CE469 Advanced Mechanics of Solids Midsem 22/2/14

- Equal marks for each question
- Show all working.
- Attempt all parts of a question in a contiguous manner, i.e., dont scatter parts of the same question all over the answerbook.
- **Only one attempt per question will be graded**. So cancel out any attempt you do not want graded. The first not-cancelled attempt will be graded by default.

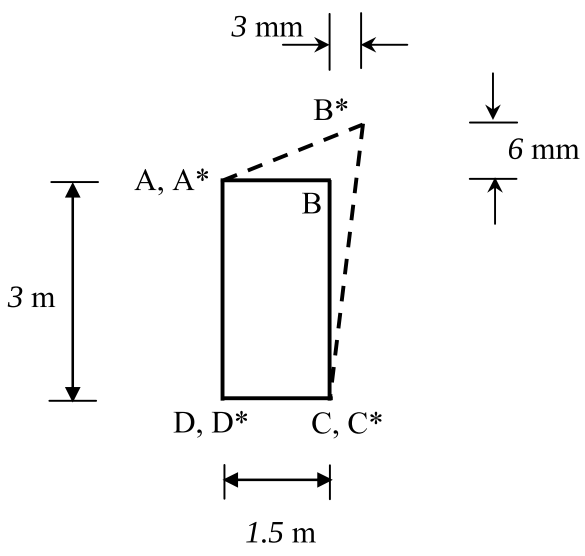
1. The principal stresses at a point *P* are 4, 5, 6. **Determine the unit normal for the plane(s) on which the normal stress is 5 and shear stress is $\frac{1}{2}$.**

2. The rectangular plate shown in **Fig. 2**, is loaded so that it is in a state of plane strain (i.e., $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$). The deformed plate is shown by dotted straight lines. Assume small displacement gradients. **At corner *B* determine**:

a) Maximum and minimum engineering extensional strains and the direction of the corresponding line elements on which they act.

b) Maximum change in angle between two elements originally at right angles. Also sketch the direction of these two elements.

Fig. 2



3. At a point on the free surface of an alloy steel (E=200000 MPa, $\nu = 0.3$), a strain rosette measures infinitesimal extensional strains of 1000E-6, 2000E-6, and 1200E-6 at respective angles of 0°, 60° and 120° with respect to ***x*** axis (measured counterclockwise). Design considerations limit the maximum normal stress to 510 MPa and maximum shearing stress to 275 MPa, the maximum normal strain to 2200E-6 and maximum shearing strain to 2500E-6. **What is your evaluation of the design?**

4. A body is subject to uniform pressure such that the state of stress throughout the body is $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p, \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$. Assume the elastic constants as E and ν. Given that the displacement and the rotation at the origin are zero, **determine the displacements as a function of** x, y, z**.**

P!1 Refer everything to p-coord system. So,

$$\underline{\underline{\sigma}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$N = \underline{n}^T \underline{\underline{\sigma}}\, \underline{n} = 4n_1^2 + 5n_2^2 + 6n_3^2 = 5 \longrightarrow ①.$$

$$S^2 = \sigma^2 - N^2 = (\sigma_{11} n_1)^2 + (\sigma_{22} n_2)^2 + (\sigma_{33} n_3)^2 - N^2$$

$$\Rightarrow \left(\tfrac{1}{2}\right)^2 = 4^2 n_1^2 + 5^2 n_2^2 + 6^2 n_3^2 - 5^2 \longrightarrow ②.$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \longrightarrow ③$$

Sol. of ①–③ is $n_1 = \pm \frac{1}{2\sqrt{2}}$, $n_2 = \pm \frac{\sqrt{3}}{2}$, $n_3 = \pm \frac{1}{2\sqrt{2}}$

i.e., 8 planes exist (only 4 of them are distinct).

P4

Note: stresses, hence strains are uniform (spatially constant)

$$e_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \left(\frac{1-2\nu}{E}\right)(-p)$$

$$= e_{yy} = e_{zz}$$

$$e_{xx} = u_{x,x} \Rightarrow u_x = \frac{(2\nu-1)}{E} p\, x + f(y,z)$$

$$e_{yy} = u_{y,y} \Rightarrow u_y = \frac{(2\nu-1)}{E} p\, y + g(x,z).$$

$$e_{zz} = u_{z,z} \Rightarrow u_z = \frac{(2\nu-1)}{E} p\, z + h(x,y)$$

$$e_{xy} = \tfrac{1}{2}(u_{x,y} + u_{y,x}) = 0 = \tfrac{1}{2}(f_{,y} + g_{,x}) \longrightarrow ①$$

$$e_{yz} = \tfrac{1}{2}(u_{y,z} + u_{z,y}) = 0 = \tfrac{1}{2}(g_{,z} + h_{,y}) \longrightarrow ②$$

$$e_{zx} = \tfrac{1}{2}(u_{z,x} + u_{x,z}) = 0 = \tfrac{1}{2}(h_{,x} + f_{,z}) \longrightarrow ③$$

$\dfrac{\partial ①}{\partial z} + \dfrac{\partial ②}{\partial x} = 0$ gives $\quad f_{,zy} + g_{,xz} + g_{,zx} + h_{,xy} = 0 \rightarrow ④$

Insert ③ in ④, get $\underbrace{(f_{,z} + h_{,x})_{,y}}_{=0 \text{ from } ③} + 2g_{,xz} = 0$

$\Rightarrow \quad g_{,xz} = 0 \quad \Rightarrow \quad g = g_1(x) + g_2(z) \longrightarrow ⑤$

Similaly you can obtain, $f_{,yz} = 0 \Rightarrow f = f_1(y) + f_2(z) \rightarrow ⑥$

& $\quad h_{,xy} = 0 \quad \Rightarrow h = h_1(x) + h_2(y) \rightarrow ⑦$

⑤, ⑥ in ① gives $\rightarrow f_1' + g_1' = 0 \Rightarrow f_1' = -g_1' = c_1$ (const)

$\Rightarrow f_1 = c_1 y + k_1 \,,\; g_1 = -c_1 x + k_2$ (k_1, k_2, are consts).

⑤, ⑦ in ② gives $\rightarrow g_2' + h_2' = 0 \Rightarrow g_2' = -h_2' = c_2$

$\Rightarrow g_2 = c_2 z + k_3 \,,\quad h_2 = -c_2 y + k_4$

⑥, ⑦ in ③ gives $\rightarrow f_2' + h_1' = 0 \Rightarrow f_2' = -h_1' = c_3$

$\Rightarrow f_2 = c_3 z + k_5 \,,\quad h_1 = -c_3 x + k_6$

So, $\quad u_x = cx + c_1 y + c_3 z + (k_1 + k_5)$

$\quad u_y = cy + c_2 z - c_1 x + (k_3 + k_2)$

$\quad u_z = cz - c_3 x - c_2 y + (k_6 + k_4)$

From zero-displ at origin $\longrightarrow (k_1 + k_5) = (k_3 + k_2) = (k_6 + k_4) = 0$

From zero rotation at origin,

$2\omega_{xy} = u_{x,y} - u_{y,x} = 2c_1 = 0$ at origin $\Rightarrow c_1 = 0 \Rightarrow \omega_{xy} = 0$ everywhere

$2\omega_{yz} = u_{y,z} - u_{z,y} = 2c_2 = 0$ at origin $\Rightarrow c_2 = 0 \Rightarrow \omega_{yz} = 0$ everywhere

$2\omega_{zx} = u_{z,x} - u_{x,z} = -2c_3 = 0$ at origin $\Rightarrow c_3 = 0 \Rightarrow \omega_{zx} = 0$ everywhere.

$\Rightarrow$ rotation is zero everywhere.

$u_x = cx \quad u_y = cy \quad u_z = cz \qquad c = \dfrac{(2\nu - 1)}{\ldots} A$

P2

$$u_1 = a_1 x_1 + b_1 x_2 + c_1 x_1 x_2$$
$$u_2 = a_2 x_1 + b_2 x_2 + c_2 x_1 x_2$$

} can write directly from Tutorial #2.

Take origin at D.

At C: $u_1 = 0 = 1.5 a_1$ At A: $u_1 = 0 = 3b_1$

$u_2 = 0 = 1.5 a_2$ $u_2 = 0 = 3b_2$

At B: $u_1 = 3E\text{-}3 = (1.5)(3)\, c_1 \Rightarrow c_1 = \frac{2}{3} E\text{-}3\ /m$

$u_2 = 6E\text{-}3 = (1.5)(3)\, c_2 \Rightarrow c_2 = \frac{4}{3} E\text{-}3\ /m$

$\Rightarrow u_1 = \left(\frac{2}{3}E\text{-}3\right) x_1 x_2$

$u_2 = \left(\frac{4}{3}E\text{-}3\right) x_1 x_2$

$e_{11} = u_{1,1} = c_1 x_2 = 2E\text{-}3$

$e_{22} = u_{2,2} = c_2 x_1 = 2E\text{-}3$

$e_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) = \frac{1}{2}\left(c_1 x_1 + c_2 x_2\right)$

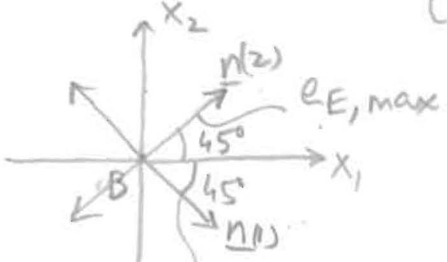
$= \frac{1}{2}\left(\frac{2}{3} * 1.5 + \frac{4}{3} * 3\right) . E\text{-}3 = 2.5 E\text{-}3$

At B, $\underline{\underline{e}} = \begin{pmatrix} 2 & 2.5 \\ 2.5 & 2 \end{pmatrix} * 10^{-3}$

$\Rightarrow (2-\lambda)^2 - 2.5^2 = (-0.5-\lambda)(4.5-\lambda) = 0$

$\lambda(1) = -0.5 E\text{-}3 = e_{E,min}$; $\lambda(2) = 4.5 E\text{-}3 = e_{E,max}$

$\underline{n}(1) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \equiv \begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix}$; $\underline{n}(2) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \equiv \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}$



$e_{12,max} = \left| \frac{\lambda(1) - \lambda(2)}{2} \right| = 2.5 E\text{-}3 \simeq \frac{\cos\theta}{2}$

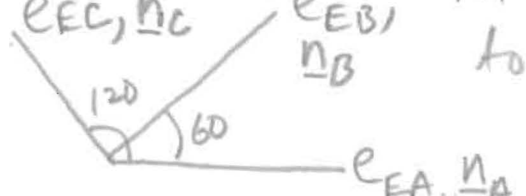
$\simeq \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{2}$

$\simeq \left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow$ max change in angle

$= \frac{\pi}{2} - \theta = 5E\text{-}3$

P3



At pt on free surface, let x_3 be normal to surface,

$$\Rightarrow \underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{e} = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix}$$

$$e_{EA} = e_{ij} n_i^{(A)} n_j^{(A)} = 10^{-3} = e_{11} \to ① \quad (\text{used } \underline{n}^{(A)} = (1\ 0\ 0)^T)$$

$$e_{EB} = e_{ij} n_i^{(B)} n_j^{(B)} = 2*10^{-3} = \frac{1}{4} e_{11} + \frac{\sqrt{3}}{2} e_{12} + \frac{3}{4} e_{22} \to ② \quad \left(\text{used } \underline{n}^{(B)} = \left(\tfrac{1}{2}\ \tfrac{\sqrt{3}}{2}\ 0\right)^T\right)$$

$$e_{EC} = e_{ij} n_i^{(C)} n_j^{(C)} = 1.2*10^{-3} = \frac{1}{4} e_{11} - \frac{\sqrt{3}}{2} e_{12} + \frac{3}{4} e_{22} \to ③ \quad \left(\text{used } \underline{n}^{(C)} = \left(-\tfrac{1}{2}\ \tfrac{\sqrt{3}}{2}\ 0\right)^T\right)$$

$$①,②,③ \longrightarrow e_{11} = 10^{-3}, \quad e_{22} = 1.8*10^{-3}, \quad e_{12} = \frac{0.8}{\sqrt{3}} * 10^{-3}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left[e_{ij} + \frac{\nu}{1-2\nu} e_{kk} \delta_{ij}\right] \longrightarrow \sigma_{33} = 0 = \frac{E}{1+\nu}\left[e_{33} + \frac{\nu}{1-2\nu} e_{kk}\right]$$

$$\Rightarrow e_{33} = -\left(\frac{\cancel{1-2\nu}}{1-\nu}\right)\left(\frac{\nu}{\cancel{1-2\nu}}\right)(e_{11} + e_{22}) = -\frac{0.3}{0.7}(2.8*10^{-3}) = -1.2*10^{-3}$$

$$e = \begin{pmatrix} 1 & 0.8/\sqrt{3} & 0 \\ 0.8/\sqrt{3} & 1.8 & 0 \\ 0 & 0 & -1.2 \end{pmatrix} * 10^{-3} \Rightarrow \left((1-\lambda)(1.8-\lambda) - \frac{0.64}{3}\right)(-1.2-\lambda) = 0$$

λ are p-strains

$$\lambda^2 - 2.8\lambda + \frac{119}{75} = 0, \quad \lambda = \frac{2.8 \pm 1.2220}{2} = (2.0110,\ 0.7890,\ -1.2) * 10^{-3}$$

p-strains
max/min normal strains.

P-stresses: align coord system along p-axes of strain, which is same as p-axes of stress. Then use CL.

$$\sigma_{11} = \frac{2E5}{1.3}\left[2.0110 + \frac{0.3}{0.4}(1.6)\right] * 10^{-3} = 4.94\,E2 = \text{p-stress}(1) = N_{max}$$

$$\sigma_{22} = \frac{2E5}{1.3}\left[0.7890 + \frac{0.3}{0.4}(1.6)\right] * 10^{-3} = 3.06\,E2 = \text{p-stress}(2) = N_{min}$$

$$\sigma_{33} = 0 = \text{p-stress}(3)$$

$$S_{max} = \left|\frac{4.94 - 0}{2}\right| * E2 = 2.47\,E2$$

Align coord system along S_{max} (say x_1') and normal to plane containing S_{max} (say x_2'). Then σ'_{12} = max shear stress component. Hence,

$e'_{12} = \frac{1+\nu}{E}\sigma'_{12} = \frac{1+\nu}{E} S_{max} = 1.6055\ E{-3}$ = max shear strain

= max change in angle between two $\perp^{ar}$ elements at P.

The two elements lie along x_1' & x_2' directions.

Alternately, you can directly use,

max shear strain $= \left|\frac{\lambda(1) - \lambda(3)}{2}\right| = \left|\frac{2.011 + 1.2}{2}\right| \times 10^{-3}$

$= 1.6055E{-3}$

Here the two elements corresponding to max shear strain (ie max change in angle lie at 45° to $\underline{n}(1)$, $\underline{n}(2)$ (p-axes of strain)