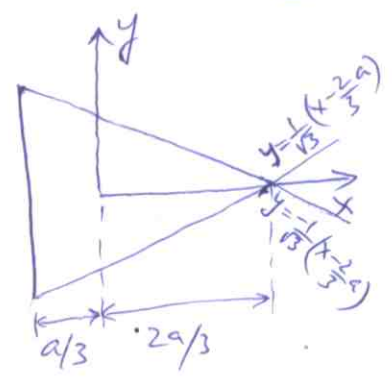P-1 $\phi = m\left[x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right]$

(a) $\nabla^2\phi = -4am = -2G\alpha$

$\Rightarrow m = \dfrac{G\alpha}{2a}$ ◄ ——————→ ①

(b) $\phi = \dfrac{G\alpha}{2a}\left[x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right]$

$\tau_{xz} = \dfrac{\partial\phi}{\partial y} = -\dfrac{G\alpha y}{a}(3x+a)$ ——————→ ②

$\tau_{yz} = -\dfrac{\partial\phi}{\partial x} = -\dfrac{G\alpha}{2a}\left[3x^2 - 2ax - 3y^2\right]$ ——→ ③

$\boxed{\tau_{xy} = \tau_{yz} = 0 \text{ at corners}}$ ◄ (can see directly from fact that const $\phi = \hat{\phi}$ line, so $\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$ = tangential to boundary — ie $\phi = \hat{\phi}$ line, so $\tau = 0$ at corners since otherwise τ would have two directions at the corners $\Rightarrow \tau_{xz} = \tau_{yz} = 0$). Can also get by substituting coordinates of corners into ② & ③.

$\boxed{\tau_{xy} = \tau_{yz} = 0 \text{ at centroid}}$ $(x=y=0)$.

(c) In order to find the maximum, it suffices to consider the side $x = -a/3$ (since all sides are symmetrically located wrt centroid). Put $x = -a/3$,

$\tau_{xz} = 0$, $\tau_{yz} = -\dfrac{G\alpha}{2a}\left[3\left(\frac{a^2}{9}\right) + \frac{2}{3}a^2 - 3y^2\right]$

max magnitude obviously occurs at $y = 0$

$\Rightarrow |\tau|_{max} = \dfrac{G\alpha a}{2}$ at $x = -\dfrac{a}{3}, \; y = 0$

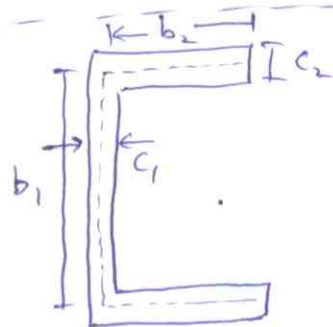Max mag, $|\tau|_{max}$, will also occur at the middle of the other two sides.

(d) $M = \dfrac{2G\alpha}{a}\displaystyle\int_{-a/3}^{2a/3}\int_{0}^{1/\sqrt{3}(x-\frac{2}{3}a)}\left(x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right)\,dy\,dx$

$= \dfrac{2G\alpha}{a\sqrt{3}}\displaystyle\int_{-a/3}^{2a/3}\left[\left(x^3 - ax^2 + \frac{4}{27}a^3\right) - \frac{1}{3}\left(x^3 - ax^2 + \frac{4}{27}a^3\right)\right]\left(x - \frac{2}{3}a\right)dx$

(diagram, top right)

$y = \frac{1}{\sqrt{3}}(x - \frac{2}{3}a)$

$y = -\frac{1}{\sqrt{3}}(x - \frac{2}{3}a)$

$a/3$ $2a/3$

$$= \frac{4G\alpha}{3a\sqrt{3}} \int_{-a/3}^{2a/3} \left(x^4 - \frac{5}{3}ax^3 + \frac{2}{3}a^2x^2 + \frac{4}{27}a^3x - \frac{8}{81}a^4 \right) dx$$

$$= \frac{4G\alpha a^4}{\sqrt{3}} \left(\frac{81}{4860} \right) = \frac{G\alpha a^4}{15\sqrt{3}}$$

$$C = \frac{M}{\alpha} = \frac{Ga^4}{15\sqrt{3}} \quad \blacktriangleleft$$
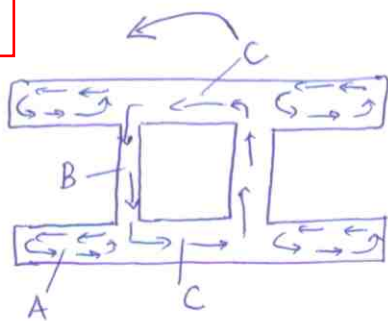

P.2 $\quad \alpha \approx \dfrac{3M}{(b_1 c_1^3 + 2b_2 c_2^3)G} = \dfrac{3 \times 600}{(0.191 \times 0.005^3 + 2 \times 0.0975 \times 0.009^3) \times 77.5 \times 10^6}$

$$= 139.9 \ rad/m$$

$\tau_{max} = \dfrac{3M(b_i)_{max}}{\sum a_i b_i^3} \qquad = \dfrac{3 \times 600 \times 0.009}{(\quad)} = 97.57 \ MPa$

Note: this should have α actually
been GPa, the $\alpha = 0.1399$
(more realistic).

Since the four limbs are identical they contribute equal torsional moments

Let M_L = torsional moments in all 4 limbs.

M_B = torsional moment in closed loop (box).

Note: $68950\,kPa = 703.6\,kgf/cm^2$

Case I: If τ_{max} occurs in limbs,

$$M_L = \frac{\overbrace{\tau_c}^{\tau_{max}} \sum a_i b_i^3}{3 b_i} = \frac{703.6 * (4 * 25.4 * 0.63^3)}{3 * 0.63} = 9455.70 \; kgf\cdot cm$$

$$\alpha_1 = \frac{3 M_L}{G \sum a_i b_i^3} = \frac{3 * 9456}{G * 25.4} = \frac{1116.85}{G}$$

Case II : If τ_{max} occurs in box, then it must occur in lower & upper horizontal legs since 't' (thickness) is smaller there.

$$\alpha_2 = \frac{a\,q}{2GA} = \frac{2 * \overbrace{\left(\frac{50.8}{0.63} + \frac{25.4}{1.27}\right)}^{a} \overbrace{(703.6 * 0.63)}^{q}}{2 * G * 50.8 * 25.4} = \frac{34.57}{G}$$

$\therefore \alpha_2 < \alpha_1 \Rightarrow \tau_{max}$ occurs as per Case II, ie in part C.

$$\tau_c = 703.6 \; kgf/cm^2 = 69023 \; kPa \; \blacktriangleleft$$

$$\tau_B = \frac{703.6}{2} \quad (\because B \text{ has twice thickness of } C) = 351.8 \; kgf/cm^2 = 34511 \; kPa \; \blacktriangleleft$$

$$\tau_A = \frac{3 M_i}{a_i b_i^2} = \frac{\alpha a_i b_i^3 G}{a_i b_i^2} = 34.57 * b_i = 21.7 \; kgf/cm^2 = 2129 \; kPa \; \blacktriangleleft$$

$$M = M_L + M_B = \frac{\alpha G}{3} \sum a_i b_i^3 + 2qA$$

$$= \frac{34.57}{3} * (4 * 25.4 * 0.63^3) + 2 * (703.6 * 0.63) * (50.8 * 25.4)$$

$$= 1.144207 * 10^6 \; kgf\cdot cm. = 112.247 \; kN\cdot m$$

$\blacktriangleright$

⑤

$$M = 2q_1 A_1 + 2q_2 A_2 + 2q_3 A_3 \longrightarrow ①$$

$$A_1 = \frac{\pi}{2}(0.127)^2 = 0.02533 \quad , \quad A_2 = A_3 = (0.254)^2 = 0.064516$$

$$a_1 = \frac{\pi(0.127)}{0.0006} + \frac{0.254}{0.0013} = 860.35 \quad , \quad a_{12} = \frac{0.254}{0.0013} = 195.38$$

$$a_2 = \frac{3*0.254}{0.0008} + \frac{0.254}{0.0013} = 1147.88 \quad , \quad a_{23} = \frac{0.254}{0.0008} = 317.5$$

$$a_3 = \frac{3*0.254}{0.0008} + \frac{0.254}{0.0010} = 1206.5$$
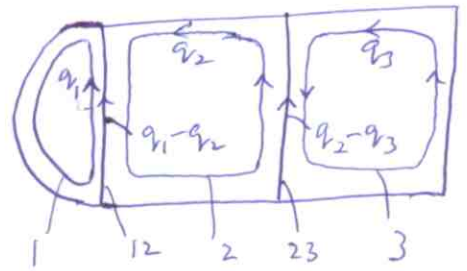
$a = \int \frac{ds}{t}$ for loop or leg as indicated.

form
$2G\alpha = \frac{1}{A_i} \oint \frac{q ds}{t}$
c_i
i^{th} closed contour.

$$\begin{cases} 2G\alpha = \frac{1}{A_1}\left[a_1 q_1 - a_{12} q_2\right] \longrightarrow ② \\[2mm] 2G\alpha = \frac{1}{A_2}\left[a_2 q_2 - a_{12}q_1 - a_{23}q_3\right] \longrightarrow ③ \\[2mm] 2G\alpha = \frac{1}{A_3}\left[a_3 q_3 - a_{23}q_2\right] \longrightarrow ④ \end{cases}$$

$$(2,3) \longrightarrow \underbrace{2G\alpha\left(A_1 a_2 + A_2 a_{12}\right)}_{K_{1\alpha}=41.68} = q_1\underbrace{\left(a_1 a_2 - a_{12}^2\right)}_{K_{11}=949413.19} + q_3\underbrace{\left(-a_{12}a_{23}\right)}_{K_{13}=-62034.61} \longrightarrow ⒤$$

$$(3,4) \longrightarrow \underbrace{2G\alpha\left(A_2 a_{23} + A_3 a_2\right)}_{K_{2\alpha}=94.54} = q_1\underbrace{\left(-a_{12}a_{23}\right)}_{K_{21}=-62034.61} + q_3\underbrace{\left(a_3 a_2 - a_{23}^2\right)}_{K_{23}=1284116.54} \longrightarrow ⒥$$

$$(⒤,⒥) \longrightarrow q_1 = 2G\alpha\underbrace{\left(\frac{K_{1\alpha}K_{23} - K_{2\alpha}K_{13}}{K_{11}K_{23} - K_{21}K_{13}}\right)}_{=C_1} = 2G\alpha\, C_1 \qquad \underset{4.887353*10^{-5}}{\downarrow}$$

$$q_3 = 2G\alpha\underbrace{\left(\frac{K_{1\alpha}K_{21} - K_{2\alpha}K_{11}}{K_{13}K_{21} - K_{23}K_{11}}\right)}_{=C_3} = 2G\alpha\, C_3 \qquad \underset{7.598423*10^{-5}}{\downarrow}$$

$$q_2 = 2G\alpha\underbrace{\left(\frac{-A_3 + a_3 C_3}{a_{23}}\right)}_{=C_2} = 2G\alpha\, C_2 \qquad \underset{8.554007*10^{-5}}{\downarrow}$$

$$M = 113000 = 4G\alpha\left(C_1 A_1 + C_2 A_2 + C_3 A_3\right)$$

$$\Rightarrow G\alpha = 2422993697$$

$$\Rightarrow q_1 = 236840.5, \quad q_2 = 414526.1, \quad q_3 = 368218.6$$

$$\tau_1 = \frac{q_1}{0.0006} = 394734*10^3\,N/m^2, \quad \tau_2 = \frac{q_2}{0.0008} = 518157*10^3\,N/m^2, \quad \tau_3 = \frac{q_3}{0.0008} = 460273*10^3\,N/m^2$$

$$\tau_4 = \frac{q_1 - q_2}{0.0013} = -136681*10^3\,N/m^2, \quad \tau_5 = \frac{q_2 - q_3}{0.0008} = 57884*10^3\,N/m^2, \quad \tau_6 = \frac{q_3}{0.0010} = 368218*10^3\,N/m^2$$
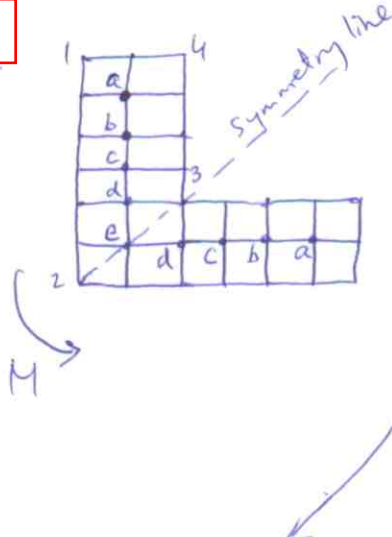
$$\nabla^2 \phi = -2G\alpha \quad \text{--- Central difference}$$

$$\phi_b - 4\phi_a = -2G\alpha h^2 \quad \longrightarrow ①$$

$$\phi_a + \phi_c - 4\phi_b = -2G\alpha h^2 \quad \longrightarrow ②$$

$$\phi_b + \phi_d - 4\phi_c = -2G\alpha h^2 \quad \longrightarrow ③$$

$$\phi_e + \phi_c - 4\phi_d = -2G\alpha h^2 \quad \longrightarrow ④$$

$$2\phi_d - 4\phi_e = -2G\alpha h^2 \quad \longrightarrow ⑤$$

We have used Central diff approx of this, ie,

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -2G\alpha h^2 \quad \text{to get ①-⑤.}$$

$$M = 2\int_A \phi \, dx \, dy \quad \longrightarrow \text{use Simpson's formula, ie,}$$

$$\int_{x_i-h}^{x_i+h}\int_{y_j-h}^{y_j+h} \phi \, dx \, dy = \frac{h^2}{9}\left[16\phi_{i,j} + 4\left(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j-1} + \phi_{i,j+1}\right) + \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\right]$$

and add integrals over all such 4-square areas.

$$M = \frac{h^2}{9}\left[\underbrace{32\phi_a + 8\phi_b + 32\phi_c + 8\phi_b + 8\phi_d + 16\phi_e + 8\phi_d}_{\text{includes contribution of symmetric counterpart of 1-2-3-4}}\right]$$

$$= \frac{16h^2}{9}\left[2\phi_a + \phi_b + 2\phi_c + \phi_d + \phi_e\right] \quad \longrightarrow ⑥$$

$(1,2) \rightarrow \phi_c - 15\phi_a = -10G\alpha h^2 \quad \longrightarrow (A)$

$(2,3,A) \rightarrow \phi_a + \phi_c + 4\phi_d - 16\phi_c = -10G\alpha h^2$

$\Rightarrow \phi_a + 4\phi_d + 15\left(10G\alpha h^2 - 15\phi_a\right) = -10G\alpha h^2$

$\Rightarrow -224\phi_a + 4\phi_d = -160G\alpha h^2 \longrightarrow (B)$

$(3,4,1,A) \rightarrow 4\phi_b - 16\phi_c + \phi_e + \phi_c = -10G\alpha h^2$

$\Rightarrow 4\left(4\phi_a - 2G\alpha h^2\right) + 15\left(10G\alpha h^2 - 15\phi_a\right) + \phi_e = -10G\alpha h^2$

$\Rightarrow -209\phi_a + \phi_e = -152 G\alpha h^2 \quad \longrightarrow (C)$

$(4,5,A) \rightarrow \phi_e + \phi_c - 8\phi_e = -6G\alpha h^2$

$ 15\phi_a - 10G\alpha h^2 - 7\phi_e = -6G\alpha h^2$

$ 15\phi_a - 7\phi_e = 4 G\alpha h^2 \longrightarrow (D)$

$$(C,D) \rightarrow (-7*209+15)^{-1} * (-7*152+4) G\alpha h^2 = \phi_a \qquad \text{⑦}$$

$$\phi_a = 0.732044 \ G\alpha h^2$$

$$\phi_b = 0.928177 \ G\alpha h^2$$

$$\phi_c = 0.980663 \ G\alpha h^2$$

$$\phi_d = 0.994475 \ G\alpha h^2$$

$$\phi_e = 0.997238 \ G\alpha h^2$$

$$\alpha = 0.0886482 \ M/G h^4 \quad \blacktriangleleft$$