CE469 Advanced Solid Mechanics Quiz-2 31/3/12

- Show all working.
- Attempt all parts of a question in a contiguous manner, i.e., dont scatter parts of the same question all over the answerbook.
- **Only one attempt per question will be graded**. So cancel out any attempt you do not want graded. The first not-cancelled attempt will be graded by default.
- **Open notes (in your handwriting only, no photocopies). No solved examples allowed.**

P1. A large plate has a small circular hole of radius a. The plate is loaded by uniform tension q applied parallel to the x- axis and by uniform pressure p applied at the hole, as shown. Determine:

(a) The stresses in the plate

(b) The least value of p for which σ_θ is tensile everywhere along the hole.

P2. The long gravity wall shown has a density ρ and is subject to uniform shear q along the right face. It is restrained at its two end faces (i.e., where z is a constant). Assuming $\sigma_{xy} = f[x]$, determine the stresses in the wall.

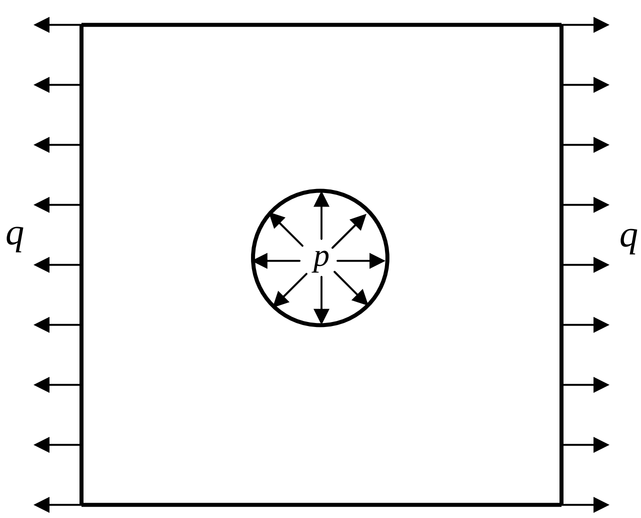


Fig. P1

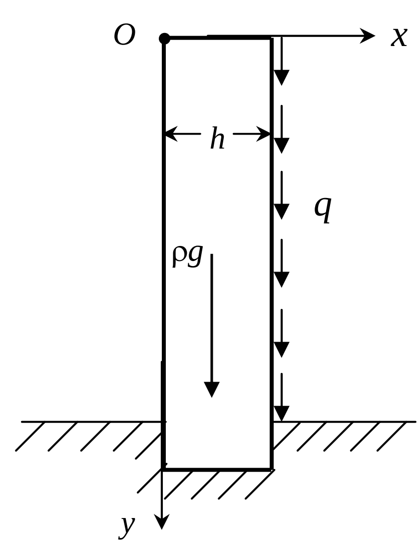
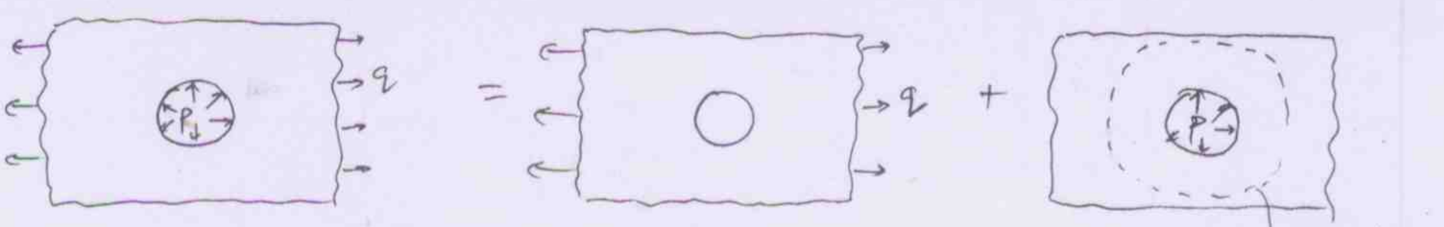


Fig. P2

P. 1

(a)



(I) plate w/hole under uniaxial tension only + (II) Thick walled cyl. with $p_i = p$, $r_0 = \infty$, $p_0 = 0$

(II) → put $p_0 = 0$, $r_0 = \infty$, $p_i = p$ in formulae,

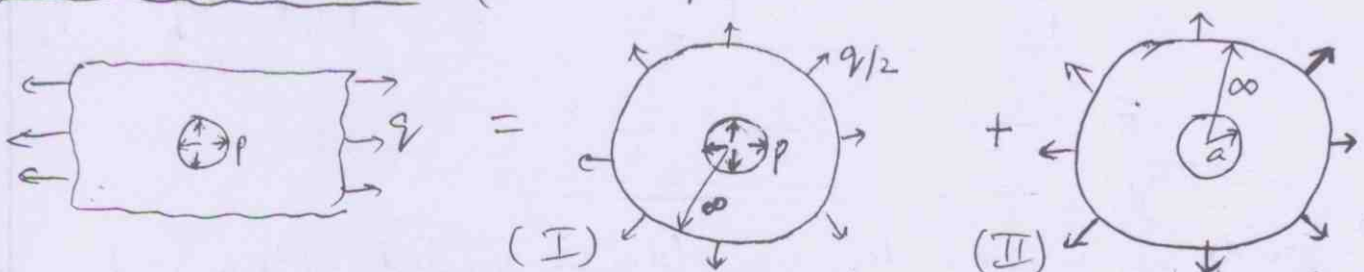
$$\sigma_{r/\theta} = \pm \frac{a^2}{r^2}(-p) \quad , \quad \sigma_{r\theta} = 0.$$

(I) → directly from formulae (class notes ...)

(I) + (II) →

$$\boxed{\begin{aligned} \sigma_r &= \frac{q}{2}\left(1-\frac{a^2}{r^2}\right) - p\frac{a^2}{r^2} + \frac{q}{2}\cos 2\theta\left(1-\frac{a^2}{r^2}\right)\left(1-3\frac{a^2}{r^2}\right) \\ \sigma_\theta &= \frac{q}{2}\left(1+\frac{a^2}{r^2}\right) + p\frac{a^2}{r^2} - \frac{q}{2}\cos 2\theta\left(1+3\frac{a^4}{r^4}\right) \\ \sigma_{r\theta} &= -\frac{q}{2}\sin 2\theta\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right) \end{aligned}} \quad \circledast$$

Alternative method : (but longer).



(I) Thickwalled cyl, $r_0 = \infty$, $p_0 = -\frac{q}{2}$, $p_i = p$.

(II) loading $\begin{cases} \sigma_r\big|_{r=\infty} = \frac{q}{2}\cos 2\theta \\ \sigma_{r\theta}\big|_{r=\infty} = -\frac{q}{2}\sin 2\theta \end{cases}$

So you can add stress functions for these two basic problems as done in class, ie,

$$\phi = \underbrace{\left(A\ln r + Br^2\right)}_{\text{(I)}} + \underbrace{\left(Cr^2 + \frac{D}{r^2} + E\right)}_{\text{(II)}}\cos 2\theta$$

$$\sigma_\theta = \phi_{,\theta\theta} = -\frac{A}{r^2} + 2B + 2C\cos 2\theta + \frac{6D}{r^4}\cos 2\theta$$

$$\sigma_r = \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} = \frac{A}{r^2} + 2B + 2C\cos 2\theta - \frac{2D}{r^4}\cos 2\theta - \frac{4}{r^2}\left(Cr^2 + \frac{D}{r^2} + E\right)\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 2\sin 2\theta\left(C - \frac{3D}{r^4} - \frac{E}{r^2}\right)$$

BC's $\quad \underline{r=a}: \ \sigma_r = -p \Rightarrow \dfrac{A}{a^2} + 2B = -p \longrightarrow (i)$

$$-2C - \frac{6D}{a^4} - \frac{4E}{a^2} = 0 \longrightarrow (ii)$$

$$\sigma_{r\theta} = 0 \Rightarrow C - \frac{3D}{a^4} - \frac{E}{a^2} = 0 \longrightarrow (iii)$$

$\underline{r=\infty}: \ \sigma_r = \dfrac{q}{2}(\cos 2\theta + 1) \Rightarrow -2C = q/2, \quad 2B = q/2 \longrightarrow (iv)$

$\sigma_{r\theta} = -\dfrac{q}{2}\sin 2\theta \Rightarrow 2C = -q/2$ — repeated.

(also if you do $\sigma_{\theta\theta} = \frac{q}{2}(1-\cos 2\theta)$ you get repeat of (iv)).

(i)–(iv) $\rightarrow$. $A = \left(-p - \frac{q}{2}\right)a^2, \ B = \frac{q}{4}, \ C = -\frac{q}{4}, \ D = -\frac{q}{4}a^4, \ E = \frac{q}{2}a^2$

$$\Rightarrow \boxed{\phi = \left(-p - \frac{q}{2}\right)a^2 \ln r + \frac{q}{4}r^2 + \left(-\frac{q}{4}r^2 - \frac{q}{4}\frac{a^4}{r^2} + \frac{q}{2}a^2\right)\cos 2\theta}$$

use this ϕ ~~to~~ get $\sigma_r, \sigma_\theta, \sigma_{r\theta}$ as in (*).

(b) $\sigma_\theta \big|_{r=a} = q + p - 2q\cos 2\theta$

least value of $p = q$ which makes $\sigma_\theta\big|_{r=a} \geq 0$

(i.e, zero for $\theta = 0, \pi$).

$\sigma_{xy} = f_1(x)$ (ie replace given $f(x)$ by $f_1(x)$).

$\Rightarrow -\phi_{,xy} = f_1(x) \rightarrow \phi_{,y} = f(x) + g_1(y) \longrightarrow \phi = y f(x) + g(y) + \cancel{h(x)}$ drop.

$\left(\because \sigma_{yy}\big|_{y=0} = 0 \Rightarrow h''(x) = 0 \Rightarrow h(x)\right.$ will not affect stresses.

$\nabla^4 \phi = 0 \Rightarrow y f^{IV} + g^{IV} = 0$

$\Rightarrow f^{IV} = k, \quad g^{IV} = -ky$

$\Rightarrow f = \frac{kx^4}{24} + Ax^3 + Bx^2 + Cx + \cancel{D}$

$g = -\frac{ky^5}{120} + Ey^4 + Fy^3 + Gy^2 + \cancel{H}y + \cancel{I}$

cancelled terms wont affect stresses so drop them. w/o loss of generality

$\Rightarrow \sigma_{xx} = -\frac{ky^3}{6} + 12Ey^2 + 6Fy + 2G - \rho g y$

$\sigma_{xy} = -\left(\frac{kx^3}{6} + 3Ax^2 + 2Bx + C\right)$

$\sigma_{yy} = y\left(\frac{kx^2}{2} + 6Ax + 2B\right) - \rho g y$

BC's : $y=0$: $\sigma_{yy} = 0 \rightarrow$ i-s.

$\sigma_{xy} = 0 \rightarrow k = A = B = C = 0 \longrightarrow ①$

$x=h$: $\sigma_{xx} = 0 \rightarrow k = E = (6F - \rho g) = G = 0 \longrightarrow ②$

$\sigma_{xy} = q \rightarrow -\left(\frac{kh^3}{6} + 3Ah^2 + 2Bh + C\right) = q \longrightarrow ③$

$x=0$: $\sigma_{xx} = 0 \rightarrow k = E = (6F - \rho g) = G = 0 \longrightarrow ④$

$\sigma_{xy} = 0 \rightarrow C = 0 \longrightarrow ⑤$

From physical considerations (ie complimentarity of σ_{xy} at $(x,y) = (0,h)$) you see that $\sigma_{xy}\big|_{y=0} = 0$ is not satisfiable. Moreover, satisfying this would result in all coeffs being zero, ie $\phi = 0$ & hence $\sigma_{xy}\big|_{x=h} = q$ not being satisfiable. Hence we relax ①, and instead satisfy,

$$\int_0^h \sigma_{xy}\, dx = 0 \Longrightarrow -\left(f(h) - f(0)\right) = 0$$

$$\Rightarrow \frac{kh^4}{24} + Ah^3 + Bh^2 + Ch = 0 \longrightarrow ⑥$$

$② - ⑥ \rightarrow A = -\frac{q}{h^2}, \; B = \frac{q}{h}, \; k = C = E = (6F - \rho g) = G = 0$

(contd on P.4 reverse)

P.2 (contd)

$$\Rightarrow \boxed{\begin{aligned}\sigma_{xx} &= 0\\ \sigma_{yy} &= y\left(-\frac{6q}{h^2}x + \frac{2q}{h}\right) - \rho g y\\ \sigma_{xy} &= -\left(-\frac{3q}{h^2}x^2 + \frac{2q}{h}x\right)\end{aligned}}$$

(4)