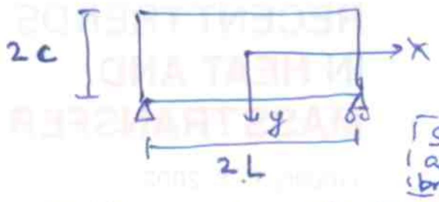
Tutorial 5 solutions

P.1 Beam with self weight only



$\psi = \rho g y$

$\phi = \phi_2$

So this page is just an academic exercise on brute force approach.

NOTE: From elementary solid mechanics, we know that $M(x)$ proportional to x^2, hence σ_{xx} proportional to $x^2 y$. So we guess that plane stress soln would also contain 'cubic' terms. Hence go up to ϕ_5 → see next page

$\sigma_{yy}\big|_{y=\pm c} = 0 \Rightarrow a_2 \mp \rho g c = 0$ → stop here since you cannot satisfy this. ($\because$ it ⇒ that $\rho g c = 0$.)

Take $\phi = \phi_2 + \phi_3$

(A) $\leftarrow \sigma_{yy}\big|_{y=\pm c} = 0 \Rightarrow a_2 \pm b_3 c \mp \rho g c = 0 \rightarrow (1) \rightarrow a_2 = 0,\ b_3 = \rho g$

$a_3 = 0 \rightarrow (2) \rightarrow a_3 = 0$

(B) $\leftarrow \sigma_{xy}\big|_{y=\pm c} = 0 \Rightarrow -b_2 \mp c_3 c = 0 \rightarrow (3) \rightarrow b_2 = c_3 = 0$

$-b_3 = 0 \rightarrow (4) \rightarrow b_3 = 0$ → conflict, even if we relax (D) & satisfy it in an $\int$ sense. → so chose $\phi = \phi_2 + \phi_3 + \phi_4$

(C) $\leftarrow \sigma_{xx}\big|_{x=\pm L} = 0 \Rightarrow c_2 \pm c_3 L = 0 \rightarrow (5) \rightarrow c_2 = c_3 = 0$

$d_3 - \rho g = 0 \rightarrow (6) \rightarrow d_3 = \rho g$

(D) $\leftarrow \sigma_{xy}\big|_{x=\pm L} = 0 \Rightarrow -b_2 \mp b_3 L = 0 \rightarrow (7) \rightarrow b_2 = b_3 = 0$

$c_3 = 0 \rightarrow (8) \rightarrow c_3 = 0$

Take $\phi = \phi_2 + \phi_3 + \phi_4$

(A*) $\leftarrow \sigma_{yy}\big|_{y=\pm c} = 0 \Rightarrow a_2 \pm b_3 c \mp \rho g c + c_4 c^2 = 0 \rightarrow (1^*)\ b_3 = \rho g,\ a_2 = -c_4 c^2$

$a_3 \pm b_4 c = 0 \rightarrow (2^*)\ a_3 = b_4 = 0$

$a_4 = 0 \rightarrow (3^*)\ a_4 = 0$

(B*) $\leftarrow \sigma_{xy}\big|_{y=\pm c} = 0 \Rightarrow -b_2 \mp c_3 c - \frac{d_4}{2} c^2 = 0 \rightarrow (4^*)\ c_3 = 0,\ b_2 = -\frac{d_4}{2} c^2$

$-b_3 \mp 2 c_4 c = 0 \rightarrow (5^*)\ b_3 = c_4 = 0$ → same conflict. so include ϕ_5

$-\frac{b_4}{2} = 0 \rightarrow (6^*)\ b_4 = 0$

(C*) $\leftarrow \sigma_{xx}\big|_{x=\pm L} = 0 \Rightarrow c_2 \pm c_3 L + c_4 L^2 = 0 \rightarrow (7^*)\ c_3 = 0,\ c_2 = -c_4 L^2$

$d_3 - \rho g \pm d_4 L \rightarrow (8^*)\ d_3 = \rho g,\ d_4 = 0$

$-(2 c_4 + a_4) = 0 \rightarrow (9^*)\ a_4 = -2 c_4$

(D*) $\leftarrow \sigma_{xy}\big|_{x=\pm L} = 0 \Rightarrow -b_2 \mp b_3 L - \frac{b_4}{2} L^2 = 0 \rightarrow (10^*)\ b_3 = 0,\ b_2 = -\frac{b_4}{2} L^2$

$-c_3 \mp 2 c_4 L = 0 \rightarrow (11^*)\ c_3 = c_4 = 0$

$-\frac{d_4}{2} = 0 \rightarrow (12^*)\ d_4 = 0$

Take $\phi = \phi_2 + \phi_3 + \phi_4 + \phi_5$ (See note on top of pg. 1. Can directly start from here).

A^{**}: $\sigma_{yy}\big|_{y=\pm c} = 0 \Rightarrow a_2 \pm b_3 c \mp \rho g c + c_4 c^2 \pm \frac{d_5}{3} c^3 = 0 \to 1^{**}$ w/ 15^{**} $\to b_3 = \rho g$, w/ 14^{**}, $a_2 = 0$

$a_3 \pm b_4 c + c_5 c^2 = 0 \to 2^{**}$ $\to b_4 = 0$, w/ 11^{**} $\to a_3 = 0$

$a_4 \pm b_5 c = 0 \to 3^{**}$ $\to a_4 = b_5 = 0$

$a_5 = 0 \to 4^{**}$ $\to a_5 = 0$

B^{**}: $\sigma_{xy}\big|_{y=\pm c} = 0 \Rightarrow -b_2 \mp c_3 c - \frac{d_4}{2} c^2 \pm \frac{1}{3}(2c_5 + 3a_5) c^3 = 0 \to 5^{**}$ **i.s.**

$-b_3 \mp 2c_4 c - d_5 c^2 = 0 \to 6^{**}$ i.s

$-\frac{b_4}{2} \mp c_5 c = 0 \to 7^{**}$ $\to b_4 = c_5 = 0$

$-\frac{1}{3} b_5 = 0 \to 8^{**}$ $\to b_5 = 0$

C^{**}: $\sigma_{xx}\big|_{x=\pm L} = 0 \Rightarrow c_2 \pm c_3 L + c_4 L^2 \pm \frac{c_5}{3} L^3 = 0 \to 9^{**}$ w/ 11^{**}, 14^{**} $\to c_2 = 0$

$d_3 - \rho g \pm d_4 L + d_5 L^2 = 0 \to 10^{**}$ w/ 15^{**} $\to d_3 = \rho g$

$-(2c_4 + a_4) \mp (2c_5 + 3a_5) L = 0 \to 11^{**}$ w/ 3^{**}, 4^{**}, $c_4 = c_5 = 0$

$-\frac{1}{3}(b_5 + 2d_5) = 0 \to 12^{**}$ w/ 3^{**} $\to d_5 = 0$

D^{**}: $\sigma_{xy}\big|_{x=\pm L} = 0 \Rightarrow -b_2 \mp b_3 L - \frac{b_4}{2} L^2 \mp \frac{1}{3} b_5 L^3 = 0 \to 13^{**}$ w/ $2^{**}, 3^{**}$ $\to b_2 = b_3 = 0$

$-c_3 \mp 2c_4 L - c_5 L^2 = 0 \to 14^{**}$ w/ 7^{**} $\to c_3 = c_4 = 0$

$-\frac{d_4}{2} \mp d_5 L = 0 \to 15^{**}$ $\to d_4 = d_5 = 0$

$\frac{1}{3}(2c_5 + 3a_5) = 0 \to 16^{**}$ i.s. (identically satisfied)

Soln of 1^{**} – 16^{**} gives all coeffs zero except $d_3 = \rho g$, $b_3 = \rho g$. However $b_3 = 0$ is obtained as a conflicting solution from 13^{**} arising from D^{**}. So we relax C^{**} & D^{**} (ie, discard 9^{**} – 16^{**}) and satisfy them in an $\int$ sense as follows:

Note: $$\sigma_{xx}\big|_{x=\pm L} = c_2 \pm c_3 L + c_4 L^2 \pm \frac{c_5}{3} L^3 + y(d_3 \pm d_4 L + d_5 L^2) + y^2\left(-(2c_4 + a_4) \mp (2c_5 + 3a_5) L\right) - \frac{1}{3}(b_5 + 2d_5) y^3 - \rho g y$$

$$\sigma_{xy}\big|_{x=\pm L} = -b_2 \mp b_3 L - \frac{b_4}{2} L^2 \mp \frac{b_5}{3} L^3 + y(-c_3 \mp 2c_4 L - c_5 L^2) + y^2\left(-\frac{d_4}{2} \mp d_5 L\right) + \frac{1}{3}(2c_5 + 3a_5) y^3$$

$$\int_{-c}^{c} \sigma_{xx}\big|_{x=\pm L} dy = 0 \Rightarrow 2c(c_2 + c_4 L^2) + \frac{2c^3}{3}(-2c_4 - a_4) = 0 \to 17^{**}$$

$$2c\left(c_3 L + \frac{c_5}{3} L^3\right) - \frac{2c^3}{3}(2c_5 + 3a_5) L = 0 \to 18^{**}$$

$$\int_{-c}^{c} \sigma_{xx}\big|_{x=\pm L} y\, dy = 0 \Rightarrow \frac{2c^3}{3}(d_3 + d_5 L^2) - \frac{2c^5}{15}(b_5 + 2d_5) - \frac{2c^3}{3} \rho g = 0 \to 19^{**}$$

$$\frac{2c^3}{3} d_4 L = 0 \to 20^{**}$$

$$\int_{-c}^{c} \sigma_{xy}\Big|_{x=\pm L} dy = \mp\frac{4cL\rho g}{2} \Rightarrow 2c\left(-b_2 - \frac{b_4}{2}L^2\right) + \frac{2c^3}{3}\left(-\frac{d_4}{2}\right) = 0 \longrightarrow 21^{**}$$

$$2c\left(b_3 L + \frac{b_5}{3}L^3\right) + \frac{2c^3}{3} d_5 L = \frac{4cL\rho g}{2} \checkmark \longrightarrow 22^{**}$$

Solution: (of $1^{**}-8^{**}$, $17^{**}-22^{**}$)

$1^{**} \longrightarrow a_2 + c_4 c^2 = 0$, $b_3 - \rho g + \frac{d_5}{3}c^2 = 0 \to 1^{**}_{(a)} \to$ w/$6^{**} \to a_2 = 0$

$2^{**} \longrightarrow a_3 + c_5 c^2 = 0$, $b_4 = 0 \to$ w/$7^{**} \to a_3 = 0$

$3^{**} \longrightarrow a_4 = b_5 = 0$.

$4^{**} \longrightarrow a_5 = 0$

$5^{**} \longrightarrow b_2 + \frac{d_4}{2}c^2 = 0$, $c_3 c - \frac{1}{3}(2c_5 + 3a_5)c^3 = 0 \to$ i.s. $\to$ w/$20^{**} \to b_2 = 0$

$6^{**} \longrightarrow c_4 = 0$, $b_3 + d_5 c^2 = 0 \to 6^{**}_{(a)}$

$7^{**} \longrightarrow b_4 = c_5 = 0$

$8^{**} \longrightarrow b_5 = 0$

$17^{**}, 3^{**}, 6^{**} \longrightarrow c_2 = 0$

$4^{**}, 7^{**}, 18^{**} \longrightarrow c_3 = 0$

$8^{**}, 19^{**} \longrightarrow d_3 = d_5\left(\frac{2c^2}{5} - L^2\right) + \rho g \to 19^{**}(a)$

$20^{**} \longrightarrow d_4 = 0$

$2^{**}, 20^{**}, 21^{**} \longrightarrow b_2 = 0$

$22^{**}, 8^{**} \longrightarrow$ yields same eqn as 1^{**}, ie 22^{**} identically satisfied using 8^{**} & 1^{**}

$$\Rightarrow a_4 = a_5 = a_2 = a_3 = b_4 = b_5 = b_2 = c_4 = c_5 = c_2 = c_3 = d_4 = 0$$

Also $1^{**}_{(a)}, 6^{**}_{(a)} \longrightarrow d_5 = -\frac{3}{2}\frac{\rho g}{c^2}$ ✓

Then $19^{**}(a) \longrightarrow d_3 = -\frac{3}{2}\rho g\left(\frac{2}{5} - \frac{L^2}{c^2}\right) + \rho g$ ✓

and $6^{**}(a) \longrightarrow b_3 = +\frac{3}{2}\rho g$ ✓

Hence, $\phi = \frac{3}{2}\rho g\frac{x^2 y}{2} + \left(\rho g - \frac{3}{2}\rho g\left(\frac{2}{5} - \frac{L^2}{c^2}\right)\right)\frac{y^3}{6} - \frac{3}{2}\frac{\rho g}{c^2}\frac{y^3 x^2}{6}$

matches Timoshenko Goodier p. 48 footnote.

stress function

$$\sigma_{xx} = \phi_{,yy} = \rho g - \frac{3}{2}\rho g\left(\frac{2}{5} - \frac{L^2}{c^2}\right)y - \frac{3}{2}\frac{\rho g}{c^2}x^2 y$$

$$\sigma_{yy} = \phi_{,xx} = \frac{3}{2}\rho g y - \frac{\rho g}{2c^2}\frac{y^3}{6}$$

$$\sigma_{xy} = -\phi_{,xy} = \frac{3}{2}\rho g x - \frac{3}{2}\frac{\rho g}{c^2}y^2 x$$

← stresses.

P. 2 $\sigma_y = x f(y)$ given.

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = x f(y)$$

$$\frac{\partial\phi}{\partial x} = \frac{x^2}{2} f(y) + g(y)$$

$$\phi = \frac{x^3}{6} f(y) + x\, g(y) + h(y) \quad \text{—— (A)}$$

Compatibility $\Rightarrow \nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2\psi$ (plane strain)

$= 0$ (constant body forces) $\rightarrow$ (B)

(A,B) $\Rightarrow 2x f'' + \frac{x^3}{6} f^{IV} + x g^{IV} + h^{IV} = 0$

$\Rightarrow$ (seperating powers of x) $h^{IV} = 0 \rightarrow h = Ey^3 + Fy^2 + \cancel{Py} + \cancel{Q}$

$f^{IV} = 0 \rightarrow f = Ay^3 + By^2 + Cy + D$

$2f'' + g^{IV} = 0 \rightarrow g^{IV} = -2(6Ay + 2B)$

$g = -12\frac{Ay^5}{120} - \frac{4By^4}{24} + Gy^3 + Hy^2 + Iy + \cancel{J}$

The scored out terms are those which dont affect stresses (ie they vanish when doing $\phi_{,xx}$, $\phi_{,yy}$ or $\phi_{,xy}$). (see eq (1))

$$\Rightarrow \phi = \frac{x^3}{6}(Ay^3 + By^2 + Cy + D) + x\left(-\frac{Ay^5}{10} - \frac{By^4}{6} + Gy^3 + Hy^2 + Iy\right) + Ey^3 + Fy^2$$

(C) $\leftarrow$

$$\begin{cases} \sigma_{xx} = \phi_{,yy} = Ax^3y + \frac{B}{2}x^3 - 2Axy^3 - 2Bxy^2 + 6Gxy + 2Hx + 6Ey + 2F - \rho g x \\ \sigma_{yy} = \phi_{,xx} = x f(y) \\ \sigma_{xy} = -\frac{x^2}{2}(3Ay^2 + 2By + C) - \left(-\frac{A}{2}y^4 - \frac{2}{3}By^3 + 3Gy^2 + 2Hy + I\right) \end{cases}$$

BC's : $\sigma_{yy}\big|_{y=-h/2} = 0$:

$$x\left(-\frac{Ah^3}{8} + \frac{Bh^2}{4} - C\frac{h}{2} + D\right) = 0 \longrightarrow (1)$$

(P.2 contd.)

$\sigma_{yy}\big|_{y=h/2} = -\rho g x$:

$$x\left(\frac{Ah^3}{8} + \frac{Bh^2}{4} + \frac{Ch}{2} + D\right) = -\rho g x \longrightarrow (2)$$

$\sigma_{xy}\big|_{y=\pm\frac{h}{2}} = 0$:

$$3A\frac{h^2}{4} + C = 0 \longrightarrow (3)$$

$$-B\frac{h}{2} = 0 \longrightarrow (4)$$

$$-\frac{A}{2}\frac{h^4}{16} + 3G\frac{h^2}{4} + I = 0 \longrightarrow (5)$$

$$-\frac{2}{3}B\frac{h^3}{8} + 2H\frac{h}{2} = 0 \longrightarrow (6)$$

obtained by separating out powers x^2 & x^0 for each of $y = \pm\frac{h}{2}$ evaluations.

$\sigma_{xx}\big|_{x=0} = 0$:

$$6E = 0 \longrightarrow (7)$$

$$2F = 0 \longrightarrow (8)$$

obt by separating powers y, y^0.

$\sigma_{xy}\big|_{x=0} = 0$: $A = B = G = H = I = 0 \longrightarrow (*)$ (obt by sep powers $y\ y'$).

→ This bc cannot be satisfied exactly. Doing so implies contradictions in (1), (2), (3), ie they cannot be simultaneously satisfied, ie satisfying (*) ⇒ (4)–(8) identically satisfied & further if (1) & (3) are satisfied then (2) is violated — which cannot be permitted ∴ height of wall is assumed much larger than thickness, so if bc on the (height) vertical surface is relaxed then St. Venant's principle implies that solution is worthless throughout. So we relax bc on thickness direction surface, ie along $x=0$ surface (top). Relaxing (*) we replace by,

$$\int_{-h/2}^{h/2} \sigma_{xy}\big|_{x=0}\, dy = 0 \ : \ -\frac{Ah^5}{160} + 6G\frac{h^3}{24} + Ih = 0 \longrightarrow (9)$$

Solution of (1) – (9) :

$$B = H = E = F = 0,\quad D = -\frac{\rho g}{2},\quad C = -\frac{3\rho g}{2h}$$

$$A = \frac{2\rho g}{h^3},\quad G = \frac{\rho g}{10h},\quad I = -\frac{\rho g h}{80}$$

(pg 6) (6)

Stresses are: Use the solution of A − I and eqn (C):

$$\sigma_{xx} = 2\rho g \frac{x^3}{h^3} y - 4\rho g \frac{x y^3}{h^3} + \frac{6}{10}\rho g \frac{xy}{h} - \rho g x$$

$$\sigma_{yy} = x\left[2\rho g \frac{y^3}{h^3} - \frac{3}{2}\rho g \frac{y}{h} - \frac{\rho g}{2}\right]$$

$$\tau_{xy} = -\frac{x^2}{2}\left(6\rho g \frac{y^2}{h^3} - \frac{3}{2}\rho g \frac{1}{h}\right) - \left(-\rho g \frac{y^4}{h^3} + \frac{3}{10}\rho g \frac{y^2}{h} - \frac{\rho g h}{80}\right)$$

P.3 For fig 3(a) given solution is converted to (x, y) system

Thus

$$\sigma_{xx} = \sigma_r \cos^2\theta = \frac{-2P x^3}{\pi(x^2+y^2)^2}$$

$$\sigma_{yy} = \sigma_r \sin^2\theta = -\frac{2P\cos\theta}{\pi r}\sin^2\theta = -\frac{2P xy^2}{\pi(x^2+y^2)^2}$$

$$\tau_{xy} = \sigma_r \sin\theta\cos\theta = -\frac{2P}{\pi r}\sin\theta\cos^2\theta = -\frac{2P x^2 y}{\pi(x^2+y^2)^2}$$

Stresses at A are obtained by superposition.

At A, $x = a\cot\alpha$, $y = 0$

$$\sigma_{xx} = \left.\frac{2P(a\cot\alpha)^3}{\pi(y^2+a^2\cot^2\alpha)^2} - \frac{4P(a\cot\alpha)^3}{\pi(a^2\cot^2\alpha+(y+a)^2)^2}\right|_{y=0}$$

$$= \frac{2P}{\pi a}\tan\alpha - \frac{4P}{\pi a}\cos^3\alpha\sin\alpha$$

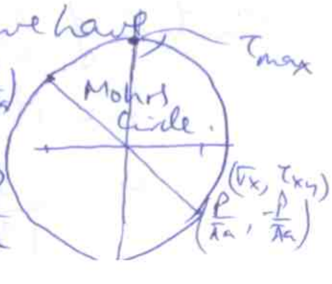
$$\sigma_{yy} = \left.\frac{2P a\cot\alpha\,(y^2)}{\pi(a^2\cot^2\alpha+y^2)^2} - \frac{4P a\cot\alpha\,(y+a)^2}{\pi(a^2\cot^2\alpha+(y+a)^2)^2}\right|_{y=0} = \frac{4P\sin^3\alpha\cos\alpha}{\pi a}$$

$$\tau_{xy} = \left.\frac{2P a^2\cot^2\alpha\,(y)}{\pi(a^2\cot^2\alpha+y^2)^2} - \frac{4P a^2\cot^2\alpha\,(y+a)}{\pi(a^2\cot^2\alpha+(y+a)^2)^2}\right|_{y=0} = \frac{-4P}{\pi a}\sin^2\alpha\cos^2\alpha$$

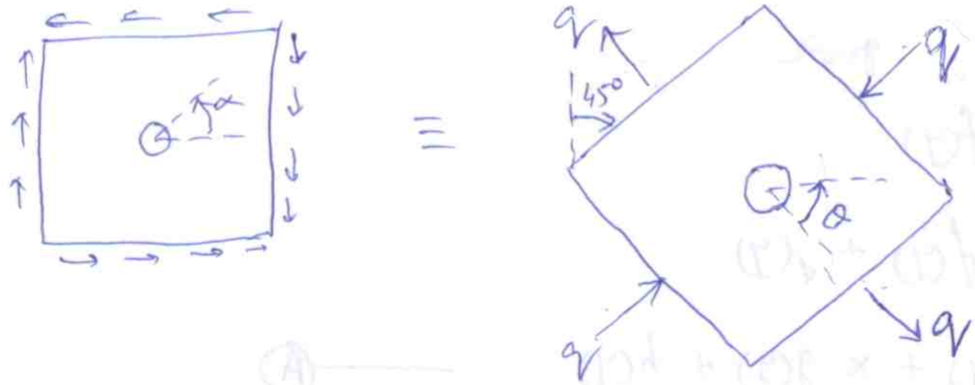
Max shear stress at A: for given $\alpha = 45^\circ$, $\cot\alpha = 1$. we have

$$\sigma_{xx} = \frac{P}{\pi a}, \quad \sigma_{yy} = \frac{-P}{\pi a}, \quad \tau_{xy} = \frac{-P}{\pi a}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{P}{\pi a}\right)^2 + \left(\frac{P}{\pi a}\right)^2} = \frac{\sqrt{2}P}{\pi a}$$



P4. This is a case of pure shear load.



For tensile load only along $\theta = 0°$.

Class solution:

$$\sigma_\theta = \frac{q}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{q}{2}\cos 2\theta\left(1+\frac{3a^4}{r^4}\right)$$

Dont need σ_r, $\sigma_{r\theta}$ since they vanish at the hole. At the hole,

$$\sigma_\theta = q - 2q\cos 2\theta$$

Superposing,

$$\sigma_\theta\Big|_{r=a} = q - 2q\cos 2\theta + \left(-q + 2q\cos 2\left(\theta - \frac{\pi}{2}\right)\right)$$

$$= -2q\cos 2\theta - 2q\cos 2\theta = -4q\cos 2\theta$$

In terms of original system, put $\alpha = \theta - \frac{\pi}{4}$

$$\Rightarrow \sigma_\theta\Big|_{r=a} = -4q\cos\left(2\alpha + \frac{\pi}{2}\right) = 4q\sin 2\alpha.$$

$$\left.\begin{array}{l}(\sigma_\theta)_{max} = 4q \text{ for } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \quad (\sigma_\theta)_{min} = -4q \text{ for } \theta = 0, \pi, \\ \sigma_\theta = 0 \text{ for } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\end{array}\right\}$$

P.5 $\phi = r^2 f(\theta)$

$$0 = \nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\underbrace{\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)}_{=(2f+2f+f'')}$$

$$= \cancel{\frac{1}{r^2}}\left(4f'' + f^{IV}\right) = 0.$$

put e^{st}, $\Rightarrow 4s^2 + s^4 = 0 \Rightarrow s = 0, 0, \pm 2i$

$$\Rightarrow f = \underbrace{A^* e^{2i\theta} + B^* e^{-2i\theta}}_{A\cos 2\theta + B\sin 2\theta} + C\theta + D$$

$$= A\cos 2\theta + B\sin 2\theta + C\theta + D.$$

$$\phi = r^2 f = r^2\left(A\cos 2\theta + B\sin 2\theta + C\theta + D\right).$$

$$\Rightarrow \sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -2A\cos 2\theta - 2B\sin 2\theta + 2C\theta + 2D$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = 2A\cos 2\theta + 2B\sin 2\theta + 2C\theta + 2D$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 2A\sin 2\theta - 2B\cos 2\theta - C$$

BC's are: $\sigma_\theta\Big|_{\theta=\pm\frac{\alpha}{2}} = 0$, $\sigma_{r\theta}\Big|_{\theta=\frac{\alpha}{2}} = q$, $\sigma_{r\theta}\Big|_{\theta=-\frac{\alpha}{2}} = -q$

$$\Rightarrow \quad A\cos\alpha \pm B\sin\alpha \pm C\frac{\alpha}{2} + 2D = 0 \longrightarrow (1,2)$$

$$A\sin\alpha - B\cos\alpha - \frac{C}{2} = q \longrightarrow (3)$$

$$-A\sin\alpha - B\cos\alpha - \frac{C}{2} = -q \longrightarrow (4)$$

$$(1-4) \Rightarrow A = \frac{q}{\sin\alpha}, \quad D = -\frac{q}{2}\cot\alpha, \quad B = 0, \quad C = 0$$

$$\Rightarrow \quad \sigma_r = -\frac{2q}{\sin\alpha}\cos 2\theta - q\cot\alpha.$$

$$\sigma_\theta = \frac{2q}{\sin\alpha}\cos 2\theta - q\cot\alpha$$

$$\sigma_{r\theta} = \frac{2q}{\sin\alpha}\sin 2\theta$$