie., single plane bending) also.

$$\Rightarrow \boxed{x_{NP} = \frac{E_1 A_1 \bar{x}_1^* + E_2 A_2 \bar{x}_2^*}{E_1 A_1 + E_2 A_2} \; ; \; y_{NP} = \frac{E_1 A_1 \bar{y}_1^* + E_2 A_2 \bar{y}_2^*}{E_1 A_1 + E_2 A_2}} \rightarrow (7)$$

So locate z-axis at (x_{NP}, y_{NP}) from O^* (the arbitrary origin). Here $(\bar{x}_1^*, \bar{y}_1^*)$, $(\bar{x}_2^*, \bar{y}_2^*)$ are centroidal coords of Areas A_1, A_2 measured from O^*, respectively.

As before, K_x, K_y determined from end moment bc's as follows

$$M_x = \int_{A_1} y \sigma_z \, dA + \int_{A_2} y \sigma_z \, dA = -E_1 K_x (I_{xy})_1 + E_1 K_y (I_x)_1$$
$$- E_2 K_x (I_{xy})_2 + E_2 K_y (I_x)_2$$

$$M_y = -\int_{A_1} x\sigma_z\, dA - \int_{A_2} x\sigma_z\, dA = E_1 K_x (I_y)_1 - E_1 K_y (I_{xy})_1 + E_2 K_x (I_y)_2 - E_2 K_y (I_{xy})_2$$

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Solving for K_x, K_y,

$$\boxed{K_x = \frac{F_3 M_x + F_1 M_y}{F_1 F_2 - F_3^2}\ ; \quad K_y = \frac{F_2 M_x + F_3 M_y}{F_1 F_2 - F_3^2}} \rightarrow (8)$$

where $F_1 = E_1 (I_x)_1 + E_2 (I_x)_2$

$F_2 = E_1 (I_y)_1 + E_2 (I_y)_2$

$F_3 = E_1 (I_{xy})_1 + E_2 (I_{xy})_2$

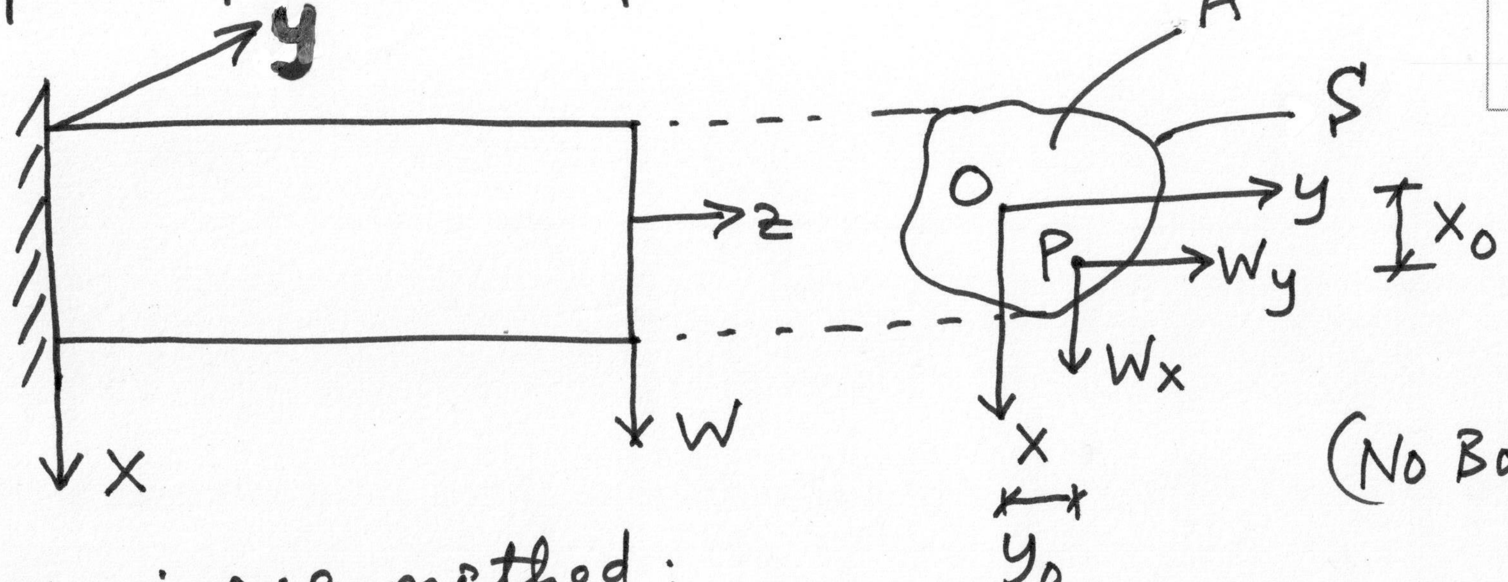
, I_x, I_y, I_{xy} are about O, ie x, y.

Now $e_x = e_y = -\nu\sigma_z = -\nu(-K_x x + K_y y)$ if $\nu_1 = \nu_2 = \nu$, else e_x, e_y discontinuous across internal boundary interface, ie delamination will occur.

Since delamination not accounted for in this formulation, result is valid for $\nu = \nu_1 = \nu_2$ only.

(II) BENDING DUE TO END LOAD

This is a 3-D Soln. Plane stress solution done in plane problems chp.



O → centroid

P → Load pt.

$\underline{W} = W_x \underline{i} + W_y \underline{j}$

(NO BODY FORCES)

Semi-inverse method:

We expect only σ_z, τ_{xz}, τ_{yz} non-zero, ie $\sigma_x = \sigma_y = \tau_{xy} = 0$

Further, $M_y = W_x(l-z)$, $M_x = -W_y(l-z)$.

From basic solid mech, σ_z proportional to $(M_x y)$ & $(M_y x)$.

Hence assume $\boxed{\sigma_z = -E(l-z)(K_x x + K_y y)} \rightarrow (1)$

in a manner similar to pure bending, K_x K_y to be determined

So, in a manner similar to pure bending, K_x, K_y will be determined from end-face b.c's for ΣF_x, ΣF_y.

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Equilibrium: $\frac{\partial \tau_{xz}}{\partial z}=0$; $\frac{\partial \tau_{yz}}{\partial z}=0$; $\rightarrow$ 2(a,b)

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0 \Rightarrow \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+E(K_x x+K_y y)=0 \rightarrow 2c$$

Lateral face b.c's: $\begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}\begin{Bmatrix} n_x \\ n_y \\ 0 \end{Bmatrix}=0 \Rightarrow \tau_{xz}n_x+\tau_{yz}n_y=0 \rightarrow 3$

End face b.c's: $$W_x=\int_A \tau_{xz}\,dA = \int_A \left[\tau_{xz}+x\underbrace{\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}\right)}_{=0}\right]dA$$

(add 3rd equil eqn)

$$=\int_A \left[\frac{\partial}{\partial x}(x\tau_{xz})+\frac{\partial}{\partial y}(x\tau_{yz})+\frac{\partial}{\partial z}(x\sigma_z)\right]dA$$

(contd)

$$W_x \underset{\text{(div. thrm)}}{=} \cancel{\int_S x(\tau_{xz} n_x + \tau_{yz} n_y)\, ds} + \int_A x \frac{\partial \sigma_z}{\partial z}\, dA$$

$=0$ (lateral face b.c.)

$$= \int_A x E (K_x x + K_y y)\, dA = E K_x I_y + E K_y I_{xy}$$

Similarly, $W_y = \int_A \tau_{yz} dA = E K_x I_{xy} + E K_y I_y$

Solving for K_x, K_y,

$$\boxed{K_x = \frac{I_x W_x - I_{xy} W_y}{E(I_x I_y - I_{xy}^2)} \; ; \; K_y = \frac{I_y W_y - I_{xy} W_x}{E(I_x I_y - I_{xy}^2)}} \rightarrow (4)$$

Put (4) in (1) and use $W_x(l-x) \equiv M_y$, $W_y(l-z) \equiv -M_x$, you get back result for pure-bending (ie σ_z as in Eq (2) on p.9).

Hence σ_z has same form as in the case of pure bending (ie (2) p.9) with M_x, M_y being B.M's at section.

$W_z = \int_A \sigma_z \, dA = 0$ is $\underline{\text{i.s.}}$ $\therefore$ origin O is the centroid.

BM at a section:

$$\left.\begin{aligned} \int_A \sigma_z y \, dA &= -E(l-z)\left[K_x I_{xy} + K_y I_x\right] = -W_y(l-z) \equiv M_x \\ -\int_A \sigma_z x \, dA &= E(l-z)\left[K_x I_y + K_y I_{xy}\right] = W_x(l-z) \equiv M_y \end{aligned}\right\} \rightarrow \text{checks out} \checkmark$$

$\boxed{\text{now } \boxed{\text{see p. 23a}}}$

Equilibrium: 2(a, b, c) satisfied by introducing stress $f^{\underline{n}}$ $\phi(x, y)$,

i.e.,

$$\boxed{\begin{aligned} \tau_{xz} &= \frac{\partial \phi}{\partial y} - \frac{1}{2} E K_x x^2 + f(y) \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} - \frac{1}{2} E K_y y^2 - g(x) \end{aligned}} \rightarrow (5)$$

Here $f(y)$, $g(x)$ introduced to provide flexibility when satisfying lateral face BC (3). $f(y)$, $g(x)$ are arbitrary.

Inteprepretation of K_x, K_y

{using S.D., equil, C.L}

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$$\frac{\partial u}{\partial z} = \frac{\tau_{xz}}{G} - \frac{\partial w}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{1}{G}\underbrace{\cancel{\frac{\partial \tau_{xz}}{\partial z}}}_{=0 \text{ (equil) } (2a)} - \frac{\partial^2 w}{\partial x \partial z} = -\frac{1}{E}\frac{\partial \sigma_z}{\partial x} = (l-z)K_x$$

$$\frac{\partial^2 u}{\partial z^2} \simeq \frac{1}{R_x} = (l-z)K_x$$, ie K_x inversely proportional to rad. of curvature in xz plane.

Similarly $\frac{\partial^2 v}{\partial z^2} = -(l-z)K_y$. , ie K_y inv. prop. to rad of curv. in yz plane.

For special case when $W_y = 0$, $I_{xy} = 0$, ie, loading parallel to p-axis,

(4) $\Rightarrow K_x = \frac{W_x}{EI_y} \Rightarrow \frac{\partial^2 u}{\partial z^2} = (l-z)\frac{W_x}{EI_y} = \frac{M_y}{EI_y} \Rightarrow \boxed{M_y = EI_y \frac{\partial^2 u}{\partial z^2}}$

{valid even when transverse load applied if its along p-axis.} ← Euler Bernoulli Law.

Lateral face b.c.

(3) $\rightarrow \tau_{xz} n_x + \tau_{yz} n_y = 0$

(5) $\Rightarrow \dfrac{\partial \phi}{\partial y}\dfrac{dy}{ds} + \dfrac{\partial \phi}{\partial x}\dfrac{dx}{ds} = \left[\frac{1}{2}EK_x x^2 - f(y)\right]\dfrac{dy}{ds} - \left[\frac{1}{2}EK_y y^2 + g(x)\right]\dfrac{dx}{ds}$

$$\boxed{\frac{d\phi}{ds} = \left[\frac{1}{2}EK_x x^2 - f(y)\right]\frac{dy}{ds} - \left[\frac{1}{2}EK_y y^2 + g(x)\right]\frac{dx}{ds} \quad \text{on } S'} \rightarrow (6)$$

<u>If possible</u> choose $g(x)$, $f(y)$ such that

(6b) $\leftarrow$

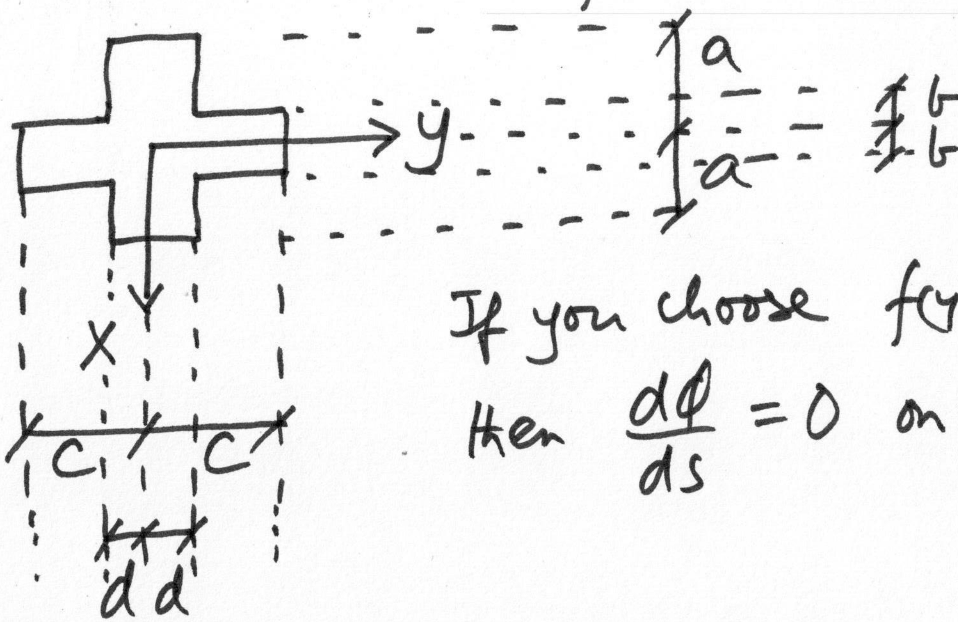
$$\boxed{\begin{aligned} g(x) &= -\frac{1}{2}EK_y y^2 \quad \text{on } S' \text{ (or part of } S \text{ for which } \tfrac{dx}{ds} \neq 0) \\ f(y) &= \frac{1}{2}EK_x x^2 \quad \text{on } S' \text{ (or part of } S \text{ for which } \tfrac{dy}{ds} \neq 0). \end{aligned}}$$

This may not always be possible. <u>But when it is</u>, <u>then</u>

$$\boxed{\frac{d\phi}{ds} = 0 \quad \text{on } S'} \rightarrow (6a) \rightarrow \text{in this case } \phi = \text{const} = 0 \text{ on } S'$$

<u>Note</u>: On parts of S' for which $\dfrac{dx}{ds} = 0$, $g(x)$ can be arbitrary. Similarly on parts of S' for which $dy/ds = 0$, $f(y)$ can be arb.

When such a judicious choice is not possible then choose f, g as arbitrary or even zero. An example is,



If you choose $f(y) = \frac{1}{2} E K_x a^2$, $g(x) = \frac{1}{2} E K_y c^2$

then $\frac{d\phi}{ds} = 0$ only on $x = \pm a$, $y = \pm c$ parts of S.

<u>Compatibility</u>: B.M compat eqns $\rightarrow \nabla^2 \sigma_{ij} + \frac{I_{1,ij}}{(1+\nu)} = 0$,

$I_1 = \sigma_z$

$$\Rightarrow \nabla^2 \tau_{yz} + \frac{E}{1+\nu} K_y = 0 \xRightarrow{5} \frac{\partial}{\partial x}(\nabla^2 \phi) = -\frac{E\nu}{1+\nu} K_y - \frac{d^2 g}{dx^2}$$

$$\nabla^2 \tau_{xz} + \frac{E}{1+\nu} K_x = 0 \xRightarrow{5} \frac{\partial}{\partial y}(\nabla^2 \phi) = \frac{E\nu}{1+\nu} K_x - \frac{d^2 f}{dy^2}$$

Other compat eqns i.s.

integrating

$$\Rightarrow \boxed{\nabla^2 \phi = -2G\nu K_y x - \frac{dg}{dx} + 2G\nu K_x y - \frac{df}{dy} - 2G\alpha} \longrightarrow 7$$

Here the constant $(-2G\alpha)$ is determined from twisting moment condition (similar as was done for $M = 2\iint \phi \, dA$ in pure torsion), ie,

$$\boxed{\iint_A (x\tau_{yz} - y\,\tau_{xz})\,dA = x_0 W_y - y_0 W_x} \rightarrow (8)$$

(ref fig. p. 20)

Summary: (i) Solve (7), (6a/6), by choosing f, g, appropriately. Get ϕ in terms of $G\alpha$.

(ii) Get τ_{xz}, τ_{yz} from (5)

(iii) Get α from (8).

Neutral plane $\rightarrow \sigma_z = 0 \Rightarrow \tan\beta = \dfrac{y}{x} = -\dfrac{K_x}{K_y}$

Note: Discarding bending terms in (7) and (5), ie $K_y = K_x = g = f = 0$, you get torsion eqn. $(\nabla^2\phi = -2G\alpha)$ So this formulation combines bending + torsion.

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Interpretation of α.

Rotation (infinitesimal) of line element in xy plane is

$$\omega_z = \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

z_1

Local twist at point (x, y) on cross-section defined as

$$\frac{\partial \omega}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z}\right) = \left(\frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y}\right) = \frac{1}{2G}\left(\frac{\partial \tau_{zy}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right)$$

$$= -\frac{1}{2G}\left(\nabla^2\phi + \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\right) = \alpha + \nu(K_y x - K_x y)$$

used 5, 7

$$\text{Mean twist} = \frac{\iint_A \frac{\partial \omega}{\partial z}\, dA}{A} = \alpha = \text{local twist at origin (centroid)}$$

{For pure torsion $K_y = K_x \Rightarrow$ local twist = constant over $A = \alpha$.}

So twisting occurs in addition to bending.

Hence, the general flexure problem can be resolved into two sub-problems, ie,

(i) A pure flexure problem with zero mean twist (ie $\alpha = 0$). Solve (7), (6a/6), (5) with $\alpha = 0$ for τ_{xz}, τ_{yz}. Then the position of the load such that $\alpha = 0$ is obtained by twisting moment equation,

$$\boxed{\iint_A (x\tau_{yz} - y\tau_{xz})\, dA = x_{CF} W_y - y_{CF} W_x} \longrightarrow (9)$$

where τ_{yz}, τ_{xz} determined for $\alpha = 0$.

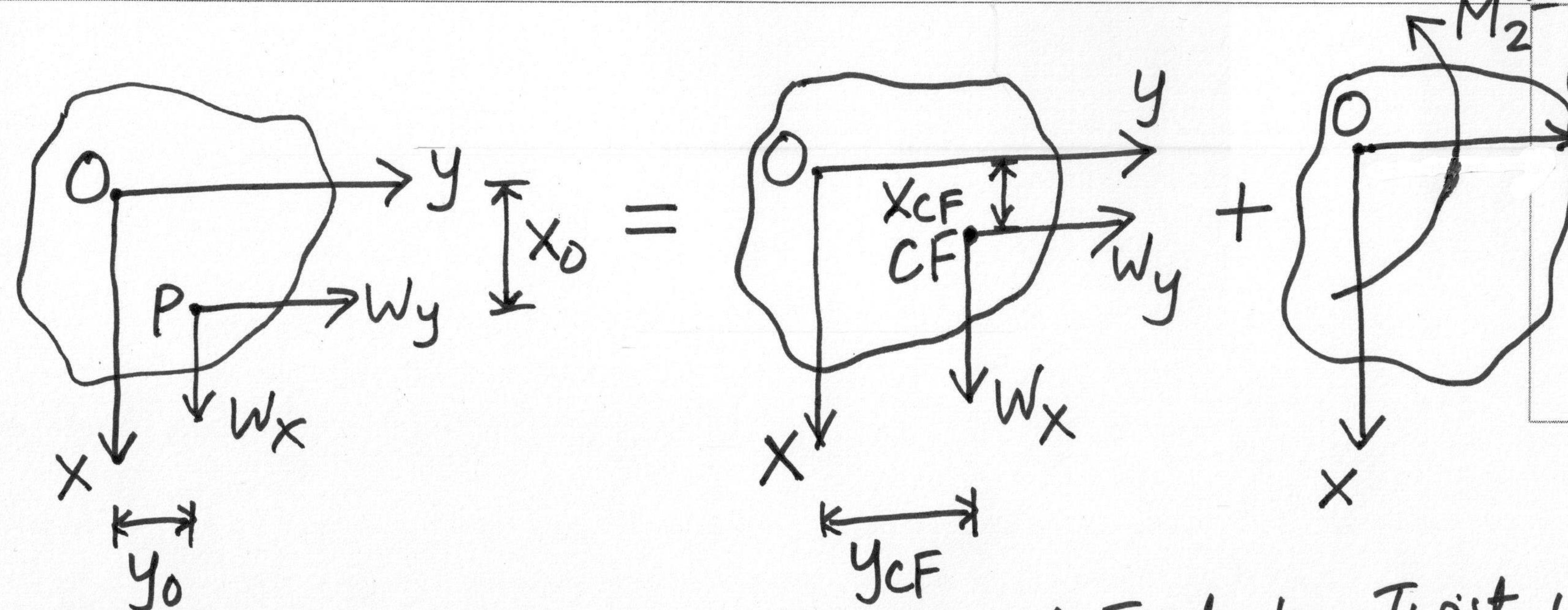
Load position (x_{CF}, y_{CF}) is called <u>Centre of Flexure</u> (CF) or <u>Shear center</u>.

(ii) A pure torsion problem having mean twist α due to applied couple

$$\boxed{M_z = W_y(x_0 - x_{CF}) - W_x(y_0 - y_{CF})} \longrightarrow (10)$$

ie, Moment of load about the CF

Here, determine τ_{xz}, τ_{yz} using $K_x = K_y = f = g = 0$ in (7), (6a/6), (5) {which is same as using Prandtl torsion formulation done previously}



Bend + Twist = Bend w/o Twist + Twist w/o Bend.

Determination of Shear Center

(9), (5) $\Rightarrow$
$$x_{CF} W_y - y_{CF} W_x = \iint_A \left[x\left(-\frac{\partial \phi}{\partial x} - g(x) - \frac{1}{2} E K_y y^2 \right) - y\left(\frac{\partial \phi}{\partial y} + f(y) - \frac{1}{2} E K_x x^2 \right) \right] dA$$

Linearity $\Rightarrow$ $\phi = W_x \phi_1 + W_y \phi_2$, where ϕ_1, ϕ_2 are solutions of (7), (6a/6) for $(W_x, W_y) = (1, 0)$ and $(W_x, W_y) = (0, 1)$, respectively.

Using $\phi = w_x \phi_1 + w_y \phi_2$ and (4) for K_x, K_y, and equating terms containing w_x and those containing w_y, you get x_{CF}, y_{CF}, as follows.

For the case $f = g = 0$, ie when its not possible to satisfy (6b) by finding suitable f, g, we have

$$\boxed{\begin{aligned} x_{CF} &= \iint_A \left[x\left(-\frac{\partial \phi_2}{\partial x} - \frac{1}{2}\frac{I_y}{\Delta} y^2 \right) - y\left(\frac{\partial \phi_2}{\partial y} + \frac{1}{2}\frac{I_{xy}}{\Delta} x^2 \right) \right] dA \\ y_{CF} &= -\iint_A \left[x\left(-\frac{\partial \phi_1}{\partial x} + \frac{1}{2}\frac{I_{xy}}{\Delta} y^2 \right) - y\left(\frac{\partial \phi_1}{\partial y} - \frac{1}{2}\frac{I_x}{\Delta} x^2 \right) \right] dA \end{aligned}} \longrightarrow (11)$$

For the case when f, g can be found to satisfy (6b), thus making (6a) valid, we have

$$x_{CF} = \iint_A \left[x\left(-\frac{\partial \phi_2}{\partial x} - \frac{1}{2}\frac{I_y}{\Delta} y^2 - g(x) \right) - y\left(\frac{\partial \phi_2}{\partial y} + \frac{1}{2}\frac{I_{xy}}{\Delta} x^2 + f(y) \right) \right] dA$$

$$= \iint_A \left[2\phi_2 + \frac{\partial}{\partial x}\left(-x\phi_2 - \frac{I_{xy}}{6\Delta} x^3 y - xy f(y) \right) + \frac{\partial}{\partial y}\left(-y\phi_2 - \frac{I_y}{6\Delta} xy^3 - xy g(x) \right) \right] dA$$

$$x_{CF} \underset{\text{(div Thrm)}}{=} \iint_A 2\phi_2 \, dA + \oint_S \left[\left(-x\phi_2 - \frac{I_{xy}}{6\Delta} x^3 y - xy f(y) \right) \underline{i} + \left(-y\phi_2 - \frac{I_y}{6\Delta} xy^3 - xy\, g(x) \right) \underline{j} \right] \cdot \underline{n} \, ds$$

6a $\Rightarrow$ $\phi = 0$ on S, ie $\phi_2 = 0$ on S, and

6b $\Rightarrow$ $xy f(y) = xy \frac{E}{2} K_x x^2 = -\frac{I_{xy}}{2\Delta} x^3 y$; $xy\, g(x) = -xy \frac{E}{2} K_y y^2 = -\frac{I_y}{2\Delta} xy^3$

where $(W_x, W_y) \equiv (0, 1)$ used in K_x, K_y, since x_{CF} obtained for $(W_x, W_y) \equiv (0,1)$.

$$\Rightarrow \boxed{x_{CF} \underset{\text{(Div Thrm)}}{=} \iint_A \left(2\phi_2 + \frac{I_{xy}}{\Delta} x^2 y + \frac{I_y}{\Delta} xy^2 \right) dA} \longrightarrow \text{11a}$$

Similarly for y_{CF}, with $(W_x, W_y) \equiv (1, 0)$, we have

$$\boxed{y_{CF} = \iint_A \left(2\phi_1 - \frac{I_x}{\Delta} x^2 y - \frac{I_{xy}}{\Delta} xy^2 \right) dA} \longrightarrow \text{11b}$$

__Note__: $\because$ ϕ (hence ϕ_1, ϕ_2) depend on geometry of section (and material properties (G, ν)), then so does x_{CF} & y_{CF}.

- For section with one axis of symmetry, Shear center lies on axis of symm.
- For section with two axes of symm, SC lies on centroid.

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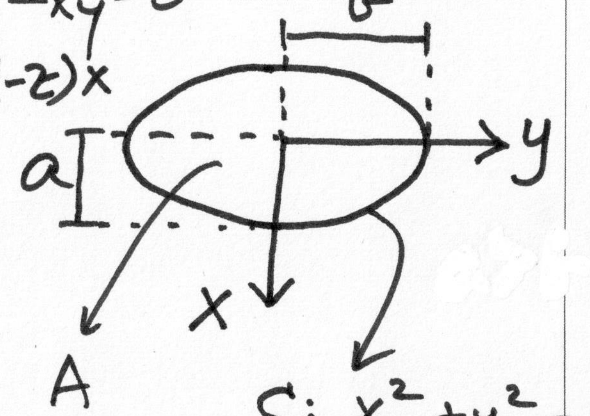
(Ex) Elliptical section.

$(W_x, 0)$ applied load at centroid. Also $I_{xy}=0$



$S: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$\Rightarrow K_y = 0, \quad \sigma_z = -E(l-z)K_x x = -\frac{4W_x(l-z)x}{\pi a^3 b}$

$\alpha = 0$, $\because$ load applied at $(0,0)$, ie S.C.

So no twisting occurs.

(6b) $\Rightarrow$ $\because K_y=0$, choose $g(x)=0$

$$f(y)\Big|_S = \frac{E}{2}K_x x^2 = \frac{W_x}{2I_y}a^2\left(1-\frac{y^2}{b^2}\right) = f(y) \text{ on } A.$$

(7) $\Rightarrow$ $\nabla^2\phi = \frac{W_x}{I_y}\left(\frac{\nu}{1+\nu}+\frac{a^2}{b^2}\right)y$ on A

(6a) $\Rightarrow$ $\phi = 0$ on S

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Try $\phi = my\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$ as solution.

$$(7) \Rightarrow \nabla^2\phi = my\left(\frac{2}{a^2}+\frac{6}{b^2}\right) = \frac{W_x}{I_y}\left(\frac{\nu}{1+\nu}+\frac{a^2}{b^2}\right)y$$

$$\phi = \frac{\left[\nu b^2+(1+\nu)a^2\right]a^2\cancel{b^2}}{\cancel{b^2}(1+\nu)\,2(b^2+3a^2)}\,\frac{W_x}{I_y}\,y\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

$$(5) \Rightarrow \tau_{xz} = \frac{2W_x}{\pi a^3 b}\,\frac{\left[2(1+\nu)a^2+b^2\right]}{(1+\nu)(3a^2+b^2)}\left[a^2-x^2-\frac{(1-2\nu)a^2y^2}{2(1+\nu)a^2+b^2}\right]$$

$$\tau_{yz} = -\frac{4W_x}{\pi a^3 b}\,\frac{\left[(1+\nu)a^2+\nu b^2\right]}{(1+\nu)(3a^2+b^2)}\,xy$$

$$\sigma_z = -\frac{4W_x}{\pi a^3 b}(l-z)x$$

τ_{yz} is max for xy max, which obviously occurs on bndry S.

$$xy\Big|_S = \underbrace{(a\cos\theta)}_{x}\underbrace{(b\sin\theta)}_{y} = \frac{ab}{2}\sin 2\theta \Rightarrow (xy)_{max} = \frac{ab}{2} \text{ for } \theta = \frac{\pi}{4}$$

(parametric eqn of ellipse)

$$\Rightarrow \boxed{(\tau_{yz})_{max} = \frac{2W_x}{A}\,\frac{b}{a}\left[\frac{(1+\nu)a^2+\nu b^2}{(1+\nu)(3a^2+b^2)}\right] \quad A = \pi ab}$$

τ_{xz} is max when $\left|\left[a^2 - x^2 - \frac{(1-2\nu)a^2}{2(1+\nu)a^2+b^2}\, y^2\right]\right|$ is max,

i.e, either max positive, $= a^2$, when $x = y = 0$,
or max negative.

For max negative, examine whether

$$\left(\frac{(1-2\nu)a^2}{2(1+\nu)a^2+b^2}\, y^2\right)_{max} \overset{?}{\gtrless} a^2, \text{ i.e, } b^2 \overset{?}{\gtrless} \frac{2(1+\nu)a^2+b^2}{(1-2\nu)}$$

$\rightarrow$ obviously "$<$" is correct $\because \nu < 0.5$, $\nu > 0$

So, $|\text{max negative}| \leqslant a^2$

Thus max τ_{xz} at $x = y = 0$, $\boxed{(\tau_{xz})_{max} = \frac{2W_x}{A}\left[\frac{2(1+\nu)a^2+b^2}{(1+\nu)(3a^2+b^2)}\right]}$

For circular section, $a = b$, from elementary beam theory, **we get**

$$(\tau_{xz})_{approx} = \frac{VQ}{I_y t} = \frac{4W_x(a^2-x^2)}{3\pi a^4} \Rightarrow \left((\tau_{xz})_{max}\right)_{approx} = \frac{4}{3}\frac{W_x}{A}$$

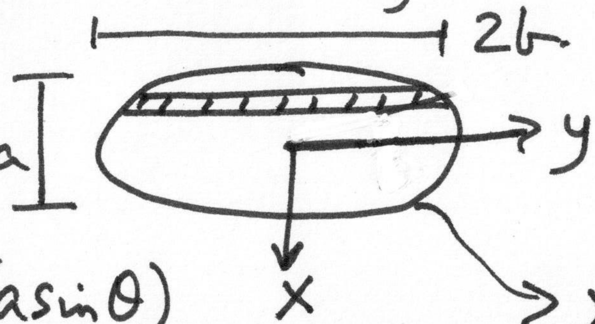
see p. 34a

$$\therefore \frac{\left((\tau_{xz})_{max}\right)_{approx}}{\left((\tau_{xz})_{max}\right)_{exact}} = \frac{3}{8}\frac{(3+2\nu)}{(1+\nu)} = 1.038 \quad (3.8\% \text{ error}).$$

for $\nu = 0.3$

Details of τ_{xz} using $\frac{VQ}{It}$ (ie elementary beam theory) for ellipse.

$V = W_x$, $\quad I = \frac{\pi}{4} a^3 b = I_y$



$$Q = \int_A x\, dA = \int_A a\sin\theta \; 2b\cos\theta \; d(a\sin\theta)$$

$\left.\begin{array}{l} x = a\sin\theta \\ y = b\cos\theta \end{array}\right\}$ parametric form of ellipse.

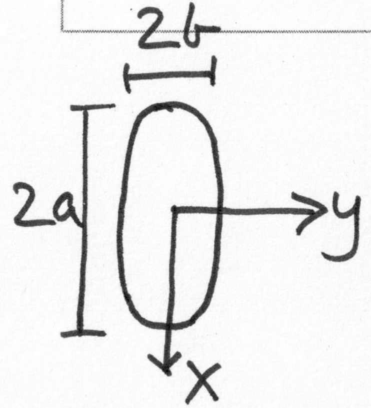
$$= \int_\theta^{\pi/2} 2a^2 b \cos^2\theta \sin\theta \, d\theta = \frac{2}{3} a^2 b \cos^3\theta$$

$$\frac{VQ}{It} = \frac{W_x \, \frac{2}{3} a^2 b \left(\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} \Big/ \sqrt{x^2 + b^2\left(1-\frac{x^2}{a^2}\right)} \right)^3}{\left(\frac{\pi}{4} a^3 b\right)\left(2b\sqrt{1-\frac{x^2}{a^2}}\right)} = \tau_{xz}$$

$$(\tau_{xz})_{max} = \frac{4}{3}\frac{W_x}{\pi a b} = \frac{4}{3}\frac{W_x}{A} \quad (\text{for } x = 0).$$

For circle, put $a = b$, $\quad \tau_{xz} = \frac{4}{3}\frac{W_x}{\pi a^2}\frac{(a^2 - x^2)}{a^2}$

Note that elementary beam theory cannot predict τ_{yz} for this case, since it assumes $\tau_{yz} \simeq 0$



<u>For $b \ll a$</u>

$(\tau_{xz})_{max} \simeq \frac{4}{3}\frac{W_x}{A}$; $(\tau_{yz})_{max} \simeq \frac{4}{3}\frac{W_x}{A}\frac{b}{2a}$

So $(\tau_{xz})_{max} \gg (\tau_{yz})_{max}$

Put $y \simeq \varepsilon$ (small), $\tau_{xz} \simeq \frac{4W_x}{3Aa^2}\left[a^2 - x^2 - O(\varepsilon^2)\right]$

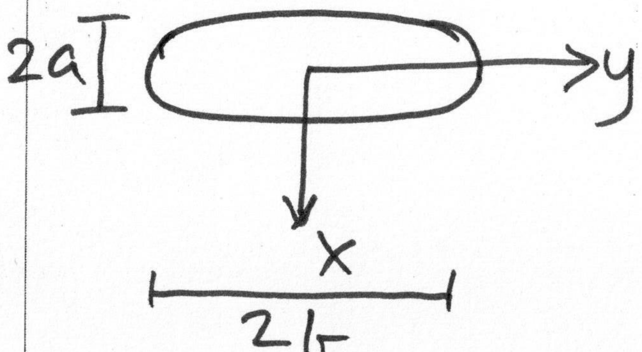
$$\tau_{yz} \simeq -\frac{4W_x}{3Aa^2} x\varepsilon = O(\varepsilon)$$

$\Rightarrow \tau_{xz} \gg \tau_{yz}$ (even for $x - a = \varepsilon$, $\tau_{xz} \simeq 2\tau_{yz}$)

From p. 34a you have same $(\tau_{xz})_{max}$ from elementary beam theory $\left(\frac{VQ}{It}\right)$. So elementary beam theory OK for $b \ll a$ case

For $b \gg a$

$$(\tau_{xz})_{max} \simeq \frac{2}{1+\nu}\frac{W_x}{A}$$



$$(\tau_{yz})_{max} \simeq \frac{2\nu}{1+\nu}\frac{W_x}{A}\frac{b}{a}$$

$$\Rightarrow (\tau_{yz})_{max} \gg (\tau_{xz})_{max}$$; ie τ_{yz} not negligible

Contrary to elementary beam theory.

Also $(\tau_{xz})_{max}$ has large error compared to that given by elementary beam theory, p. 34a.

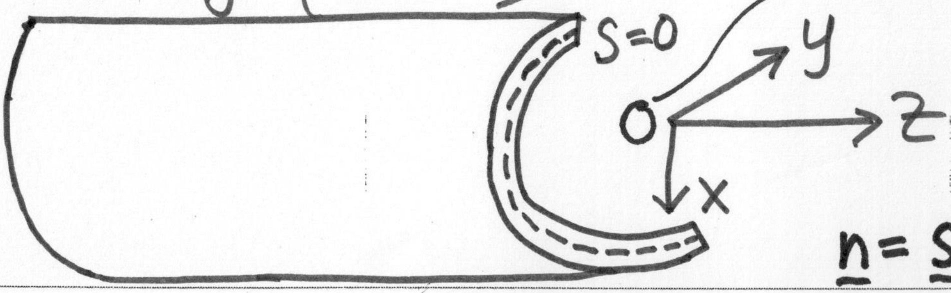
$\{$SO ELEMENTARY BEAM THEORY DOES NOT WORK WHEN $b \gg a$.

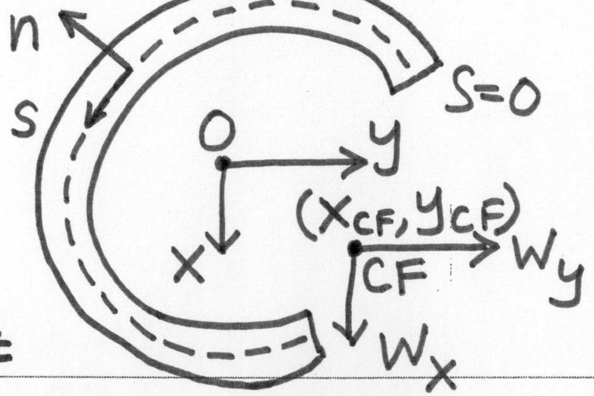
This is to be expected $\because$ $\frac{VQ}{It}$ assumes τ_{xz} constant over dimension t, which is OK when $b \ll a$ but not OK when $b \gg a$

SHEAR STRESSES IN OPEN THIN-WALLED BEAMS — 1-D Shear Flows

IIT Bombay

- σ_z from ② p.9 holds for pure bending case as well as bending due to end loads (see ①, ④ pp. 20 & 22) for which shear force is constant wrt z.
- However, assume that ② p9 holds for general loading for which SF varies with z.
- Assume shear stresses constant thru thickness, hence they act along ℄
- Assume only σ_z, τ_{xz}, τ_{yz} non-zero (ie, τ_{sz} non-zero).
- Assume end loads (W_x, W_y) applied thru CF, so no twisting $(\alpha = 0)$.

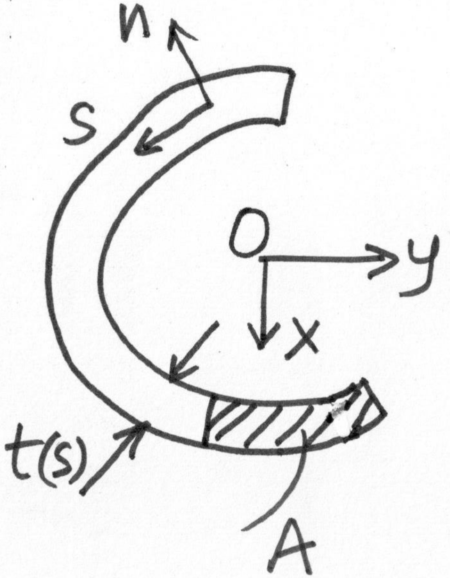


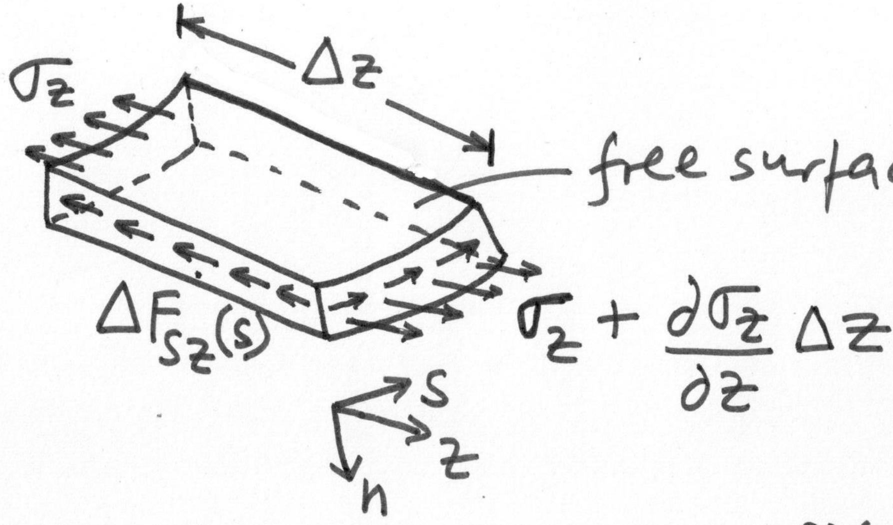


$\underline{n} = \underline{s} \times \underline{e}_z$

$$\sigma_z = -\frac{(M_x I_{xy} + M_y I_x)}{\Delta} x + \frac{(M_x I_y + M_y I_{xy})}{\Delta} y$$

② p.9 repeated.





$$\Sigma F_z = 0 \Rightarrow \Delta F_{sz}(s) - 0 = \tau_{sz}\, t\, dz - 0 = \iint_A \left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\Delta z - \sigma_z\right) dA$$

$$\left.\frac{\Delta F_{sz}}{\Delta z}\right|_{\Delta z \to 0} = q_{sz} = \tau_{sz}\, t = \iint_A \frac{\partial \sigma_z}{\partial z}\, dA$$

$\hookrightarrow$ SHEAR FLOW

$$\frac{\partial \sigma_z}{\partial z} = -\frac{\left(\frac{\partial M_x}{\partial z} I_{xy} + \frac{\partial M_y}{\partial z} I_x\right)}{\Delta} x + \frac{\left(\frac{\partial M_x}{\partial z} I_y + \frac{\partial M_y}{\partial z} I_{xy}\right)}{\Delta} y$$

ref p.3

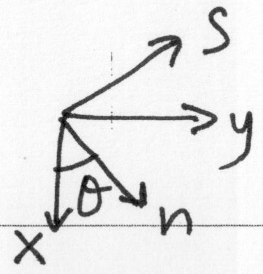
$$\frac{\partial \sigma_z}{\partial z} = -\frac{(V_y I_{xy} - V_x I_x)}{\Delta} x + \frac{(V_y I_y - V_x I_{xy})}{\Delta} y$$

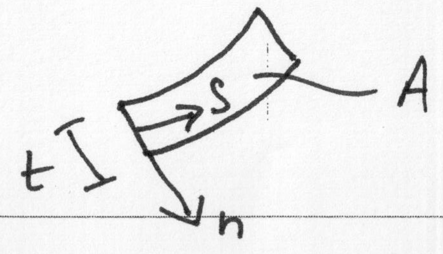
$$\Rightarrow q_{sz} = \frac{(-V_y I_{xy} + V_x I_x)}{\Delta} Q_y + \frac{(V_y I_y - V_x I_{xy})}{\Delta} Q_x$$

where $Q_y = \iint_A x\,dA$, $Q_x = \iint_A y\,dA$, $\Delta = I_x I_y - I_{xy}^2$

$$\boxed{\begin{aligned} \tau_{sz} &= \frac{q_{sz}}{t} = \frac{(-V_y I_{xy} + V_x I_x)}{t\Delta} Q_y + \frac{(V_y I_y - V_x I_{xy})}{t\Delta} Q_x \\ \tau_{xz} &= -\tau_{sz} n_y = \frac{(V_y I_{xy} - V_x I_x)}{t\Delta} Q_y n_y + \frac{(V_x I_{xy} - V_y I_y)}{t\Delta} Q_x n_y \\ \tau_{yz} &= \tau_{sz} n_x = \frac{(-V_x I_{xy} + V_x I_x)}{t\Delta} Q_y n_x + \frac{(V_y I_y - V_x I_{xy})}{t\Delta} Q_x n_x \end{aligned}} \quad (12)$$

Gives shear stresses due to bending only, ie no twisting ($\alpha = 0$)





Locating Shear Center

Use (9), (12) (i.e. equate moments due to end loads with moment caused by τ_{xz}, τ_{yz}, due to bending w/o twisting). Since we are applying this for case where shear force varies with z, so put $V_x \equiv W_x$, $V_y \equiv W_y$. Get x_{cf} for $(W_x, W_y) \equiv (0,1)$ and y_{cf} for $(W_x, W_y) \equiv (1, 0)$. Use $dA = t\,ds$. Thus,

(9), (12) $\Rightarrow$

$$x_{cf} = -\frac{I_{xy}}{\Delta}\int_C Q_y (x n_x + y n_y) ds + \frac{I_y}{\Delta}\int_C Q_x (x n_x + y n_y) ds$$

$$= -\frac{I_{xy}}{\Delta}\int_C Q_y (x\,dy - y\,dx) + \frac{I_y}{\Delta}\int_C Q_x (x\,dy - y\,dx) ds$$

(13)

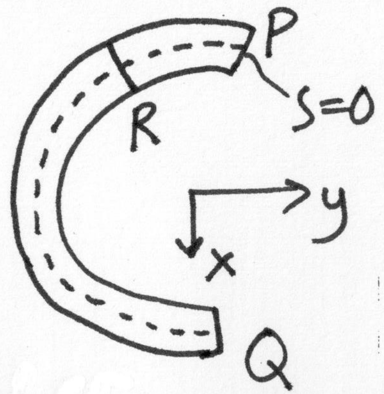
$$y_{cf} = -\frac{I_x}{\Delta}\int_C Q_y (x n_x + y n_y) + \frac{I_{xy}}{\Delta}\int_C Q_x (x n_x + y n_y) ds$$

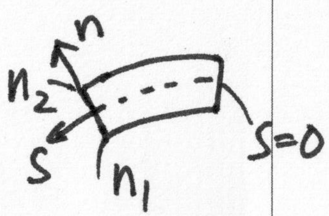
$$= -\frac{I_x}{\Delta}\int_C Q_y (x\,dy - y\,dx) + \frac{I_{xy}}{\Delta}\int_C Q_x (x\,dy - y\,dx) ds.$$

- Contour integrals evaluated from $s=0$ to $s=s_{max}$
- When (W_x, W_y) applied at CF, no twisting occurs ($\alpha=0$) although moment about CG is $M_o = (W_y x_{CF} - W_x y_{CF})$. When (W_x, W_y) applied at CG (0) then $M_o=0$ but twisting occurs ($\alpha \neq 0$). This assumes $CG \neq CF$.

<u>Guidelines for computing Q_x, Q_y</u>

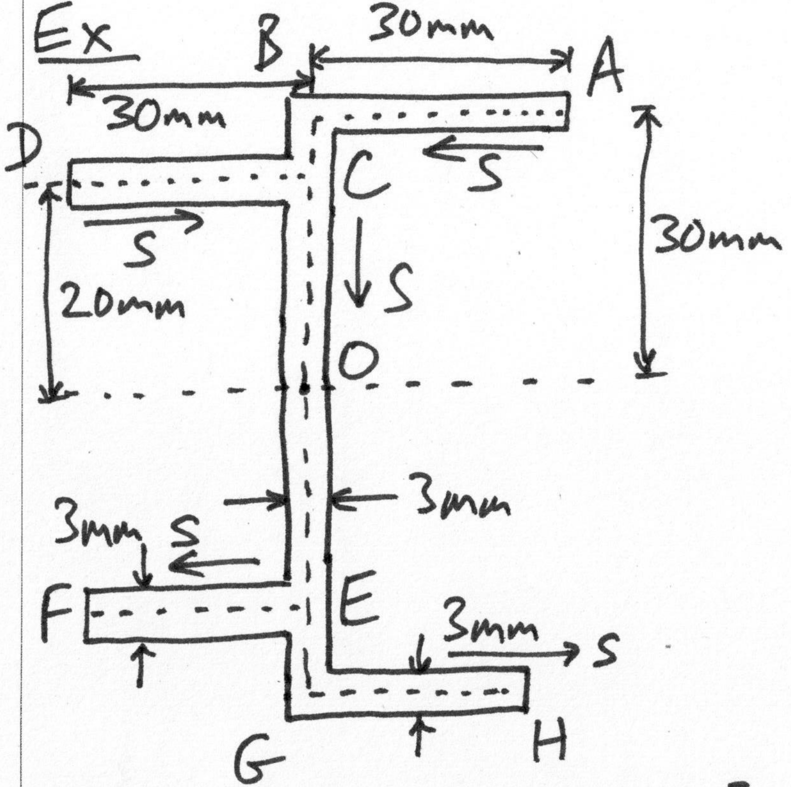
- Q_x, Q_y can be positive/negative. $Q_x = \iint_A y\,dA$; $Q_y = \iint_A x\,dA$
- $dA = ds\,dn$.
  Integration limits for s are from cut section to free surface.
  Limits for n are from min to max,
  ie n_1 to n_2.





$\underline{s} \times \underline{e}_z = \underline{n}$

- Does not matter whether you consider area RP or RQ. The Q's will be same.



Symmetric section. O is CG.
So, $x_{CF} = 0$. Find y_{CF}.

$$y_{CF} = -\frac{I_x}{\Delta}\int Q_y\,(x\,dy - y\,dx)$$

$(\because I_{xy} = 0)$.

So Q_x not required.
But for demonstration we will compute all Q_x, Q_y.

$$I_x = \frac{1}{12}\left\{63.3^3 + 2.3.60^3 - 2.3.3^3\right\}$$

$$I_y = \frac{1}{12}\left\{63^3.3 + 4.3^3.30 - 2.3^3.3\right\} + 2.28{\cdot}5.3.(30^2+20^2)$$

$$Q_x^{AB} = \iint y\,ds\,dn = \int_{-28.5}^{-31.5}\int_{y}^{30} y(-dy)(-dx) = \frac{(30^2-y^2)}{2}(-3) = -f(y)$$

$$Q_x^{BC} = \int_{x}^{-30}\int_{1.5}^{-1.5} y(-dy)\,dx + Q_x^{AB}\Big|_{y=0} = -\frac{3}{2}(30)^2 = -C_1$$

$$Q_x^{DC} = \int_{-21.5}^{-18.5}\int_{y}^{-30} y\,dy\,dx = \left(\frac{30^2-y^2}{2}\right)(3) = f(y)$$

$$Q_x^{FE} = \int_{21.5}^{18.5}\int_{y}^{-30} y(-dy)(-dx) = \left(\frac{30^2-y^2}{2}\right)(-3) = -f(y)$$

$$Q_x^{GH} = \int_{28.5}^{31.5}\int_{y}^{30} y\,dy\,dx = \left(\frac{30^2-y^2}{2}\right)(3) = f(y)$$

$$Q_x^{CE} = \int_{x}^{-20}\int_{1.5}^{-1.5} y(-dy)\,dx + Q_x^{BC}\Big|_{x=-20} + Q_x^{DC}\Big|_{y=0} = -\frac{3}{2}(30)^2 + \frac{3}{2}(30)^2 = 0$$

$$Q_x^{EG} = \int_{x}^{30}\int_{1.5}^{-1.5} y(-dy)(dx) + Q_x^{GH}\Big|_{y=0} = \frac{3}{2}\cdot 30^2 = C_1$$

$$Q_y^{AB} = \int_{28.5}^{-31.5}\int_{y}^{30} x(-dy)(-dx) = \left(\frac{31.5^2-28.5^2}{2}\right)(30-y) = g(y)$$

$$Q_y^{BC} = \int_{x}^{-30}\int_{1.5}^{-1.5} x(-dy)(dx) + Q_x^{AB}\Big|_{y=0} = \left(\frac{30^2-x^2}{2}\right)(3) + \left(\frac{31.5^2-28.5^2}{2}\right)(30) = p(x)$$

$$Q_y^{DC} = \int_{-21.5}^{-18.5}\int_{y}^{-30} x\,dy\,dx = \left(\frac{21.5^2 - 18.5^2}{2}\right)(30+y) = h(y)$$

$$Q_y^{FE} = \int_{21.5}^{18.5}\int_{y}^{-30} x(-dy)(-dx) = \left(\frac{21.5^2 - 18.5^2}{2}\right)(30+y) = h(y)$$

$$Q_y^{GH} = \int_{28.5}^{31.5}\int_{y}^{30} x\,dy\,dx = \left(\frac{31.5^2 - 28.5^2}{2}\right)(30-y) = g(y)$$

$$Q_y^{EG} = \int_{x}^{30}\int_{1.5}^{-1.5} x(-dy)(dx) + Q_y^{GH}\Big|_{y=0} = \left(\frac{30^2 - x^2}{2}\right)(3) + \left(\frac{31.5^2 - 28.5^2}{2}\right)(30) = p(x)$$

$$Q_y^{CE} = \int_{x}^{-20}\int_{1.5}^{-1.5} x(-dy)(dx) + Q_y^{BC}\Big|_{x=-20} + Q_y^{DC}\Big|_{y=0} = \left(\frac{20^2 - x^2}{2}\right)(3) + \left(\frac{30^2 - 20^2}{2}\right)(3)$$
$$+\left(\frac{31.5^2 - 28.5^2}{2}\right)(30) + \left(\frac{21.5^2 - 18.5^2}{2}\right)(30) = q(x)$$

$$\int_C Q_y(x\,dy - y\,dx) = \int_{A\to B} Q_y\,x\,dy + \int_{B\to C} Q_y(-y\,dx) + \boxed{\int_{D\to C} Q_y\,x\,dy} + \int_{C\to E} Q_y(-y\,dx) + \int_{E\to G} Q_y(-y\,dx)$$
$$+ \int_{E\to F} Q_y\,x\,dy + \int_{G\to H} Q_y\,x\,dy$$

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$$\int_C Q_y (x\,dy - y\,dx) = \int_{30}^{0} g(y)*(-30)\,dy + \int_{-30}^{-20} p(x)*(0)\,dx$$

$$+ \int_{-30}^{0} h(y)*(-20)\,dy + \int_{-20}^{20} q(x)*(-0)\,dx + \int_{20}^{30} p(x)*(0)\,dx$$

$$+ \int_{0}^{-30} h(y)*(20)\,dy + \int_{0}^{30} g(y)*(30)\,dy$$

$$= 2\left[30*\left(\frac{31.5^2-28.5^2}{2}\right)\left(30*30-\frac{30^2}{2}\right) + 20*\left(\frac{21.5^2-18.5^2}{2}\right)\left(30*(-30)+\frac{30^2}{2}\right)\right.$$

$$= 135000$$

ie to left of O

$$\Rightarrow y_{CF} = -\frac{1350000}{285068.25} = -4.7357 \text{ mm} \quad \text{(from O, ie CG)}$$

Q: What if we reverse direction of 's' in leg DC. Will it affect y_{CF}??

A: $$Q_y^{DC} = \int_{-18.5}^{-21.5}\int_{y}^{-30} x(-dy)(-dx) = -h(y)$$

note this change

(C→D)

$$\int Q_y\, x\,dy = \int_{0}^{-30} -h(y)*(-20)\,dy = \int_{-30}^{0} h(y)*(-20)\,dy = \text{same as dotted circled term above.}$$

SO y_{CF} will not change

This is the advantage of this seemingly lengthy method, ie, no matter what direction of s you assume, as long as you are consistent with the limits of integration for the Q's (ie, from cut section to free surface and min to max 'n'), and for the $\int_C Q_y(x\,dy - y\,dx)$ and $\int_C Q_x(x\,dy - y\,dx)$ (ie along increasing 's'), you simply cannot go wrong.

This method is advantageous for non-symmetric sections and it is programmable. — as opposed to short-cut but more physical methods.

<u>Note:</u> For Q_x^{CE}, when reversing 's' in DC, we would have,

$$Q_x^{CE} = \int_x^{-20} \int_{1.5}^{-1.5} y(-dy)\,dx + Q_x^{BC}\Big|_{x=-20} - Q_x^{DC}\Big|_{y=0} = 0 + (-C_1) - (-f(y))\Big|_{y=0} = 0,$$

since 'S' in DC is in opp. dir.

as before

Simplified Solution

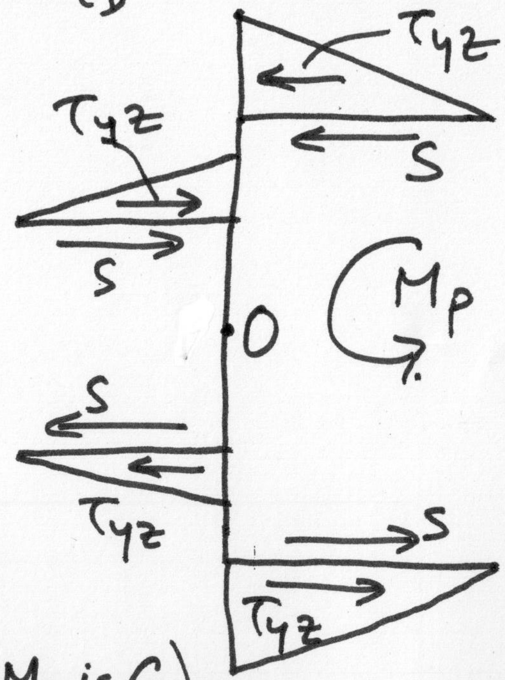
$$\textcircled{12} \Rightarrow (\tau_{yz})_{AB} = \frac{V_x}{I_y t}(Q_y n_x)_{AB} = \frac{V_x}{I_y t}\, g(y)(-1)$$

$$(\tau_{yz})_{GH} = \frac{V_x}{I_y t}(Q_y n_x)_{GH} = \frac{V_x}{I_y t}\, g(y)(+1) = -(\tau_{yz})_{AB}$$

$$(\tau_{yz})_{CD} = \frac{V_x}{I_y t}(Q_y n_x)_{CD} = \frac{V_x}{I_y t}\, h(y)(+1)$$

$$(\tau_{yz})_{EF} = \frac{V_x}{I_y t}(Q_y n_x)_{EF} = \frac{V_x}{I_y t}\, h(y)(-1) = -(\tau_{yz})_{CD}$$

$g(y)$, $h(y)$ linear in y (p. 44)



$$M_O = \left\{\frac{1}{2}\cdot 30\cdot \left(\frac{31.5^2 - 28.5^2}{2}\right)\cdot 30\cdot 2\cdot 30 - \frac{1}{2}\cdot 30\cdot\left(\frac{21.5^2 - 18.5^2}{2}\right)\cdot 30\cdot 2\cdot 20\right\} * \frac{tV_x}{I_y t} = 1350000\,\frac{V_x}{I_y}$$

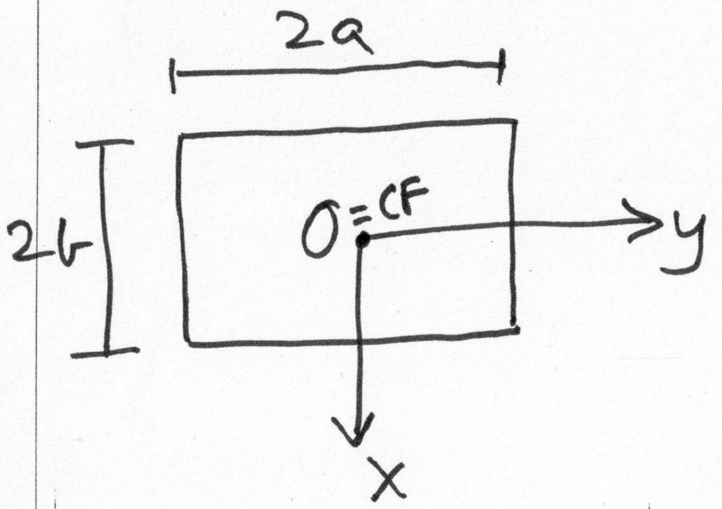
Equating M_O due to shear stresses with moment due to applied V_x,

$$1350000\,\frac{\cancel{V_x}}{\underset{285068.25}{I_y}} = \cancel{V_x}\cdot y_{CF} \Rightarrow y_{CF} = 4.735$$

(to left of O ∵ M_p is ↺)

Shear center of section with ALL legs intersecting at a point, is the point of intersection.

FINITE DIFFERENCE SOLUTION FOR BENDING WITH END LOADS FOR RECTANGLE



If load (W_x, W_y) applied at (x_0, y_0) find twisting moment M_z from (10) p. 28 with $(x_{CF}, y_{CF}) = (0,0)$, and solve torsional shear stresses τ_{xz}, τ_{yz} due to M_z by FDM for torsion.

So here we consider $(x_0, y_0) = (0,0)$ ie load thru CF, and solve bending problem for τ_{xz}, τ_{yz} (ie $\alpha = 0$)

(6b) $\Rightarrow$ $$\left.\begin{array}{l} f(y) = \frac{1}{2} E K_x b^2 = \frac{1}{2}\frac{W_x}{I_y} b^2 \\ g(x) = -\frac{1}{2} E K_y a^2 = -\frac{1}{2}\frac{W_y}{I_x} a^2 \end{array}\right\} \Rightarrow \phi_{S'} = 0 \rightarrow \text{(6a)}$$

(7) $\Rightarrow$ $$\nabla^2 \phi = -2G\nu\left[\frac{W_y}{I_x} x - \frac{W_x}{I_y} y\right]$$

Boundary S' defined by $(y^2 - a^2)(x^2 - b^2) = 0$ $\longrightarrow$ Laplacian not linear in x, y.

So choosing $\phi = m\, p(x)\left[(y^2-a^2)(x^2-b^2)\right]$ won't work.

Hence do FDM.

$$\nabla^2 \phi_{i,j} = \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -2G\nu\left(\frac{W_y}{I_x} x_{i,j} - \frac{W_x}{I_y} y_{i,j}\right)$$

$\rightarrow$ n eqns for n interior nodes

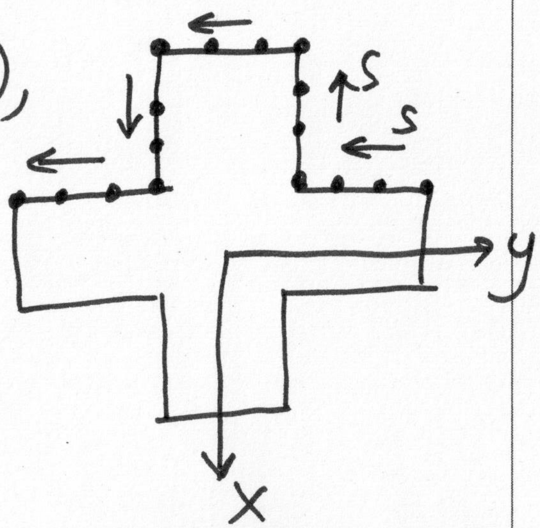
and $\phi_{i,j} = 0$ on S'.

So solve the n eqns, then get stresses.

Fourier series soln also exists for rectangle (see TIMOSHENKO, GOODIER).

For general section, best to take $f = g = 0$, and do FDM discretization of (6) $\because$ (6a) invalid.

For section with only horizontal & vertical boundaries this discretization is relatively easy, $\because \frac{d\phi}{ds} = \left(\frac{d\phi}{dx} \text{ or } \frac{d\phi}{dy}\right)$ for which you can do CDM for boundary nodes other than corner nodes.



<u>For interior nodes on boundary</u> (ie not corners),

$$\frac{d\phi}{ds} = \frac{d\phi}{dx} = \pm\frac{1}{2} E K_y y^2 \text{ on vertical boundaries}$$

(used $\frac{dy}{ds} = 0$, $\frac{dx}{ds} = \pm 1$)

$$\frac{d\phi}{ds} = \frac{d\phi}{dy} = \pm\frac{1}{2} E K_x x^2 \text{ on horizontal boundaries}$$

(used $\frac{dx}{ds} = 0$, $\frac{dy}{ds} = \pm 1$)

Above can be discretized by CDM. $\longrightarrow$ get 'm' eqns for m such nodes.

For boundary nodes at corners, do BDM or ForwDM for $\frac{d\phi}{dx}$ & $\frac{d\phi}{dy}$ and add these eqns, ie $\left(\frac{d\phi}{dx} + \frac{d\phi}{dy}\right)_{i,j} = \pm\frac{1}{2} E K_y y^2 \pm \frac{1}{2} E K_x x^2 \longrightarrow$ 'p' such eqns.

Then solve $(n+m+p)$ eqns for $(n+m+p)$ unknowns.