CE469 Advanced Solid Mechanics Midsem 25/2/12

- Show all working.
- Attempt all parts of a question in a contiguous manner, i.e., dont scatter parts of the same question all over the answerbook.
- **Only one attempt per question will be graded**. So cancel out any attempt you do not want graded. The first not-cancelled attempt will be graded by default.
- **Open notes (in your handwriting only, no photocopies). No solved examples allowed.**

P1. A pure shear state of stress referred to the *xyz* coordinate system is given as

$$\sigma_{ij} = \begin{pmatrix} a & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & b \end{pmatrix}$$

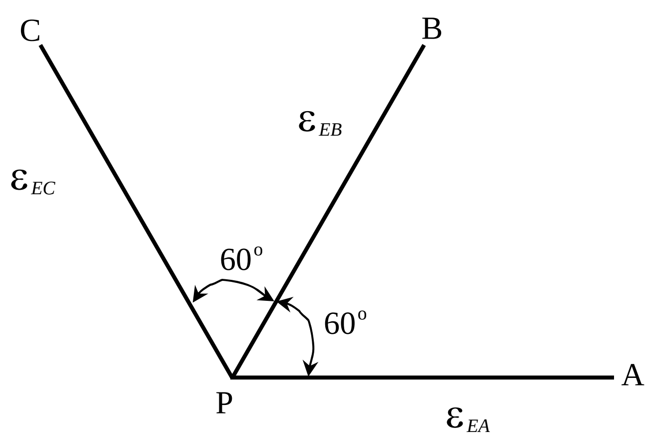
Thus, there exists a transformation to a coordinate system *XYZ* such that the stress matrix in the *XYZ* system has zero diagonal entries. In addition, it is also given that there exist plane(s) on which the total stress is zero. **Determine** :

(i) *a*, *b*, and direction cosines (referred to *xyz* coordinate system) of the plane(s) on which the total stress is zero.

(ii) The transformation from *xyz* system to *XYZ* system.

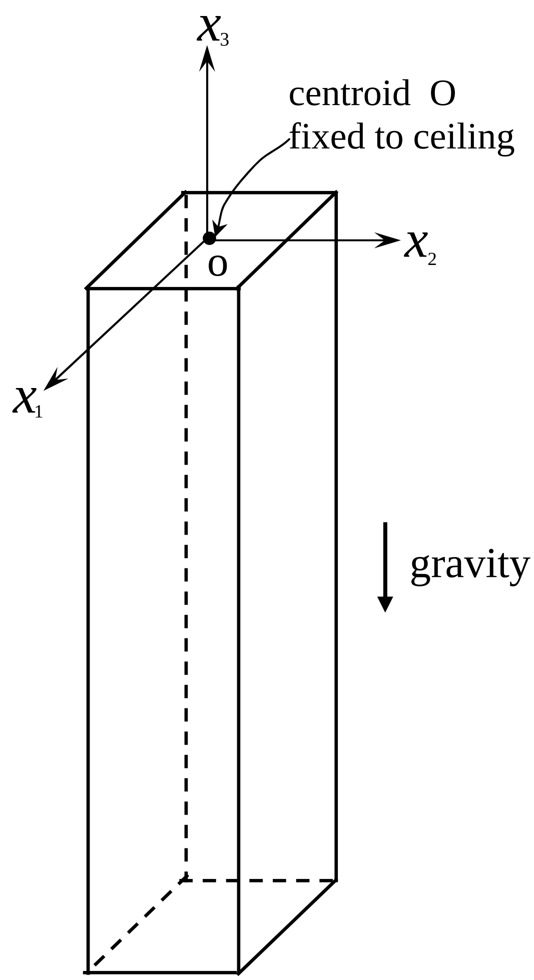
P2. A long solid cylinder is restrained from longitudinal displacement at its two ends. It is subject to loads that are perpendicular to its longitudinal axis and that do not vary along the longitudinal axis. A strain rosette, having geometry as shown, is embedded at point *P* lying away from the two ends, so as to measure extensional strains along directions shown which are perpendicular to the longitudinal axis. The measured strains are $\varepsilon_{EA} = 200$, $\varepsilon_{EB} = -100$, $\varepsilon_{EC} = 400$ μmm/mm. Material properties are $E = 210 \times 10^9$ N/m^2 and $\nu = 0.25$. Assuming infinitesimal displacement gradients, determine the principal stresses at *P*.

Fig. P2



P3. Consider a bar of uniform cross-section suspended vertically by **fixing** the **centroid** of the upper base, as shown. Thus, an arbitrary line element at the centroid of the upper base has **zero rotation**. **Obtain the stresses, strains, and displacements**. (Hint: Assume a suitable distribution of stresses σ_{ij} such that only one stress component is non-zero and it varies only with the x_3 coordinate, the variation being linear. Check that your stress distribution satisfies boundary conditions, equilibrium, compatibility. Then, use this distribution to obtain strains and hence displacements).

Fig. P3



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P.1 (a) $\underline{t} = 0$ on $\underline{n} = (n_1, n_2, n_3)$, $\underline{\underline{\sigma}} = \begin{pmatrix} a & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & b \end{pmatrix}$ given

$$\Rightarrow a n_1 + 2 n_2 + n_3 = 0 \quad - (1)$$
$$2 n_1 + \quad n_3 = 0 \quad - (2)$$
$$n_1 + n_2 + b n_3 = 0 \quad - (3)$$

Pure shear $\Rightarrow a + b = 0 \quad - (4)$

also, $n_1^2 + n_2^2 + n_3^2 = 1 \quad - (5)$

$(1)-(5) \Rightarrow n_3 = -2n_1$

$$n_2 = -n_1 - b n_3 = n_1(-1+2b)$$
$$n_1(a - 2 + 4b - 2) = 0 \Rightarrow a + 4b = 4$$
$$\Rightarrow \boxed{b = \frac{4}{3}, \quad a = -\frac{4}{3}}$$
$$n_1^2(1 + 4b^2 + 1 - 4b + 4) = 1$$
$$\boxed{n_1 = \pm\sqrt{\frac{9}{70}}, \quad n_2 = \pm\sqrt{\frac{25}{70}}, \quad n_3 = \mp\sqrt{\frac{36}{70}}}$$

(b) Do rotation abt x_2 axis (ie $\sigma_{22} = \sigma_{22}' = 0$) so that $\sigma_{11}' = 0$. Then $\because$ σ_{ii} invariant, $\sigma_{33}' = 0$. So find required θ.

$$\begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} a & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & b \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} = \sigma_{ij}'$$

$$\left.\begin{aligned} \sigma_{11}' &= a c^2\theta + 2 s\theta c\theta + b s^2\theta = 0 \quad - (6) \\ \sigma_{22}' &= a s^2\theta - 2 s\theta c\theta + b c^2\theta = 0 \quad - (7) \end{aligned}\right\} \text{use any one to get } \theta$$

$$a(c^2\theta - s^2\theta) = -2 s\theta c\theta$$
$$\tan 2\theta = -a \Rightarrow \boxed{\theta = -\frac{1}{2}\tan^{-1}(a)}$$

P2 $\varepsilon_{i3} = 0$ (plane strain)

Linear theory $\rightarrow$ $\varepsilon_E = \varepsilon_{ij} n_i n_j$

$n_i = (1, 0, 0) \rightarrow \varepsilon_{EA} = \varepsilon_{11} = 200 * 10^{6}$

$n_i = (0.5, \frac{\sqrt{3}}{2}, 0) \rightarrow \varepsilon_{EB} = \varepsilon_{11} n_1 n_1 + \varepsilon_{22} n_2 n_2 + 2\varepsilon_{12} n_1 n_2$

$= 0.75\,\varepsilon_{22} + 0.866\,\varepsilon_{12} = -150 * 10^{-6}$

$n_i = (-0.5, \frac{\sqrt{3}}{2}, 0) \rightarrow \varepsilon_{EC} = 0.75\,\varepsilon_{22} - 0.866\,\varepsilon_{12} = 350 * 10^{-6}$

$\varepsilon_{12} = -288.68 * 10^{-6},\ \varepsilon_{22} = 133.33 * 10^{-6} \leftarrow$ Solve

Principal strains $\rightarrow$ $10^{-6} \det \begin{vmatrix} (200-\varepsilon) & -288.68 & 0 \\ -288.68 & (133.33-\varepsilon) & 0 \\ 0 & 0 & -\varepsilon \end{vmatrix} = 0.$

$\Rightarrow \varepsilon(3) = 0$

$(200-\varepsilon)(133.33-\varepsilon) - 288.68^2 = 0$

$\varepsilon(1) = 457.26 * 10^{-6}$

$\varepsilon(2) = -123.94 * 10^{-6}$

Principal stresses $\rightarrow$ $\left[\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\underbrace{\cancel{\sigma_{kk}}}_{\frac{E}{1-2\nu}\varepsilon_{kk}} - \delta_{ij}\,\varepsilon\right] n_j = 0.$

p-stress $\Rightarrow \sigma = \frac{E}{1+\nu}\left(\frac{\nu}{E}\cdot\frac{E}{1-2\nu}\cdot\varepsilon_{kk} + \varepsilon\right)$ p-strain.

get same result by referring ε_{ij} to p-axes system & then using const. law ($\because$ p-axes of $\underline{\underline{\sigma}}$ & $\underline{\underline{\varepsilon}}$ are same for isotropic material)

$$\boxed{\begin{aligned} \sigma(1) &= \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \varepsilon(1)\right) = 104.8 * 10^{6}\ \mathrm{N/m^2} \\ \sigma(2) &= \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \varepsilon(2)\right) = 7.18 * 10^{6}\ \mathrm{N/m^2} \\ \sigma(3) &= \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \varepsilon(3)\right) = 28 * 10^{6}\ \mathrm{N/m^2} \end{aligned}}$$

P.3. Obviously σ_{33} is the (only) non-zero component of stress.

Choose $\sigma_{33} = k(L - x_3)$ ($\because$ given that its linear variation).

At $x_3 = 0$, $\sigma_{33} A = kLA = \text{weight} = AL\rho g$ (ie static equil)

$\Rightarrow k = \rho g$

So, $\sigma_{33} = \rho g (L - x_3)$

BC's : $\sigma_{33}\big|_{x_3 = L} = 0 \rightarrow$ satisfied. All other σ_{ij}'s zero everywhere. So assumed σ_{33} is correct.

$$\varepsilon_{11} = \varepsilon_{22} = -\frac{\nu}{E}\sigma_{33} = -\frac{\nu \rho g}{E}(L - x_3)$$

$$\varepsilon_{33} = \frac{\sigma_{33}}{E} = \frac{\rho g}{E}(L - x_3)$$

$$\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} = 0.$$

$$u_{1,1} = -\frac{\nu \rho g}{E}(L - x_3) \Rightarrow u_1 = -\frac{\nu \rho g}{E}(L - x_3)x_1 + F_1(x_2, x_3) \quad \text{— (1)}$$

$$u_{2,2} = -\frac{\nu \rho g}{E}(L - x_3) \Rightarrow u_2 = -\frac{\nu \rho g}{E}(L - x_3)x_2 + F_2(x_1, x_3) \quad \text{— (2)}$$

$$u_{3,3} = \frac{\rho g}{E}(L - x_3) \Rightarrow u_3 = \frac{\rho g}{E}\left(L - \frac{x_3}{2}\right)x_3 + F_3(x_1, x_2) \quad \text{— (3)}$$

$$\varepsilon_{12} = 0 = F_{1,2} + F_{2,1} \quad \text{— (4)}$$

$$\varepsilon_{23} = 0 = \frac{\nu \rho g}{E}x_2 + F_{2,3} + F_{3,2} \quad \text{— (5)}$$

$$\varepsilon_{13} = 0 = \frac{\nu \rho g}{E}x_1 + F_{1,3} + F_{3,1} \quad \text{— (6)}$$

$$((5)_{,1} + (6)_{,2}) \text{ with } (4)_{,3} \Rightarrow F_{3,12} = 0. \quad \text{— (7)}$$

$$(5)_{,2} = 0 = \frac{\nu \rho g}{E} + F_{3,22} \quad \text{— (8)}$$

$$(6)_{,1} = 0 = \frac{\nu \rho g}{E} + F_{3,11} \quad \text{— (9)}$$

$$(4)_{,1} = 0 = F_{2,11} \quad \text{— (10)}$$

$$(4)_{,2} = 0 = F_{1,22} \quad \text{— (11)}$$

$(5)_{,3} = 0 = F_{2,33}$ — (12)

$(6)_{,3} = 0 = F_{1,33}$ — (13)

$(5)_{,1} = 0$ with (7) $\Rightarrow F_{2,13} = 0$ — (14)

$(6)_{,2} = 0$ with (7) $\Rightarrow F_{1,23} = 0$ — (15)

(11), (13), (15) $\Longrightarrow F_1 = a x_2 + b x_3 + c$ — (17)

(10), (12), (14) $\Rightarrow F_2 = d x_1 + e x_3 + f$ — (18)

(7), (8), (9) $\Rightarrow F_3 = -\frac{1}{2}\frac{\nu \rho g}{E}\left(x_1^2 + x_2^2\right)$ — (19)

(4), (17), (18) $\Rightarrow a = -d$ — (20)

$u_1 = u_2 = u_3 = 0$ at $x_1 = x_2 = x_3 = 0 \Rightarrow \boxed{c = f = 0}$

$$\omega_{13} = \frac{1}{2}\left(\frac{\nu \rho g}{E} x_1 + b + \frac{\nu \rho g}{E} x_1\right)$$

$$\omega_{23} = \frac{1}{2}\left(\frac{\nu \rho g}{E} x_2 + e + \frac{\nu \rho g}{E} x_2\right)$$

$$\omega_{12} = \frac{1}{2}(a - d)$$

$\omega_{13}\big|_{x_1 = x_2 = x_3 = 0} = 0 \Rightarrow \boxed{b = 0}$

$\omega_{23}\big|_{x_1 = x_2 = x_3 = 0} = 0 \Rightarrow \boxed{e = 0}$

$\omega_{12}\big|_{x_1 = x_2 = x_3} = 0 \Rightarrow a = d$ — (21)

(20), (21) $\Rightarrow \boxed{a = d = 0}$

$$\Rightarrow \boxed{\begin{aligned} u_1 &= -\frac{\nu \rho g}{E}(L - x_3)\, x_1 \\ u_2 &= -\frac{\nu \rho g}{E}(L - x_3)\, x_2 \\ u_3 &= \frac{\rho g}{E}\left[\left(L - \frac{x_3}{2}\right)x_3 - \frac{1}{2}\nu\left(x_1^2 + x_2^2\right)\right] \end{aligned}}$$

Thus at $x_3 = 0$

$u_1 = u_2 = 0 \ \forall\, x_1, x_2$

but $u_3 = 0$ only for $x_1 = x_2 = 0$

Solution for σ_{33} valid away from $x_3 = 0$ end